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Where Does it All End? Boundaries Beyond Euclidean Space

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WHERE DOES IT ALL END? BOUNDARIES BEYOND EUCLIDEAN SPACE

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Preface:

Euclidean space, named for the ancient Greek geometer Euclid, is in some sense the home of mathematics. Mathematicians have been studying the structure and properties of this place for over two thousand years, so they feel at home here. Furthermore, it is a very smooth, homogeneous, friendly place in which to work, where their geometric intuition serves as a dependable guide. If you studied geometry in high school (which would have been Euclidean geometry), then you are familiar with this place. The plane in which you drew your figures is two-dimensional Euclidean space. However, in the early part of the nineteenth century, mathematicians found that Euclidean space has some dark corners that Euclid did not foresee. In fact, there are many subspaces of Euclidean space that bear very little family resemblance.

In these pathological subspaces, they found that their deeply seated intuition was sometimes misleading or even useless. This situation demanded an extension of familiar concepts and definitions, such as what constitutes a boundary point. A large and useful class of spaces they encountered that allowed them to extend the utility of their classical intuition was the class of manifolds. A manifold is a space that can be very unruly on the large scale, but on the small scale resembles Euclidean space. To be a bit more precise, given any point of a manifold, one can enclose it in a sphere (perhaps a very small sphere) inside of which the space is indistinguishable from Euclidean space. Take as an example a basketball and suppose that a recent technological breakthrough has provided us with a shrink ray. It would be possible for us to reduce our size to such a point that, regardless of where we were to stand on the ball, its curvature would be imperceptible to us and so it would appear flat (much as the earth appears flat to us, though we now know it to be a globe). Thus, on the small, local scale the ball resembles two-dimensional Euclidean space at every point making it a two-dimensional manifold, despite the fact that on the large, global scale it is a three-dimensional object.

This local resemblance to Euclidean space allows us to extend fairly easily many of the concepts formerly applied to Euclidean space, such as the idea of a boundary point. Consider a square cut from the plane, taken to include its edges. There is an intuitively clear difference between a point that is on the edge of the square and a point that is not. To formalize this intuitive difference, we observe that the interior of an arbitrarily small circle centered around a non-edge point resembles all of Euclidean space. The same cannot be said of points on the edge of the square. The interiors of arbitrarily small circles centered about edge points resemble Euclidean "half-space", that is Euclidean space that extends infinitely in all dimensions but one, where it is cropped. Since this observation involves only local properties, it may be applied to manifolds. We can in essence say that a point of a manifold is an edge point if locally it resembles Euclidean half-space, and a non-edge point if locally it resembles whole Euclidean space.

Let us apply this definition to a soup can. If we take any point on the side it should be apparent that, much as with the basketball, we can enclose it in a circle sufficiently small as to make the curvature of the can imperceptible within it. Thus the side of the can be made to resemble the plane. A point taken from the top is only different in that the circle enclosing it can be larger, since the top is already flat. If we take a point on the circle dividing the top from the side, though, we find that this is not the case. Whether we start drawing our circle on the side of the top, we find that even a circle small enough to mask the curvature will hang off the edge of the can and thus resemble Euclidean half-space. Therefore, the can is a two-dimensional manifold with boundary, and the boundary consists of the two circles separating the top and bottom from the side. Although many manifolds have a much more complex structure than our examples of the square in the plane or the cylinder in three-space, the definition of boundary point sketched above continues on where our intuition becomes unclear.

However, there are many interesting spaces that are not manifolds, that is, they do not resemble Euclidean space at all locally (see fig. 2). In many of these examples, we have no intuitive basis for labeling a point boundary or interior, and so must devise formal definitions that both capture and extend our intuitive notions. My thesis explores three proposed solutions for this problem of generalizing the idea of boundary point.

Introduction:

Given a topological space, it seems intuitively clear that there is a fundamental difference between boundary points and interior points, but what properties differentiate these two? In the theory of manifolds there is a well-defined notion of which points are boundary points and which are not. We can, in essence, say that a point of an n-dimensional manifold is a boundary point if it resembles half Euclidean n-space locally, and a point is not on the boundary if it resembles all of Euclidean n-space locally. This definition is somewhat restrictive because it requires the space under consideration to be locally like Euclidean space and so does not deal with intrinsic properties of a given point. Furthermore, there are many spaces of interest that do not resemble Euclidean space locally, and so are not subject to the previous definition.

Homotopy affords us a generalization from manifolds to spaces that are locally arcwise connected. In the early nineteen thirties Hopf and Pannwitz advanced the idea of stabil and labil points, here referred to as homotopically stabil and homotopically labil [Hopf]. Roughly speaking, a point is homotopically labil if one can continuously deform the neighborhoods containing the point, while leaving the rest of the space undisturbed, in such a way that the resulting images of the neighborhoods no longer contain the point. This definition replaces the requirement of Euclidean neighborhoods with the requirement of arcwise connected neighborhoods, which is less stringent, and so is in fact an improvement in generality over the manifold definition.

In considering the work of Hopf and Pannwitz, Borsuk and Jaworowski developed a definition of boundary point in the middle of the twentieth century that further relaxed the requirements on the space [Borsuk]. Again speaking roughly, a point is labil if one can find continuous images of the entire space containing the point so that the images do not contain the point and every point of the space is moved "very little." This definition only requires a notion of distance, and so is applicable to any metric space regardless of its connectedness.

In this paper, we first present a brief overview of topology and homotopy to familiarize the reader with the subjects. Next we introduce some preliminary definitions and results in topology to set the stage for a more formal discussion of the boundary definitions. Thirdly, we present the boundary definitions in full formality, and argue that they are indeed successively more general. This is followed by a series of examples further illustrating the interplay between the definitions. Finally, we explore what sorts of processes preserve the properties of homotopic lability and lability, and discuss briefly a cohomological definition of boundary and interior that reflects more recent progress in generality.

The Definitions:

Let us begin with our definition of what it is to be on the boundary of such a manifold. In the following definition the set

$$\mathbf{H}^k = \{x \in \mathbf{R}^n \mid x_k \geq 0 \text{ and } x_{k+1} = \ldots = x_n = 0\}.$$

Definition B1: Given a k-dimensional manifold M, the *boundary of M* is the set of all points x for which there exists an open set U containing x, an open set $V \subset \mathbf{R}^n$, and a homeomorphism h: $U \longrightarrow V$ such tha $h(U \cap M) = V \cap (\mathbf{H}^k \times \{0\}) = \{x \in V \mid x_k \geq 0$ and $x_{k+1} = \ldots = x_n = 0\}$ and $h(x) = (x_1, x_2, \ldots, x_{k-1}, 0)$. A manifold for which the boundary is nonempty is called a *manifold with boundary* [Spivak 113].

That is, a point of a k-manifold is on the boundary if it has a neighborhood homeomorphic to half Euclidean k-space, and the homeomorphism maps it onto a point on the boundary of the half space. Again we observe that this definition is somewhat unsatisfactory in its indirectness and limited applicability. Next we move on to our homotopic definition first given by Hopf and Pannwitz [Hopf].

Definition B2: A point a of a space S is *homotopically labil* whenever for every neighborhood U of a there exists a function $F: S \times I \longrightarrow S$ (where I is the unit interval [0,1]) which is continuous and satisfies the following conditions:

1. $F(x,0) = x$ for every $x \in S$
2. $F(x,t) = x$ for every $(x,t) \in (S \setminus U) \times I$
3. $F(x,t) \in U$ for every $(x,t) \in U \times I$
4. $F(x,1) \neq a$ for every $x \in S$.

A point that is not homotopically labil is *homotopically stabil*. Note that homotopical lability is a local property, in that if a is a homotopically labil point of a space S and b is a point of a space T and there exists a homeomorphism h that takes a neighborhood U_0 of a onto a neighborhood V_0 of b such that $h(a) = b$, then b is homotopically labil in T.

Our final definition was motivated by the observation that, for a metric space, conditions equivalent to 1-4 may be given in terms of the space's metric as follows:

Definition B2': A point a of a metric space M is *homotopically labil* if for every $\varepsilon > 0$, there exists a mapping $g: (M \times I) \longrightarrow M$ (where again I is the unit interval) satisfying the following conditions:

1'. $g(x,0) = x$ for every $x \in M$

2'. $\delta(x, g(x,t)) < \varepsilon$ for every $(x,t) \in (M \times I)$

3'. $g(x,1) \neq a$ for every $x \in M$.

It should be clear that a point $a \in M$ that is homotopically labil by **B2** is also homotopically labil by **B2'**, but the skeptical reader will require some support for the claim that the converse is true. Suppose then that under **B2'** a is homotopically labil, $g(x,t)$ is a mapping satisfying 1'-3' for some $\varepsilon > 0$, and that U is a neighborhood of a such that $\delta(x,a) < 3\varepsilon$ implies $x \in U$. Define F: $(M \times I) \longrightarrow M$ thus:

$$F(x,t) = \begin{cases} g(x,t) & \text{for } \delta(x,a) \le \varepsilon,\ 0 \le t \le 1 \\ g[x,\ t(2 - (\delta(x,a)/\varepsilon))] & \text{for } \varepsilon \le \delta(x,a) \le 2\varepsilon,\ 0 \le t \le 1 \\ x & \text{for } \delta(x,a) \ge 2\varepsilon,\ 0 \le t \le 1 \text{ [Borsuk].} \end{cases}$$

Having made this realization, Borsuk and Jaworowski saw a way to further generalize this concept of homotopic lability by divorcing it from homotopy. In their paper they introduce the notion of *lability*, and define it thus:

Definition B3: A point p of a metric space S is *labil* whenever there exists for every $\varepsilon > 0$, a mapping g: S ⟶ S such that

1a. $\delta(x,g(x))<\varepsilon$ for every x S

2a. $g(x)$ J p for every x S.

A point that is not labil is stabil [Borsuk].

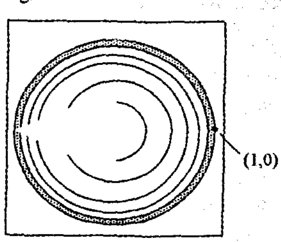
We note here that in contrast to homotopic lability, lability is not a local property. Consider the following sets in the Euclidean plane.

$S_0 = \{(x,y) |\ x^2 + y^2 = 1\}$

$S_n = \{(x,y) |\ x = (1 - 1/n)\cos\theta,\ y = (1 - 1/n)\sin\theta;\ |\theta| \le (\pi - 1/n)\}$ for $n \in \mathbf{N}$

$S = \cup_{n=0}^{\infty} S_n$ $\qquad$ $T = S \cap \{(x,y) | x \ge 0\}$ (see Fig. 1)[Borsuk].

Fig. 1



Obviously, S and T are identical about the point (1,0), yet this point is stabil in S and labil in T.

Equivalences and Divergences:

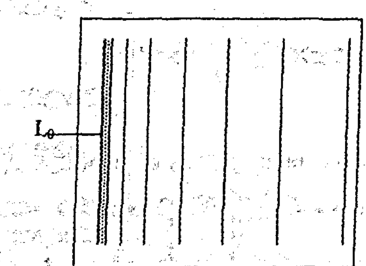
We can show without much work that our manifold definition implies our homotopic definition, which in turn implies our metric definition. For the first implication, assume that an n-dimensional manifold M is locally homeomorphic to half of Euclidean n-space about the point p. Given some $\varepsilon > 0$ let

$U_0 = \varnothing$, $U_t = \{(x,y) |\ \delta(x,p) \le \varepsilon t\}$, and $U = \cup_{t \in [0,1]} U_t$.

For each t, define $r(x,t) = r_t(x)$ as the retraction mapping r_t: M ⟶ $(M \backslash U_t)$. It should be clear that $r(x,t)$ satisfies conditions 1-4 for the neighborhood U of the point p in the space M, and thus p is homotopically labil. To see that our homotopic definition implies our metric definition, simply observe that if g is a mapping satisfying conditions 1¢-3¢ for a point a of a metric space L, then setting $f(x) = g(x,1)$ gives a mapping satisfying conditions 1a and 2a for $a \in$ L.

Obviously the converse is not true in general for the manifold and homotopic definitions, since a point labil in our homotopic sense may not have a single neighborhood homeomorphic to half- or whole Euclidean space. Our next example shows that the converse is also false in general for our homotopic and metric definitions. For the following of example, consider the set given as a subset of the Euclidean plane with the induced topology.

Fig. 2



Let $N = \{1/n \mid n \in \mathbf{N}\}$, $J = [-1,1]$, $S = N \times J$, $L_0 = S \backslash (N \times J)$ and $L_k = \{1/k\} \times J$ (see Fig 2). First observe that any point of L_0 fails to have connected neighborhoods, and so by theorem 4 cannot be homeomorphic to Euclidean space. Thus our manifold definition is of no service to us. We claim that any non-endpoint p of L_0 is homotopically stabil, but labil. To see that p is homotopically stabil, assume that there exists a function f satisfying conditions 1-4 for p and some neighborhood U of p, and consider the images of L_0 under f. If we assume that f does not map L_0 onto itself, then f maps L_0 onto some L_k and we have a situation depicted in figure 2a. Here we see that the continuous image of L_0, a connected set, is disconnected, and this contradicts the fact that connectedness is a continuous invariant. Hence, it must be that f maps L_0 onto itself. However, condition 4 requires that p not be an element of $f(U,1)$, and so we arrive at the situation depicted in figure 2b. Again we have L_0 mapped into a disconnected set, namely $L_0 \backslash \{p\}$, which is another contradiction. Thus we must conclude that no such f exists and thus p is homotopically stabil.

Figure 2a.

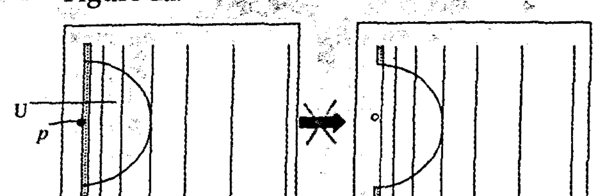
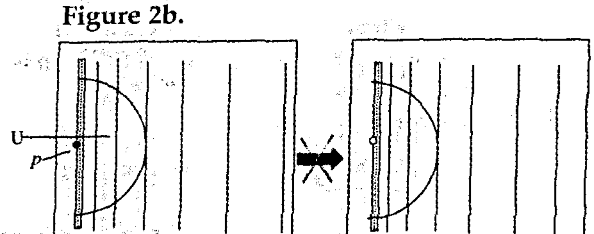


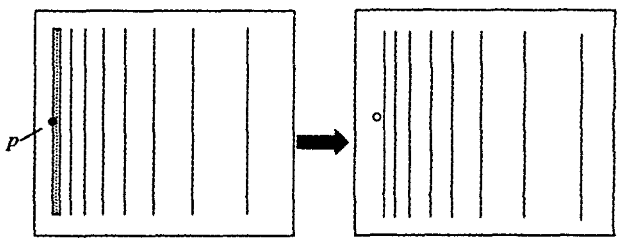
Figure 2b.



The problem here arises from the fact that, for homotopic lability, the space must remain fixed outside the neighborhood U. For lability there is no such requirement, so we are free to use a restriction of a projection mapping to map onto some segment L_{k*} the segment L_0 and all segments L_k where $k > k^*$, as depicted in figure 2c. More specifically, let ε be an arbitrary positive real number, choose an $n \in \mathbf{N}$ such that $n > 1.\varepsilon$, and define f_e: S⟶S as follows:

$$f_e(0,s) = (1/n,s) \text{ and } f_e(1/k,s) = \begin{cases} (1/k,s) & \text{for } k \le n \\ (1/n,s) & \text{for } k > n. \end{cases}$$

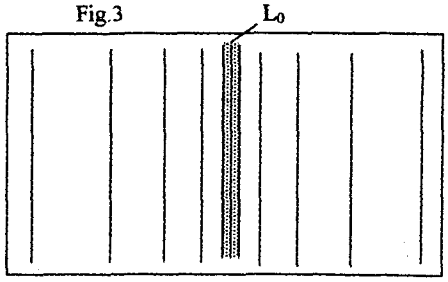
This mapping demonstrates the lability of p (see fig 2c.)



).

A slight modification of S indicates that our more general definitions sometimes defy our intuition of what a boundary point is. Let $N^* = \{1/n | n \in \mathbf{Z}\}$, and define L_0*, S*, and L_k* analogously to L_0, S, and L_k (see Fig. 3). Note that the points of L_0* fail to have Euclidean neighborhoods as well. An argument similar to that of our previous example shows that the non-endpoints of L_0* are homotopically stabil as might be expected, but a simple modification to f_e shows that these points are still labil, despite the fact that any arc drawn from such a point to a simple closed curve enclosing the space intersects the space in at least one other point.

Fig.3

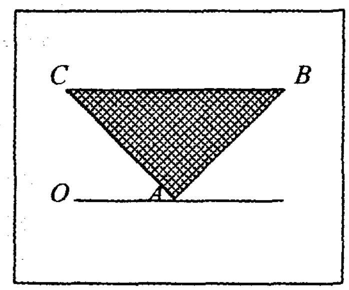


Before we move on to our next topic we note that for a rather large class of examples, the notions of homotopical lability and lability do coincide: when the space under consideration is an absolute neighborhood retract.

Property Preserving Processes:

A natural question to ask is 'when are our boundary points preserved?' In the example depicted in figure 2, we see that a sequence of homotopically labil points (i.e. the endpoints of each L_k) converges to a homotopically labil point (i.e. the endpoint of L_0). Is the limit of a sequence of labil (or homotopically labil) points labil (or homotopically) labil in general? Consider in the plane the triangle with vertices $A = (1/2,0)$, $B = (1,1)$, and $C = (0,1)$, and let X be the union of the set of all points on or contained by the triangle and the segment of the x-axis between 0 and 1 (see Fig 5).

Fig. 5



A sequence of retraction mappings almost identical to the one at the beginning of the previous section shows that any non-endpoint of the line segment AB is homotopically labil (and thus labil), and the segment contains a sequence of points approaching A. However, another connectedness argument shows that A is in fact stabil (and thus homotopically stabil).

Borsuk and Jaworowski prove in their paper on lability that the stability of a point, and also the homotopic stability, are invariant under Cartesian division but leave the question of Cartesian multiplication open. It is fairly easy to see that lability is invariant under this operation, and in fact that a sufficient condition for a point $(a,b) \in$ A x B to be labil is for a to be labil in A or b to be labil in B. For if we have a point $a \in$ A that is labil in A and f is a function satisfying conditions 1a and 2a for a, then by defining g: A x B⟶A x B as $g(x,y) = (f(x),y)$ we obtain a function satisfying 1a and 2a for any point in A x B of the form (a,y). Similarly, we can argue that if $b \in$ B is labil in B, then any point in A x B of the form (x,b) is labil.

Soon after the paper of Borsuk and Jaworowski appeared, Noguchi supplied an answer to the question of the invariance of homotopic stability under Cartesian multiplication in two papers published in 1954 and 1955. In the first paper, Noguchi supplied the following homological characterizations of homotopically labil and stabil points.

Theorem N1: Let A be a complex. A point a of A is homotopically labil if and only if there exists a contractible neighborhood complex of a [Noguchi 1954].

Theorem N2: Let A be a complex. A point a of A is homotopically stabil if and only if there exists a neighborhood complex of *a* which is not contractible [Noguchi 1954].

In his paper of 1955, Noguchi used these characterizations along with the homological properties of joins to show that in fact homotopic stability is not invariant under Cartesian multiplication, except in the special case of homogeneous complexes [Noguchi 1955].

A Final Generalization:

Before we conclude we discuss one further generalization of the boundary point concept due to J. Lawson and B. Madison, who published a paper on the subject in 1970. Our homotopic definition replaced the requirement of Euclidean neighborhoods with that of arcwise connected neighborhoods; our metric definition freed us from that restriction by using only a notion of distance; our final definition lacks even that requirement making it the most general to date. Lawson and Madison gave and investigated two definitions in terms of cohomology, which they call peripheral and marginal.

Conclusion:

We have now presented three successively more general notions of a boundary point. In manifolds, we have identified boundary points as those points having neighborhoods homeomorphic to half-Euclidean space. In locally arcwise connected spaces we have that a point is homotopically labil if we can deform the neighborhoods of the point in the space continuously in such a way that the result does not contain the point, but the complements of the neighborhoods remain unmoved. For any metric space we say that a point is labil if there exist continuous images of the space containing the point that do not contain the point and that do not move any point far.

We then argued that each of these definitions generalizes the one before it. A pair of examples showed that these definitions are not equivalent, and that in at least one instance the more general definitions can have rather counterintuitive consequences. After an investigation into various operations applied to labil and stabil points we learned that Cartesian division preserves both homotopic lability and lability, that Cartesian multiplication preserves lability but not homotopic lability, and that the limit process preserves neither lability nor homotopical lability. Finally, we directed the interested reader to more work in improving the generality of the boundary concept. We hope that our discussion will be as illuminating for others as it has been for us.

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Faculty comment:

Dr. Bernard Madison made the following remarks about Mr. Thompson's work:

> Jonathan's research is in an area that overlaps into point-set topology and algebraic topology. Basically, the goal of work in this area is to determine structures and properties of spaces, e.g. subsets of Euclidean space, using algebraic constructs. The particular problem that Jonathan studied was distinguishing boundary points from interior points in spaces that are different from spaces that are like Euclidean spaces locally. Locally Euclidean spaces such as a circular disk are called manifolds and the notions of boundary and interior are well known and reasonably obvious. This interior versus boundary problem was studied in the 1930's and 1950's by several European mathematicians and resurfaced in the 1960's and 1970's because of relevance to work in topological algebra.
>
> We have no undergraduate course work here in any area of topology so Jonathan had to learn a considerable body of material as background. Jonathan's work centers on two different concepts of boundary points, one defined in terms of metrics and one defined in terms of homotopies. These notions were introduced by H. Hopf and E. PannwiLz (1933) and K. Borsuk and J. Jaworowski (1952). Later, A. D. Wallace, K. H. Hofinann, P. S. Mostert, J. D. Lawson and I expanded these definitions using cohomology structures and applied the results to topological algebra structures. Jonathan's major creative contribution was to describe and analyze several fairly complex subspaces of the plane that provide examples that refine and distinguish between the metric and homotopy definitions of boundary. His apparent understanding of these examples is impressive for an undergraduate, and his exposition of his understanding is extraordinary. Jonathan's introduction to his paper is an excellent effort to convey understanding of his work to non-experts.