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3-Uniform 4-Path Decompositions of Complete 3-Uniform Hypergraphs

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3-Uniform 4-Path Decompositions of Complete 3-Uniform Hypergraphs

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By

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Abstract

The complete 3-uniform hypergraph of order v is denoted as $K_v^{(3)}$ and consists of vertex set V with size v and edge set E, containing all 3-element subsets of V. We consider a 3-uniform hypergraph P_7, a path with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and edge set $\{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_4, v_5, v_6\}, \{v_5, v_6, v_7\}\}$. We provide the necessary and sufficient conditions for the existence of a decomposition of $K_v^{(3)}$ into isomorphic copies of P_7.

Contents

1 Introduction

Graph theory is an important area of study in mathematics that has applications in a variety of fields. Informally, a graph is a collection of vertices, sometimes called nodes, that are connected to each other. Graphs can be used in a wide variety of settings. They can be used to model web design, by representing webpages as vertices and the the hyperlinks between them as the connections. Graphs can also be used to design a tournament in which every participant plays against every other participant exactly once. In this case, every participant is a node and the matches they play are the connections between them. This notion of a graph can also be extended to allow a single connection to contain more than two nodes.

A hypergraph is a structure for which we define the connections to be between subsets of nodes, rather than strictly two. This, too, has applications in various fields. In civil engineering, we may want to consider a collection of cities that are all located on the same highway. In this case, the nodes are the cities and the shared highway is the connection between them. We are interested in understanding hypergraphs as a useful tool to organize and optimize information.

Graph decomposition is an area of research that seeks to partition a graph into smaller subgraphs, where the union of these subgraphs equals the original graph. Of particular interest is the decomposition of the complete k-uniform graph of order v, denoted $K_v^{(k)}$, into isomorphic copies of a particular subgraph, denoted H. The problem of determining all values of v for which an H-decomposition of $K_v^{(k)}$ exists is called the spectrum problem for H, and finding such a decomposition is known as settling the spectrum problem. To settle the spectrum problem for the hypergraph P_7 (see Figure 1), we seek to prove the following result:

Main Theorem. *There exists a P_7-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, 2, 4,$ or 6 (mod 8) and $v \geq 7$.*

We demonstrate the necessity of this condition in Section 5 using combinatorics, and its
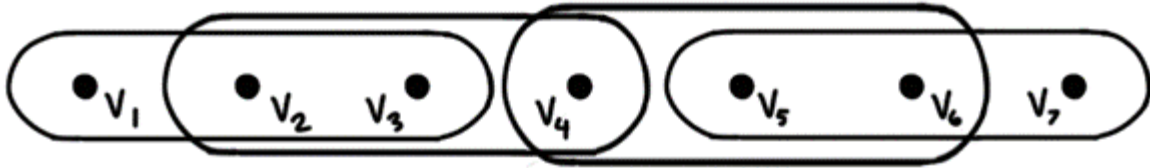
Figure 1: The hypergraph P_7

sufficiency in Section 3 by decomposing $K_v^{(3)}$ into nine smaller constituent hypergraphs. We find that if we can decompose these nine subgraphs individually, then we can decompose $K_v^{(3)}$. These results are supported by Section 4, in which we provide specific examples of such decompositions.

2   Background

2.1   Hypergraphs

A *graph* H is defined as the ordered pair $(V(H), E(H))$, where $V(H)$ denotes the set of *vertices* of the graph and $E(H)$ denotes the set of *edges*, or a collection of 2-element subsets of $V(H)$. We call $|V(H)|$ the *order* of H and $|E(H)|$ its *size*. Visually, graphs are seen as a collection of vertices connected to one another by lines (edges). Examples of graphs are shown in Figure 2.

A *hypergraph* is a generalization of graphs where we no longer require that the elements of $E(H)$ be 2-element subsets of $V(H)$, and instead let the edges in a hypergraph join any number of vertices together. If every edge in a hypergraph contains exactly k vertices, we say that it is *k-uniform*. Thus, a graph can also be called a 2-uniform hypergraph. The number of edges a given vertex is contained in is called the *degree* of that vertex. Two vertices are *adjacent* if they are contained by the same edge.

The *complete* 3-uniform hypergraph of order v, $K_v^{(3)}$, is the hypergraph whose edge
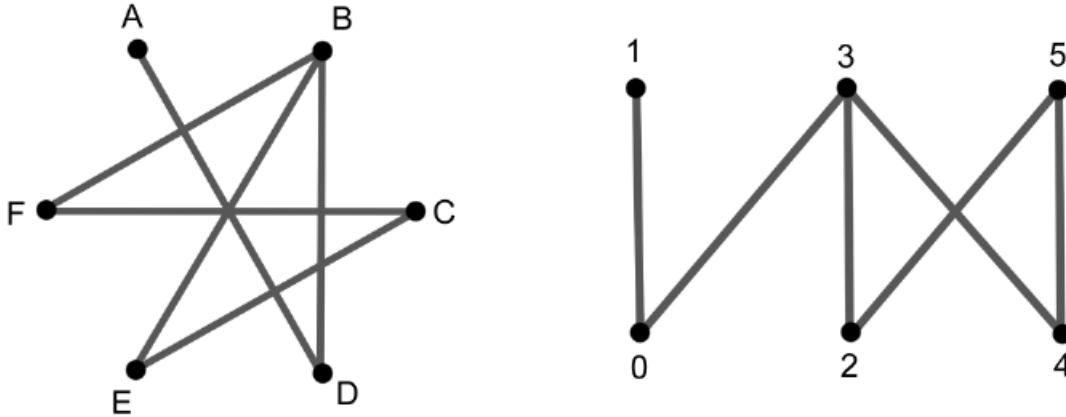
6

Figure 2: Examples of graphs

set contains every 3-element subset of the vertex set. That is, for some set of v vertices, we include every possible edge between them. There are some particularly useful classes of hypergraphs for decomposing $K_v^{(3)}$. The complete *multipartite* 3-uniform hypergraph is defined with vertex set $V = U_1 \cup U_2 \cup U_3$, where U_1, U_2, U_3 are pairwise disjoint sets and with edge set consisting of all 3-element subsets of V having exactly one vertex from each U_i. We denote a multipartite hypergraph as $K_{n_1,n_2,n_3}^{(3)}$, where $n_i = |U_i|$.

A similar structure is called *'multipartite-like,'* a hypergraph consisting of vertex set $V = U_1 \cup U_2$, where U_1 and U_2 are pairwise disjoint sets and edge set consisting of all 3-element subsets of V having at least one vertex from each U_i. We denote a multipartite-like hypergraph as $L_{n_1,n_2}^{(3)}$, where $n_i = |U_i|$.

Vertex coloring is when we assign a color to each vertex, and this coloring is called *proper* if adjacent vertices have different colors. This is closely tied to the idea of multipartite. A 3-uniform hypergraph is multipartite if and only if it is 3-colorable. If we assign a unique color to each of $U_1 \cup U_2 \cup U_3$, then every edge would contain three unique colors and thus adjacent vertices would be distinct. Likewise, the colors in a 3-colorable hypergraph determine the partition $V = U_1 \cup U_2 \cup U_3$.

Two hypergraphs G and H are *isomorphic* to one another if there is a bijection from

$V(G)$ to $V(H)$ that preserves the structure of the graph. That is, by a scheme of relabeling, G is identical to H and vice versa. Isomorphism preserves such adjacency.

Recall the notion of the spectrum problem from Section 1. There are many classes of hypergraphs whose spectrum problems have been studied. Among these are *paths*, which are graphs with two terminal vertices, and the vertices in between can be arranged in order. A *cycle* is a path where the terminal vertices are the same, while every other vertex is distinct. A path with n edges is called an n-path, and a cycle with n edges is called an n-cycle. Other types of graphs include stars and forests. In this thesis we will focus on a specific path denoted P_7, a 4-path of order 7. This hypergraph is depicted in Figure 1.

2.2   Difference Classes and Edge Orbits

In order to settle the spectrum problem for our hypergraph, we have several tools to increase efficiency for finding decompositions. First, we partition $K_v^{(3)}$ into several smaller pairwise edge-disjoint hypergraphs. This partitioning is described in detail in Lemma 2 and Lemma 3. This creates a much more approachable problem as it requires several solutions for small hypergraphs rather than one solution for a possibly very large hypergraph, which are found by hand.

As these decompositions are found manually, we need methods to keep track of the edges in our decomposition. First, we use a labeling scheme for the hypergraph we are trying to decompose. For the complete hypergraph of order n, we label the vertex set $V(K_v^{(3)}) = \{v_1, v_2, ..., v_n\}$ as $\{0, 1, 2, ..., n-1\}$, considered modulo n, which allows us to perform operations on the vertex indices and find patterns among the decompositions. The primary operation we use is known as *clicking*. Let $e = \{v_1, v_2, v_3\}$ be some edge in $V(K_v^{(3)})$. When we click e, we apply the isomorphism $i \mapsto i + 1$ on the indices of the vertices of e. Clicking e once, we obtain a new edge $e + 1 = \{v_1 + 1, v_2 + 1, v_3 + 1\}$. We define an *edge orbit* as the set $\{e + i : i \in \mathbb{Z}_v\}$. Edge orbits partition the set $E(K_v^{(3)})$. Out of convention, we typically order the vertices in an edge lexicographically.

8

After applying this labeling scheme, we seek to find a decomposition of $K_v^{(3)}$ by using every $e = \{v_1, v_2, v_3\} \in V(K_v^{(3)})$ exactly once to create a series of isomorphic copies of the graph we are interested in. Since the edge orbits nicely partition the edge set, we can use this to our advantage to make decomposition simpler. For some edge $e = \{v_1, v_2, v_3\}$ where $v_1 < v_2 < v_3$, we define a *difference vector* as the 3-tuple (d_1, d_2, d_3) where $d_1 = v_2 - v_1, d_2 = v_3 - v_2$, and $d_3 = v_1 - v_3 \pmod{n}$. This vector denotes the distances between the vertices of an edge. Two difference vectors are equivalent if they can be obtained from each other by cyclic permutation. Note that the difference vector for e and $e + 1$ are equivalent.

Lemma 1. *Edges e and e' are in the same edge orbit if and only if their difference vectors are equivalent.*

Proof. We prove the forward direction. Let e and e' be two edges from the same edge orbit in $K_v^{(3)}$. Recall that an edge orbit is the set $\{e + i : i \in \mathbb{Z}_v\}$. Then we define $e = \{v_1, v_2, v_3\}$ and $e' = \{v_1 + i, v_2 + i, v_3 + i\}$ for some $i \in \mathbb{Z}_v$. The difference vector for e is $\big(v_2 - v_1, v_3 - v_2, v_1 - v_3 \pmod{n}\big)$ and the difference vector for e' is $\big((v_2 + i) - (v_1 + i), (v_3 + i) - (v_2 + i), (v_1 + i) - (v_3 + i) \pmod{n}\big)$. Simplifying, we find that the difference vector for $e' = \big(v_2 - v_1, v_3 - v_2, v_1 - v_3 \pmod{n}\big)$, the same as e.

We prove the other direction. Let $e = \{v_1, v_2, v_3\}$ and $e' = \{u_1, u_2, u_3\}$ be two edges with the same difference vector, (d_1, d_2, d_3). Thus e' must be equal to $e + i$ for some $i \in \mathbb{Z}_v$ in order to preserve the distance between each pair of vertices. Suppose $u_1 = v_1 + i$. Then we have $u_1 - v_1 = i$ and thus $u_2 = u_1 + d_1 = u_1 + (v_2 - v_1) = v_2 + i$. Similarly, $u_3 = u_2 + d_2 = u_2 + (v_3 - v_2) = v_3 + i$. Thus $e' = e + i$ for some $i \in \mathbb{Z}_v$ and the two edges are from the same edge orbit. $\square$

We name *difference classes* by the difference vector that represents them. Consider the edge $\{0, 1, 4\}$ in $K_{10}^{(3)}$. By clicking, we obtain the following edges:

$$\{1,2,5\}, \{2,3,6\}, \{3,4,7\}, \{4,5,8\}, \{5,6,9\}, \{6,7,0\}, \{7,8,1\}, \{8,9,2\}, \{9,0,3\}, \{0,1,4\}.$$

Finding the differences between the vertices of $\{0, 1, 4\}$, we obtain $(1 - 0, 4 - 1, 0 - 4$ $(\mod 10)) = (1, 3, 6)$. Similarly, the differences between the vertices of the rest of the edges result in a vector of $(1, 3, 6)$. We group all of these edges into the difference class represented by the vector $(1, 3, 6)$. The hypergraph $K_{10}^{(3)}$ has $\binom{n}{3} = 120$ edges that are partitioned into 12 orbits of size 10, similar to the previous example.

2.3 Cyclic and R-Pyramidal Decomposition

Difference classes allow us to find a smaller subset of representative copies of our graph for decomposition. If we choose one edge from four distinct difference classes, we can automatically obtain $v - 1$ additional copies by clicking through each edge in the orbits. We call this a *cyclic decomposition*.

In order to use cyclic decomposition, we work under two assumptions. Firstly, we must choose exactly one edge from each difference class so that clicking does not repeat edges that have already been used. Secondly, we assume that all edge orbits have the same cardinality. If either of these assumptions are not true, we cannot produce $n - 1$ additional copies by clicking and must use another method.

In most cases, as in $K_{10}^{(3)}$, we expect edge orbits to have cardinality v. It turns out that this is dependent only on $\gcd(v, 3)$, since we are working with a 3-uniform graph. Consider the example of $K_9^{(3)}$. Our difference classes are as follows:

$$(1, 1, 7), (1, 2, 6), (1, 3, 5), (1, 4, 4), (1, 5, 3), (1, 6, 2), (2, 2, 5), (2, 3, 4), (2, 4, 3), (3, 3, 3).$$

The difference class $(3, 3, 3)$ has the following edges, obtained by clicking modulo 9:

$$\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 0\}, \{4, 7, 1\}, \{5, 8, 2\}, \{6, 0, 3\}, \{7, 1, 4\}, \{8, 2, 5\}.$$

We notice that since edges are not an ordered subset, the edges $\{0, 3, 6\}, \{3, 6, 0\}, \{6, 0, 3\}$ are the same. The same holds true for $\{1, 4, 7\}, \{4, 7, 1\}, \{7, 1, 4\}$ and $\{2, 5, 8\}, \{5, 8, 2\}, \{8, 2, 5\}$.

Thus, the difference class $(3, 3, 3)$ in $K_9^{(3)}$ is an example of a *short orbit*, $\{\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}\}$. This phenomenon happens when v and 3 are not coprime, as there must exist a difference class $(v/3, v/3, v/3)$ with an orbit of length $v/3$. This result can be anticipated as $K_9^{(3)}$ contains $\binom{9}{3} = 84$ edges, which is not divisible by 9. Rather, we have 9 full orbits of size 9, and 1 short orbit of size 3.

To handle such cases, we re-label our vertices to keep one or more of them fixed. For some graph $K_v^{(3)}$, let $V(K_v^{(3)}) = \mathbb{Z}_{v-r} \cup I_r$ for $I_r = \{\infty_1, ..., \infty_r\}$. Let $e = \{v_1, v_2, v_3\}$ be some edge in $K_v^{(3)}$. Using this new vertex set, when we click e, we redefine our isomorphism as $i \mapsto i + 1$ in $V(e)$ with $\infty_i + 1 = \infty_i$. We do this to force orbits to be the same size. For example, by relabeling $V(K_9^{(3)}) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$, we obtain the following difference classes:

$$(1, 1, 5), (1, 2, 4), (1, 3, 3), (1, 4, 2), (2, 2, 3), (1, 6, \infty_1),$$

$$(2, 5, \infty_1), (3, 4, \infty_1), (1, 6, \infty_2), (2, 5, \infty_2), (3, 4, \infty_2), (7, \infty_1, \infty_2).$$

Note that we consider fixed points as placeholders and use $\mathbb{Z}_{v-r}$ to calculate the differences between the remaining vertices. Now, rather than having 9 orbits of size 9 and 1 orbit of size 3, we have 12 orbits of size 7. This allows us to find decomposition in a manner similar to cyclic decomposition, called *r-pyramidal decomposition* for $I_r = \{\infty_1, ..., \infty_r\}$. We also often use r-pyramidal decomposition when the number of difference classes is not divisible by 4, even if $\gcd(v, 3) = 1$. Recall that our requirements for cyclic decomposition includes that we choose exactly one edge from each difference class. Since we have 4 edges in P_7, we want to be able to exhaust all of our difference classes in this manner, so we can use fixed points to force the number of difference classes to be divisible by 4.

Finally, we can use similar methods for multipartite and multipartite-like graphs, though we must first partition our vertices to reflect the structure of such graphs. For a multipartite-like graph $L_{n,n}^{(3)}$, let $V(L_{n,n}^{(3)}) = \mathbb{Z}_{2n}$ with the vertices partitioned into the evens and odds. Recall that for a multipartite-like graph $L_{U_1,U_2}^{(3)}$, each edge must contain at least 1 vertex

from each U_i. With this partitioning, we can preserve the multipartite-like structure by ensuring that no edges contain vertices that are exclusively even or odd. To do this, we simply omit any difference classes in $L_{n,n}$ that have exclusively even numbers.

For example, consider $V(L_{4,4}^{(3)}) = \mathbb{Z}_8$ with vertex partition as the evens and odds. Using methods as we have done previously, our difference classes would be as follows:

$$\{(1,1,6),(1,2,5),(1,3,4),(1,4,3),(1,5,2),(2,2,4),(2,3,3)\}.$$

Since the difference class represented by $(2,2,4)$ would cause all of our vertices to be exclusively even or odd, we omit it. We use a similar process for multipartite graphs. We partition the vertices of $K_{n_1,n_2,n_3}^{(3)}$ modulo 3, omitting any difference classes whose representative difference vector contains multiples of 3.

3   Decomposing Hypergraphs

We can decompose $K_v^{(3)}$ into smaller hypergraphs for which it is easier to find P_7-decompositions. Similar decompositions appear in literature for other hypergraphs, such as those those by Akin, et al. [1].

Lemma 2. *Let x be a positive integer.*

1. *If $r \in \{0,1,2\}$, then $K_{8x+r}^{(3)}$ decomposes into the following hypergraphs:*

$$K_{8x+r}^{(3)} = xK_{8+r}^{(3)} \cup \binom{x}{2}\left(K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}\right) \cup \binom{x}{3}K_{8,8,8}^{(3)}.$$

2. *If $r \in \{4,6\}$, then $K_{8x+r}^{(3)}$ decomposes into the following hypergraphs:*

$$K_{8x+r}^{(3)} = K_{8+r}^{(3)} \cup (x-1)K_8^{(3)} \cup (x-1)L_{8+r,8}^{(3)} \cup \binom{x-1}{2}\left(K_{8+r,8,8}^{(3)} \cup L_{8,8}^{(3)}\right) \cup \binom{x-1}{3}K_{8,8,8}^{(3)}.$$

Proof. This result follows from representing the vertex set of $K_v^{(3)}$ as an edge-disjoint union

of these pieces.

First we prove 1. Consider the case $K_v^{(3)} = K_{8x+0}^{(3)}$. Then we can decompose $K_v^{(3)}$ into the following:

$$K_{8x+0}^{(3)} = xK_8^{(3)} \cup \binom{x}{2}L_{8,8}^{(3)} \cup \binom{x}{3}K_{8,8,8}^{(3)}.$$

Indeed, we represent the complete graph visually as x groups of 8 vertices each. The complete graph contains every possible edge between 3 vertices. There are three types of edges in this case: those contained fully within a single group of 8, edges between two groups of 8, and edges between three groups of 8. Since there are x groups, the first type of edge is handled by the term $xK_8^{(3)}$. Edges between two groups must use one vertex from one group and two from the other. Thus this case is handled by the term $\binom{x}{2}L_{8,8}^{(3)}$. Finally, the final case is handled by $\binom{x}{3}K_{8,8,8}^{(3)}$ in a similar manner.

Consider when $r = 1$. Then we can decompose $K_v^{(3)}$ into the following:

$$K_{8x+1}^{(3)} = xK_9^{(3)} \cup \binom{x}{2}\left(K_{1,8,8}^{(3)} \cup L_{8,8}^{(3)}\right) \cup \binom{x}{3}K_{8,8,8}^{(3)}.$$

Once again, we represent the vertices as x groups of 8, with one leftover vertex, which we leave outside of the groups of 8. In addition to the edges described in the $r = 0$ case, we must now account for the edges between the leftover vertex and one group of 8, as well as the edges between the leftover vertex and two groups of 8. These are represented by $xK_9^{(3)}$ and $\binom{x}{2}K_{1,8,8}^{(3)}$ respectively. Notice that $K_9^{(3)}$ contains both the edges within a group of 8 as well as the edges between it and the leftover vertex. Thus the edges in $K_8^{(3)}$ as shown in the $r = 0$ case are included in $K_9^{(3)}$.

The same procedure also works for $r = 2$, where there are now two leftover vertices to consider.

Now we prove 2. Because now $r \geq 3$, we must also consider edges contained fully within the leftover vertices. To simplify this, we include the leftover vertices with a single group of

8. This creates one group of $8 + r$ vertices and $x - 1$ groups of 8. Then for $r = 4$ we can decompose $K_v^{(3)}$ into the following:

$$K_{8x+4}^{(3)} = K_{12}^{(3)} \cup (x - 1)K_8^{(3)} \cup (x - 1)L_{12,8}^{(3)} \cup \binom{x - 1}{2} \left(K_{12,8,8}^{(3)} \cup L_{8,8}^{(3)} \right) \cup \binom{x - 1}{3} K_{8,8,8}^{(3)}.$$

Here we are considering the edges within the group of $8 + r$ and the edges within one of the $x - 1$ groups of 8, as well as the edges between any two or three groups. This is similar to the case $r = 0$, but we must now consider the two possible number of vertices in each group.

The case $r = 6$ follows from the same process. $\qquad\square$

We further decompose the hypergraphs $L_{8,8}^{(3)}$, $K_{8,8,8}^{(3)}$, $K_{1,8,8}^{(3)}$, $K_{2,8,8}^{(3)}$, $K_{12,8,8}^{(3)}$, $K_{14,8,8}^{(3)}$, $L_{12,8}^{(3)}$, and $L_{14,8}^{(3)}$ from Lemma 2 into smaller hypergraphs.

Lemma 3. *Let x be a positive integer and suppose $r \in \{0, 1, 2, 4, 6\}$. A P_7-decomposition of $K_{8x+r}^{(3)}$ exists provided there is a P_7-decomposition of each of the following hypergraphs:*

$$K_8^{(3)}, \; K_9^{(3)}, \; K_{10}^{(3)}, \; K_{12}^{(3)}, \; K_{14}^{(3)}, \; L_{4,4}^{(3)}, \; L_{4,6}^{(3)}, \; K_{2,4,4}^{(3)}, \; K_{1,4,4}^{(3)} \cup L_{4,4}^{(3)}.$$

Proof. Any multipartite hypergraph $K_{n_1,n_2,n_3}^{(3)}$ can be built by $K_{m_1,m_2,m_3}^{(3)}$ given m_i divides n_i. We can subdivide any of the n_i into equal-sized subsets of order m_i, then consider the edges between the subgroups. Thus $K_{8,8,8}^{(3)} = 16K_{2,4,4}^{(3)}$, $K_{2,8,8}^{(3)} = 4K_{2,4,4}^{(3)}$, $K_{12,8,8}^{(3)} = 24K_{2,4,4}^{(3)}$ and $K_{14,8,8}^{(3)} = 28K_{2,4,4}^{(3)}$ can all be built by the smaller hypergraph $K_{2,4,4}^{(3)}$.

We represent $L_{8,8}^{(3)}$ as two disjoint groups of 8 vertices, with no edge contained fully in a single group. We partition each group of 8 into two groups of 4. Preserving the original $L_{8,8}^{(3)}$ structure, we now consider the edges between a group of 4 on one side and a group of 4 on the other. We also consider the edges between two groups of 4 on one side and one group of 4 on the other side. This means $L_{8,8}^{(3)} = 4L_{4,4}^{(3)} \cup 4K_{4,4,4}^{(3)}$. As described above, the $K_{4,4,4}^{(3)}$ term can be further decomposed into $2K_{2,4,4}^{(3)}$, and hence $L_{8,8}^{(3)} = 4L_{4,4}^{(3)} \cup 8K_{2,4,4}^{(3)}$. The hypergraph $K_{1,8,8}^{(3)}$ can be decomposed in the same manner, but with an additional $4K_{1,4,4}^{(3)}$.

14

We represent $L_{12,8}^{(3)}$ as two disjoint groups of 12 and 8 vertices, with no edge contained fully in a single group. Subdividing both into groups of order 4 and recounting the edges, we have $9K_{4,4,4}^{(3)} \cup 6L_{4,4}^{(3)}$. However, as described previously, $9K_{4,4,4}^{(3)}$ decomposes into $18K_{2,4,4}^{(3)}$, and thus $6L_{12,8}^{(3)} = 18K_{2,4,4}^{(3)} \cup 6L_{4,4}^{(3)}$.

Finally, we represent $L_{14,8}^{(3)}$ as two disjoint groups of 14 and 8 vertices, with no edge contained fully in a single group. We divide the group of 12 into subgroups of 6, 4, and 4, and the group of 8 into groups of 4. Recounting the edges, we have $4K_{4,4,4}^{(3)} \cup 5K_{6,4,4}^{(3)} \cup 2L_{4,6}^{(3)} \cup 4L_{4,4}^{(3)}$. Further decomposing $4K_{4,4,4}^{(3)}$ into $8K_{2,4,4}^{(3)}$ and $5K_{6,4,4}^{(3)}$ into $15K_{2,4,4}^{(3)}$, we are left with $L_{14,8}^{(3)} = 23K_{2,4,4}^{(3)} \cup 2L_{4,6}^{(3)} \cup 4L_{4,4}^{(3)}$.

Thus, a P_7-decomposition of $K_v^{(3)}$ can be shown by decomposing each of

$$K_8^{(3)}, \ K_9^{(3)}, \ K_{10}^{(3)}, \ K_{12}^{(3)}, \ K_{14}^{(3)}, \ L_{4,4}^{(3)}, \ L_{4,6}^{(3)}, \ K_{2,4,4}^{(3)}, \ K_{1,4,4}^{(3)} \cup L_{4,4}^{(3)}$$

into copies of P_7. $\qquad\square$

Hence, the sufficiency of the Main Theorem is established by exhibiting P_7-decompositions for the 9 hypergraphs listed in Lemma 3.

4   Providing P_7-Decompositions for Lemma 3

To describe the decompositions, we use the notation $H[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$, a particular isomorphic copy of P_7, which uses the vertices $v_1, v_2, ..., v_7$ in the relevant hypergraph. This is to denote what is known as an H-block. Ideally, we describe the decomposition by giving a collection B of starters, i.e. we give orbit representatives of H-blocks for the click action. However, in some cases we have to consider multiple actions and produce multiple collections $B, B^*, ...$ to produce a decomposition.

The first example is shown in the figures below. We have two H-blocks, or starters. If we add 1 (mod 7) to each vertex in Figure 3 and Figure 5, or rotate the edges by clicking

in Figure 4 and Figure 6, we obtain every possible edge in $K_8^{(3)}$.
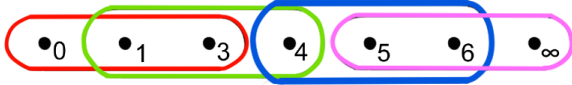

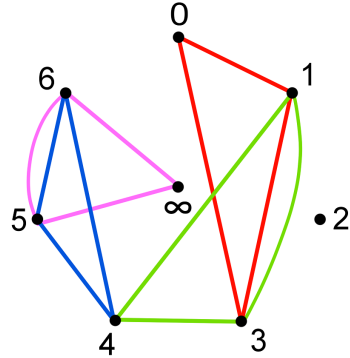Figure 3: $K_8^{(3)}$ first starter

Figure 4: $K_8^{(3)}$ first starter
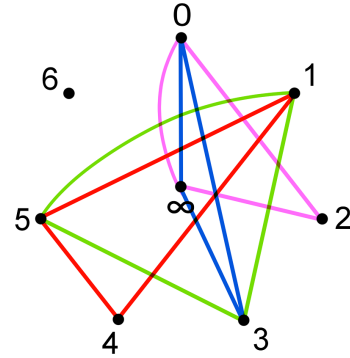

Figure 5: $K_8^{(3)}$ second starter

Figure 6: $K_8^{(3)}$ second starter

Here we supply explicit decompositions for those hypergraphs listed in Lemma 3.

4.1 K_8

Let $V\left(K_8^{(3)}\right) = \mathbb{Z}_7 \cup \{\infty\}$ and let

$$B = \left\{H[0, 1, 3, 4, 5, 6, \infty], H[4, 1, 5, 3, \infty, 0, 2]\right\}.$$

There are 56 edges in $K_8^{(3)}$. Trying to use $\mathbb{Z}_8$ as our vertex set results in 7 difference classes of size 8. Since 7 is not divisible by the number of edges in P_7, we cannot cyclically decompose $K_8^{(3)}$ using $\mathbb{Z}_8$ and instead use a fixed point. The difference classes in $\mathbb{Z}_7 \cup \{\infty\}$

are as follows:

$$(1, 1, 5), (1, 2, 4), (1, 3, 3), (1, 4, 2), (2, 2, 3), (\infty, 1, 6), (\infty, 2, 5), (\infty, 3, 4).$$

We now have 8 difference classes of size 7. In Figure 3, the difference classes of each from left to right are $(1, 2, 4)$, $(1, 4, 2)$, $(1, 1, 5)$, and $(\infty, 1, 6)$. In Figure 5, these difference classes are $(1, 3, 3)$, $(2, 2, 3)$, $(\infty, 3, 4)$, and $(\infty, 2, 5)$. Since every difference class is the same size and each is represented exactly once in B, we can obtain the remaining edges by clicking as a 1-pyramidal decomposition.

Then a P_7-decomposition of $K_8^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod 7$ on the vertices.

4.2 K_9

Let $V\left(K_9^{(3)}\right) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$ and let

$$B = \big\{H[\infty_1, \infty_2, 0, 1, 2, 3, 6], \, H[\infty_1, 0, 2, \infty_2, 1, 4, 6], \, H[0, \infty_1, 1, 4, 3, 6, 5]\big\}.$$

Recall from Section 2 that using $\mathbb{Z}_9$ as our vertex set for $K_9^{(3)}$ results in a short orbit. We also find that using $\mathbb{Z}_8 \cup \{\infty\}$ results again in a short orbit for the difference class $(\infty, 4, 4)$. Finally, we arrive at $\mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$, which partitions the 84 edges into 12 difference classes of size 7. Recall from Section 2 that these difference classes are:

$$\{(1, 1, 5), (1, 2, 4), (1, 3, 3), (1, 4, 2), (2, 2, 3), (1, 6, \infty_1),$$
$$(2, 5, \infty_1), (3, 4, \infty_1), (1, 6, \infty_2), (2, 5, \infty_2), (3, 4, \infty_2), (7, \infty_1, \infty_2)\}$$

The difference classes in the first starter from left to right are $(7, \infty_1, \infty_2)$, $(\infty_2, 1, 6)$, $(1, 1, 5)$, and $(1, 3, 3)$. The difference classes in the first starter are $(\infty_1, 2, 5)$, $(\infty_2, 2, 5)$,

$(\infty_2, 3, 4)$, and $(2, 2, 3)$. The difference classes from the third starter are $(\infty_1, 1, 6)$, $(\infty_1, 3, 4)$, $(1, 2, 4)$, and $(1, 4, 2)$. Since every difference class is the same size and each is represented exactly once in B, we can obtain the remaining edges by clicking as a 2-pyramidal decomposition.

Then a P_7-decomposition of $K_9^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty_i$, for $i \in \{1, 2\}$ and $j \mapsto j + 1 \pmod 7$ on the vertices.

The following examples are done in a similar manner and thus we only list the starters and action on the vertices.

4.3 K_{10}

Let $V\left(K_{10}^{(3)}\right) = \mathbb{Z}_{10}$ and let

$$B = \big\{ H[8, 0, 7, 4, 2, 9, 1], \ H[8, 0, 2, 6, 9, 4, 5], \ H[9, 0, 1, 4, 7, 8, 3] \big\}.$$

Then a P_7-decomposition of $K_{10}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $j \mapsto j + 1 \pmod{10}$ on the vertices.

4.4 K_{12}

Let $V\left(K_{12}^{(3)}\right) = \mathbb{Z}_{11} \cup \{\infty\}$ and let

$$B = \big\{ H[10, 0, 1, \infty, 2, 6, 9], \ H[\infty, 0, 2, 4, 1, 7, 10], \ H[3, \infty, 0, 5, 4, 9, 10],$$
$$H[6, 0, 2, 7, 3, 8, 5], \ H[1, 0, 3, 4, 6, 7, 10] \big\}.$$

Then a P_7-decomposition of $K_{12}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{11}$ on the vertices.

4.5 K_{14}

Let $V\left(K_{14}^{(3)}\right) = \mathbb{Z}_{13} \cup \{\infty\}$ and let

$$B = \{H[1, \infty, 0, 6, 7, 8, 2],\ H[2, \infty, 0, 5, 4, 9, 3],\ H[3, \infty, 0, 4, 5, 11, 2],\ H[0, 1, 9, 11, 5, 8, 4],$$
$$H[3, 0, 2, 6, 4, 8, 1],\ H[2, 0, 8, 11, 1, 6, 10],\ H[1, 0, 3, 4, 8, 10, 2]\}.$$

Then a P_7-decomposition of $K_{14}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{13}$ on the vertices.

4.6 $L_{4,4}$

Let $V\left(L_{4,4}^{(3)}\right) = \mathbb{Z}_6 \cup \{\infty_1, \infty_2\}$ with vertex partition $\{\{0, 2, 4, \infty_2\}, \{1, 3, 5, \infty_1\}\}$. Then

$$B = \{H[2, 3, 4, \infty_1, \infty_2, 0, 1]\},$$
$$B^* = \{H[5, 0, 2, \infty_1, 1, 4, 3], H[4, 2, 5, \infty_2, 1, 3, 0]\}.$$

In this case, we are not able to decompose the hypergraph cyclically or r-pyramidally. Here, we have partitioned the H-blocks into B and B^* with unique actions on each. The difference classes of $\mathbb{Z}_6 \cup \{\infty_1, \infty_2\}$ are

$$\{(1, 1, 4), (1, 2, 3), (1, 3, 2), (\infty_1, 1, 5), (\infty_2, 1, 5),$$
$$(\infty_1, 2, 4), (\infty_2, 2, 4), (\infty_1, 3, 3), (\infty_2, 3, 3), (\infty_1, \infty_2, 6)\},$$

where $(\infty_1, 2, 4), (\infty_2, 2, 4), (\infty_1, 3, 3), (\infty_2, 3, 3)$ have orbits of size 3 and the rest have orbits of size 6. The difference classes in B from left to right are $(1, 1, 4), (\infty_1, 1, 5), (\infty_1, \infty_2, 6), (\infty_2, 1, 5)$. Thus the edges in these difference classes can be obtained in an r-pyramidal manner by clicking. For the remaining difference classes, we click the edges two at a time by adding 2 to each vertex. This allows us to use half of the full orbits $(1, 2, 3)$ and $(1, 3, 2)$ in each starter

19

along with the short orbits.

Thus a P_7-decomposition of $L_{4,4}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty_i \mapsto \infty_i$ for $i \in \{1,2\}$ and $j \mapsto j+1 \pmod 6$, and the orbits of the P_7-blocks in B^* under the action of the map $\infty_i \mapsto \infty_i$ for $i \in \{1,2\}$ and $j \mapsto j+2 \pmod 6$.

4.7 $K_{2,4,4}$

Let $V\left(K_{2,4,4}^{(3)}\right) = \mathbb{Z}_8 \cup \{\infty_1, \infty_2\}$ with vertex partition $\{\{\infty_1, \infty_2\},\ \{0,2,4,6\},\ \{1,3,5,7\}\}$ and let

$$B = \big\{ H[1, \infty_1, 0, 3, 4, \infty_2, 7] \big\}.$$

Then an P_7-decomposition of $K_{2,4,4}^{(3)}$ consists of the orbit of the H-block in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2\}$ and $j \mapsto j+1 \pmod 8$ on the vertices.

4.8 $L_{4,6}$

Let $V\left(L_{4,6}^{(3)}\right) = \mathbb{Z}_6 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ with the obvious vertex partition and let

$$B = \big\{ H[1, \infty_4, 2, 4, 0, \infty_3, 5],\ H[1, \infty_1, 2, 4, 0, \infty_2, 5] \big\},$$

$$B^* = \big\{ H[\infty_1, 0, 3, \infty_2, 1, 4, \infty_3],\ H[\infty_2, 2, 5, \infty_4, 0, 3, \infty_3],\ H[\infty_4, 1, 4, \infty_1, 2, 5, \infty_3] \big\},$$

$$B^{**} = \big\{ H[0, \infty_1, \infty_2, 1, \infty_3, \infty_4, 2],\ H[0, \infty_1, \infty_3, 1, \infty_2, \infty_4, 2],\ H[0, \infty_1, \infty_4, 1, \infty_2, \infty_3, 2] \big\}.$$

Then a P_7-decomposition of $L_{4,6}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2,3,4\}$ and $j \mapsto j+1 \pmod 6$, the H-blocks in B^*, and the orbits of the H-blocks in B^{**} under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2,3,4\}$ and $j \mapsto j+2 \pmod 6$.

4.9 $K_{1.4,4} \cup L_{4,4}$

Let $V\left(K_{1,4,4}^{(3)} \cup L_{4,4}^{(3)}\right) = \mathbb{Z}_8 \cup \{\infty\}$ with vertex partition $\{\{\infty\}, \{0, 2, 4, 6\}, \{1, 3, 5, 7\}\}$ and let

$$B = \{H[1, \infty, 0, 3, 2, 4, 7], \ H[3, 1, 4, 0, 5, 6, 2]\}.$$

Then a P_7-decomposition of $K_{1,4,4}^{(3)} \cup L_{4,4}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod 8$ on the vertices.

5 Proof of Main Theorem

Here we prove the main theorem, which is restated below for the convenience of the reader.

Main Theorem. *There exists a P_7-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, 2, 4,$ or 6 (mod 8) and $v \geq 7$.*

Proof. It remains to show necessity. Sufficiency is established by Lemma 3 and the explicit P_7-decompositions in Section 4. We require that $v \equiv 0, 1, 2, 4$ or $6 \pmod 8$ and $v \geq 7$ based on the following conditions:

1. Order condition: $v \geq 7$

2. Size condition: $4 \mid \binom{v}{3}$

Since there must be at least 7 vertices in $K_v^{(3)}$ to embed a single subgraph of order 7, we have the order condition. The size condition follows because we need the number of edges in $K_v^{(3)}$ to be divisible by the number of edges in P_7. Our goal is to fully decompose $K_v^{(3)}$ into copies of P_7, so there cannot be any edges in $K_v^{(3)}$ left over. Since our graph is 3-uniform, we have 3 vertices per edge and thus the number of edges in $K_v^{(3)}$ is $\binom{v}{3}$. The hypergraph P_7 has size 4, so $\binom{v}{3}$ must be divisible by 4.

We have $\binom{v}{3} = \frac{v(v-1)(v-2)}{6}$, which tells us that 24 must divide $v(v-1)(v-2)$. Since we know that at least one of these terms must also be divisible by 3, we have that $8 \mid v(v-1)(v-2)$. Checking values modulo 8, we find that the size condition holds precisely when $v \equiv 0, 1, 2, 4$, or 6 (mod 8). Combining these two conditions, we arrive at the Main Theorem. $\qquad\square$

6   Conclusion

By providing the examples in Section 4, we successfully provided a P_7-decomposition for the complete 3-uniform hypergraph $K_v^{(3)}$ when $v \equiv 0, 1, 2, 4$ or 6 (mod 8) and $v \geq 7$ and have settled the spectrum problem for P_7. These results parallel those of similar problems for different hypergraphs, such as the cycles by Akin, et al. in [1], Bunge, et al. in [2], and Bunge, et al. in [3]. Many of these graphs were studied by the Illinois State University Math REU. The results of P_7 open the door to research on similar graphs, such as a loose 4-path of order 9 or tight 4-path of order 6, shown in Figure 7.
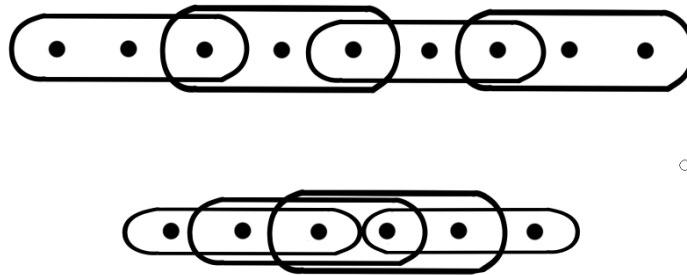

Figure 7: Loose 9-Path and Tight 6-Path

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