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Interpolation Problems and the Characterization of the Hilbert Function

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Interpolation Problems and the Characterization of the Hilbert Function

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1 Introduction

In mathematics, it is often useful to approximate the values of functions that are either too awkward and difficult to evaluate or not readily differentiable or integrable. To approximate its values, we attempt to replace such functions with more well-behaving examples such as polynomials or trigonometric functions. This is accomplished by constructing a well-behaving function that agrees with the values of a discrete set of arguments from the poorly-behaving function. Specifically, the problem of interpolation as defined in [18] is: "Given the values of a function for a finite set of arguments, to approximate the value of the function for some intermediate argument."

In the case of using polynomial approximations, it is already known that for a polynomial f of one variable over the algebraically closed field $\mathbb{C}$ and any set of r points on the affine complex line $\mathbb{A}^1$ with multiplicities $m_1, m_2..., m_r$, f is completely determined by its zeros and the vanishing conditions up to its $m_i - 1$ derivative for each point. It is then natural to consider the case in higher dimensions corresponding to polynomials in several variables.

An active area of research is analyzing interpolation with multivariate polynomials. In this case, the problem is much more difficult so the existing research primarily focuses on the easier case where the ambient space is the projective rather than the affine space (which is analogous to using the more varied tools in complex analysis to study integrals of real functions), and the number of times a polynomial passes through a set of discrete points is the same for every point in the set. In other words, the multiplicity for each point is equal. Even with these simplifications, the problems have proved to be incredibly hard, and a large body of literature is dedicated to investigating them.

For this thesis, we will classify polynomials in three variables passing through a set of discrete points using an abstract algebraic structure known as an ideal. Then, we will analyze these ideals and specifically provide structural and numerical information. That is, we characterize their Hilbert Functions, which in our setting, are functions describing the number of linearly independent polynomials passing through the set of points with given multiplicity. Specifically, we will also see that in these cases, there is an expected "maximal" Hilbert Function value, and the main goal is to determine if these ideals have the "maximal" Hilbert Function or not.

The Hilbert Function of an ideal defined by any set of $r \leq 9$ points is well known from [16]. Additionally, [11] determines that all possible Hilbert Functions for 6 points arise from only 11 different configurations. This work is even further expanded in [7] where the authors find 29 different configurations for 7 points, 143 different types for 8 points, and prove that there are infinite configurations for 9 points or more. However, the methods used in these publications utilize very advanced theoretical approaches from algebraic geometry. The strategies used in this thesis aim to answer special cases of the above question using elementary techniques. In other words, instead of developing and explaining several years' worth of existing mathematical theory, we will analyze the properties of these polynomial rings and ideals directly, working closely with their generators and calculating their Hilbert Functions explicitly. Using such elementary techniques, we will reprove the following results:

- For 3 points each with multiplicity 3, the Hilbert Function is maximal. For 3 points each with multiplicity $m > 3$, the Hilbert Function is not maximal.

- For 4 points each with multiplicity m, the Hilbert Function is always maximal.
- For 5 points each with multiplicity $m > 1$, the Hilbert Function is never maximal.
- For 6 points each with multiplicity m where $m = 4, 6, 8$, the Hilbert function is maximal.

1.1 Connections to Symbolic Rees Algebras and Hilbert's 14th Problem

In 1900, German mathematician David Hilbert presented a list of 23 problems all unsolved at the time which he considered to be fundamental for the development of mathematics in the 20th century. Particularly, Hilbert's 14th problem asks about the finite generation of certain algebras. More precisely, Hilbert conjectured that for a field k and a sub-field of the field of rational functions in n variables K, the k-Algebra $R := K \bigcap k[x_1, ..., x_n]$ would be finitely generated over k. Although Masayoshi Nagata disproved this conjecture in 1958, the question sparked active areas of research into exploring other algebras which may be finitely generated. One such example with relatively limited literature is the symbolic Rees Algebras of an ideal I in a Noetherian ring R, denoted $\mathcal{R}_s(I)$.

More information and formal definitions on the properties and structure of the symbolic Rees Algebras can be found in [10]. More importantly, it is known from [15, Lem 4] and [8, Thm 3.2] that $\mathcal{R}_s(I)$ is a finitely generated R-algebra if and only if there exists a n such that $(I^{(n)})^t = I^{(nt)}$ for all $t \geq 1$. We will later define terms such as "ideal," "ring," and "Noetherian" in the context of polynomials and points more rigorously and show that the ring R that we are working with within this thesis is indeed Noetherian, so a secondary goal of this research is also to find a n that satisfies that above criteria for a finite generation. Specifically, we will prove the following results:

- $n = 2$ suffices for 3 and 5 points.
- Any n suffices for 4 points.
- $n = 10$ suffices for 6 points.

To summarize, we will primarily study two questions. First, by linear algebra, there is an expected number of linearly independent equations of given degrees passing through a set of "random" points with given multiplicity which is represented by the Hilbert Function. We study this question for sets of points of small size (up to 6 points). Secondly, from Nagata's results, for these ideals I we know there exist some n such that $(I^{(n)})^t = I^{(nt)}$ for every t. We study these values of n.

2 Preliminaries/Background

We will begin by explaining the important notations and theorems necessary to prove the key results in this thesis.

2.1 Projective Plane

We first describe the projective plane. In $\mathbb{R}^2$, points are notated using the tuple (x, y) where $x, y \in \mathbb{R}$. In $\mathbb{P}^2(\mathbb{R})$, points are determined using three coordinates (x, y, z) where $x, y, z \in \mathbb{R}$.

Definition 2.1 ($\mathbb{R}$-Projective Plane, [3])**.** *For any $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$, we define the equivalence relation $\sim$ where $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if there is a nonzero number $\lambda \in \mathbb{R}$ such that $(x_1, y_1, z_1) = \lambda(x_2, y_2, z_2)$. Then, $\mathbb{P}^2(\mathbb{R}) = (\mathbb{R}^3 - \{(0, 0, 0)\})/\sim$*

The "$/\sim$" part of the definition means that in $\mathbb{P}^2(\mathbb{R})$, the points $(1, 1, 1), (2, 2, 2)$, and (π, π, π) are all equal in $\mathbb{P}^2(\mathbb{R})$. Furthermore, it is important to state that $(0, 0, 0)$ is not a point in $\mathbb{P}^2(\mathbb{R})$. A line in $\mathbb{P}^2(\mathbb{R})$ is defined as the set of points $(x, y, z) \in \mathbb{P}^2(\mathbb{R})$ that satisfies an equation of the form $Ax + By + Cz = 0$ where $A, B, C \in \mathbb{R}$.

For the purposes of this thesis, we will specifically use the projective plane over $\mathbb{C}$, denoted $\mathbb{P}^2(\mathbb{C})$. In this set, the definition is identical except a point (x, y, z) has coordinates in $\mathbb{C}$ instead of $\mathbb{R}$. Unless otherwise specified, we will simplify $\mathbb{P}^2(\mathbb{C})$ to just $\mathbb{P}^2$. The reason why we decide to use $\mathbb{C}$ instead of $\mathbb{R}$ is because we will later see that there is a relationship between points and equations primarily motivated by the zeros of the equations. In $\mathbb{R}$, we have the issue where some equations cannot be associated with any points ($x^2 + 1$ has no zeros for example). However, in $\mathbb{C}$, there is a property known as algebraic closure, where every equation will have a zero.

One of the most important fundamental properties of the projective plane is the property of duality. The property states that any true statement regarding lines and points still remains true if every instance of the word "line" was replaced with "point" and vice versa. For example, notice that in the standard $\mathbb{R}^2$ space, duality does not hold. We know that for every two points in $\mathbb{R}^2$, there exists a unique line that passes through them. However, because of the existence of parallel lines, there exist two lines that do not intersect at a single point. Since duality holds in $\mathbb{P}^2$, any two distinct lines will intersect at a unique point, meaning that parallel lines do not exist in the projective plane.

2.2 Rings, Ideals, and Varieties

Now that we have established the projective plane, we now want to consider the set of polynomial equations with coefficients in $\mathbb{P}^2$. To do this, we will first formalize the definitions of abstract algebraic structures that generalize these polynomials.

Definition 2.2 (Binary Operation, [5])**.** *A binary operation $\star$ on a set R is a function $\star : R \times R \to R$ where for any $x, y \in R$, $\star(x, y) = x \star y \in R$.*

Definition 2.3 (Rings, [5])**.** *A set R with two binary operations $+$ and $\times$ (called addition and multiplication) is a ring if R satisfies the following proprieties:*

1. *$+$ is associative: for all $x, y, z \in R$, $(x + y) + z = x + (y + z)$.*

2. *$+$ is commutative: for all $x, y \in R$, $x + y = y + x$*

3. *There is an additive identity: there exists an element* $0 \in R$ *such that for all* $x \in R$, $x + 0 = 0 + x = x$

4. *There is an additive inverse: for every* $x \in R$ *there exists an element* $-x \in R$ *such that* $x + (-x) = (-x) + x = 0$.

5. $\times$ *is associative: for all* $x, y, z \in R$, $(x \times y) \times z = x \times (y \times z)$.

6. *Left distribution holds: for all* $x, y, z \in R$, $x \times (y + z) = (x \times y) + (x \times z)$

7. *Right distribution holds: for all* $x, y, z \in R$ $(x + y) \times z = (x \times z) + (y \times z)$,

Additionally, a ring R *is commutative if* $\times$ *is commutative and* R *is a ring with identity if there exists a* $1 \in R$ *not equal to the additive identity* $0 \in R$ *such that for all* $x \in R$, $1 \times x = x \times 1 = x$. *We will often denote* $x \times y$ *as just* xy. *Lastly, in a commutative ring with identity, a ring element* x *is called a unit if there exists another element* y *such that* $xy = 1$. *If every nonzero ring element is a unit, then we call the ring a field.*

Common examples of rings with standard addition and multiplication operations include the integers, denoted $\mathbb{Z}$, and polynomials in one variable with real number coefficients, denoted $\mathbb{R}[x]$. Intuitively, a ring R is simply a collection of objects with binary operations that satisfy certain rules. These rules state that we can add, subtract, and multiply ring elements but not divide them unless the element is a unit. Although the most abstract definition of a ring does not necessitate that it be commutative or have an identity, for the remainder of this thesis all rings will be commutative rings with identity. Furthermore, it is often useful to consider a subset of the entire ring. In particular, a subring I is a subset of R that also satisfies the conditions for being a ring under the same binary operations as R. Furthermore, we call a subring I an ideal of R if it satisfies the following additional properties.

Definition 2.4 (Ideal, [5])**.** *Let* R *be a ring and let* $r \in R$. *Let* I *be a subset of* R. *Define* $rI = \{ra : a \in I\}$ *and* $Ir = \{ar : a \in I\}$. I *is an ideal of* R *if* I *satisfies the following properties*

1. I *is a subring of* R.

2. $rI \subseteq I$.

3. $Ir \subseteq I$.

Additionally, if $r_1, r_2 ... r_n \in R$, *then we call* $(r_1, r_2, ..., r_n)$ *the ideal generated by* $\{r_1, ..., r_n\}$, *which is the smallest ideal that contains* $\{r_1, ..., r_n\}$. *Ideals generated by one element are called principal ideals.*

For example, consider the ring $\mathbb{Z}$. Then, for any $n \in \mathbb{Z}$, the principal ideal (n) consists of all the elements in $\mathbb{Z}$ that are a multiple of n. In fact, all ideals in $\mathbb{Z}$ are principal. Furthermore, for any ring, there is precisely one ideal that contains one element, (0). This is sometimes called the trivial ideal.

Next, we will define some common ways ideals are used in other structures.

Definition 2.5 (Quotient Ring, [5])**.** *Let R be a ring and I be an ideal of R. A coset is a subset of R of the form $r + I = \{r + i : i \in I\}$ for some $r \in R$. If I is an ideal, then the following two operations are well-defined.*

1. *For any $r, s \in R$, $(r + I) + (s + I) = (r + s) + I$*
2. *For any $r, s \in R$, $(r + I)(s + I) = (rs) + I$.*

With these two operations, the collection of all cosets also forms a ring. We denote this ring using R/I and call it a quotient ring.

Definition 2.6 (Maximal Ideal, [5])**.** *Let R be a ring and $\mathfrak{m}$ be an ideal of R. We say that $\mathfrak{m}$ is a maximal ideal if $\mathfrak{m} \neq R$ and the only ideals containing $\mathfrak{m}$ are $\mathfrak{m}$ and R.*

Consider the ring $\mathbb{R}[x, y]$. The ideal (x) is an example of a non-maximal ideal because $(x) \subset (x, y)$. However, (x, y) is a maximal ideal. To see this, suppose there is an ideal properly containing (x, y), call it $\mathfrak{m}$. If we take any element $z \notin (x, y)$, then z must be a constant. But, by the closure of multiplication, $1 = z(1/z) \in \mathfrak{m}$. But this implies that $\mathfrak{m} = R$, so (x, y) must be maximal.

Definition 2.7 (Ideal Operations, [5])**.** *Let R be a ring and I, J be ideals of R. We define:*

- *The sum of I and J by $I + J = \{i + j : i \in I, j \in J\}$*
- *The product of I and J by IJ is the set of all finite sums of elements of the form ij with $i \in I$ and $j \in J$*
- *For any $n \geq 1$, the n power of I, denoted I^n is the set of all finite sums of elements of the form $i_1 i_2 ... i_n$ with $i_k \in I$ for every $1 \leq k \leq n$*

The following is another useful tool to help calculate intersections of ideals directly.

Proposition 2.8 (Modularity Law, [5])**.** *Let R be a ring and I, J, K be ideals of R. Further, suppose $J \subseteq I$. Then, $I \bigcap (J + K) = J + (I \bigcap K)$.*

Definition 2.9 (Integral Domain, [5])**.** *Let R be a commutative ring with identity. R is an integral domain if it has no zero divisors. That is, if $x, y \in R$ and $xy = 0$, then either $x = 0$ or $y = 0$.*

One of the most obvious examples of an integral domain is $\mathbb{Z}$. Indeed, the multiplication of any two nonzero numbers will always be nonzero. However, the quotient ring $\mathbb{Z}/(4)$ is not an integral domain. This is because $2+(4) \neq 0+(4)$ but $(2+(4))(2+(4)) = 4+(4) = 0+(4)$.

Definition 2.10 (Unique Factorization Domain, [5])**.** *Let R be an integral domain. Let $r \in R$ be a nonzero element that is not a unit. Call an element x_i irreducible if it cannot be written as a product of elements in R. Then, R is a Unique Factorization Domain (UFD) if r satisfies the following two properties.*

1. *r can be rewritten as a finite product of irreducibles: $r = x_1 x_2 ... x_n$*

2. *The product is unique up to units. That is, if* $r = y_1 y_2 ... y_m$ *for some other set of irreducible elements* $\{y_i\}$*, then* $n = m$ *and the factors can be reordered such that* $x_i = u_i y_i$ *for some unit* u_i *for all* $1 \leq i \leq n$*.*

The rings we will specifically use are polynomial rings over a field which are UFDs. In more detail, let $R = \mathbb{C}[x, y, z]$ be the set of all polynomials with three variables with coefficients in $\mathbb{C}$. Then, since standard polynomial addition and multiplication satisfy the ring properties in Definition 2.3, R is a ring. Additionally, since polynomial multiplication is indeed commutative, and the constant polynomial 1 is the multiplicative identity, R is a commutative ring with identity. Lastly, from [5, Sec 9.3, Thm 7], R is also a UFD.

The ring $R = \mathbb{C}[x, y]/(x^2 - y^3)$ is an example of a non-UFD. To see this, consider the elements x and y. These elements do not have inverses in R, so they are not units and there is no way to write x and y as a product of non-units since they both have degree 1. However in R, we have $x^2 = y^3$. So, we have an element in R with two different factorizations.

We now combine the concepts defined above to describe projective varieties. First, to establish some intuition, consider the polynomial equation $x - y^2 = 0$. If we take solutions in $\mathbb{C}^2$, the point $(4, 2)$ is a solution to this equation. However, we must be more careful in $\mathbb{P}^2$. For example, in $\mathbb{P}^2$, $(4, 2) = (8, 4)$, but $8 - 16 = -8 \neq 0$ so $(8, 4)$ is not a solution. To avoid this, we use homogeneous polynomials, which is a polynomial that has the same total degree in every term. Examples of homogeneous polynomials include $x + y$, $xy - x^2$, $y^{30} + x^{29}y$. Non-examples include $x - y^2$, $y^2 + x^4$, $y^3 - x^2$.

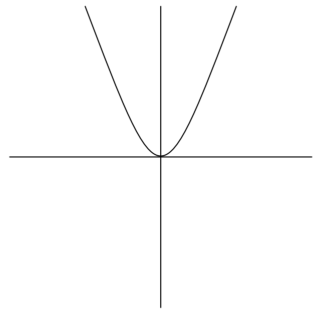
Definition 2.11 (Homogeneous Ideal, [3])**.** *I is a called a homogeneous ideal of* $\mathbb{C}[x, y, z]$ *if* $I = (f_1, f_2, ..., f_n)$ *where* $f_1, ..., f_n$ *are all homogeneous polynomials.*

Using the examples from above, the ideal $(x + y, xy - x^2, y^{30} + x^{29}y)$ is an example of a homogeneous ideal, where $(x + y, x - y^2)$ is not. Notice that in order to be a homogeneous ideal, each generator does not necessarily have to be homogeneous of the same degree.

Next, we formally define projective varieties. One can think of these objects as the points that solve a system of homogeneous equations. The goal of studying projective varieties is to extend linear algebra to homogeneous equations that are not necessarily linear.

Definition 2.12 (Projective Varieties, [3])**.** *If* f *is a homogeneous polynomial in* $R = \mathbb{C}[x, y, z]$*, then* $\boldsymbol{V}(f) = \{(x, y, z) \in \mathbb{P}^2 : f(x, y, z) = 0\}$*.*
If I *is a homogeneous ideal generated by* $f_1, ..., f_n$*, then* $\boldsymbol{V}(I) = \boldsymbol{V}(f_1, ..., f_n) = \{(x, y, z) \in \mathbb{P}^2 : f_i(x, y, z) = 0, 1 \leq i \leq n\}$*.*
We call these sets projective varieties.

The easiest examples of projective varieties are of the form $\mathbf{V}(f_1, f_2, ... f_n)$ where each f_i is a homogeneous degree 1 equation. In this case, the variety is no different than solving the null space of a system of linear equations in linear algebra. For example, $\mathbf{V}(x, y) = \{(0, 0, 1)\}$. For the projective plane $\mathbb{P}^2(\mathbb{R})$, it is generally difficult to sketch the variety because of the complicated definitions. However, recall that points in the projective plane are equivalence classes, so any point of the form (x, y, z) is equivalent to a point of the form $(x', y', 1)$. So, one can always set a variable equal to 1 to visualize at least part of the variety. The sketch below shows the variety $\mathbf{V}(yz - x^2)$ by setting $z = 1$.



Definition 2.13 (Defining Ideal, [12])**.** *Let x be a point in $\mathbb{P}^2$. Then, $\boldsymbol{I}(x) = \{f \in \mathbb{C}[x, y, z] : f(x) = 0\}$.*
If X is a set of r points in $\mathbb{P}^2$, then $\boldsymbol{I}(X) = \{f \in \mathbb{C}[x, y, z] : f(x_i) = 0, x_i \in X, 1 \leq i \leq r\}$.
We call $\boldsymbol{I}(x)$ and $\boldsymbol{I}(X)$ the defining ideal of x and X respectively.

Let $R = \mathbb{C}[x, y, z]$, and X is a set of r points in $\mathbb{P}^2$. For $1 \leq i \leq r$, if P_i is the defining ideal of the i^{th} point in X, then a consequence of Zariski and Nagata's Theorem described in [6, Sec 3.9] states that the defining ideal of X is $\bigcap_{1 \leq i \leq r} P_i$, and if we let $I = \mathbf{I}(X)$, then the m^{th} symbolic power of I, denoted $I^{(m)}$, is equal to $\bigcap_{1 \leq i \leq r} P_i^m$. Geometrically, $I^{(m)}$ is the homogeneous ideal consisting of all the homogeneous polynomials in R that passes through each point of X at least m times. Note that this is different from ordinary powers as defined in Definition 2.7, where $I^m = (\bigcap_{1 \leq i \leq r} P_i)^m$. For every $1 \leq i \leq r$, if $x \in I^m$, then $x \in P_i^m$, so $x \in I^{(m)}$. This implies that $I^m \subseteq I^{(m)}$, but the reverse inclusion is not always true.

2.3 Advanced Ring/Ideal Theory and the Algebra-Geometry Map

Notice that in the case of polynomial rings, we can write the defining ideal of a set of points as an intersection of the defining ideals of each individual point. To get a more robust understanding of the properties of these intersecting ideals and how they connect to points in $\mathbb{P}^2$, we will need to first introduce a few more advanced definitions and properties of rings and ideals.

Definition 2.14 (Prime and Primary Ideals, [5])**.** *Let R be a ring and P, Q be ideals of R. P is a prime ideal if $xy \in P$ implies that either $x \in P$ or $y \in P$. Q is a primary ideal if $xy \in Q$ implies that either $x \in Q$ or $y^n \in Q$ for some $n \in \mathbb{N}$. All prime ideals are primary. Additionally, all maximal ideals are prime.*

The ring $\mathbb{Z}$, the ideal (2), or the set of even integers, is a prime ideal. This is due to the simple fact that a product of any two numbers resulting in an even number implies immediately that one of the numbers must be even. However, the ideal (4) is not prime because $2 \notin (4)$ but $2 * 2 = 4 \in (4)$

Definition 2.15 (Radical of an Ideal, [5])**.** *Let I be an ideal of a ring R. Denote $\sqrt{I} = \{x \in R : x^n \in I, n \in \mathbb{N}\}$. If $\sqrt{I} = I$, then I is a radical ideal.*

In $\mathbb{Z}$, radical ideals are precisely the ideals generated by square free numbers. That is, in the prime decomposition of any number, every prime component is raised only to the power of 1. For example, $10 = 2 * 5$ is square free but $45 = 3^2 * 5$ is not. In order to calculate the radical of an ideal (n) for some n with prime decomposition ${p_1}^{q_1}{p_2}^{q_2}...{p_r}^{q_r}$, one simply needs to take the ideal $(p_1 p_2 ... p_r)$.

Definition 2.16 (P-Primary Ideal, [5]). *Let R be a ring and P, Q be ideals of R. Suppose P is prime and Q is primary. If $\sqrt{Q} = P$, then Q is a P-primary ideal.*

Although we have seen that (4) is not a prime ideal, it is a primary ideal. This is because if a xy is a multiple of 4, then either x is a multiple of 4 or y is a multiple 2. On the other hand, (10) is not a primary ideal because $2 * 5 \in (10)$ but $2 \notin (10)$ and $5^n \notin (10)$ for any $n \in \mathbb{N}$.

Proposition 2.17. *Here are some useful properties of prime and primary ideals.*

1. *[17] Suppose $n \in \mathbb{N}$. $I = \bigcap_{1 \leq i \leq n} P_i$ where each P_i is a prime ideal if and only if I is a radical ideal.*
2. *[5] Every prime ideal is radical.*
3. *[5] If Q is a primary ideal, then $\sqrt{Q}$ is a prime ideal. In other words, Q is $\sqrt{Q}$-Primary.*
4. *[5] If P is a prime ideal that contains I^n for some ideal I and natural number n, then, P also contains I.*

It is important to note that the powers of prime ideals are not P-primary in general. However, in our case where $R = \mathbb{C}[x, y, z]$ and P is the defining prime ideal of a point, [6, Thm 3.14] shows that P^n for $n \in \mathbb{N}$ is indeed a P-primary ideal.

From the examples above working in $\mathbb{Z}$, we have seen that prime ideals correlate to prime numbers, primary ideals correlate to powers of primes, and radical ideals correlate to square-free numbers. Furthermore, just as how any number can be decomposed into prime numbers, we also want to generalize when ideals in general also decompose into their primary components. It turns out that rings in which this occurs have a very special property.

Definition 2.18 (Noetherian Rings, [5]). *Let R be a ring. R is Noetherian if for every increasing chain of ideals in R eventually stops, that is if $I_1 \subset I_2 \subset ... \subset I_k \subset ...$, then there exists a $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geq k$.*

Theorem 2.19 (Hilbert Basis Theorem, [5]). *If R is a Noetherian ring, then $R[X]$, where X is a set of variables, is also a Noetherian ring.*

Definition 2.20 (Primary Decomposition, [17]). *Let I be an ideal of a ring R. The primary decomposition of I is an expression such that $I = Q_1 \bigcap Q_2 ... \bigcap Q_k$ where:*

1. *k is a finite natural number.*
2. *For all $1 \leq i \leq k$, Q_i is a primary ideal.*
3. *$I \subset \bigcap_{1 \leq i \leq k, i \neq j} Q_i$ for every $1 \leq j \leq k$. This property states that no Q_i is redundant.*

4. $\sqrt{Q_i} \neq \sqrt{Q_j}$ *for all* $1 \leq i, j \leq k$*. This property states each* Q_i *is contained in a distinct prime ideal.*

Theorem 2.21 (Primary Decomposition Existence, [17])**.** *In a Noetherian ring* R*, every ideal has a primary decomposition.*

It is often difficult to calculate the primary decomposition of a specific ideal in general. However, in cases where we do know the primary decomposition, we gain a lot of additional information about the ideal. In particular, we can intuitively think of the ideal in terms of the prime ideals that contain it.

Definition 2.22 (Associated Primes, [17])**.** *Let* I *be an ideal in a Noetherian ring* R *with the primary decomposition* $Q_1 \bigcap Q_2 \bigcap ... \bigcap Q_k$ *for some* $k \in \mathbb{N}$*. Then, the associated primes of* I*, denoted* $Ass(R/I)$ *is the set* $\{\sqrt{Q_i} : 1 \leq i \leq k\}$*.*

Definition 2.23 (Minimal Primes, [6])**.** *Let* P *be a prime ideal that contains another ideal* I *in a Noetherian ring* R*. Then,* P *is a minimal prime over* I *if for any prime* Q *such that* $I \subseteq Q \subseteq P$*, we have that* $Q = P$*. The set of all minimal primes over* I *is denoted* $Min(I)$*.*

Proposition 2.24 (Minimal Primes are a Subset of Associated Primes, [6])**.** *Let* I *be an ideal of a Noetherian ring* R*. Then,* $Min(I) \subseteq Ass(R/I)$

Definition 2.25 (Irreducible Varieties, [17])**.** *Let* X *be a nonempty variety in* $\mathbb{P}^2$*. Then, if* $X = X_1 \bigcup X_2$*, then* $X = X_1$ *or* $X = X_2$*. Equivalently,* X *cannot be written as the union of two strictly smaller projective varieties.*

We will soon see in the following theorem that irreducible varieties will correspond to radical prime ideals using the functions defined in Definitions 2.12 and 2.13.

But first, we note that the set of complex numbers can easily be shown to be a commutative ring with 1. Additionally, for every nonzero element $x \in \mathbb{C}$, $1/x \in \mathbb{C}$ so there always exists an element $y \in \mathbb{C}$ such that $xy = 1$. If x is a nonzero element in some ideal I of $\mathbb{C}$, then by property 2 in Definition 2.4, $(1/x)x = 1 \in I$ which also implies $\mathbb{C} \subseteq I$. Therefore, the only ideals of $\mathbb{C}$ are 0 and $\mathbb{C}$, so $\mathbb{C}$ is Noetherian. By the Hilbert Basis Theorem, we now know that $\mathbb{C}[x, y, z]$ is a Noetherian ring. Lastly, Proposition 2.19 implies that every ideal of $\mathbb{C}[x, y, z]$ indeed has a primary decomposition.

We are now ready to state the key relationship between ideals (equations) in $\mathbb{C}[x, y, z]$ and varieties (points) in $\mathbb{P}^2$.

Theorem 2.26 (Hilbert's Nullstellensatz ($\mathbb{C}$-Projective Version), [3])**.** *Let* $\mathcal{X} = \{X \subseteq \mathbb{P}^2 : X \text{ is a variety}\}$ *and* $\mathcal{I} = \{I \subseteq \mathbb{C}[x, y, z] : I \text{ is a radical homogeneous ideal}\}$*. Let* $\mathcal{X}' = \{X \subseteq \mathbb{P}^2 : X \text{ is a irreducible variety}\}$ *and* $\mathcal{I}' = \{I \subseteq \mathbb{C}[x, y, z] : I \text{ is a radical homogeneous prime ideal}\}$*. Then, there exist bijective correspondences between*

$$\mathcal{X} \leftrightarrow \mathcal{I} \text{ and } \mathcal{X}' \leftrightarrow \mathcal{I}'$$

under the maps $\boldsymbol{V}$ *and* $\boldsymbol{I}$ *as defined in Definitions 2.12 and 2.13.*

Intuitively, this means that if J is a homogeneous prime ideal, then $\mathbf{I}(\mathbf{V}(J)) = J$ and if X is a variety, then $\mathbf{V}(\mathbf{I}(X)) = X$. The same bijection holds if J is a homogeneous radical prime ideal and X is an irreducible variety.

With Hilbert's Nullstellensatz, we now have a clear picture of the properties of the defining ideal of a finite set of points in $\mathbb{P}^2$. If X is this set of points, then each $x \in X$ is a projective variety since it can be written as $\mathbf{V}(f, g)$ where f and g are any two distinct lines the intersect at x. Furthermore, x is clearly an irreducible variety since it is a single point. By the Nullstellensatz, this implies that $\mathbf{I}(x)$ is a prime ideal and $\mathbf{I}(x) = (f, g)$. Recall that the defining ideal of X can be written as the intersection of each $\mathbf{I}(x)$ by Zariski and Nagata's Theorem. So the defining ideal of a finite set of points in $\mathbb{P}^2$ is simply the intersection of all the defining ideals of each point generated by two lines that pass through the point.

2.4 Localization

We often want to show when two ideals are equal to each other. When we know the associated primes of an ideal, we can use a powerful tool known as localization to prove this.

Definition 2.27 (Unit of a Ring, [5])**.** *An element u of a ring R is a unit if there exists another element $u^{-1} \in R$ such that $uu^{-1} = 1$.*

Definition 2.28 (Ring of Fractions, [5])**.** *Let D be a multiplicative closed subset of a ring R containing 1. R localized at D or $R_D = \{\frac{a}{b} : a \in R, b \in D\}$. R_D is a ring where every element of D is a unit. When I is an ideal of R, I_P is defined similarly as $\{\frac{a}{b} : a \in I, b \in D\}$*

Remark 2.29 (Localization at P)**.** *The above definition is slightly different when considering prime ideals. If P is a prime ideal of a ring R, then $R \setminus P$ is a multiplicatively closed subset of R that contains 1. We denote R_P as R localized at P even though the multiplicatively closed subset is $R \setminus P$.*

Proposition 2.30 (Localization Properties, [6])**.** *If I, J are ideals in a Noetherian ring R, then $I \subseteq J$ if and only if $I_P \subseteq J_P$ for all $P \in Ass(R/J)$.*

When localizing an ideal I at P, also notice that if I is not contained in P, then by definition, there is an element $x \in I$ that becomes a unit in I_P. Since I_P contains a unit, we automatically know that it is equal to R_P.

2.5 Vector Spaces and the Hilbert Function

We are finally now ready to introduce the Hilbert Function. The Hilbert Function is a powerful tool that uses linear algebra in the setting of projective varieties and homogeneous ideals. In particular, it gives us a better understanding of homogeneous ideals I by studying the graded quotient rings R/I. We will first define what it means for a ring to be graded.

Definition 2.31 (Direct Sum, [5])**.** *Let R_i be a collection of rings for some index set I. Then, $\bigoplus_{i \in I} R_i = \{(r_i)$ ordered tuples $: r_i \in R_i$, and only finitely many of the r_i components are nonzero$\}$.*

Definition 2.32 (Graded Ring, [5])**.** *A ring R is graded if $R = \bigoplus_{i \in I} R_i$ for some index set I and for all $i, j \in I$, $R_i R_j = \{r_i r_j : r_i \in R_i, r_j \in R_j\} \subseteq R_{i+j}$.*

In the case where R is a polynomial ring. The most natural grading for R is by degree. The set R_i is the set of all homogeneous polynomials of degree i in R. Therefore, if $R = \mathbb{C}[x, y, z]$, then R is a graded ring by degree. Furthermore, R has an additional structure where one can view it alternatively as a $\mathbb{C}$-vector space. This alternative algebraic interpretation on R will be essential to properly define the Hilbert Function.

Definition 2.33 (Field, [5])**.** *Let F be a commutative ring with identity denoted with 1 where 1 is not equal to the additive identity of F. If for every nonzero $x \in R$ there exists an element $x^{-1} \in R$ such that $x^{-1}x = xx^{-1} = 1$, then F is a field.*

Definition 2.34 (Vector Space and Dimension, [5])**.** *Let V be a set and F a field. Let $+ : V \times V \to V$ (notice this map is a binary operation) and $\times : F \times V \to V$ be two maps. For all $\alpha \in F$, $v \in V$, we will write $\times(\alpha, v)$ as αv. V is a vector space if it satisfies the following properties:*

1. *$+$ satisfies the first 4 properties from Definition 2.3 on V.*
2. *$\times$ is associative: For all $\alpha, \beta \in F$, $v \in V$, $(\alpha\beta)v = \alpha(\beta v)$.*
3. *Field distribution holds: For all $\alpha, \beta \in F$, $v \in V$, $(\alpha + \beta)v = \alpha v + \beta v$.*
4. *Vector distribution holds: For all $\alpha \in F$, $v, w \in V$, $\alpha(v + w) = \alpha v + \alpha w$.*
5. *Identity in the field is consistent: If 1 is the multiplicative identity in F, then $1v = v$ for all $v \in V$*

Let $\{v_i : i \in I \subseteq \mathbb{N}\}$ be a list of vectors in V. If the following two properties are true, then we call $\{v_i\}$ a basis for V.

1. *$\{v_i\}$ is a set of linearly independent vectors: if $\sum_{i \in I} c_i v_i = 0$, then for each $i \in I$, $c_i = 0$.*
2. *$\{v_i\}$ is a spanning set: $V = \{\sum_{i \in I} c_i v_i : c_i \in F\}$.*

If the number of vectors in $\{v_i\}$ is a basis and is finite, we call V a finite n^{th} dimensional vector space where n is the number of basis vectors. We denote this as $\dim_F(V) = n$.

Definition 2.35 (Hilbert Function, [12])**.** *Let $R = \bigoplus_{i \in I} R_i$ be a graded ring with the additional finite vector space structure over $\mathbb{C}$. Then, the Hilbert Function of R is defined as the function $H_R : \mathbb{N} \to \mathbb{N}$ where $H_R(d) = \dim_{\mathbb{C}}(R_d)$*

When $R = \mathbb{C}[x, y, z]$ and I is the defining ideal for some set of r points in $\mathbb{P}^2$, then it is known from [12] that $H_{R/I^{(m)}}(d) \leq \min\{\binom{2+d}{d}, r\binom{m+1}{2}\}$. In particular, $\binom{2+d}{d} = H_R(d)$ and we call $r\binom{m+1}{2}$ the multiplicity of $R/I^{(m)}$ and denote it by $e(R/I^{(m)})$. When $H_{R/I^{(m)}}(d) = \min\{\binom{2+d}{d}, e(R/I^{(m)})\}$ for all $d \in \mathbb{N}$, we say that $H_{R/I^{(m)}}$ has the expected (or maximal) Hilbert Function. Furthermore, it is also known from [12, Thm C.7] that $H_{R/I^{(m)}}$ is always maximal if $m = 1$ for any $r \in \mathbb{N}$.

Lastly, the following is a short list of important notational shorthands regarding the Hilbert Function that will be used frequently throughout the remainder of the thesis.

- $\alpha(I^{(m)})$ denotes the degree of the smallest degree homogeneous polynomial in $I^{(m)}$. This is also called the initial degree.
- $\epsilon(R/I^{(m)})$ denotes the smallest $d \in \mathbb{N}$ such that $H_R(d) > e(R/I^{(m)})$.

2.6 Short Exact Sequences

One of the most effective ways to calculate the Hilbert Function of a graded ring is by using a short exact sequence. To define a short exact sequence, we first define a ring homomorphism.

Definition 2.36 (Ring Homomorphism and Kernel, [5]). *Let R and S be rings. A ring homomorphism is a map $\phi : R \to S$ such that*

1. *$\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in R$.*
2. *$\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in R$.*

The kernel of ϕ, denoted $\ker \phi$, is the set of elements of R that map to the additive identity (denoted 0) of S. Furthermore, if R and S are graded rings, then ϕ is a graded ring homomorphism if ϕ also respects the grading, i.e., if $\phi(R_i) \subseteq S_i$ for all i in some index set.

Definition 2.37 (Short Exact Sequence, [5]). *Let A, B, C be rings and $\phi : A \to B$ and $\tau : B \to C$ be ring homomorphisms. A short exact sequence is the sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\tau} C \to 0$ where*

1. *ϕ is injective.*
2. *τ is surjective.*
3. *$\phi(A) = \ker \tau$.*

Proposition 2.38 (Additivity of the Hilbert Function, [12]). *Let A, B, C be graded rings and let $\phi : A \to B$, $\tau : B \to C$ be graded ring homomorphisms such that the rings form a graded short exact sequence. Suppose in addition that A, B, C has a vector space structure. Then, $H_B(d) = H_A(d) + H_C(d)$ for all $d \in \mathbb{N}$.*

The next definition is analogous to division in ideals and will play a key role in developing useful short exact sequences.

Definition 2.39 (Colon Notation, [5]). *Let I be an ideal of a ring R. Let $f \in R$. Then $I : f = \{r \in R : fr \in I\}$.*

For example, suppose I is an ideal of a ring R and $I = P_1 \bigcap P_2$ where P_1 and P_2 are prime ideals. Suppose further that $f \in P_1$ but $f \notin P_2$. By definition, it is trivial to show that $I : f = (P_1 : f) \bigcap (P_2 : f)$. If $f \in P_1$, then by the closure of multiplication in ideals, $P_1 : f = R$. And since P_2 is prime, for any $r \in R$ such that $rf \in P_2$, since $f \notin P_2$, $r \in P_2$. So, $P_2 : f = P_2$. So $I : f = P_2$. This also demonstrates why it is often useful to study ideals in terms of their primary components.

Proposition 2.40 (Colon Property). *Let I, J, H be ideals of a ring R. Let $f \in R$. Then, if $I = (fJ, H)$, then $I : f = (J, H : f)$.*

Proof. The reverse inclusion is clear. For the forward inclusion, let $a \in I : f$, then $af = (fJ, H)$, so there exists a $j \in J$ and $h \in H$ such that $af = fj + h$. Then, $f(a - j) = h$ and let $h' = a - j$. Since $a - j = h' \in H : f$, then $a = j + h'$ so $a \in (J, H : f)$. $\square$

Proposition 2.41 (Fundamental Short Exact Sequence for Polynomial Rings, [6]). *Let R be a polynomial ring and I an ideal of R and $f \in R$. Then, the following sequence is a short exact sequence:*

$0 \to R/(I : f) \xrightarrow{\phi} R/I \xrightarrow{\tau} R/(f, I) \to 0$ *where ϕ multiplies an element in $R/I : f$ by f and τ is the standard projection map taking $r + I \to r + (I, f)$.*

Proposition 2.42 (Connection between Associated Primes and Short Exact Sequences, [6]). *Let A, B, C be rings and $0 \to A \to B \to C \to 0$ be a short exact sequence. Then, $Ass(A) \subseteq Ass(B) \subseteq Ass(A) \bigcup Ass(C)$.*

Since a graded ring homomorphism is a homomorphism that preserves the grading, in order to use Propositions 2.41 and 2.38, we need to shift the degree of $R/(I : f)$ by $-\deg(f)$, which we denote $R/(I : f)(-\deg(f))$. For example, an element a in A has degree $\deg(a) + \deg(f)$ in $R/(I : f)(-\deg(f))$. This also implies that $H_{R/(I:f)(-\deg(f))}(d) = H_{R/(I:f)}(d - \deg(f))$. Then, the previous propositions imply that in order to calculate the Hilbert function of R/I, it is sufficient to calculate the Hilbert function of $R/(I : f)(-\deg(f))$ and $R/(I, f)$.

2.7 Important Theorems and Properties

We end this section by listing some important miscellaneous theorems that will be used to prove the later results.

Theorem 2.43 (Hilbert Function Properties, [12]). *Let $R = \mathbb{C}[x, y, z]$, X a set of r points in $\mathbb{P}^2$, and I the defining ideal of X.*

1. *As an immediate consequence of Proposition 2.38, for all $m, d \in \mathbb{N}$, $H_{R/I^{(m)}}(d) = H_R(d) - H_{I^{(m)}}(d)$.*
2. *$H_{R/I^{(m)}}$ is non-decreasing for all $m \in \mathbb{N}$. This implies that in order to prove that $H_{R/I^{(m)}}$ has the expected Hilbert Function, it suffices to check there that there are no equations in of degree $\epsilon(R/I^{(m)})-1$ in $I^{(m)}$ and that there exists at most $H_R(\epsilon(R/I^{(m)}))-e(R/I^{(m)})$ equations of degree $\epsilon(R/I^{(m)})$ in $I^{(m)}$. More generally, if $I = P_1 \bigcap \ldots \bigcap P_r$ defines a set of r points, then $H_{R/(P_1^{m_1} \cap P_2^{m_2} \cap \ldots \cap P_r^{m_r})}$ is non-decreasing for all $m_i \in \mathbb{N}$*
3. *If J, K are any ideals of R, and $J \subseteq K$, then $J = K$ if and only if $H_{R/J}(d) = H_{R/K}(d)$ for all $d \in \mathbb{N}$.*

Theorem 2.44 (Bezout, [3]). *Let $R = \mathbb{C}[x, y, z]$. Let f, g be homogeneous polynomials in R with $\deg(f) = d$ and $\deg(g) = m$. If $\gcd(f, g) = 1$, then $|\boldsymbol{V}(f) \cap \boldsymbol{V}(g)| = md$.*

The definition of “a set of general points” formalizes the algebraic geometry version of “a set of random points.” However, the definition is very technical, and it involves a few notions from topology. It requires one to first defined the so-called “Zariski topology,” which goes well beyond the purpose of this thesis. The interested reader can find the precise definition of general points in [12, Definition D.2]. For the purpose of this thesis, we will use the following simplified version (which is a very special case of “a set of general points”), and by abuse of notation, we will still call it “a set of general points.”

Definition 2.45 (General Points). *Let X be a set of points in $\mathbb{P}^2$ and p be some point in X. X is general if for any $d \in \mathbb{N}$, any homogeneous equation f of degree d will have less than $\binom{d+2}{d}$ points satisfying $f(p) = 0$. That is if no 3 points lie on a line, no 6 points lie on a quadratic, no 10 points of X are on a cubic, etc.*

The next theorem is a celebrated theorem completely solving the problem of when the symbolic square of a set of general points has the expected Hilbert Function.

Theorem 2.46 (Alexander-Hirschowitz (AH-Theorem), [12]). *Let $d \in \mathbb{N}$, X be a general set of r points in $\mathbb{P}^2$. Let I be the defining ideal of X. Then, $H_{R/I^{(2)}}(d)$ has the expected Hilbert Function for all $r \in \mathbb{N}$ except for $r = 2$ and $r = 5$.*

Theorem 2.47 (Huneke's Criterion, [14]). *Let X be a set of r points in $\mathbb{P}^2$ and let I be the defining ideal of X. Let $k \in \mathbb{N}$. Then, if there exists a $f, g \in I^{(k)}$ such that $\gcd(f, g) = 1$ and $\deg(f)\deg(g) = rk^2$, then $I^{(kt)} = (I^{(k)})^t$ for all $t \in \mathbb{N}$.*

Theorem 2.48 (Waldschmidt-Skoda Constant, [1]). *Define $\hat{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}$.*

1. *For all $m, t \in \mathbb{N}$, $\frac{\alpha(I^{(mt)})}{mt} \leq \frac{\alpha(I^{(m)})}{m}$.*
2. *$\hat{\alpha}(I) \leq \frac{\alpha(I^{(m)})}{m}$ for all $m \in \mathbb{N}$.*
3. *If there exists a m_0 such that $(I^{(m_0)})^t = I^{(m_0 t)}$ for all $t \in \mathbb{N}$, then, $\hat{\alpha}(I) = \frac{\alpha(I^{(m_0)})}{m_0}$.*

Definition 2.49 (Exceptional Curve, [2]). *Let X be a set of r general points in $\mathbb{P}^2$ and let P_i be the defining ideal of each point $p_i \in X$. Let f be an irreducible equation of homogeneous degree d. Let n_i be the number such that $f \in P_i^{n_i}$ but $f \notin P_i^{n_i+1}$. Then, f is exceptional if $d^2 - \sum_{i=1}^r n_i^2 = -3d + \sum_{i=1}^r n_i = -1$.*

Theorem 2.50 (SHGH Conjecture, [2]). *For a set of general points in $\mathbb{P}^2$ with defining ideal I, $H_{R/I}(d)$ does not have the expected maximal value if and only if every homogeneous degree d element in I is a multiple of some f^m where $m > 1$ and f is an exceptional curve.*

Although this statement is technically a conjecture, it has been proven for any r points up to $m \leq 12$ in [4, Prop 5.1].

3 Standard Bezout Theorem Argument

Most of the key results will rely on the following argument using Bezout Theorem. In this section, we outline this argument and will implicitly refer to it in the next sections. For brevity, an arbitrary equation of degree n will be denoted as f_n and P_i will denote the defining ideal of the i-th point in a set of X of r general points. Unless otherwise stated, this notation will be consistent for the remainder of the thesis.

Theorem 3.1. *Let X be a set of r general points and I its defining ideal. Suppose the following:*

- *For some $m, n \in \mathbb{N}$, $f_n \in I^{(m)} = \bigcap_{1 \le i \le r} P_i^m$.*
- *There exists an irreducible equation g with a degree $d < n$ such that $g \in \bigcap_{j \in J}(P_j \setminus P_j^2)$ where $J \subseteq \{1, 2, ..., r\}$.*
- *$|\boldsymbol{V}(g) \bigcap \boldsymbol{V}(f_n)| > dn$, i.e. the assumptions for the contrapositive of Bezout Theorem are satisfied.*

Then, $f_n = gf_{n-d}$ where $f_{n-d} \in \bigcap_{1 \le i \le r} P_i^a$ where $a = (m-1)$ if $i \in J$ and $a = m$ otherwise.

Proof. Since the assumptions for Bezout Theorem (2.44) are satisfied, $\gcd(f_n, g) > 1$. But since g is irreducible, this implies that $\gcd(f_n, g) = g$, so $g | f_n$. Therefore, $f_n = gf_{n-k}$ and $f_{n-k} \in I^{(m)} : g$. We now prove that $I^{(m)} : g = \bigcap_{1 \le i \le r} P_i^a$

Consider $I^{(m)} : g = \bigcap_{1 \le i \le r}(P_i^m : g)$. If $i \notin J$, then $g \notin P_i$. If $x \in P_i^m$, then by the properties of an ideal, $xg \in P_i^m$ so $x \in P_i^m : g$. Now suppose $x \notin P_i^m$, and assume for contradiction that $xg \in P_i^m$. Since P_i^m is a primary ideal and $x \notin P_i^m$, then this implies that $g^\eta \in P_i^m$ for some $\eta \in \mathbb{N}$. But this implies that $g \in \sqrt{P_i^m} = P_i$ by Proposition 2.17, which is a contradiction. This shows that $P_i^m : g = P_i^m$.

Now suppose $i \in J$. This means that $g \in P_i$. So $gP_i^{m-1} \subseteq P_i P_i^{m-1} = P_i^m$, which implies $P_i^{m-1} \subseteq P_i^m : g$. To prove the reverse inclusion, it is sufficient by Proposition 2.30 to show that $(P_i^m : g)_P \subseteq (P_i^{m-1})_P$ for all $P \in \mathrm{Ass}(R/P_i^{m-1})$. By Proposition 2.17 and Definition 2.22, $\mathrm{Ass}(R/P_i^{m-1}) = \{P_i\}$. Since $g \in P_i \setminus P_i^2$, then $\frac{g}{1} = g \in (P_i)_{P_i} \setminus (P_i)_{P_i}^2$. Let $(f_1, ..., f_k)$ be the finite number of elements that generate $(P_i)_{P_i}$. Since $g \in (P_i)_{P_i}$, $g = \alpha_1 f_1 + ... + \alpha_k f_k$. But since $g \notin (P_i)_{P_i}^2$, there exists some α_η such that $\alpha_\eta \notin (P_i)_{P_i}$. Without loss of generality, assume $\eta = 1$. Then, by Remark 2.29, α_1 is a unit. Therefore, $f_1 = \frac{g - (\alpha_2 f_2 + ... + \alpha_k f_k)}{\alpha_1}$ so $(P_i)_{P_i} = (g, f_2, ..., f_k)$. This implies that $(P_i^m : g)_{P_i} = (g, f_2, ..., f_k)^m : g = (g, f_2, ..., f_k)^{m-1} = (P_i^{m-1})_{P_i}$.

So, $I^{(m)} : g = \bigcap_{1 \le i \le r} P_i^a$ and since $f_{n-d} \in I^{(m)} : g$, then $f_{n-d} \in \bigcap_{1 \le i \le r} P_i^a$. □

4 Results

4.1 3 Point Case

Lemma 4.1. *Let X be a set of 3 general points and I its defining ideal. Let $\ell_1 \in P_1 \bigcap P_2$, $\ell_2 \in P_2 \bigcap P_3$, and $\ell_3 \in P_3 \bigcap P_1$ where each ℓ_i is a degree 1 homogeneous equation. Then, $I = (\ell_1\ell_2, \ell_2\ell_3, \ell_1\ell_3)$.*

Proof. Recall that $I = P_1 \bigcap P_2 \bigcap P_3$ and we can write $P_1 = (\ell_1, \ell_3)$, $P_2 = (\ell_1, \ell_2)$, $P_3 = (\ell_2, \ell_3)$. By Proposition 2.8, $P_1 \bigcap P_2 = (\ell_1) + [(\ell_2) \bigcap (\ell_1, \ell_3)]$. But ℓ_2 is principal, so everything in $[(\ell_2) \bigcap (\ell_1, \ell_3)]$ is a multiple of ℓ_2 so the ideal becomes $(\ell_1) + \ell_2((\ell_1, \ell_3) : \ell_2))$. Furthermore, $(\ell_1, \ell_3) = P_1$ is prime and $\ell_2 \notin P_1$ so $(\ell_1) + \ell_2((\ell_1, \ell_3) : \ell_2)) = (\ell_1) + \ell_2(\ell_1, \ell_3) = (\ell_1, \ell_2\ell_3)$. We then take $(\ell_1, \ell_2\ell_3) \bigcap P_3 = (\ell_1, \ell_2\ell_3) \bigcap (\ell_2, \ell_3)$. After noticing that $(\ell_2\ell_3) \subseteq (\ell_2, \ell_3)$, we can apply Proposition 2.8 again and after a similar argument, we get that $I = (\ell_1\ell_2, \ell_2\ell_3, \ell_1\ell_3)$ as desired.

□

Lemma 4.2. *Let X be a set of 3 general points and I its defining ideal. Let ℓ_1, ℓ_2, ℓ_3 be the same as above. Then, $\ell_1\ell_2\ell_3 \in I^{(2)}$ and no equation in degree 2 lies in $I^{(2)}$.*

Proof. The first part is clear from the definition of each ℓ_i for $i = 1, 2, 3$. The second part follows from the AH Theorem (2.46). That is, we know $R/I^{(2)}$ has the expected Hilbert Function, so $H_{R/I^{(2)}} = H_R - H_{I^{(2)}} = 1, 3, 6, 9, 9, \ldots$. Since $H_{R/I^{(2)}}(2) = H_R(2) = \binom{2+2}{2} = 6$, there are no equations of degree 2 in $I^{(2)}$. □

As a brief remark, the previous lemma and the fact that $I^2 \subseteq I^{(2)}$ gives the intuition that $I^{(2)} = (\ell_1\ell_2\ell_3, \ell_1^2\ell_2^2, \ell_1^2\ell_3^2, \ell_2^2\ell_3^2)$. Although we won't prove it here, it turns out that this equality is indeed true.

Theorem 4.3. *Let X be a set of 3 general points in $\mathbb{P}^2$ and I its defining ideal. Then $R/I^{(m)}$ has the expected Hilbert Function at $m = 3$, but $R/I^{(m)}$ does not have the expected Hilbert Function at $m = 4$ and $m = 5$.*

From Definition 2.35, we recall the expected Hilbert Function for $R/I^{(m)}$ when $m = 3, 4, 5$.

- $H_{R/I^{(3)}} = 1, 3, 6, 10, 15, 18, 18, 18, \ldots$
- $H_{R/I^{(4)}} = 1, 3, 6, 10, 15, 21, 28, 30, 30, 30, \ldots$
- $H_{R/I^{(5)}} = 1, 3, 6, 10, 15, 21, 28, 36, 45, 45, 45, \ldots$

Proof. We will look at each case individually. For each case, we let $\ell_1 \in P_1 \bigcap P_2$, $\ell_2 \in P_2 \bigcap P_3$, and $\ell_3 \in P_3 \bigcap P_1$ as in Lemma 4.1 and 4.2.

1. $m = 3$. Using part 2 of Theorem 2.43, it suffices to prove that there are no equations of degree 4 in $I^{(3)}$ and at most 3 linearly independent equations in degree 5 in $I^{(3)}$. Assume for contradiction that there exists an equation of degree 4 in $I^{(3)}$, let us denote it as f_4. Then, by Bezout Theorem (2.44), since $\deg(f_4)\deg(\ell_i) < 6 \leq |\mathbf{V}(f_4) \bigcap \mathbf{V}(\ell_i)|$ for $i = 1, 2, 3$, one can write $f_4 = \ell_1\ell_2\ell_3 f_1$ where $f_1 \in I$. But recall that R/I has the expected Hilbert Function (since $I = I^{(1)}$), there does not exist any equation of degree 1 in I, so there is no equation of degree 4 in $I^{(m)}$.

   Let f_5 be an equation of degree 5 in $I^{(3)}$. Then $\deg(f_5)\deg(\ell_i) < 6 \leq |\mathbf{V}(f_5) \bigcap \mathbf{V}(\ell_i)|$ for $i = 1, 2, 3$, so $f_5 = \ell_1\ell_2\ell_3 f_2$ for some $f_2 \in I$. But since R/I has the expected Hilbert Function, there exists at most 3 linearly independent equations of degree 2 in I, so there exists at most 3 linearly independent equations of degree 5 in $I^{(3)}$.

2. $m = 4$

   Using part 2 of Theorem 2.43, to prove that $H_{R/I^{(4)}}$ does not have the expected Hilbert Function, we show there exists an equation of degree 6 in $I^{(4)}$. Since $H_{R/I^{(2)}}$ has the expected Hilbert Function, we know there exists one equation of degree 3 in $I^{(2)}$. So there exists a f_6 in I^4 where $f_6 = \ell_1\ell_2\ell_3 f_3$ and f_3 is in $I^{(2)}$.

3. $m = 5$

   Similarly to the above, to prove that $H_{R/I^{(5)}}$ does not have the expected Hilbert Function, we show there exists an equation of degree 8 in $I^{(5)}$. Since $H_{R/I^{(3)}}$ has the expected Hilbert Function, we know there exist 3 linearly independent equations of degree 5 in $I^{(3)}$. Let f_5 any of these 3 equations. We know that $\ell_1\ell_2\ell_3 \in I^{(2)}$, so $f_5\ell_1\ell_2\ell_3$ is an equation of degree 8 in $I^{(3)}I^{(2)} \subseteq I^{(5)}$.

□

Theorem 4.4. *For all $m \in \mathbb{N}$, define $g : \mathbb{N} \to \mathbb{N}$ to be $g(m) = \frac{3}{2}m$ if m is even, or $g(m) = \frac{1}{2} + \frac{3m}{2}$ if m is odd. Let X be a set of 3 points in $\mathbb{P}^2$ and I its defining ideal. Then $\alpha(I^{(m)}) = g(m)$. Furthermore for all $m \geq 6$, $\binom{2+g(m)}{2} < e(R/I^{(m)})$ so $R/I^{(m)}$ does not have the expected Hilbert Function for all $m \geq 6$.*

Proof. Let us recall from the Lemma 4.1 that $\ell_1 \in P_1 \bigcap P_2$, $\ell_2 \in P_2 \bigcap P_3$, and $\ell_3 \in P_3 \bigcap P_1$. Furthermore, note that $g(m-2) = g(m) - 3$ for all $m \in \mathbb{N}$. From Lemmas 4.1 and 4.2, we know there exists an equation of degree $g(1) = 2$ in I and there exists an equation of degree $g(2) = 3$ in $I^{(2)}$. Recall that we will use f_n to denote a homogeneous equation of degree n. If there exists a $k \in \mathbb{N}$ such that $f_{g(k-2)} \in I^{(k-2)}$, then $f_{g(k)} \in I^{(k)}$ since $f_{g(k)} = \ell_1\ell_2\ell_3 f_{g(k)-3}$. Furthermore, Lemma 4.1 also shows there exist no equations of degree $g(1)-1 = 1$ in I, and Lemma 4.2 shows there exist no equations of degree $g(2) - 1 = 2$ in $I^{(2)}$. Now, assume for contradiction that there exists a k such that there does not exist any $f_{g(k-2)-1} \in I^{(k-2)}$ but that there does exist a $f_{g(k)-1} \in I^{(k)}$. By Bezout Theorem (2.44), we know for any $f \in I^{(m)}$, $|V(f) \bigcap V(\ell_i)| \geq 2m$ for all $i = 1, 2, 3$. For all $m \geq 3$, we note that $(g(m) - 1) < 2m$. So by Bezout Theorem, $f_{g(k)} = \ell_1\ell_2\ell_3 f_{g(k-2)-1}$, where $f_{g(k-2)-1} \in I^{(k-2)}$. This gives a contradiction to the inductive hypothesis. Therefore, $\alpha(I^{(m)}) = g(m)$.

To prove the second part, we note that when m is even, $\binom{2+g(m)}{2} = \frac{\frac{9}{4}m^2+\frac{9}{2}m+2}{2}$. When m is odd, $\binom{2+g(m)}{2} = \frac{\frac{9}{4}m^2+6m+\frac{15}{4}}{2}$. Lastly, $e(R/I^{(m)}) = \frac{3m^2+3m}{2}$. Define the following:

- $f(m) = \frac{3m^2+3m}{2}$,
- $f_e(m) = \frac{\frac{9}{4}m^2+\frac{9}{2}m+2}{2}$, and
- $f_o(m) = \frac{\frac{9}{4}m^2+6m+\frac{15}{4}}{2}$.

At $m = 6$, $f > f_e$ and $f > f_o$. Assume that there exists a $k \in \mathbb{N}$ such that $f(k) > f_e(k)$. Then, $f(k+1) = \frac{3(k+1)^2+3(k+1)}{2} = \frac{3k^2+3k}{2} + \frac{6k+6}{2}$. Similarly, $f_e(k+1) = \frac{\frac{9}{4}(k+1)^2+\frac{9}{2}(k+1)+2}{2} = \frac{\frac{9}{4}k^2+\frac{9}{2}k+2}{2} + \frac{\frac{9}{2}k+\frac{27}{4}}{2}$. Since for all $k \geq 6$, $\frac{6k+6}{2} > \frac{\frac{9}{2}k+\frac{27}{4}}{2}$, using the inductive hypothesis we know

$f(k+1) > f_e(k+1)$. So by induction, $f(m) > f_e(m)$ for all even m. An identical argument holds for all odd m using f_o instead. So for all $m \geq 6$, $\binom{2+g(m)}{2} < e(R/I^{(m)})$. Recall that in order for R/I to have the expected Hilbert Function, $H_{(R/I)}(d) = \min\{\binom{2+d}{d}, e(R/I^{(m)})\}$. Given that we have just shown $\binom{2+g(m)}{2} < e(R/I^{(m)})$, if we assume for contradiction that $R/I^{(m)}$ does have the expected Hilbert Function, then we know $H_{R/I^{(m)}}(g(m)) = \binom{2+g(m)}{2}$. But this is equal to $H_R(g(m))$, which is a contradiction because there do indeed exist equations in $I^{(m)}$ at $d = g(m)$. That is, $H_{I^{(m)}}(g(m)) \neq 0$ and by part 1 of Theorem 2.43, $H_{R/I^{(m)}}(g(m)) = H_R(g(m)) - H_{I^{(m)}}(g(m)) \neq H_R(g(m))$.

□

Next, we want to address the secondary research question by determining which symbolic powers $I^{(m)}$ of I we do have $(I^{(m)})^t = I^{(mt)}$. Huneke's criterion (2.47) oftentimes can be used to get a starting point in these kinds of investigations.

Theorem 4.5. *For 3 general points, Huneke's Criterion holds for $k = 2$.*

Proof. To prove the proposition, we must find homogeneous polynomials $f, g \in I^{(2)}$ such that $\deg(f)\deg(g) = 3k^2 = 12$ and $\gcd(f, g) = 1$. For the remainder of the proof, let $\ell_{12} \in P_1 \bigcap P_2$, $\ell_{13} \in P_1 \bigcap P_3$, $\ell_{23} \in P_2 \bigcap P_3$, $\ell_1 \in P_1 \setminus (P_2 \bigcup P_3)$, $\ell_2 \in P_2 \setminus (P_1 \bigcup P_3)$, and $\ell_3 \in P_3 \setminus (P_1 \bigcup P_2)$ where each ℓ_i and ℓ_{ij} are degree 1 homogeneous equations. Define $f_3 := \ell_{12}\ell_{13}\ell_{23} \in I^{(2)}$. Therefore, we want to find a quartic $f_4 \in I^{(2)}$ such that $\gcd(f_3, f_4) = 1$.

Let $g_1 := \ell_{12}\ell_{23}\ell_1\ell_3 \in I^{(2)}$ and $g_2 := l_{13}^2 l_2^2 \in I^{(2)}$. In a UFD, $g_1 = \ell_{12}\ell_{23}\ell_1\ell_3$ and $g_2 = \ell_{13}^2\ell_2^2$ are unique irreducible factorizations, and since $\ell_{12}, \ell_{13}, \ell_{23}, \ell_1, \ell_2, \ell_3$ define distinct lines, $\gcd(g_1, g_2) = 1$. If $\gcd(f_3, g_1) = 1$ or $\gcd(f_3, g_2) = 1$, then we are done. Otherwise, assume $\gcd(f_3, g_1) > 1$ and $\gcd(f_3, g_2) > 1$. Let $f_\alpha = g_1 + \alpha g_2$ and $f_\beta = g_1 + \beta g_2$ for some $\alpha \neq \beta \in \mathbb{C}$. Intuitively, what we are trying to show is that there exists some equation of degree 4 as a linear combination of g_1 and g_2 such that it is coprime with f_3. To do this, we will show how all elements in the set $\{f_\alpha : \alpha \in \mathbb{C}\}$ are coprime with each other. Now, we can go on with the proof.

Consider $g = \gcd(f_\alpha, f_\beta)$. We will show that $g = 1$. Assume for contradiction that $g > 1$. Then, $g|f_\alpha$ and $g|f_\beta$. So, $g|(f_\alpha - f_\beta)$ and $g|(f_\alpha - \frac{\alpha}{\beta}f_\beta)$. So, $g|(\alpha - \beta)g_2$ and $g|(1 - \frac{\alpha}{\beta})g_1$ which implies $g|g_1$ and $g|g_2$. But $\gcd(g_1, g_2) = 1$, so $g = 1$. Consider the set $\{f_\alpha : \alpha \in \mathbb{C}\}$. For each f_α in the set, $\gcd(f_\alpha, f_\beta) = 1$ for any other $\beta \neq \alpha$. This implies there are an infinite number of elements in $\{f_\alpha : \alpha \in \mathbb{C}\}$ with each element having its own distinct factorization. However, since f_3 is an equation with a finite number of irreducible factors, there must exist some $\gamma \in \mathbb{C}$ such that $\gcd(f_3, f_\gamma) = 1$. Since $f_\gamma \in I^{(2)}$ and has degree 4, Huneke's Criterion applies for $k = 2$. □

Theorem 4.6. *For 3 general points, for all $q \in \mathbb{N}$ and $t \geq 2 \in \mathbb{N}$, $(I^{(2q+1)})^t \neq I^{(t(2q+1))}$ and $(I^{(2q)})^t = I^{(2tq)}$.*

Proof. Since $2q+1$ is odd, by Theorem 4.4, $\alpha(I^{(2q+1)}) = 2 + 3q$. This implies $\alpha((I^{(2q+1)})^t) = 3qt + 2t$. Now consider $I^{t(2q+1)}$. If t is even then by Theorem 4.4, $\alpha(I^{t(2q+1)}) = 3qt + \frac{3}{2}t$. Otherwise, if t is odd, then $\alpha(I^{t(2q+1)}) = 3qt + \frac{3}{2}t + \frac{1}{2}$. Notice that since $3qt + 2t \neq 3qt + \frac{3}{2}t$ for all $t \geq 2$ and $3qt + 2t = 3qt + \frac{3}{2}t + \frac{1}{2}$ if and only if $t = 1$, $\alpha((I^{(2q+1)})^t) \neq \alpha(I^{t(2q+1)})$. Therefore, $(I^{(2q+1)})^t \neq I^{t(2q+1)}$

Lastly, $(I^{(2q)})^t = I^{t(2q)}$ follows immediately from Huneke's Criterion for $k = 2$. □

4.2 4 Point Case

To solve the case for 4 points, we utilize some nice properties of ideals which are complete intersections. Essentially, when an ideal is a complete intersection, $I^m = I^{(m)}$. That is, its ordinary and symbolic power is equal. The proof is relatively complex, and a precise proof of a more general statement can be found in [13, Thm 16.2].

Definition 4.7 (Complete Intersection)**.** *Let R be a ring and I an ideal of R. I is a complete intersection if I is a proper ideal of R and it is generated by a regular sequence, i.e. a sequence of elements $x_1, x_2, ..., x_n \in R$ where for each $1 \leq i \leq n$, x_i is a nonzero divisor in $R/(x_1, x_2, ..., x_{i-1})$.*

For example, if R is a UFD (e.g. $R = \mathbb{C}[x, y, z]$, as in our case), and let f, g be any two polynomials with $(f, g) \neq R$, then (f, g) is a complete intersection if and only if $\gcd(f, g) = 1$.

Theorem 4.8. *Let X be a set of 4 general points in $\mathbb{P}^2$ and I its defining ideal. Then, the initial degree of $R/I^{(m)}$ is $2m$ for all $m \in \mathbb{N}$. Furthermore, $R/I^{(m)}$ has the expected Hilbert Function for all $m \in \mathbb{N}$.*

Proof. We first show that $2m$ is the degree to when we expect the Hilbert Function to differ from H_R provided that it is maximal. Then, we can apply part 2 of Theorem 2.43. Set $d = 2m$. Then, $\binom{1+d}{2} = 2m^2+m$, $e(R/I^{(m)}) = 4\binom{m+1}{2} = 2m^2+2m$, and $\binom{2+d}{2} = 2m^2+3m+1$, Since $2m^2+m < 2m^2+2m < 2m^2+3m+1$ for all $m \in \mathbb{N}$, $\binom{1+d}{2} < 4\binom{m+1}{2} = 2m^2+2m < \binom{2+d}{2}$. Therefore, $2m = \epsilon(R/I^{(m)})$.

To show that $H_{R/I^{(m)}}$ has the expected Hilbert Function for all $m \in \mathbb{N}$, it suffices to prove there exists no equations of degree $2m-1$ in $I^{(m)}$ and that there are at most $\binom{2+2m}{2} - e(R/I^{(m)}) = m+1$ linearly independent equations of degree $2m$. Let $\ell_1 \in P_1 \bigcap P_2$, $\ell_2 \in P_3 \bigcap P_4$, $\ell_3 \in P_1 \bigcap P_3$, $\ell_4 \in P_2 \bigcap P_4$ where each ℓ_i is a degree 1 homogeneous equation.

Let $q_1 = \ell_1\ell_2, q_2 = \ell_3\ell_4$. For any of the four points, its defining ideal P_i can be defined as the intersection of two nonparallel lines. In particular, $P_1 = (\ell_1, \ell_3)$. We see that ℓ_1 and ℓ_3 are not parallel (which means they are distinct lines) because $\ell_1 \in P_1 \bigcap P_2$ and $\ell_3 \in P_1 \bigcap P_3$, and three points cannot lie on the same line. Similarly, $P_2 = (\ell_1, \ell_4)$, $P_3 = (\ell_2, \ell_3)$, $P_4 = (\ell_2, \ell_4)$. Therefore, $I = P_1 \bigcap P_2 \bigcap P_3 \bigcap P_4 = (\ell_1, \ell_3) \bigcap (\ell_1, \ell_4) \bigcap (\ell_2, \ell_3) \bigcap (\ell_2, \ell_4) = (\ell_1\ell_2, \ell_3\ell_4) = (q_1, q_2)$. Since we are working in a UFD, $q_1 = \ell_1\ell_2$ and $q_2 = \ell_3\ell_4$ are unique irreducible factorizations. Therefore, since each ℓ_i are distinct lines, $\gcd(q_1, q_2) = 1$.

Therefore, since I is a complete intersection ideal, meaning $I^{(m)} = I^m$ for all $m \in \mathbb{N}$. $I^{(1)} = I$ is generated by two equations. So, $(q_1, q_2)^m$ will be generated by equations of the form $q_1^i q_2^j$ where $i + j = m$, and $i, j \geq 0$. With combinatorics, the number of solutions to $i + j = m$ is precisely $\binom{m+1}{1} = m+1$. Furthermore, this also explicitly proves that $\alpha(R/I^{(m)}) = 2m$ since the generators are of degree 2. So, there exist no equations of degree $2m - 1$.

To show that the $m+1$ generators of $I^{(m)}$ are linearly independent, we assume for contradiction that there exists a sequence of constants c_i (with at least one nonzero) such that $\sum_{i=0}^m c_i q_1^i q_2^j = 0$ where for all i, $j = m-i$. Then, we dehomgenize with respect to q_1, meaning $0 = \sum_{i=0}^m c_i q_1^i q_2^j = q_1^m \sum_{i=0}^m c_i (\frac{q_2}{q_1})^j$. If we let $t = \frac{q_2}{q_1}$, then we have a single variable polynomial with coefficients in the algebraically closed field $\mathbb{C}$. So, the polynomial splits completely into linear terms, which implies that $(t - c) = 0$ for some $c \in \mathbb{C}$. After rehomogenizing, this implies $q_2 = cq_1$, which contradicts the fact that $\gcd(q_1, q_2) = 1$. Therefore, the $m + 1$ generators of $I^{(m)}$ are linearly independent.

$\square$

4.3 5 Point Case

Theorem 4.9. *Let X be a set of 5 general points in $\mathbb{P}^2$ and I its defining ideal. Then, $R/I^{(m)}$ does not have the expected Hilbert function for any $m \geq 2$.*

Proof. From [12, Thm C.7] and part 1 of Theorem 2.43, we know that $H_{R/I}(d) = H_R(d) - H_I(d)$ and that there is precisely one homogeneous polynomial of degree 2 in I. Therefore, there exists a polynomial of degree $2m$ in I^m. Indeed, since $I^m \subseteq I^{(m)}$, there exists at least one homogeneous polynomial of degree $2m$ in $I^{(m)}$. This implies that $H_{I^{(m)}}(2m) \geq 1$, so $H_{R/I^{(m)}}(2m) = H_R(2m) - H_{I^{(m)}}(2m) \leq \binom{2m+2}{2} - 1$. So, $H_{R/I^{(m)}}(2m) < \binom{n+d}{n} = \binom{2m+2}{2}$.

We now note that $e(R/I^{(m)}) = 5\binom{2+m-1}{2} = \frac{5m^2+5m}{2}$. Furthermore, note that $\binom{2m+2}{2} = \frac{4m^2+6m+2}{2}$. We will show by induction that $5\binom{2+m-1}{2} \geq \binom{2m+2}{2}$ for all $m \geq 2$. At $m = 2$, $5\binom{2+m-1}{2} = \binom{2m+2}{2}$. For the inductive step, assume that for some $k \in \mathbb{N}$, $\frac{5k^2+5k}{2} \geq \frac{4k^2+6k+2}{2}$. Then,

- $\frac{5(k+1)^2+5(k+1)}{2} = \frac{5k^2+5k+(10k+10)}{2}$ and
- $\frac{4(k+1)^2+6(k+1)+2}{2} = \frac{4k^2+6k+2+(8k+10)}{2}$.

Since $\frac{10k+10}{2} > \frac{8k+10}{2}$, then by the inductive hypothesis, $\frac{5(k+1)^2+5(k+1)}{2} \geq \frac{4(k+1)^2+6(k+1)+2}{2}$. Therefore, $5\binom{2+m-1}{2} \geq \binom{2m+2}{2}$ for all $m \geq 2$. So, $H_{R/I^{(m)}}(2m) < \binom{2m+2}{2} \leq e(R/I^{(m)})$. This implies $H_{R/I^{(m)}}$ does not have the expected Hilbert Function for all $m \geq 2$. $\square$

Theorem 4.10. *Let X be a set of 5 general points in $\mathbb{P}^2$ and I its defining ideal. Then, $\alpha(I^m) = 2m$ for all $m \in \mathbb{N}$.*

Proof. We already know that there exists an equation of degree $2m$ in $I^{(m)}$ for all $m \in \mathbb{N}$. Furthermore, because the points are taken to be general there exists an irreducible quadratic $q \in I$. We also know that there does not exist any equation of degree 1 in I. Now, assume for contradiction that there exists a k such that there does not exist any $f_{2(k-1)-1} \in I^{(k-1)}$ but that there exists a $f_{2k-1} \in I^{(k)}$. Note that $|\mathbf{V}(f_{2k-1}) \bigcap \mathbf{V}(q)| \geq 5k$. Since $\deg(f_{2k-1})\deg(q) = 4k-2$, by Bezout Theorem, $f_{2k-1} = qf_{2(k-1)-1}$ where $f_{2(k-1)-1} \in I^{k-1}$, which contradicts the inductive hypothesis. $\square$

Theorem 4.11. *For 5 general points, Huneke's Criterion holds for $k = 2$.*

Proof. To prove the proposition, we must find $f, g \in I^{(2)}$ such that $\deg(f)\deg(g) = 5k^2 = 20$ and $\gcd(f, g) = 1$. Let q be the irreducible quadratic in I and $\ell_{ij} \in P_i \bigcap P_j$ be degree 1 homogeneous equations for $1 \leq i, j \leq 5, i \neq j$. Note that $f_4 := q^2 \in I^{(2)}$. So, we need to find a $f_5 \in I^{(2)}$ such that $\gcd(f_4, f_5) = 1$.

Let $f_5 = \ell_{12}\ell_{23}\ell_{34}\ell_{45}\ell_{51} \in I^{(2)}$. In a UFD, $f_4 = q^2$ and $f_5 = \ell_{12}\ell_{23}\ell_{34}\ell_{45}\ell_{51}$ have unique irreducible factorizations. Since $\ell_{12}\ell_{23}\ell_{34}\ell_{45}\ell_{51}, q$ are distinct, $\gcd(f_4, f_5) = 1$. Therefore, Huneke's Criterion applies for $k = 2$. $\square$

4.4 6 Point Case

In the case of 6 points, Bezout Theorem becomes far less useful since it is much more difficult for the assumptions of the theorem to be satisfied. Furthermore, we will later see that Huneke's Criterion also fails for small k. This makes it much more difficult to answer in full generality if a set of 6 general points has the expected Hilbert Function. However, using some more advanced techniques involving properties including Artinian Reductions, unmixed ideals, and short exact sequences, we are able to prove some interesting results showing $(I^{(2)})^k = I^{(2k)}$ when $k = 2, 3, 4$. We will first introduce these new tools exclusive to the 6 point case.

Definition 4.12 (Height of an Ideal, [6])**.** *If I is a prime ideal, the height of I, denoted $ht(I)$ is the maximum length of a chain of prime ideals descending from I. If I is not prime, then let $ht(I)$ be the minimum of all $ht(P)$ where P is a prime containing I. Equivalently, it is the minimum of $\{ht(P_i) : P_i \in Ass(R/I)\}$.*

Definition 4.13 (Unmixed Ideal for Polynomial Rings, [13])**.** *Let $R = \mathbb{C}[x, y, z]$. Then, an ideal I of R is unmixed of height c if $c = ht(P_i)$ for all $P_i \in Ass(R/I)$.*

Proposition 4.14 (Determining Height 1 Ideals, [6])**.** *Let $R = \mathbb{C}[x, y, z]$. Then, an ideal I of R has height 1 if and only if there exists a non-unit $f \in R$ such that $I = fJ$ for some other ideal J.*

The following is a nice result of the previous definition. If $I \subset \mathbb{C}[x, y, z]$ is an unmixed ideal of height 2, then for all $P \in \text{Ass}(R/I)$, $\text{ht}(P) = 2$. This automatically shows that the maximal ideal (x, y, z), which has height 3, is not an associated prime of I. Since we are working with homogeneous ideals, it turns out that (x, y, z) is indeed the only maximal ideal and also the only ideal of height 3. Furthermore, the definition also implies that $\text{Ass}(R/I) \subseteq \text{Min}(I)$ so as a consequence of Proposition 2.24, we actually get that the two sets are equal for unmixed ideals.

The next definition and proposition are motivated by some more intricate results involving very special types of rings known as Cohen-Macaulay (CM) rings. To fully describe it here would be unnecessarily arduous for the main results of this thesis, so the reader may refer to chapter 6, sections 16 and 17 of [13] for precise definitions and properties.

Definition 4.15 (Hilbert Function of the Artinian Reduction)**.** *Let $R = \mathbb{C}[x, y, z]$, I be a defining ideal of a set of general points, and $H_{R/I}$ be the Hilbert Function of R/I. The Hilbert Function of the Artinian Reduction of R/I, denoted $H_{A(R/I)}$, is defined as $H_{A(R/I)}(d) = 1$ when $d = 0$ and $H_{A(R/I)}(d) = H_{R/I}(d) - H_{R/I}(d-1)$ otherwise.*

The easiest way to comprehend the relatively complicated definition above is with an explicit example. Consider $I = P_1 \bigcap P_2 \bigcap P_3 \bigcap P_4 \bigcap P_5 \bigcap P_6$. That is, I is the defining ideal for 6 general points of multiplicity 1. We know that $H_{R/I} = 1, 3, 6, 6, ...$ Then, $H_{A(R/I)} = 1, 2, 3, 0, 0, ...$ Intuitively, the Artinian Reduction starts at 1 and then is the subsequent first difference of the original Hilbert Function.

Proposition 4.16 (Complete Intersection and Artinian Reduction)**.** *Let $R = \mathbb{C}[x, y, z]$ and $I = (f, g)$ be a complete intersection. Let $a = \deg f$ and $b = \deg g$ and assume $a \leq b$. Then,*

- $H_{A(R/I)}(d) = d + 1$ *when* $0 \leq d \leq a - 1$,

- $H_{A(R/I)}(d) = a$ *when* $a \leq d \leq b - 1$, *and*

- $H_{A(R/I)}(d) = \max(a - (b + 1 - d), 0)$ *when* $d \geq b$.

Once again, we will consider an explicit example. Let $R = \mathbb{C}[x, y, z]$ and suppose $I = (f, g)$ is a complete intersection with $\deg f = 3$ and $\deg g = 7$. Then, $H_{A(R/I)} = 1, 2, 3, 3, 3, 3, 3, 2, 1, 0, 0, \ldots$. Intuitively, the proposition above states that the Artinian Reduction will increment upwards by 1, remain constant from $d = a$, and will increment back down at $d = b$. Also notice that knowing the Artinian Reduction also immediately determines the Hilbert Function. In the example above, $H_{R/I} = 1, 2, 5, 8, 11, 14, 17, 19, 20, 20, \ldots$. We are now ready to begin proving results for 6 points.

Lemma 4.17. *Fix* j *for some* $1 \leq j \leq 6$. *For a general set of* 6 *points in* $\mathbb{P}^2$, *any quadratic in* $\bigcap_{1 \leq i \leq 6, i \neq j} P_i$ *is irreducible.*

Proof. Suppose $q \in \bigcap_{1 \leq i \leq 6, i \neq j} P_i$ is reducible. Then $q = \ell_1 \ell_2$ where ℓ_1 and ℓ_2 are equations are degree 1. However, since $q \in \bigcap_{1 \leq i \leq 6, i \neq j} P_i$, either ℓ_1 or ℓ_2 is an element in 3 of the 5 intersecting P_i's (three points lie on the same line). But in a general set of 6 points, any three points do not lie on the same line, so q must be irreducible. □

For the remainder of the section, let $q_j \in \bigcap_{1 \leq i \leq 6, i \neq j} P_i$ for some $1 \leq j \leq 6$, and let $\ell_{ij} \in P_i \bigcap P_j$. By Lemma 4.17, each q_j is irreducible. Next, in order to fully utilize these new techniques, we need to work explicitly with the generators of different ideals. We first determine the generators of $I^{(2)}$.

Theorem 4.18. *Let* X *be a set of* 6 *general points in* $\mathbb{P}^2$ *and* I *its defining ideal. Then,* $I^{(2)} = (q_1 q_2 \ell_{12}, q_1 q_3 \ell_{13}, q_2 q_3 \ell_{23}, q_1 q_2 q_3)$.

Proof. Set $K = (q_1 q_2 \ell_{12}, q_1 q_3 \ell_{13}, q_2 q_3 \ell_{23}, q_1 q_2 q_3)$. By definition of each q_i and ℓ_{ij}, it is clear that every generator of K passes through the 6 points twice, so $K \subseteq I^{(2)}$. Therefore, by part 3 Theorem 2.43, it suffices to show that K and $I^{(2)}$ have the same Hilbert Function for all $d \in \mathbb{N}$. To show this, consider the following short exact sequences all in the form of Proposition 2.41:

- $0 \to R/(q_1 q_2, q_1 \ell_{13}, q_2 \ell_{23})(-2) = R/(K : q_3)(-2) \to R/K \to R/(K, q_3) = R/(q_1 q_2 \ell_{12}, q_3) \to 0$ and

- $0 \to R/(q_1, \ell_{23})(-4) = R/((K : q_3) : q_2) \to R/(K : q_3)(-2) \to R/(K : q_3, q_2)(-2) = R/(q_1 \ell_{13}, q_2)(-2) \to 0$

By additivity of the Hilbert Function (2.38), $H_{R/K} = H_{R/(q_1 q_2 \ell_{12}, q_3)} + H_{R/(q_1 \ell_{13}, q_2)(-2)} + H_{R/(q_1, \ell_{23})(-4)}$. All of the ideals in the sum are complete intersections, so applying Proposition 4.16, we get that as we increment d,

- $H_{A(R/(q_1 q_2 \ell_{12}, q_3))} = 1, 2, 2, 2, 2, 1, 0, 0, \ldots$. Therefore,

- $H_{R/(q_1 q_2 \ell_{12}, q_3)} = 1, 3, 5, 7, 9, 10, 10, \ldots$

- $H_{A(R/(q_1\ell_{13},q_2)(-2))} = 0,0,1,2,2,1,0,0,....$ Therefore,
- $H_{R/(q_1\ell_{13},q_2)(-2)} = 0,0,1,3,5,6,6,....$
- $H_{A(R/(q_1,\ell_{23})(-4))} = 0,0,0,0,1,0,0,....$ Therefore,
- $H_{R/(q_1,\ell_{23})(-4)} = 0,0,0,0,1,2,2,...$

Then we add the respective Hilbert Functions.

$$\begin{array}{rcrrrrrrrl}
H_{R/(q_1q_2\ell_{12},q_3)} &=& 1, & 3, & 5, & 7, & 9, & 10, & 10, & ... \\
H_{R/(q_1\ell_{13},q_2)(-2)} &=& 0, & 0, & 1, & 3, & 5, & 6, & 6, & ... \\
H_{R/(q_1,\ell_{23})(-4)} &=& 0, & 0, & 0, & 0, & 1, & 2, & 2, & ... \\
\hline
H_{(R/K)} &=& 1, & 3, & 6, & 10, & 15, & 18, & 18, & ....
\end{array}$$

Given that $e(R/I^{(2)}) = 18$ and by the AH Theorem we know that $R/I^{(2)}$ has the expected maximal Hilbert Function, we have that $K = I^{(2)}$. $\square$

We now move on to the main results of this section where we prove that $(I^{(2)})^n = I^{(2n)}$ for $n = 2,3,4$.

Lemma 4.19. *Let q_j and ℓ_{ij} be the same as before. Then, there exists lines a_1, a_2, a_3 and units u_1, u_2, u_3 such that $q_1 = u_1(a_1\ell_{23} + \ell_{12}\ell_{23})$, $q_2 = u_2(a_2\ell_{13} + \ell_{12}\ell_{23})$, and $q_3 = u_3(a_3\ell_{12} + \ell_{13}\ell_{23})$. Furthermore, a_1, a_2, ℓ_{12} are linearly independent.*

Proof. Recall that P_1 is the defining ideal of a point, so it is generated by two lines that pass through the point. In other words, $P_1 = (\ell_{12}, \ell_{13})$. Similarly, $P_2 = (\ell_{12}, \ell_{23})$ and $P_3 = (\ell_{13}, \ell_{23})$. Since by definition, $q_1 \in P_2 \bigcap P_3 = (\ell_{23}, \ell_{12}\ell_{13})$, $q_1 = a\ell_{23} + c\ell_{12}\ell_{13}$ for some line a and constant c. Since q_1 is irreducible, we can assume that $c > 0$ and since it is a constant, it is a unit. Therefore, $q_1 = c(ac^{-1}\ell_{23} + \ell_{12}\ell_{13})$. We then get the desired result by setting $u_1 = c$ and $a_1 = ac^{-1}$. The proof for q_2 and q_3 are identical.

We now show that a_1, a_2, ℓ_{12} are linearly independent. Because our sets of points are always general, it is sufficient to prove that these lines are linearly independent for one set of points. We use the following points in $\mathbb{P}^2$:

- $P_1 = (y, z)$ corresponding to the point $(1,0,0)$.
- $P_2 = (x, z)$ corresponding to the point $(0,1,0)$.
- $P_3 = (x, y)$ corresponding to the point $(0,0,1)$.
- $P_4 = (x-y, y-z)$ corresponding to the point $(1,1,1)$.
- $P_5 = (2y-x, 3z-2y)$ corresponding to the point $(1,2,3)$.
- $P_6 = (4y+x, 6z-4y)$ corresponding to the point $(-1,4,6)$.

Using the computation software from [9], we can calculate the intersection of ideals explicitly and get that $l_{12} = z$, $a_1 = -7x + 17y - 11z$, and $a_2 = \frac{1}{11}(19x - 14y - 16z)$. To show that these lines are linearly independent, we calculate the rank of the following matrix with entries corresponding to the basis $\{x, y, z\}$ using row operations:

$$\begin{bmatrix} 0 & 0 & 1 \\ -7 & 17 & -11 \\ \frac{19}{11} & \frac{-14}{11} & \frac{-16}{11} \end{bmatrix} \to \begin{bmatrix} -7 & 17 & -11 \\ \frac{19}{11} & \frac{-14}{11} & \frac{-16}{11} \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -7 & 17 & -11 \\ 0 & \frac{225}{77} & \frac{-321}{77} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the matrix has the maximal rank, the a_1, a_2, ℓ_{12} are linearly independent.

□

Theorem 4.20. *For a general set of* 6 *points let I be its defining ideal. Then,* $(I^{(2)})^2 = I^{(4)}$.

Proof. For the remainder of the proof, let $q_j \in \bigcap_{1\le i\le 6, i\ne j} P_i$ for some $1 \le j \le 6$ where q_j is a degree 2 homogeneous equation, and let $\ell_{ij} \in P_i \bigcap P_j$ where ℓ_{ij} is a degree 1 homogeneous equation. By Lemma 4.17, each q_j is irreducible. Let $J = (q_1q_2\ell_{12}, q_1q_3\ell_{13}, q_2q_3\ell_{23})$. Note that $J \subseteq I^{(2)} \subseteq I^{(4)}$. We will prove that $J^2 = I^{(4)}$. To do this, we will show that $J^2 = I^{(4)} \bigcap H$ where H is either a (x, y, z)-primary ideal or 0 and then prove that H must be zero by showing J^2 is unmixed.

We first show that $J = I^{(2)} \bigcap H'$ where H' is a (x, y, z)-primary ideal. To begin, first note by Proposition 4.14, since there isn't any non-unit equation f that divides the three generators of J, we know $\mathrm{ht}(J) \ge 2$. This implies that the prime ideals in $\mathrm{Ass}(R/J)$ have height 2 or 3.

Next, consider the element $q_1q_2q_3$. Any element in J can be written in the form $f_1q_1q_2\ell_{12} + f_2q_1q_3\ell_{13} + f_3q_2q_3\ell_{23}$ for some $f_1, f_2, f_3 \in R$. Furthermore, in order for the degrees to be consistent, f_1, f_2, f_3 must also all be degree 1, so they are all also irreducible. Assume for contradiction that $q_1q_2q_3 \in J$, that is $q_1q_2q_3 = f_1q_1q_2\ell_{12} + f_2q_1q_3\ell_{13} + f_3q_2q_3\ell_{23}$. But this implies that $f_3q_2q_3\ell_{23} = q_1(q_2q_3 + f_1q_2\ell_{12} + f_2q_3\ell_{13})$. That is, $f_3q_2q_3\ell_{23} \in (q_1)$. Since $R = \mathbb{C}[x, y, z]$ is a UFD, this is a contradiction, so $q_1q_2q_3 \notin J$. However, by Proposition 2.40, $J : q_1q_2q_3 = (\ell_{12}, \ell_{13}, \ell_{23})$. For a general set of points, $\ell_{12}, \ell_{13}, \ell_{23}$ will be linearly independent, so with a change of variables we have that $(\ell_{12}, \ell_{13}, \ell_{23}) = (x, y, z)$. This implies that $q_1q_2q_3(x, y, z) \subseteq J$. If we then localize at a non-maximal prime ideal P, we see that $q_1q_2q_3(x, y, z)_P \subseteq J_P$. Since (x, y, z) is a maximal ideal, there exists some element $f \in (x, y, z) \setminus P$, so $(x, y, z)_P$ contains a unit f, so $(x, y, z)_P = R_P$. This shows that $q_1q_2q_3 \in J_P$. Recall that $I^{(2)} = (J, q_1q_2q_3)$, so $I^{(2)}_P = (J, q_1q_2q_3)_P = J_P$.

Given that $I^{(2)} \ne J$, we know that J must have a height 3 ideal in its primary decomposition. In other words, there must be a (x, y, z)-primary ideal in the primary decomposition of J, call it H'. Rewriting J as $G' \bigcap H'$ where G' is the intersection of the height 2 primary components of J, we see that $I^{(2)}_P = J_P = G'_P$ for all non-maximal primes P. Therefore, by Proposition 2.30, $G' = I^{(2)}$. So, $J = I^{(2)} \bigcap H'$.

We will use this fact to show that $J^2 = I^{(4)} \bigcap H$. Since $J^2 = (q_1^2q_2^2\ell_{12}^2, q_1^2q_3^2\ell_{13}^2, q_2^2q_3^2\ell_{23}^2, q_1^2q_2q_3\ell_{12}\ell_{13}, q_1q_2^2q_3\ell_{12}\ell_{23}, q_1q_2q_3^2\ell_{13}\ell_{23})$, by proposition 4.14, J^2 is also a height 2 ideal. Recall that H is a (x, y, z)-primary ideal, and rewrite $J^2 = G \bigcap H$ where G is the intersection of the height 2 primary components of J^2. If $H = 0$, then we are done, so assume that H is a (x, y, z)-primary ideal. By part 4 of Proposition 2.17, any non-maximal ideal P (so P has height 2) containing J^2 also contains J. Since J has height 2, any prime of height 2 containing J is a minimal prime so $P \in \mathrm{Min}(J) \subseteq \mathrm{Ass}(R/J)$. Since $\mathrm{Ass}(R/J) = \{P_1, P_2, ..., P_6, (x, y, z)\}$ and P is non-maximal, $P = P_i$ for some $1 \le i \le 6$. This implies that the associated primes of R/J^2 can only be (x, y, z) or the associated primes of R/I.

Let $P = P_i$ for some $1 \leq i \leq 6$. Then, from the result we have just proven, $G_P = J_P^2 = (J_P)^2 = (I_P^{(2)})^2 = (P_P^2)^2 = P_P^4 = I_P^{(4)}$. So by Proposition 2.30, $G = I^{(4)}$ and $J^2 = I^{(4)} \bigcap H$. Therefore, it suffices to show that J^2 is unmixed of height 2. To do this, recall that complete intersections of two elements are unmixed of height 2 by Proposition 4.14 and the observation that they are not the maximal ideal (because the Hilbert Function will always be different from $H_{(x,y,z)}$). Using the notation from Proposition 2.42, B will be unmixed of height h if A and C are unmixed of height h.

Consider the following list of short exact sequences. All are in the form of Proposition 2.41.

- $0 \to R/(J^2 : q_3)(-2) \to R/J^2 \to R/(J^2, q_3) \to 0$.
  Let $A_3 = J^2 : q_3$.
- $0 \to R/(A_3 : q_2)(-2) \to R/A_3 \to R/(A_3, q_2) \to 0$.
  Let $A_{32} = A_3 : q_2$.
- $0 \to R/(A_{32} : q_1)(-2) \to R/A_{32} \to R/(A_{32}, q_1) \to 0$.
  Let $A_{321} = A_{32} : q_1$.

By direct calculations and Proposition 2.40, we see that:

- $A_3 = J^2 : q_3 = (q_1^2 q_2^2 \ell_{12}^2, q_1^2 q_3 \ell_{13}^2, q_2^2 q_3 \ell_{23}^2, q_1^2 q_2 l_{12} \ell_{13}, q_1 q_2 q_3 \ell_{13} \ell_{23}, q_1 q_2^2 \ell_{12} \ell_{13})$,
  $(J^2, q_3) = (q_1^2 q_2^2 \ell_{12}^2, q_3)$.
- $A_{32} = A_3 : q_2 = (q_1^2 q_2 \ell_{12}^2, q_1^2 q_3 \ell_{13}^2, q_2 q_3 \ell_{23}^2, q_1^2 \ell_{12} \ell_{13}, q_1 q_3 \ell_{13} \ell_{23}, q_1 q_2 \ell_{12} \ell_{13})$,
  $(A_3, q_2) = (q_1^2 q_3 \ell_{13}^2, q_2)$.
- $A_{321} = A_{32} : q_1 = (q_1 q_2 \ell_{12}^2, q_1 q_3 \ell_{13}^2, q_2 q_3 \ell_{23}^2, q_1 \ell_{12} \ell_{13}, q_3 \ell_{13} \ell_{23}, q_2 \ell_{12} \ell_{13})$,
  $(A_{32}, q_1) = (q_2 q_3 \ell_{23}^2, q_1)$.

From these calculations, it is clear that J^2 is unmixed of height 2 if A_{321} is also unmixed of height 2. To show A_{321} is also unmixed, let $B = (q_1 \ell_{12} \ell_{13}, q_2 \ell_{12} \ell_{23}, q_3 \ell_{13} \ell_{23}) \subseteq A_{321}$ and we consider another set of short exact sequences.

- $0 \to R/(B : \ell_{23})(-1) \to R/B \to R/(B, \ell_{23}) \to 0$.
  Let $B_3 = B : \ell_{23} = (q_1 \ell_{12} \ell_{13}, q_3 \ell_{13}, q_2 \ell_{12})$ and note that $(B, \ell_{23}) = (q_1 \ell_{12} \ell_{13}, \ell_{23})$.
- $0 \to R/(B_3 : \ell_{13})(-1) \to R/B_3 \to R/(B_3, \ell_{13}) \to 0$.
  Let $B_{32} = B_3 : \ell_{13} = (q_1 \ell_{12} \ell_{13}, q_3 \ell_{13}, q_2 \ell_{12})$ and note that $(B_3, \ell_{13}) = (q_2 \ell_{12}, \ell_{13})$.
- $0 \to R/(B_{32} : \ell_{12})(-1) \to R/B_{32} \to R/(B_{32}, \ell_{12}) \to 0$.
  Let $B_{321} = B_{32} : \ell_{12} = (q_1, q_2, q_3)$ and note that $(B_{32}, \ell_{12}) = (q_3, \ell_{12})$.

If A_{321} is not unmixed, we can rewrite $A_{321} = C_1 \bigcap C_2$ where C_1 are the height 2 primary components of A_{321} and C_2 is a (x, y, z)-primary component. We will end up showing that this subset B is precisely equal to C_1. We first show that B is indeed unmixed of height 2. Using the Proposition 2.42 as above, this follows if B_{321} is unmixed of height 2. Indeed,

$B_{321} = (q_1, q_2, q_3) \subseteq P_4 \bigcap P_5 \bigcap P_6$. By definition, q_1, q_2, q_3 are linearly independent equations, and since Lemma 4.1 implies that the defining ideal of 3 general points is generated by 3 linearly independent equations of degree 2, we have that $B_{321} = (q_1, q_2, q_3) = P_4 \bigcap P_5 \bigcap P_6$.

Next, we directly calculate $\mathrm{Ass}(R/C_1)$. Note that $A_{321} = J^2 : q_1q_2q_3 = (I^{(4)} : q_1q_2q_3) \bigcap (H : q_1q_2q_3) = (\bigcap_{1\leq i\leq 6}(P_i^4) : q_1q_2q_3) \bigcap (H')$. Where $H' := H : q_1q_2q_3$ is a (x, y, z)-primary ideal. Let $P = P_i$ where $i = 1, 2, 3$. It is clear that $P^2 \subseteq P^4 : q_1q_2q_3$. For the reverse inclusion, note that since q_1, q_2, q_3 are irreducible, they are elements of $P \setminus P^2$ provided that $i \neq q_i$. Given that $P^2 = P^{(2)}$ by the definition of symbolic powers, this is equivalent to saying that q_1, q_2, q_3 each pass through P precisely once. So, if $x \in P^4 : q_1q_2q_3$, then $xq_1q_2q_3 \in P^4 = P^{(4)}$. Since $q_1q_2q_3$ passes through P twice, x must pass through P at least 2 times, so $x \in P^{(2)} = P^2$. Now let $P = P_i$ for $i = 4, 5, 6$. Once again, it is clear that $P \subseteq P^4 : q_1q_2q_3$. For the reverse inclusion, we make the same argument as above. In this case, $q_1q_2q_3$ passes through P three times, so x must pass through P at least once. So, $x \in P$. This implies that $C_1 = P_1^2 \bigcap P_2^2 \bigcap P_3^2 \bigcap P_4 \bigcap P_5 \bigcap P_6$ and $\mathrm{Ass}(R/C_1) = \{P_1, P_2, ..., P_6\}$.

We now prove that $\mathrm{Ass}(R/B) = \mathrm{Ass}(R/C_1)$. Since we have proven that B is unmixed of height 2, the inclusion $\mathrm{Ass}(R/C_1) \subseteq \mathrm{Ass}(R/B)$ is clear since $\mathrm{Ass}(R/B) = \mathrm{Min}(B)$ and P_i is a height 2 ideal containing B. By Proposition 2.42, we can write $\mathrm{Ass}(R/B) \subseteq \mathrm{Ass}(R/(B, \ell_{23})) \bigcup \mathrm{Ass}(R/(B_3, \ell_{13})) \bigcup \mathrm{Ass}(R/(B_{32}, \ell_{12})) \bigcup \mathrm{Ass}(R/B_{321})$. We can calculate each of the associated primes of each of the ideals in the union explicitly. We have already shown $\mathrm{Ass}(R/B_{321}) = \{P_4, P_5, P_6\}$. The remaining ideals are all complete intersections of two elements, so they are unmixed of height 2 and the associated primes are equal to their minimal primes. $(B_{32}, \ell_{12}) = (q_3, \ell_{12})$ and $q_3 \in P_1 \bigcap P_2$ and $\ell_{12} \in P_1 \bigcap P_2$. So, P_1 and P_2 are minimal primes over (B_{32}, ℓ_{12}) and therefore $\{P_1, P_2\} \subseteq \mathrm{Ass}(R/(B_{32}, \ell_{12}))$. If there was another prime in $P \in \mathrm{Ass}(R/(B_{32}, \ell_{12}))$, then $q_3, \ell_{12} \in P$. Recall from Hilbert's Nullstellensatz (2.26) that P corresponds to a point in $\mathbb{P}^2$, so q_3 and ℓ_{12} will intersect at some other point. But this implies that $|\mathbf{V}(q_3) \bigcap \mathbf{V}(\ell_{12})| > 2$ which by Bezout Theorem (2.44) implies that ℓ_{12} divides q_3. This is a contradiction since q_3 is irreducible. So, $\{P_1, P_2\} = \mathrm{Ass}(R/(B_{32}, \ell_{12}))$. The other ideals are similar. $(B_3, \ell_{13}) = (q_2\ell_{12}, \ell_{13})$. $q_2\ell_{12}, \ell_{13} \in P_1 \bigcap P_3$ so $\{P_1, P_3\} \subseteq \mathrm{Ass}(R/(B_3, \ell_{13}))$. No other prime can be in $\mathrm{Ass}(R/(B_3, \ell_{13}))$ since that would imply that $|\mathbf{V}(q_2\ell_{12}) \bigcap \mathbf{V}(\ell_{13})| > 3$ which contradicts Bezout Theorem. $(B, \ell_{23}) = (q_1\ell_{12}\ell_{13}, \ell_{23})$. $q_1\ell_{12}\ell_{13}, \ell_{23} \in P_2 \bigcap P_3$ so $\{P_2, P_3\} \subseteq \mathrm{Ass}(R/(B, \ell_{23}))$. No other prime can be in $\mathrm{Ass}(R/(B, \ell_{23}))$ since that would imply that $|\mathbf{V}(q_1\ell_{12}\ell_{13}) \bigcap \mathbf{V}(\ell_{23})| > 4$ which again contradicts Bezout Theorem. So, $\mathrm{Ass}(R/B) \subseteq \mathrm{Ass}(R/C_1)$.

Lastly, since $\mathrm{Ass}(R/B) = \mathrm{Ass}(R/C_1)$, by Proposition 2.30, it suffices to show that $B_P = (C_1)_P$ for all $P \in \mathrm{Ass}(R/C_1) = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ to conclude $B = C_1$. The inclusion $B \subseteq C_1$ is clear by definition. For the reverse inclusion, let $P = P_1$. Then, $B_P = (q_1\ell_{12}\ell_{13}, q_2\ell_{12}\ell_{23}, q_3\ell_{13}\ell_{23})_P$. Using the fact that q_1, ℓ_{23} are units in R_P and further simplifying using Lemma 4.19, we get that $B_P = (\ell_{12}\ell_{13}, q_2\ell_{12}, q_3\ell_{13})_P = (\ell_{12}\ell_{13}, (a_2\ell_{13} + \ell_{12}\ell_{23})\ell_{12}, (a_3\ell_{12} + \ell_{13}\ell_{23})\ell_{13})_P = (\ell_{12}\ell_{13}, \ell_{12}^2, \ell_{13}^2)_P = P_P^2 = (C_1)_P$. By symmetry of the generators of B, the cases where $P = P_2$ and $P = P_3$ are proved identically. Now let $P = P_i$ where $i = 4, 5, 6$. Then, since $\ell_{12}, \ell_{13}, \ell_{23}$ are units in R_P, $B_P = (q_1, q_2, q_3)_P = (P_4 \bigcap P_5 \bigcap P_6)_P \subseteq P_P = (C_1)_P$.

□

Lemma 4.21. *Let J, q_j and ℓ_{ij} be the same as before. Then, $J^3 : q_1q_2\ell_{12} = J^2$.*

Proof. Since $q_1q_2\ell_{12} \in J$, the inclusion $J^2 \subseteq J^3 : q_1q_3\ell_{12}$ is trivial. For the reverse inclusion, we know that $J^2 = I^{(4)}$ and $\operatorname{Ass}(R/I^{(4)}) = \{P_i\}$ for $1 \leq i \leq 6$. So, by Proposition 2.30, it suffices to show that $J^3_{P_i} : q_1q_2\ell_{12} \subseteq J^2_{P_i}$ for all i. We first note that regardless of which i is chosen, $J^2_{P_i} = I^{(4)}_{P_i} = P^4_{P_i}$ and $J^3_{P_i} = (J_{P_i})^3 = (I^{(2)}_{P_i})^3 = (P^2_{i\,P_i})^3 = P^6_{i\,P_i}$. Additionally, by the definition of symbolic powers, if P is a prime ideal, then its primary decomposition is simply itself, so its ordinary power is equal to its symbolic power.

Now consider $P = P_1$. Since $q_1 \notin P$, q_1 is a unit in R_P. So, $J^3_P : q_1q_2\ell_{12} = P^6_P : q_2\ell_{12}$. Since q_2 and ℓ_{12} are irreducible, both are elements of $P \setminus P^2$. Since $P^2 = P^{(2)}$, this is equivalent to saying that q_2 and ℓ_{12} passes through P precisely once. So, if $x \in P^6 : q_2\ell_{12}$ then $xq_2\ell_{12} \in P^6 = P^{(6)}$. Since $xq_2\ell_{12}$ passes through P at least 6 times, then then x must pass through P at least 4 times. That is, $x \in P^{(4)} = P^4$. Finally, since $J^2_P = I^{(4)}_P = P^4_P$, the reverse inclusion when localizing at P_1 is proved.

The argument for P_2 is identical by replacing q_1 with q_2. Lastly, the argument for $P = P_i$ for $i = 3, 4, 5, 6$ follows by noticing that $\ell_{12} \notin P$, so ℓ_{12} is a unit. Therefore, we have that $J^3_P : q_1q_2\ell_{12} = P^6_P : q_1q_2$, and we use a similar argument as above. Then, we have shown that $J^3_{P_i} : q_1q_2\ell_{12} \subseteq J^2_{P_i}$ for all i. $\square$

Lemma 4.22 (SHGH Conjecture Results)**.** *Using the SHGH Conjecture, $R/I^{(m)}$ has the expected Hilbert Function for $m = 4, 6, 8$.*

Proof. Let q_j and ℓ_{ij} have the same definition as above and note from Definition 2.49 that they are indeed exceptional. Recall from Proposition 2.43 that to prove $R/I^{(m)}$ has the expected Hilbert Function, we show that there are no equations of degree $\epsilon(R/I^{(m)}) - 1$ in $I^{(m)}$ and there are at most $H_R - e(R/I^m)$ equations of degree $\epsilon(R/I^{(m)})$ in $I^{(m)}$. We will show the first part using the standard Bezout Theorem argument. However, we will use the SHGH Conjecture (2.50) for the second part.

- $m = 4$

  Note that $e(R/I^{(m)}) = 60$ and $\epsilon(R/I^{(m)}) = 10$. Suppose for contradiction there exists an equation of degree 9 in $I^{(m)}$, call it f_9. Since $|\mathbf{V}(f_9) \bigcap \mathbf{V}(q_1)| = 20 > 18$, we can rewrite $f_9 = q_1 f_7$ where $f_7 \in P_1^4 \bigcap P_2^3 \bigcap P_3^3 \bigcap P_4^3 \bigcap P_5^3 \bigcap P_6^3$. Notice again that $|\mathbf{V}(f_7) \bigcap \mathbf{V}(q_1)| \geq 15 > 14$ so we have that $f_7 = q_1 f_5$ where $f_5 \in P_1^4 \bigcap P_2^2 \bigcap P_3^2 \bigcap P_4^2 \bigcap P_5^2 \bigcap P_6^2$. Continuing this argument, we eventually see that $f_9 = q_1^2 q_2^2 f_1$ where $f_1 \in P_1^2 \bigcap P_2^2$. This is a contradiction since f_1 is linear and therefore cannot pass through P_1 and P_2 twice. So there are no equations of degree 9 in $I^{(m)}$.

  Consider the following equations in of degree 10 in $I^{(m)}$: $q_1q_2q_3q_4q_5, q_1q_3q_4q_5q_6, q_1q_2q_4q_5q_6, q_1q_2q_3q_5q_6, q_1q_2q_3q_4q_6, q_2q_3q_4q_5q_6$. For any irreducible quadratic passing through 5 points or line passing through 2 points, one can find an equation from the list above that is not a multiple of the chosen exceptional equation. So by the SHGH conjecture, $I^{(m)}$ has no more than 6 equations.

- $m = 6$

  Note that $e(R/I^{(m)}) = 126$ and $\epsilon(R/I^{(m)}) = 15$. Suppose for contradiction there exists an equation of degree 14 in $I^{(m)}$, call it f_{14}. By the same argument as above, we can write $f_{14} = q_1^2 q_2^2 q_3^2 f_2$ where $f_2 \in P_1^2 \bigcap P_2^2 \bigcap P_3^2$. This is a contradiction by Lemma 4.2. So there are no equations of degree 14 in $I^{(m)}$.

Consider the following equations in of degree 15 in $I^{(m)}$: $q_1^2 q_2 q_3 q_4 q_5 \ell_{12}\ell_{13}\ell_{45}, q_1 q_3^2 q_4 q_5 q_6 \ell_{34}\ell_{35}\ell_{16}$, $q_1 q_2 q_4^2 q_5 q_6 \ell_{14}\ell_{24}\ell_{56}, q_1 q_2 q_3 q_5^2 q_6 \ell_{15}\ell_{25}\ell_{36}, q_1 q_2 q_3 q_4 q_6^2 \ell_{16}\ell_{12}\ell_{34}, q_2^2 q_3 q_4 q_5 q_6 \ell_{23}\ell_{24}\ell_{56}$. By the same argument as above, we can apply the SHGH conjecture and conclude that $I^{(m)}$ has no more than 10 equations of degree 15.

- $m = 8$

  Note that $e(R/I^{(m)}) = 216$ and $\epsilon(R/I^{(m)}) = 20$. Suppose for contradiction there exists an equation of degree 19 in $I^{(m)}$, call it f_{19}. By the same argument as above, we can write $f_{19} = q_1^2 q_2^2 q_3^2 q_4^2 f_3$ where $f_3 \in P_1^2 \bigcap P_2^2 \bigcap P_3^2 \bigcap P_4^2$. This is a contradiction since the AH Theorem (2.46) implies that $\alpha(I^{(2)}) = 4$ for $r = 4$ points. So there are no equations of degree 19 in $I^{(m)}$.

  Consider the following equations in of degree 20 in $I^{(m)}$: $(q_1 q_2 q_3 q_4 q_5)^2, (q_1 q_3 q_4 q_5 q_6)^2$, $(q_1 q_2 q_4 q_5 q_6)^2, (q_1 q_2 q_3 q_5 q_6)^2, (q_1 q_2 q_3 q_4 q_6)^2, (q_2 q_3 q_4 q_5 q_6)^2$. By the same argument as above, we can apply the SHGH conjecture and conclude that $I^{(m)}$ has no more than 15 equations of degree 20.

□

Theorem 4.23. *For a general set of* 6 *points let* I *be its defining ideal. Then,* $(I^{(2)})^3 = I^{(6)}$ *and* $(I^{(2)})^4 = I^{(8)}$.

Proof. We will first show that $(I^{(2)})^3 = I^{(6)}$. By the Lemma 4.22, we know that $R/I^{(6)}$ has the expected Hilbert Function. Since it is clear that $J^3 \subseteq (I^{(2)})^3 \subseteq I^{(6)}$, it to show by part 3 of Theorem 2.43 that $H_{R/J^3} = H_{R/I^{(6)}}$. To show this, we first consider the following short exact sequences:

- $0 \to R/(J^3 : q_1 q_2 \ell_{12})(-5) \to R/J^3 \to R/(J^3, q_1 q_2 \ell_{12}) \to 0$.
  Let $K_0 = (J^3, q_1 q_2 \ell_{12})$.
- $0 \to R/(K_0 : q_3^3)(-6) \to R/K_0 \to R/(K_0, q_3^3) \to 0$.
  Let $K_1 = K_0 : q_3^3$.
- $0 \to R/(K_1 : q_1)(-8) \to R/K_1(-6) \to R/(K_1, q_1)(-6) \to 0$.
  Let $K_2 = K_1 : q_1$.
- $0 \to R/(K_2 : q_2)(-10) \to R/K_2(-8) \to R/(K_2, q_2)(-8) \to 0$.
  Let $K_3 = K_2 : q_2$.
- $0 \to R/(K_3 : \ell_{13}^2 \ell_{23}^2)(-14) \to R/K_3(-10) \to R/(K_3, \ell_{23}^2 \ell_{13}^2)(-10) \to 0$.
  Let $K_4 = K_3 : \ell_{13}^2 \ell_{23}^2$.

By Proposition 2.38, $H_{R/J^3} = H_{R/(J^3 : q_1 q_2 l_{12})(-5)} + H_{R/(K_0, q_3^3)} + H_{R/(K_1, q_1)(-6)} + H_{R/(K_2, q_2)(-8)} + H_{R/(K_3, \ell_{23}^2 \ell_{13}^2)(-10)} + H_{R/K_4(-14)}$. We now calculate each of the relevant ideals directly.

- $(J^3 : q_1 q_2 \ell_{12}) = J^2$ by Lemma 4.21.

- $K_0 = (q_1q_2\ell_{12}, q_3^3(q_1\ell_{13}, q_2\ell_{23})^3)$ so $(K_0, q_3^3) = (q_1q_2\ell_{12}, q_3^3)$.
- $K_1 = (q_1q_2\ell_{12}, q_1^3\ell_{13}^3, q_1^2q_2\ell_{13}^2\ell_{23}, q_1q_2^2\ell_{13}\ell_{23}^2, q_2^3\ell_{23}^3)$ so $(K_1, q_1) = (q_2^3\ell_{23}^3, q_1)$.
- $K_2 = (q_2\ell_{12}, q_1^2\ell_{13}^3, q_1q_2\ell_{13}^2\ell_{23}, q_2^2\ell_{13}\ell_{23}^2, q_2^3\ell_{23}^3)$ so $(K_2, q_2) = (q_1^2\ell_{13}^3, q_2)$.
- $K_3 = (\ell_{12}, q_1^2\ell_{13}^3, q_1\ell_{13}^2\ell_{23}, q_2\ell_{13}\ell_{23}^2, q_2^2\ell_{23}^3)$. Using Lemma 4.19, we can rewrite K_3 as $(\ell_{12}, a_1\ell_{23}^2\ell_{13}^2, a_2\ell_{23}^2\ell_{13}^2)$. So, $K_4 = (\ell_{12}, a_1, a_2)$ and $(K_3, \ell_{23}^2\ell_{13}^2) = (\ell_{23}^2\ell_{13}^2, \ell_{12})$.

We also know the Hilbert Function for all of the relevant ideals as well. $H_{R/(J^3:q_1q_2l_{12})(-5)}$ follows from $R/I^{(4)}$ having the expected Hilbert Function by Lemma 4.22, and $H_{R/K_4(-14)}$ follows from Lemma 4.19 which states that $K_4 = (x, y, z)$. The remaining ideals are complete intersections, so we will utilize Proposition 4.16. For these ideals, the technique is identical to the calculations in Theorem 4.18, so for brevity, the Artinian reductions will not be explicitly written below.

$H_{R/(J^3:q_1q_2l_{12})(-5)}$	=	0,	0,	0,	0,	0,	1,	3,	6,	10,	15,	21,	28,	36,	45,	55,	60,	60, …
$H_{R/(K_0,q_3^3)}$	=	1,	3,	6,	10,	15,	20,	24,	27,	29,	30,	30,	…					
$H_{R/(K_1,q_1)(-6)}$	=	0,	0,	0,	0,	0,	0,	1,	3,	5,	7,	9,	11,	13,	15,	17,	18,	18, …
$H_{R/(K_2,q_2)(-8)}$	=	0,	0,	0,	0,	0,	0,	0,	0,	1,	3,	5,	7,	9,	11,	13,	14,	14, …
$H_{R/(K_3,\ell_{23}^2\ell_{13}^2)(-10)}$	=	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	1,	2,	3,	4,	4,	…	
$H_{R/K_4(-14)}$	=	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	1,	0,	0, …
H_{R/J^3}	=	1,	3,	6,	10,	15,	21,	28,	36,	45,	55,	66,	78,	91,	105,	120,	126,	126, …

From the last row of the table, we see that $H_{R/J^3} = H_{R/I^{(6)}}$ so $(I^{(2)})^3 = I^{(6)}$.

Since now that we know that $J^3 = I^{(6)}$, by a similar argument as Lemma 4.21, we have that $J^4 : q_1q_2\ell_{12} = J^3$. The techniques to prove used to show $(I^{(2)})^4 = I^{(8)}$ are identical to the above argument so for conciseness, we will simply write the relevant short exact sequences and the appropriate Hilbert Functions.

- $0 \to R/(J^4 : q_1q_2\ell_{12})(-5) \to R/J^4 \to R/(J^4, q_1q_2l_{12}) \to 0$.
  Let $K_0 = (J^4, q_1q_2\ell_{12})$.
- $0 \to R/(K_0 : q_3^4)(-8) \to R/K_0 \to R/(K_0, q_3^4) \to 0$.
  Let $K_1 = K_0 : q_3^4$.
- $0 \to R/(K_1 : q_1)(-10) \to R/K_1(-8) \to R/(K_1, q_1)(-8) \to 0$.
  Let $K_2 = K_1 : q_1$.
- $0 \to R/(K_2 : q_2)(-12) \to R/K_2(-10) \to R/(K_2, q_2)(-10) \to 0$.
  Let $K_3 = K_2 : q_2$.
- $0 \to R/(K_3 : \ell_{13}^3\ell_{23}^3)(-18) \to R/K_3(-12) \to R/(K_3, \ell_{23}^3\ell_{13}^3)(-12) \to 0$.
  Let $K_4 = K_3 : \ell_{13}^3\ell_{23}^3$.

We note that $H_{R/J^4} = H_{R/J^3(-5)} + H_{R/(K_0,q_3^4)} + H_{R/(K_1,q_1)(-8)} + H_{R/(K_2,q_2)(-10)} + H_{R/(K_3,\ell_{23}^3\ell_{13}^3)(-12)} + H_{R/K_4(-18)}$.

$H_{R/J^3(-5)} =$	0,	0,	0,	0,	0,	1,	3,	6,	10,	15,	21,	28,	36,	45,	55,	66,	78,	91,	105,	126,	126,	…
$H_{R/(K_0,q_3^4)} =$	1,	3,	6,	10,	15,	20,	25,	30,	34,	37,	39,	40,	40,	…								
$H_{R/(K_1,q_1)(-8)} =$	0,	0,	0,	0,	0,	0,	0,	0,	1,	3,	5,	7,	9,	11,	13,	15,	17,	19,	21,	23,	24,	24,
$H_{R/(K_2,q_2)(-10)} =$	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	1,	3,	5,	7,	9,	11,	13,	15,	17,	19,	20,	20,
$H_{R/(K_3,\ell_{23}^3\ell_{13}^3)(-12)} =$	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	1,	2,	3,	4,	5,	6,	6,	…		
$H_{R/K_4(-18)} =$	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	1,	2,	0,	0,
$H_{R/J^4} =$	1,	3,	6,	10,	15,	21,	28,	36,	45,	55,	66,	78,	91,	105,	120,	136,	153,	171,	190,	210,	216,	…

So, we see that $H_{R/J^4} = H_{R/I^{(8)}}$ so $(I^{(2)})^4 = I^{(8)}$.

$\square$

We conclude this section with a brief analysis of the usability of Huneke's Criterion. As previously mentioned, Huneke's Criterion fails for small k. However, one can use it for $k = 10$, and we prove that it is indeed the smallest k that holds.

Theorem 4.24. *For 6 general points, Huneke's Criterion holds for $k = 10$ and cannot be applied for $k \leq 9$.*

Proof. As before, let $q_j \in \bigcap_{1 \leq i \leq 6, i \neq j} P_i$ and let $\ell_{ij} \in P_i \bigcap P_j$.

We first show that Huneke's Criterion is applicable when $k = 10$. To show this, we must first find $f, g \in I^{(10)}$ such that $\deg(f)\deg(g) = 6k^2 = 600$ and $\gcd(f, g) = 1$. I claim that $\alpha(I^{(10)}) = 24$. To show this, note that $q_1^2q_2^2q_3^2q_4^2q_5^2q_6^2$ is an equation of degree 24 in $I^{(10)}$. Now assume for contradiction that there exists a $f_{23} \in I^{(10)}$. Then, Bezout Theorem implies that $f_{23} = q_1^2q_2^2q_3^2q_4^2q_5^2q_6f_1$ where $f_1 \in P_1 \bigcap P_2 \bigcap P_3 \bigcap P_4 \bigcap P_5$. This is a contradiction for a general set of 5 points, so there does not exist an equation of degree 23 in $I^{(10)}$.

Lastly, we need to show that if $f_{24} = q_1^2q_2^2q_3^2q_4^2q_5^2q_6^2$, then $\gcd(f_{24}, f_{25}) = 1$ for some $f_{25} \in I^{(10)}$. Let $g_1 = q_1q_2\ell \in I^{(2)}$ and let $g_2 = q_3q_4\ell' \in I^{(2)}$. In a UFD, $g_1 = q_1q_2\ell_{12}$ and $g_2 = q_3q_4\ell_{34}$ are unique irreducible factorizations, and since $q_1, q_2, q_3, q_4, \ell_{12}, \ell_{34}$ are distinct, $\gcd(g_1, g_2) = 1$. Then, this implies that for $g_1^5, g_2^5 \in I^{(10)}$, $\gcd(g_1^5, g_2^5) = 1$. If $\gcd(f_{24}, g_1^5) = 1$ or $\gcd(f_{24}, g_2^5) = 1$, then we are done. Otherwise, assume $\gcd(f_{24}, g_1^5) > 1$ and $\gcd(f_{24}, g_2^5) > 1$. Let $f_\alpha = g_1^5 + \alpha g_2^5$ and $f_\beta = g_1^5 + \beta g_2^5$ for some $\alpha, \beta \in \mathbb{C}$. Consider $g = \gcd(f_\alpha, f_\beta)$. We will show that $g = 1$. Assume for contradiction that $g > 1$. Then, $g|f_\alpha$ and $g|f_\beta$. So, $g|(f_\alpha - f_\beta)$ and $g|(f_\alpha - \frac{\alpha}{\beta}f_\beta)$. So, $g|(\alpha - \beta)g_2^5$ and $g|(1 - \frac{\alpha}{\beta})g_1^5$ which implies $g|g_1^5$ and $g|g_2^5$. But $\gcd(g_1^5, g_2^5) = 1$, so $g = 1$. Consider the set $\{f_\alpha : \alpha \in \mathbb{C}\}$. For each f_α in the set, $\gcd(f_\alpha, f_\beta) = 1$ for any other $\beta \neq \alpha$. This implies there are an infinite number of elements in $\{f_\alpha : \alpha \in \mathbb{C}\}$ with each element having its own distinct factorization. However, since f_{24} is an equation with a finite number of irreducible factors, there must exist some γ such that $\gcd(f_{24}, f_\gamma) = 1$. Since $f_\gamma \in I^{(10)}$ and has degree 25, Huneke's Criterion applies for $k = 10$.

To show that Huneke's Criterion fails when $2 \leq k \leq 9$, we recall the following facts from Theorem 2.48.

1. For all $m, t \in \mathbb{N}$, $\frac{\alpha(I^{(mt)})}{mt} \leq \frac{\alpha(I^{(m)})}{m}$

2. Define $\hat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}$ and $\hat{\alpha}(I) \leq \frac{\alpha(I^{(m)})}{m}$ for all $m \in \mathbb{N}$.

3. If there exists a m_0 such that $(I^{(m_0)})^t = I^{(m_0t)}$ for all $t \in \mathbb{N}$, then, $\hat{\alpha}(I) = \frac{\alpha(I^{(m_0)})}{m_0}$.

Since Huneke's Criterion holds for $k = 10$ and that $\alpha(I^{(10)}) = 24$, by fact (3), $\hat{\alpha}(I) = 2.4$. Then, fact (2) implies the following

- $\alpha(I^{(2)}) \geq 4.8$ so $\alpha(I^{(2)}) \geq 5$,
- $\alpha(I^{(3)}) \geq 7.2$ so $\alpha(I^{(3)}) \geq 8$,
- $\alpha(I^{(4)}) \geq 9.6$ so $\alpha(I^{(4)}) \geq 10$,
- $\alpha(I^{(5)}) \geq 12$,
- $\alpha(I^{(6)}) \geq 14.4$ so $\alpha(I^{(6)}) \geq 15$,
- $\alpha(I^{(7)}) \geq 16.8$ so $\alpha(I^{(7)}) \geq 17$,
- $\alpha(I^{(8)}) \geq 19.2$ so $\alpha(I^{(8)}) \geq 20$,
- $\alpha(I^{(9)}) \geq 21.6$ so $\alpha(I^{(9)}) \geq 22$.

Consider the set $A := \{(a,b) : ab = 6k^2, a,b \in \mathbb{N}\}$. A is a finite set, so let $(a',b') = \min\{|a-b| : (a,b) \in A\}$. It suffices to prove that there do not exist equations of degree $\min\{a',b'\}$. Without loss of generality, if $a' < b'$, then $\max\{a,b\} > a'$ for all $(a,b) \in A$. So if there do not exist equations of degree a', then there do not exist equations of degree $\max\{a,b\}$ for all $(a,b) \in A$. The following table shows that for all $2 \leq k \leq 9$, $\alpha(I^{(k)}) \geq a'$, so Huneke's Criterion fails.

k	$6k^2$	(a',b')
2	24	(4,6)
3	54	(6,9)
4	96	(8,12)
5	150	(10,15)
6	216	(12,18)
7	294	(14,21)
8	384	(16,24)
9	486	(18,27)

□

5 Conclusion and Areas of Further Research

To summarize, in the case of 3,4, and 5 general points on the projective plane, we have determined if the Hilbert Function of the defining ideal is maximal or not for any multiplicity m. In regards to finding a k such that $(I^{(k)})^t = I^{(kt)}$ for all $t \in \mathbb{N}$, $k = 2$ suffices for 3,4, and 5 points. The case for 6 points is particularly interesting. We could only determine the Hilbert Function for low multiplicities using a specific conjecture, but doing so showed that $(I^{(2)})^t = (I^{(2t)})$ for $t = 2,3,4$. This gives some intuition that $k = 2$ also works in the case of 6 points, but unfortunately, this equality explicitly fails at $t = 5$. Nevertheless, $k = 10$ is also the point that Huneke's Criterion can be applied. In some sense, $I^{(10)}$ has some special numerical oddities that could be further explored.

For an even broader picture of more general problems, recall that the Alexander-Hirschowitz Theorem proved in full generality the expected growth of the Hilbert Function in the case of double points. In that theorem, the multiplicity of the points was fixed, but the dimension of the projective space and the number of points were variable. An analogous theorem would involve fixing the dimension of the projective space to be two but instead allow multiplicity and the number of points to vary. Our results aim to provide an intuitive start to this problem by analyzing specific cases involving a very small finite set of points, but future researchers should search for a proof that considers any number of points. Naturally, if the above theorem can be proven, one may then also consider the most general case where all three parameters can vary. Specifically, answering if one can show if the Hilbert Function for a defining ideal is maximal for any number of points, any dimension of the projective space, and any multiplicity.

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