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Holomorphic Hardy Space Representations for Convex Domains in Cn

Jennifer West Paulk
University of Arkansas, Fayetteville

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A dissertation in partial fulfillment
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Jennifer Paulk
Mississippi State University
Bachelor of Science in Secondary Education, 2003
University of Arkansas
Master of Science in Mathematics, 2005

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University of Arkansas

ABSTRACT

This thesis deals with Hardy Spaces of holomorphic functions for a domain in $\mathbb{C}^n$ when the complex dimension n is greater than or equal to two. The results we obtain are analogous to well known theorems in one complex variable. The domains we are concerned with are strongly convex with real boundary of class C^2. We obtain integral representations utilizing the Leray kernel for H^1 functions on such domains D. Next we define an operator to prove the non-tangential limits of a function in $H^p(D)$ integrated against any Lipschitz function is also in $H^p(D)$, once again utilizing the Leray kernel. This result yields a separation of singularities for any function f in the H^p space on domain D.

This dissertation is approved for
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Dissertation Director

Dr. Loredana Lanzani

Thesis Committee:

Dr. Luca Capogna

Dr. Phillip S. Harrington

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1 Introduction

This thesis deals with Hardy Spaces of holomorphic functions for a domain in $\mathbb{C}^n$ when the complex dimension n is greater than or equal to two. The theory is quite classical in one complex variable, that is, for domains in $\mathbb{C}$. In particular, using the complexified form of Stokes' Theorem one obtains the Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \int\limits_{\zeta \in bD} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in D \tag{1}$$

which is true for any simply connected domain D that is, say, of class C^1, and any function f that is holomorphic in D and continuous up to the boundary of D. This formula is true in the more general, and in fact optimal, case when f is in the holomorphic Hardy space $H^1(D)$ and D is simply connected and rectifiable, see [5]. Formula (1) has many applications in complex analysis, mainly due to the fact that the scalar part of the Cauchy kernel:

$$C(z, \zeta) = \frac{d\zeta}{2\pi i(\zeta - z)}, \tag{2}$$

that is, the function:

$$\frac{1}{2\pi i(\zeta - z)}$$

is holomorphic with respect to z in domain D. The objective of this thesis is to study higher dimensional versions of the Cauchy formula and of the Cauchy kernel.

A very natural generalization of the one dimensional Cauchy kernel is obtained by first rewriting it as:

$$C(\zeta, z) = \frac{\overline{\zeta} - \overline{z}}{2\pi i|\zeta - z|^2} d\zeta, \tag{3}$$

where $\overline{\zeta}$ stands for the complex conjugate of ζ, and $|\zeta|^2$ is the square of the absolute value of ζ. It is this form of the Cauchy kernel that naturally generalizes for higher dimensions to:

$$K_{0,0}(\zeta, z) \tag{4}$$

$$= \sum_{j=1}^{n} C_{n,j} \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^{2n}} d\zeta_1 \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge [d\overline{\zeta}_j] \wedge \cdots \wedge d\overline{\zeta}_n \tag{5}$$

$$\text{where} \quad C_{n,j} = \frac{(n-1)!(-1)^{\frac{n(n-1)}{2}}(-1)^{j-1}}{(2\pi i)^n}. \tag{6}$$

This kernel is known in the literature as the Martinelli-Bochner kernel; it was independently discovered by Martinelli (1938, [16]) and Bochner (1943, [2]). Proceeding in the same manner as the proof for the original Cauchy formula (that is, by way of Stokes' Theorem) we see that the formula (4) leads to

$$f(z) = \int\limits_{\zeta \in bD} f(z) K_{0,0}(\zeta, z)\,, \quad z \in D \tag{7}$$

for all f holomorphic in D and continuous up to the boundary of D, and for all bounded and simply connected domains $D \subset \mathbb{C}^n$ such that Stokes' Theorem holds for the closure of said domain, e.g., for D Lipschitz. Contrarily to the one dimensional case, the Martinelli-Bochner kernel (4) is easily seen not to be holomorphic in D

when n is greater than or equal to two. And so the need arises for a kernel in higher dimensions that is holomorphic for z in D and satisfies the reducing formula (7) for holomorphic functions.

As is well known in the literature, see e.g. [16], the construction of holomorphic kernels brings geometric obstructions that require restricting so, the class of domains to satisfy some form or other of "convexity." Here we will be concerned with the case when D is strongly convex and of class C^2, that is, when (any) defining function of D has real Hessian that is strictly positive definite when acting on (any) real tangent vector, at any boundary point p in the boundary of D. In this case, a kernel was first introduced by J. Leray in 1959 [15]

$$L(\zeta, z) = \frac{1}{(2\pi i)^n} \frac{\partial\rho(\zeta) \wedge (\overline{\partial}\partial\rho)^{n-1}(\zeta)}{\langle \partial\rho(\zeta), \zeta - z\rangle^n} \quad z \in D,\, \zeta \in bD. \tag{8}$$

Here ρ is any defining function for the domain D, and the convexity assumption on D guarantees that the denominator does not vanish. Also notice that this kernel is holomorphic with respect to z in D, it is a rational function of z.

This thesis makes use of the Leray kernel to prove some integral representation formulas for functions in Hardy spaces in higher complex dimensions on a domain $D \subset \mathbb{C}^n$ that is strongly convex with boundary a real manifold of class C^2. As an application, we obtain a separation of singularities for functions in Hardy spaces on such domains. These results parallel those of E. Stout [19] in the 1970's where H^p functions where studied on strongly *pseudoconvex* domains. Stout's results make use of the Henkin-Ramirez kernel which is holomorphic on strongly psuedoconvex domains in $\mathbb{C}^n$ with boundary a real manifold of class C^3. As is well known, strong convexity implies strong pseudoconvexity. Comparing this thesis with Stout's results we see a relaxation of the domain in amount of convexity but stronger assumption on the boundary (Stout's results) versus a stronger requirement on the domain's convexity but more relaxed boundary condition (this thesis). This exemplifies a "robbing from Peter to pay Paul, who in turn repays Peter" dichotomy that exists in several complex variables anytime an attempt is made to reduce the ambient domain's boundary regularity.

The advantage of working with the Leray kernel (8) is that it is globally defined, that is, it is well-defined for all ζ in the boundary of our domain and for all z in the domain. In contrast, for the Henkin-Ramirez kernel used by Stout, z in domain D must be sufficiently close to ζ in the boundary. Thus, the Leray kernel has the same spirit as the Cauchy kernel in one complex variable and it has the advantage of being technically less demanding as other kernels in several complex variables.

1.1 Statement of Main results

The following are the main results that are proved in this thesis. In what follows we let D_ε denote a family of domains that are obtained by "shrinking" the original domain D by a small amount $\varepsilon > 0$. The precise definitions are given in section 1.2.

Theorem 1.1. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with bD a real manifold of class C^2. Fix $z \in D$. Hence, there is $\varepsilon(z) > 0$ such that for all $\varepsilon < \varepsilon(z)$, and for all $f \in H^1(D)$, we have*

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{\zeta \in bD_\varepsilon} f(\zeta) j^* \left(\frac{\partial\lambda(\zeta) \wedge (\bar{\partial}\partial\lambda(\zeta))^{n-1}}{\langle \partial\lambda(\zeta), \zeta - z \rangle^n} \right) \quad z \in D_\varepsilon.$$

Here, λ is any defining function for D and j^* denotes the pull back under the inclusion $j : bD_\varepsilon \hookrightarrow \mathbb{C}^n$.

Theorem 1.2. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with bD a real manifold of class C^2. If $f \in H^1(D)$ and $z \in D$, then*

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega) j^* \left(\frac{\partial\lambda(\omega) \wedge (\bar{\partial}\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z \rangle^n} \right)$$

where $f^+(\omega)$ is the non-tangential limit of f at $\omega \in bD$.

Theorem 1.3. *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain of class C^2. Suppose γ is defined and satisfies a Lipschitz condition on $\mathbb{C}^n$. For $g \in L^1(bD, d\sigma)$ define the operator*

$$Tg(z) = \frac{1}{2\pi i} \int\limits_{\omega \in bD} g(\omega)\{\gamma(\omega) - \gamma(z)\} j^* \left(\frac{\partial\lambda(\omega) \wedge (\overline{\partial}\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n} \right), \quad z \in D. \qquad (9)$$

Then T has the following properties:

1. *If $g \in L^q(bD, d\sigma)$, $q > 2n$, then $Tg \in L^\infty(D)$. Here σ denotes surface measure on bD.*

2. *If $g \in L^p(bd, d\sigma)$, $1 \le p < \infty$, then*

$$\sup_{\varepsilon > 0} \int\limits_{\omega \in bD_\varepsilon} |Tg(\omega)|^p d\sigma_\varepsilon(\omega) \ < \ \infty$$

   Here, σ_ε denotes surface measure on bD_ε.

Theorem 1.4. *Suppose $f \in H^p(D)$, $1 \le p \le \infty$, and suppose that γ is defined and satisfies a Lipschitz condition on $\mathbb{C}^n$, that is*

$$|\gamma(z) - \gamma(\zeta)| \ \le \ C|z - \zeta| \ \text{for all } \zeta, z \ \in \mathbb{C}^n$$

then the function defined by

$$F(z) = \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega)\gamma(\omega) \ j^* \left(\frac{\partial\lambda(\omega) \wedge (\bar{\partial}\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n} \right), z \in D$$

belongs to $H^p(D)$.

Corollary 1.5. *Suppose* $f \in H^p(D)$ *and* $U = (U_1, ..., U_q)$ *is a finite open cover of* bD*. Then there exists* $f_1 ... f_q \in H^p(D)$ *such that*

$$f = f_1 + \cdots + f_q.$$

Furthermore, for all $j = 1, ..., q$ *there is an open neighborhood of* $bD \backslash U_j$, $W(bD \backslash U_j)$ *such that* $f_j \in \mathcal{O}(W(bD \backslash U_j))$.

1.2 Preliminaries

Definition 1.6. *Let D be a domain in $\mathbb{C}^n$. A real-valued function λ is a defining function for D if λ is defined in a neighborhood U of D and it satisfies the following conditions:*

1. $\lambda(z) < 0 \;\Leftrightarrow\; z \in D$

2. $bD = \{z \,\big|\, \lambda(z) = 0\}$

Moreover, we say that D is of class C^1 if λ is continuously differentiable and we have:

$$\bigtriangledown\lambda(z) \neq 0 \;\; \textit{for all} \;\; z \in bD$$

Definition 1.7. *A domain D is said to be class C^2 if there exists a defining function λ such that $\lambda \in C^2(U)$ for U as in definition (1.6).*

Definition 1.8. *For $\varepsilon > 0$ fixed and D as in definition (1.6), we define*

$$D_\varepsilon := \{z \in D \big| \lambda(z) < -\varepsilon\}$$

Lemma 1.9. *For D and D_ε as in definition (1.6) and definition (1.8), respectively, the following are true:*

1. *If $\varepsilon << \varepsilon_\circ$, then $\lambda_\varepsilon(z) := \lambda(z) + \varepsilon$ is a defining function for D_ε.*

2. *If $\varepsilon_1 < \varepsilon_2$, then $D_{\varepsilon_2} \subset D_{\varepsilon_1}$*

3. $D_\varepsilon \nearrow D$ *as* $\varepsilon \to 0$

Proof. We will first prove that $\lambda_\varepsilon(z) := \lambda(z) + \varepsilon$ is a defining function for D_ε for $0 < \varepsilon \ll \varepsilon_\circ$.

1. Let $z \in D_\varepsilon$. Then

   $\lambda_\varepsilon(z) = \lambda(z) + \varepsilon \;<\; -\varepsilon + \varepsilon \;=\; 0$

2. Notice that

$$\lambda_\varepsilon(z) = 0 \quad \Longleftrightarrow \quad \lambda(z) = \; -\varepsilon.$$

   It follows that the bD_ε is precisely the set $\{z \in D \mid \lambda_\varepsilon(z) = 0\}$.

3. For $z \in D$, we have

$$\triangledown\lambda_\varepsilon(z) = \triangledown\lambda(z).$$

By condition three along with the continuity of $\triangledown\lambda$, it follows that $\triangledown\lambda(z) \neq 0$ for all $z \in U(bD)$. In particular, $\triangledown\lambda_\varepsilon(z) \neq 0$ for all $z \in U(bD_\varepsilon)$. Thus, λ_ε is a defining function for D_ε.

Now suppose $\varepsilon_1 < \varepsilon_2$. Then $-\varepsilon_2 < -\varepsilon_1$. So we see for all $z \in D$

$\lambda(z) < -\varepsilon_2 < -\varepsilon_1 \;\;\Rightarrow\;\; \lambda(z) \;<\; -\varepsilon_1.$

Thus, $D_{\varepsilon_2} \subset D_{\varepsilon_1}$.

Recall that for $z \in D$

$\lambda_\varepsilon(z) := \lambda(z) + \varepsilon$ and so we have that

$$\lambda_\varepsilon(z) < -\varepsilon \quad \Longleftrightarrow \quad \lambda_(z) + \varepsilon < 0$$

$$\Rightarrow \quad \lambda(z) < 0 \ \text{ as } \ \varepsilon \to 0$$

Thus, $D_\varepsilon \nearrow D$ as $\varepsilon \to 0$. □

Definition 1.10. *Let $D \subset \mathbb{C}^n$ be open. A function $f : D \to \mathbb{C}$ is holomorphic in D, denoted $f \in \mathcal{O}(D)$, if $f \in C^1(\overline{D})$ and*

$$\frac{\partial f}{\partial \overline{z}_j} = 0 \ \textit{for all } 1 \leq j \leq n \textit{ and } z \in D$$

where

$$\frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - \frac{1}{i}\frac{\partial f}{\partial x_{n+j}}\right), \quad j = 1, ..., n.$$

Definition 1.11. *A function $u : D \to \mathbb{R}$ is harmonic in D, denoted $u \in Harm(D)$, if $u \in C^2(D)$ and*

$$\triangle u = \sum_{j=1}^{2n} \frac{\partial^2 u}{\partial x_j \partial x_j} = 0 \ \ \textit{for all } z \in D$$

Definition 1.12. *Let $p > 0$ and $f : D \to \mathbb{C}$ be given. We say that $f \in H^p(D)$ if and only if $f \in \mathcal{O}(D)$ and*

$$\sup_{\varepsilon > 0} \int\limits_{bD_\varepsilon} |f(\omega)|^p d\sigma_\varepsilon(\omega) \ < \infty$$

where D is as in definition (1.6) and we have set D_ε as in Definition (1.8).

Definition 1.13. *Let* $p > 0$ *and* $u : D \to \mathbb{R}$ *be given. We say* $u \in Harm^p(D)$ *if and only if* $u \in Harm(D)$ *and*

$$\sup_{\varepsilon>0} \int_{bD_\varepsilon} |u(\omega)|^p d\sigma_\varepsilon(\omega) < \infty$$

where D_ε *is as above.*

Note that $f : D \to \mathbb{C}$ can be decomposed as $f = u + iv$, where $u = Ref$, is the real part of f and $v = Imf$ is the imaginary part of f.

Remark 1.14. *Since* $|Ref| \leq |f|$ *and* $|Imf| \leq |f|$ *, then* $f \in H^p(D)$ *implies:*

1. $Ref \in Harm^p(D)$

2. $Imf \in Harm^p(D)$.

In particular, we have $\mathcal{O}(D) \subset Harm(D)$ *and* $H^p(D) \subset Harm^p(D)$ *for all* p *and the second inclusion is strict.*

Theorem 1.15. *Let $D \subseteq \mathbb{R}^N$ be a bounded domain of class C^2 and let u be harmonic on D.*

Let $1 \leq p < \infty$. the following are equivalent:

1. *$u \in Harm^p(D)$*

2. *There exists $\tilde{u} \in L^p(bD, d\sigma)$ such that*

$$u(z) = \int_{\omega \in bD} P(z,\omega)\tilde{u}(\omega)d\sigma(\omega)$$

3. *$|u|^p$ has a harmonic majorant on D.*

Proof. We will proceed as in the proof in [7, theorem 8.3.6].

2. $\Rightarrow$ 3.

If $p > 1$, let

$$h(z) = \int_{\omega \in bD} P(z,\omega)|\tilde{u}(\omega)|^p d\sigma(\omega).$$

Then, treating $P(z,\cdot)d\sigma$ as a positive measure of total mass 1, we have

$$\begin{aligned} |u(z)|^p &= \left| \int_{\omega \in BD} P(z,\omega)|\tilde{u}(\omega)|d\sigma(\omega) \right|^p \\ &\leq \int_{\omega \in bD} P(z,\omega)|\tilde{u}(\omega)|^p d\sigma(\omega) \text{ via Jensen} \end{aligned}$$

The proof for $p = 1$ is similar.

3. $\Rightarrow$ 1.

If $\varepsilon > 0$ is small, $z_\circ$ is fixed, and G_D is the Green's function for D, then $G_D(z_\circ, \cdot)$ has nonvanishing gradient near bD (use Hopf's Lemma). Therefore,

$$\tilde{D}_\varepsilon \equiv \left\{ z \in D \middle| -G_D(z_\circ, \cdot) < -\varepsilon \right\}$$

are well-defined domains for ε small. Moreover, the Poisson kernel for $\tilde{D}_\varepsilon$ is $P_\varepsilon(z,\omega) = -\nu_\omega^\varepsilon G_D(z_\circ, \omega)$. Here ν_ω^ε is the normal to $b\tilde{D}_\varepsilon$ at $\omega \in b\tilde{D}_\varepsilon$. Assume that $\varepsilon > 0$ is so small that $z_\circ \in \tilde{D}_\varepsilon$. So if h is the harmonic majorant for $|u|^p$, then

$$h(z_\circ) = \int\limits_{\omega \in \tilde{D}_\varepsilon} -\nu_\omega^\varepsilon G_D(z_\circ, \omega) h(\omega) d\sigma(\omega).$$

Let $\pi_\varepsilon : b\tilde{D}_\varepsilon \to bD$ be normal projection for ε small. Then

$$-\nu_\omega^\varepsilon G_D(z_\circ, \cdot) \to -\nu_\omega G_D(z_\circ, \cdot)$$

uniformly on bD as $\varepsilon \to 0^+$. By [7,(8.2.1)], $-\nu_\omega G_D(z_., \cdot) \geq C_{z_\circ} > 0$ for some constant $C_{z_\circ}$. Thus $-\nu_\omega^\varepsilon G_D(z_., \pi_\varepsilon^{-1}(\cdot))$ are all bounded below by $\frac{C_{z_.}}{2}$ of ε is small enough. As a result, [7,(8.3.6.4)] yeilds

$$\int\limits_{b\tilde{D}_\varepsilon} h(\omega) d\sigma(\omega) \leq \frac{2h(z_\circ)}{C_{z_\circ}}.$$

for $\varepsilon > 0$ small. In conclusion,

$$\int\limits_{b\tilde{D}_\varepsilon} |u(\omega)|^p d\sigma(\omega) \leq \frac{2h(z_\circ)}{C_{z_\circ}}.$$

1. ⇒ 2.

Let D_j be as in [7, equation (8.3.3)] through [7, equation (8.3.5)]. Fix j. Define on D_j the functions $u_\varepsilon(z) = f(z - \varepsilon\nu_j),\ 0 < \varepsilon < \varepsilon_\circ$. Then the hypothesis and (a small modification of) lemma 8.3.2 show that $\{u_\varepsilon\}$ forms a bounded subset of $L^p(bD_j)$. If $p > 1$, let $\tilde{u}_j \in L^p(bD_\varepsilon)$ be a weak-* accumulation point (for the case $p = 1$ replace $\tilde{u}_j$ by a Borel measure $\tilde{\mu}_j$.) The crucial observation at this point is that u is the Poisson integral of $\tilde{u}_j$ on D_j. Therefore, u on D_j is completely determined by $\tilde{u}_j$ and conversely. A moment's reflection shows that $\tilde{u}_j = u_k$ almost everywhere $[d\sigma]$ in $bD_j \cap bD_k \cap bD$ so that $\tilde{u} \equiv \tilde{u}_j$ on $bD_j \cap bD$ is well defined. By appealing to a partition of unity on bD that is subordinate to the open cover induced by the (relative) interiors of the sets $bD_j \cap bD$, we see that $u_\varepsilon = u \circ \pi_\varepsilon^{-1}$ converges weak-* to $\tilde{u}$ on bD when $p > 1$ (respectively $u_\varepsilon \to \tilde{\mu}$ weak-* when $p = 1$). □

Definition 1.16. *Let D be a domain in $\mathbb{C}^n$. For each $p \in bD$, and for each $\alpha > 0$ we define the approach region $\Gamma_\alpha(p)$ to be*

$$\Gamma_\alpha(p) := \{z \in D \mid \ |z - p| \leq (1 + \alpha) dist(z, bD)\}$$

where $dist(z, bD)$ denotes the Euclidean distance in $\mathbb{R}^{2n}$ of z for the boundary of D.

Theorem 1.17. *Suppose $u \in Harm(D)$ and there exists $\tilde{u} \in L^p(bD, d\sigma)$ such that*

$$u(z) = \int\limits_{\omega \in bD} P(z,\omega)\tilde{u}(\omega)d\sigma(\omega).$$

Then the non-tangential limit of u:

$$u^+(\omega) = \lim_{\substack{z \to \omega \\ z \in \Gamma_\alpha(\omega)}} u(z)$$

exists for a.e. $w \in bD$.

For the proof of Theorem (1.17) we refer to [18, Theorem I.5.4]

Corollary 1.18. *Suppose $f \in H^p(D)$. Then there exists*

$$f^+(\omega) = \lim_{\substack{z \to \omega \\ z \in \Gamma_\alpha(\omega)}} f(z) \;\; a.e.\, \omega \in bD$$

Proof. Let $f \in H^p(D)$ such that $f := u + iv$. Then we have that $u \in Har^p(D)$ and $v \in Har^p(D)$ by remark 1.14. Theorem 1.15 gives us that

1. $u(z) = \int\limits_{\omega \in bD} P(z,\omega)\tilde{u}d\sigma(\omega)$
2. $v(z) = \int\limits_{\omega \in bD} P(z,\omega)\tilde{v}d\sigma(\omega).$

Theorem 1.17 yields

1. $u^+(\omega) = \lim\limits_{\substack{z \to \omega \\ z \in \Gamma_\alpha(\omega)}} u(z)$ a.e. $\omega \in bD$
2. $v^+(\omega) = \lim\limits_{\substack{z \to \omega \\ z \in \Gamma_\alpha(\omega)}} v(z)$ a.e. $\omega \in bD$

Putting this together we obtain

$$\begin{aligned} u^+(\omega) + iv^+(\omega) &= \lim_{\substack{z\to\omega \\ z\in\Gamma_\alpha(\omega)}} (u(z) + iv(z)) \text{ a.e } \omega \in bD \\ &= \lim_{\substack{z\to\omega \\ z\in\Gamma_\alpha(\omega)}} f(z) \text{ a.e } \omega \in bD \\ &=: f^+(z). \end{aligned}$$

□

Definition 1.19. *For $u : D \to \mathbb{R}$ the non-tangential maximal function of u is:*

$$Mu(\omega) := \sup_{z\in\Gamma_\alpha(\omega)} |u(z)|, \quad \omega \in bD.$$

Theorem 1.20. *For $p > 0$, D a bounded domain of class C^2 and $u \in Harm(D)$ we have*

$$Mu \in L^p(bD, d\sigma)$$

Here, σ denotes surface measure for bD.

For the proof of this theorem we refer to [18, Theorem I.5.3]

Corollary 1.21. *For $p > 0$, D a bounded domain of class C^2 and $f \in H^p(D)$ we have*

$$Mf \in L^p(bD, d\sigma).$$

2 Review of Cauchy-Fantappié Forms of Order 0

Lemma 2.1. *Let* $W(\zeta) = \sum_{j=1}^{n} \omega_j(\zeta)(\zeta_j - z_j)$ *be a* $C^2_{0,1}$ *form on a given set* $U \subset \mathbb{C}^n$. *Suppose there is* $z \notin U$ *such that*

$$\langle W(\zeta), \zeta - z \rangle = \sum_{j=1}^{n} \omega_j(\zeta)(\zeta_j - z_j) = 1, \;\; for\, \zeta \in U. \tag{10}$$

Then the $(n, n-1)$*-form*

$$\Omega_\circ(W) = (2\pi i)^{-n} W \wedge (\overline{\partial} W)^{n-1}$$

satisfies

$$d\Omega_\circ(W) = \overline{\partial}_\zeta \Omega_\circ(W) = 0 \;\; on\ U.$$

Proof. We will proceed as in the proof of [13, Lemma IV.3.1]. Since $\Omega_\circ$ is of maximal type n with respect to ζ, it is immediate that

$$\begin{aligned}
(2\pi i)^n d\Omega_\circ(W) &= (2\pi i)^n \overline{\partial} \Omega_\circ(W) \\
&= \left(\overline{\partial} W\right)^n \\
&= \left(\sum_{j=1}^{n} \overline{\partial} \omega_j \wedge d\zeta_j \right)^n \\
&= n! \left(\overline{\partial}_\zeta \omega_1 \wedge d\zeta_1\right) \wedge \cdots \wedge \left(\overline{\partial}_\zeta \omega_n \wedge d\zeta_n\right).
\end{aligned} \tag{11}$$

Applying $\overline{\partial}$ to identity (10) we get

$$\sum_{j=1}^{n} \overline{\partial} \omega_j(\zeta) \cdot (\zeta_j - z_j) = 0.$$

So, for $\zeta \neq z$, the set $\{\overline{\partial}_\zeta \omega_1(\zeta), ..., \overline{\partial}_\zeta \omega_n(\zeta)\}$ is linearly independent, which implies that (11) is 0, as desired. □

Definition 2.2. *A generating form W on $U \subset \mathbb{C}^n$ (for the point z) is a $C^1_{0,1}$ form on U which satisfies (10).*

Definition 2.3. *With same hypothesis as Lemma (2.1) , the $(n, n-1)$-form*

$$\Omega_\circ(W) = (2\pi i)^{-n} W \wedge \left(\overline{\partial}_\zeta W\right)^{n-1} \quad \zeta \in U$$

is called the Cauchy-Fantappié form of order 0 generated by W at z.

Lemma 2.4. *Suppose $\Omega_\circ(W)$ is a Cauchy-Fantappié form and g is any C^1 function. Then*

$$\Omega_\circ(gW) = g^n \Omega_\circ(W).$$

Proof. Notice

$$gW \wedge \overline{\partial}_\zeta(gW) \;=\; gW \wedge \left(\overline{\partial}_\zeta g \wedge W \;+\; g\overline{\partial}_\zeta W\right) = gW \wedge g\overline{\partial}_\zeta W.$$

So

$$\begin{aligned}
\Omega_\circ(gW) &= (2\pi i)^{-n} gW \wedge \left(\overline{\partial}_\zeta gW\right)^{n-1} \\
&= (2\pi i)^{-n} gW \wedge g^{n-1} \left(\overline{\partial}_\zeta W\right)^{n-1} \\
&= (2\pi i)^{-n} g^n W \wedge \left(\overline{\partial}_\zeta W\right)^{n-1} \\
&= g^n \Omega_\circ(W)
\end{aligned}$$

$\square$

3 Proof of Main Results

Theorem 3.1. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with bD a real manifold of class C^2. Fix $z \in D$. Hence, there is $\varepsilon(z) > 0$ such that for all $\varepsilon < \varepsilon(z)$, and for all $f \in H^1(D)$, we have*

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{\zeta \in bD_\varepsilon} f(\zeta) j^* \left(\frac{\partial\lambda(\zeta) \wedge (\bar{\partial}\partial\lambda(\zeta))^{n-1}}{\langle \partial\lambda(\zeta), \zeta - z \rangle^n} \right) \quad z \in D_\varepsilon.$$

Here, λ is any defining function for D and j^* denotes the pull back under the inclusion $j : bD_\varepsilon \hookrightarrow \mathbb{C}^n$.

Proof. Fix $z \in D$ and let $\varepsilon(z) > 0$ be such that $z \in D_\varepsilon$ for all $\varepsilon < \varepsilon(z)$. Now $D_{\varepsilon_\circ} \subset\subset D$ and $f \in \mathcal{O}(D_{\varepsilon_\circ} \cap C(\bar{D_{\varepsilon_\circ}})$ as $f \in H^1(D)$, for all $\varepsilon < \varepsilon(z)$

Let $\chi(\zeta, z) := \chi(|\zeta - z|) \in C^\infty_\circ(\mathbb{C}^n)$ be a real-valued function such that

$$\chi(|\zeta - z|) = \begin{cases} 0 & \zeta \in U'(bD_{\varepsilon_\circ}), \\ 1 & \zeta \in D \backslash \overline{U'(bD)}. \end{cases}$$

For ζ, z define

$$W(\zeta, z) := \chi(|\zeta - z|) \frac{\partial_\zeta \beta}{\beta}(\zeta, z) + (1 - \chi(|\zeta - z|)) \frac{\partial\lambda(\zeta)}{\langle \partial\lambda(\zeta), \zeta - z \rangle}$$

where

$$\beta(\zeta, z) = |\zeta - z|^2 \quad \zeta, z \in \mathbb{C}^n. \tag{12}$$

Since

$$\beta(\zeta,z) = |\zeta - z|^2 = (\zeta_1 - z_1)(\overline{\zeta_1 - z_1}) + \cdots + (\zeta_n - z_n)(\overline{\zeta_n - z_n})$$

$$= (\zeta_1 - z_1)(\overline{\zeta_1} - \overline{z_1}) + \cdots + (\zeta_n - z_n)(\overline{\zeta_n} - \overline{z_n})$$

it follows that

$$\begin{aligned}\partial_\zeta \beta &= \sum_{j=1}^{n} \frac{\partial}{\partial \zeta_j}\left[(\zeta_1 - z_1)(\overline{\zeta_1 - z_1}) + \cdots + (\zeta_n - z_n)(\overline{\zeta_n - z_n})\right] d\zeta_j \\ &= (\overline{\zeta_1} - \overline{z_1})d\zeta_1 + \cdots + (\overline{\zeta_n} - \overline{z_n})d\zeta_n\end{aligned}$$

So

$$\langle \partial_\zeta \beta, \zeta - z\rangle = (\zeta_1 - z_1)(\overline{\zeta_1} - \overline{z_1}) + \cdots + (\zeta_n - z_n)(\overline{\zeta_n} - \overline{z_n}) = \beta(\zeta, z)$$

Then

$$\begin{aligned}&\langle W(\zeta,z), \zeta - z\rangle \\ =\ &\left\langle \chi(|\zeta - z|)\frac{\partial_\zeta \beta}{\beta} + (1 - \chi(|\zeta - z|))\frac{\partial\lambda(\zeta)}{\langle \partial\lambda(\zeta), \zeta - z\rangle}, \zeta - z\right\rangle \\ =\ &\left\langle \chi(|\zeta - z|)\frac{\partial_\zeta \beta}{\beta}, \zeta - z\right\rangle + \left\langle (1 - \chi(|\zeta - z|))\frac{\partial\lambda(\zeta)}{\langle \partial\lambda(\zeta), \zeta - z\rangle}, \zeta - z\right\rangle \\ =\ &\frac{\chi(|\zeta - z|)}{\beta}\langle \partial_\zeta\beta, \zeta - z\rangle + \frac{(1 - \chi(|\zeta - z|))}{\langle \partial\lambda(\zeta), \zeta - z\rangle}\langle \partial\lambda(\zeta), \zeta - z\rangle \\ =\ &\chi(|\zeta - z|) + 1 - \chi(|\zeta - z|) \equiv 1 \quad \forall \zeta \in \overline{D}_{\varepsilon_\circ}\setminus\{z\}.\end{aligned}$$

This computation shows that $W(\zeta,z)$ is a generating form for z on $\overline{D}_{\varepsilon_\circ}$.

Now letting

$$\Omega_\circ(W(\zeta,z)) = \frac{1}{(2\pi i)^n} W(\zeta,z) \wedge (\bar{\partial}_\zeta W(\zeta,z))^{n-1},$$

we have that $\Omega_\circ$ is a Cauchy-Fantappié form of type $(n, n-1)$ in the variable ζ. So in particular

$$d\Omega_\circ(W(\zeta,z) = \bar{\partial}\Omega_\circ(W(\zeta,z),$$

and by lemma 2.1 we have:

$$\bar{\partial}\Omega_\circ(W(\zeta,z) = 0$$

For $0 < \varepsilon < \varepsilon_\circ$

$$D_{\varepsilon,\delta} := \left\{ \zeta \in D_{\varepsilon_\circ} \;\middle|\; |\zeta - z| > \frac{\delta}{2} \right\}. \tag{13}$$

Recall that

$$df(\zeta) \wedge \Omega_\circ(W(\zeta,z)) = \partial f(\zeta) \wedge \Omega_\circ(W(\zeta,z)) + \overline{\partial} f(\zeta) \wedge \Omega_\circ(W(\zeta,z)).$$

By type considerations it follows:

$$\partial f(\zeta) \wedge \Omega_\circ(W(\zeta,z)) = 0.$$

We now recall Stokes' Theorem:

$$\int_{\zeta \in bD} j^*\omega(\zeta) = \int_{\zeta \in D} d\omega(\zeta)$$

which holds for any differential form $\omega \in C^1_{2n-1}(\overline{D})$ with $j : bD \hookrightarrow \mathbb{C}^n$ denoting the inclusion. Thus, applying Stokes' Theorem to $\omega(z) = f(\zeta)\Omega_\circ(W(\zeta, z)$ and the domain $D_{\varepsilon,\delta}$ we obtain:

$$\int_{\zeta\in bD_{\varepsilon,\delta}} f(\zeta)j^*\left(\Omega_\circ(W_\circ(\zeta,z))\right)$$
$$= \int_{\zeta\in D_{\varepsilon,\delta}} d(f(\zeta)\Omega_\circ(W(\zeta,z)))$$
$$= \int_{\zeta\in D_{\varepsilon,\delta}} \overline{\partial} f(\zeta)\wedge\Omega_\circ\left(W(\zeta,z)\right) = 0$$

since $f \in H^1(D)$ is, in particular, holomorphic in D. On the other hand, we have:

$$\int_{\zeta\in bD_{\varepsilon,\delta}} f(\zeta)j^*\left(\Omega_\circ(W(\zeta,z))\right)$$
$$= \int_{\zeta\in bD_{\varepsilon_\circ}} f(\zeta)j^*\Omega_\circ(W(\zeta,z)) - \int_{|\zeta-z|=\frac{\delta}{2}} f(\zeta)j^*\Omega_\circ(W(\zeta,z))$$
$$= \int_{\zeta\in bD_\varepsilon} f(\zeta)j^*\Omega_\circ\left(\frac{\partial\lambda(\zeta)}{\langle\partial\lambda(\zeta),\zeta-z\rangle}\right)$$
$$- \int_{|\zeta-z|=\frac{\delta}{2}} f(\zeta)j^*\Omega_\circ\left(\left(\frac{\partial_\zeta\beta}{\beta}(\zeta,z)\right)\right),$$

where β is as in equation(12). Combining these identities we obtain:

$$\begin{aligned}
&\frac{1}{(2\pi i)^n} \int\limits_{\zeta\in bD_\varepsilon} f(\zeta) j^* \left(\frac{\partial\lambda(\zeta) \wedge (\overline{\partial}\partial\lambda(\zeta))^{n-1}}{\langle \partial\lambda(\zeta), \zeta - z\rangle^n} \right) \\
= &\int\limits_{|\zeta - z| = \frac{\delta}{2}} f(\zeta) j^* \Omega_\circ \left(\left(\frac{\partial\beta}{\beta}(\zeta, z) \right) \right) \\
= &\int\limits_{|\zeta - z| = \frac{\delta}{2}} (f(\zeta) - f(z)) j^* \Omega_\circ \left(\left(\frac{\partial\beta}{\beta}(\zeta, z) \right) \right) \\
&+ \int\limits_{|\zeta - z| = \frac{\delta}{2}} f(z) j^* \Omega_\circ \left(\left(\frac{\partial\beta}{\beta}(\zeta, z) \right) \right) \\
=: &\ A_\delta + B_\delta
\end{aligned}$$

Concerning term B_δ, by [17, Lemma IV.1.2] we have:

$$\int_{|\zeta - z| = \frac{\delta}{2}} j^* \Omega_\circ \left(\left(\frac{\partial_\zeta\beta}{\beta}(\zeta, z) \right) \right) = 1$$

and it follows that

$$B_\delta = \int\limits_{|\zeta - z| = \frac{\delta}{2}} f(z) \Omega_\circ \left(\left(\frac{\partial_\zeta\beta}{\beta}(\zeta, z) \right) \right) = f(z)$$

holds for each fixed $z \in D_\varepsilon$ and for each $\delta > 0$. We now investigate term A_δ. First note that by [17, Lemma VII.3.9] in the special case when $D = \mathbb{B}_{\frac{\delta}{2}}(z)$, so that for $\rho(\zeta) = |\zeta - z|^2 - \left(\frac{\delta}{2}\right)^2$ and $\|d\rho\| = 2\delta$, it follows that:

$$j^* \Omega_\circ \left(\left(\frac{\partial\beta}{\beta}(\zeta, z) \right) \right) = C_n \frac{d\sigma(z)}{\delta^{2n-1}} \quad \text{for each } \|z\| < \frac{\delta}{2}$$

where σ denotes surface measure.

Putting all of this together we obtain:

$$\begin{aligned} 0 \le |A_\delta| &\le \frac{C_n}{\delta^{2n-1}} \int\limits_{|\zeta - z| = \frac{\delta}{2}} |f(\zeta) - f(z)| d\sigma \\ &\le \max\left\{ f(\zeta) - f(z) \,\middle|\, |\zeta - z| = \frac{\delta}{2} \right\} \frac{C_n}{\delta^{2n-1}} \sigma\left(b\mathbb{B}_{\frac{\delta}{2}}\right)(z) \\ &\le C_n \max\left\{ f(\zeta) - f(z) \,\middle|\, |\zeta - z| = \frac{\delta}{2} \right\} \end{aligned}$$

and by the uniform continuity of f on $\overline{\mathbb{B}_{\frac{\delta}{2}}(z)}$, it follows that the latter tends to 0 as $\delta \to 0$.

So we have that:

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{\zeta \in bD_\varepsilon} f(\zeta) j^* \left(\frac{\partial\lambda(\zeta) \wedge (\bar{\partial}\partial\lambda(\zeta))^{n-1}}{\langle \partial\lambda(\zeta), \zeta - z \rangle^n} \right)$$

□

Theorem 3.2. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with bD a real manifold of class C^2. If $f \in H^1(D)$ and $z \in D$, then*

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega) j^* \left(\frac{\partial\lambda(\omega) \wedge (\bar{\partial}\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z \rangle^n} \right)$$

where $f^+(\omega)$ is the non-tangential limit of f at $\omega \in bD$.

Proof. Fix $z \in D$ and let $\varepsilon(z) > 0$ be such that $z \in D_\varepsilon$ for all $\varepsilon < \varepsilon(z)$. Now $D_\varepsilon \subset\subset D$ and $f \in \mathcal{O}(D_\varepsilon \cap C(\overline{D_\varepsilon})$ as $f \in H^1(D)$, for all $\varepsilon < \varepsilon(z)$

Let $\pi_\varepsilon : bD \to bD_\varepsilon$ be a diffeomorphism whose inverse $\pi_\varepsilon^{-1} : bD \to bD_\varepsilon$ is given by the formula

$$\pi_\varepsilon^{-1}(\omega) = \zeta := \omega_\varepsilon = \omega - \varepsilon \cdot \nu(\omega) \tag{14}$$

where $\nu(\omega)$ is the outward unit normal vector to bD at $\omega \in bD$. Since D is of class C^2, it follows that π_ε and π_ε^{-1} are of class C^1. Note that ω_ε lies along the inner normal direction to bD at ω, so in particular we have that

$$\omega_\varepsilon \to \omega \text{ as } \varepsilon \to 0$$

with non-tangential convergence. We now inspect

$$\frac{f(\omega_\varepsilon)}{\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle^n},$$

with $\omega \in bD$, $z \in D$, and $\omega_\varepsilon = \pi_\varepsilon^{-1}(\omega)$. Let $dist^2(z, bD) := C_z$. Then for each $\omega \in bD$ we have $C_z \leq |\omega - z|^2$ as $\omega \in bD$ and $z \in D$.

By the strong convexity of D we have:

$$|\langle \partial\lambda(\omega), \omega - z\rangle| \geq C|\omega - z|^2 \text{ for all } \omega \in bD.$$

See [17, Theorem IV.3.2.4]. We now continue as follows.

$$\begin{aligned}
& |\,\langle \partial\lambda(\omega), \omega - z\rangle\,| \\
= \;& |\,\langle \partial\lambda(\omega_\varepsilon), \omega - z\rangle + \langle \partial\lambda(\omega) - \partial\lambda(\omega_\varepsilon), \omega - z\rangle\,| \\
\leq \;& |\,\langle \partial\lambda(\omega_\varepsilon), \omega - z\rangle\,| + |\,\langle \partial\lambda(\omega) - \partial\lambda(\omega_\varepsilon), \omega - z\rangle\,| \\
= \;& |\,\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle + \langle \partial\lambda(\omega_\varepsilon), \omega - \omega_\varepsilon\rangle\,| + |\,\langle \partial\lambda(\omega) - \partial\lambda(\omega_\varepsilon), \omega - z\rangle\,| \\
\leq \;& |\,\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle\,| + |\,\langle \partial\lambda(\omega_\varepsilon), \omega - \omega_\varepsilon\rangle\,| + |\,\langle \partial\lambda(\omega) - \partial\lambda(\omega_\varepsilon), \omega - z\rangle\,|,
\end{aligned}$$

and applying the Cauchy Schwartz Inequality we obtain that the latter is bounded by

$$|\,\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle\,| + |\partial\lambda(\omega_\varepsilon)| \cdot |\omega - \omega_\varepsilon| + |\partial\lambda(\omega) - \partial\lambda(\omega_\varepsilon)| \cdot |\omega - z|$$

Since D is of class C^2 (and bounded), in particular, we have that $\partial\lambda$ is Lipschitz continuous on a (bounded) neighborhood of bD and so

$$|\partial\lambda(\omega) - \partial\lambda(\omega_\varepsilon)| \leq C_2|\omega - \omega_\varepsilon|,$$

and by the definition of ω_ε we have:

$$|\omega - \omega_\varepsilon| \approx \varepsilon.$$

Also, since bD is compact (and so is $\overline{U(bD)}$) we have:

$$|\partial\lambda(\omega_\varepsilon)| \cdot |\omega - \omega_\varepsilon| \leq C_1|\omega - \omega_\varepsilon| \leq C_1\varepsilon.$$

Combining these with all of the above we obtain:

$$
\begin{aligned}
& |\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle| + |\partial\lambda(\omega_\varepsilon)|\cdot|\omega-\omega_\varepsilon| + |\partial\lambda(\omega)-\partial\lambda(\omega_\varepsilon)|\cdot|\omega - z| \\
\leq\; & |\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle| + C_1\varepsilon + C_2|\omega-\omega_\varepsilon||\omega - z \\
\leq\; & |\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle| + C_1\varepsilon + C_2\ diam(D)\varepsilon \\
=\; & |\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle| + C\varepsilon.
\end{aligned}
$$

Putting all of this together we conclude:

$$C_z - C\varepsilon \leq |\langle \partial\lambda(\omega-\varepsilon), \omega_\varepsilon - z\rangle|, \quad \text{for all} \ \ \omega \in bD.$$

Let $\varepsilon_1 = \frac{1}{2}C_z$. Then $\forall\ 0 < \varepsilon < \varepsilon_1$ we have $\frac{1}{2}C_z = \frac{1}{2}C_z - C\varepsilon_1 < C_z - \varepsilon C$,and it follows

$|\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle| \geq C_z - \varepsilon C > \frac{1}{2}C_z$ for all $\omega \in bD$ and for all

$varepsilon < \varepsilon_1$. We have proved

$$\frac{1}{|\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle|^n} \leq C_z$$

for all $\omega \in bD$ and for all $0 < \varepsilon < \varepsilon_1$. Since $f \in H^1(D)$ and ω_ε belongs to (any) non-tangential approach region at ω we have that $|f(\omega_\varepsilon)| \leq M(f)(\omega)$, and we conclude that

$$\left|\frac{f(\omega_\varepsilon)}{\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle|^n}\right| \leq C_z M(f)(\omega)$$

holds for all $\omega \in bD$, for every $0 < \varepsilon < \varepsilon_1$. Since $f \in H^1(D)$ we also have:

$$f(\omega_\varepsilon) \to f^+(\omega) \text{ a.e. } \omega \in bD \text{ as } \varepsilon \to 0$$

and we conclude

$$\frac{f(\omega_\varepsilon)}{\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle^n} \quad \longrightarrow \quad \frac{f^+(\omega)}{\langle \partial\lambda(\omega), \omega - z\rangle^n} \quad \text{a.e. } \omega \in bD \quad \text{as } \ \varepsilon \to 0.$$

In particular, we also have:

$$\frac{f(\omega_\varepsilon)\cdot h(\omega_\varepsilon)}{\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle^n} \quad \longrightarrow \quad \frac{f^+(\omega)\cdot h(\omega)}{\langle \partial\lambda(\omega), \omega - z\rangle} \quad \text{a.e. } \omega \in bD \quad \text{as } \ \varepsilon \to 0.$$

for every $h \in C(\overline{U(bD)})$. And since $\pi_\varepsilon^{-1}(\omega) \to \omega$ as $\varepsilon \to 0$ for every $\omega \in bD$ (see equation (14)), we also have:

$$\left(\pi_\varepsilon^{-1}\right)^* \left(\frac{f(\omega_\varepsilon)\cdot h(\omega_\varepsilon)}{\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle^n} d\sigma_\varepsilon(\omega_\varepsilon)\right) \longrightarrow \frac{f^+(\omega)\cdot h(\omega)}{\langle \partial\lambda(\omega), \omega - z\rangle^n} d\sigma(\omega) \quad \text{as } \ \varepsilon \to 0. \qquad (15)$$

Next, we observe that since

$$\lambda_\varepsilon = \lambda + \varepsilon$$

is a defining function for bD_ε (see section 1.2), then in particular we have:

$$\partial\lambda_\varepsilon = \partial\lambda \text{ , and}$$

$$\partial\lambda_\varepsilon \wedge (\overline{\partial}\partial\lambda_\varepsilon)^{n-1} = \partial\lambda \wedge (\overline{\partial}\partial\lambda)^{n-1} \qquad (16)$$

By identity (16) and Theorem 3.1 we have:

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{\omega_\varepsilon \in bD_\varepsilon} f(\omega_\varepsilon) \cdot \frac{j^*\left(\partial\lambda_\varepsilon(\omega_\varepsilon) \wedge \left(\overline{\partial}\partial\lambda_\varepsilon(\omega_\varepsilon)\right)^{n-1}(\omega_\varepsilon)\right)}{\langle \partial\lambda_\varepsilon(\omega_\varepsilon), \omega_\varepsilon - z\rangle^n}$$

Moreover, by [17, Lemma VII.3.9] together with 16 we have:

$$\frac{1}{(2\pi i)^n} j^* \left(\partial\lambda_\varepsilon(\omega_\varepsilon) \wedge \left(\overline{\partial}\partial\lambda_\varepsilon(\omega_\varepsilon)\right)^{n-1}(\omega)\right) = h_\varepsilon(\omega_\varepsilon) \cdot d\sigma_\varepsilon(\omega_\varepsilon)$$

where

$$h_\varepsilon \in C^\circ(\overline{U(bD_\varepsilon)}), \quad \text{satisfies } 0 < C_3 < h_\varepsilon(\omega_\varepsilon) < C_4 \quad \text{for all } \varepsilon << \varepsilon_\circ, \tag{17}$$

for all $\omega_\varepsilon \in bD_\varepsilon$, see [17, VII (3.21)] for an explicit formula for h_ε.

Thus

$$f(z) = \int\limits_{\omega_\varepsilon \in bD_\varepsilon} \frac{f(\omega_\varepsilon) h_\varepsilon(\omega_\varepsilon)}{\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle^n} d\sigma_\varepsilon(\omega_\varepsilon). \tag{18}$$

Next, by the change of variables formula for the C^1-diffeomorphism $\pi_\varepsilon^{-1} : bD \to bD_\varepsilon$, see e.g. [17,III(1.25)], we have

$$\int\limits_{\omega_\varepsilon \in bD_\varepsilon} \frac{f(\omega_\varepsilon) h_\varepsilon(\omega_\varepsilon)}{\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle^n} d\sigma_\varepsilon(\omega_\varepsilon) = \int\limits_{\omega \in bD} \left(\pi_\varepsilon^{-1}\right)^* \left(\frac{f(\omega_\varepsilon) h_\varepsilon(\omega_\varepsilon) d\sigma_\varepsilon(\omega_\varepsilon)}{\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle^n}\right). \tag{19}$$

On account of (15) and (17) we also have: (recalling that $\omega_\varepsilon = \pi_\varepsilon^{-1}(\omega)$)

$$\left| \left(\pi_\varepsilon^{-1}\right)^* \left(\frac{f(\omega_\varepsilon) h_\circ(\omega_\varepsilon) d\sigma_\varepsilon(\omega_\varepsilon)}{\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle^n}\right) \right| \leq C_z \cdot M(f)(\omega)\dot{h}_\circ \tag{20}$$

where $M(f)$ is the non-tangential maximal function of f (see section 1.2). Now because $f \in H^1(D)$ we know that

$$M(f) \in L^1(bD, d\sigma),$$

(see again section 1.2). Combining equations (15), (18), (19), (20) we see that by the Lebesgue Dominated Convergence theorem it follows that for each $0 < \varepsilon < \varepsilon_1$

$$\begin{aligned} f(z) &= \int\limits_{\omega_\varepsilon \in bD} \left(\pi_\varepsilon^{-1}\right)^* \left(\frac{f(\omega_\varepsilon) h_\circ(\omega_\varepsilon) d\sigma_\varepsilon(\omega_\varepsilon)}{\langle \partial\lambda(\omega_\varepsilon), \omega_\varepsilon - z\rangle^n} \right) \\ &\longrightarrow \int\limits_{\omega \in bD} \frac{f^+(\omega) h_\circ(\omega) d\sigma(\omega)}{\langle \partial\lambda(\omega), \omega - z\rangle^n} \quad \text{as } \varepsilon \to 0 \\ &= \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega) j^* \left(\frac{\partial\lambda(\omega) \wedge \left(\overline{\partial}\partial\lambda(\omega)\right)^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle} \right) \end{aligned}$$

where the last identity is due to [17, Lemma VII.3.9]. □

Theorem 3.3. *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain of class C^2. Suppose γ is defined and satisfies a Lipschitz condition on $\mathbb{C}^n$. For $g \in L^1(bD, d\sigma)$ define the operator*

$$Tg(z) = \frac{1}{2\pi i} \int\limits_{\omega \in bD} g(\omega)\{\gamma(\omega) - \gamma(z)\} j^* \left(\frac{\partial\lambda(\omega) \wedge (\overline{\partial}\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n} \right), \quad z \in D. \qquad (21)$$

Then T has the following properties:

1. *If $g \in L^q(bD, d\sigma)$, $q > 2n$, then $Tg \in L^\infty(D)$. Here σ denotes surface measure on bD.*

2. *If $g \in L^p(bd, d\sigma)$, $1 \le p < \infty$, then*

$$\sup_{\varepsilon > 0} \int\limits_{\omega \in bD_\varepsilon} |Tg(\omega)|^p d\sigma_\varepsilon(\omega) \; < \; \infty$$

   Here, σ_ε denotes surface measure on bD_ε.

Proof. Assume for the time being that 1. and 2. below are true:

1. $T : L^\infty(bD) \to L^\infty(D)$ is bounded for the time being. That is

   (a) $\| Tf \|_{L^\infty(D)} \leq C_1 \| f \|_{L^\infty(bD)}$.

   Note that the above implies that $T : L^\infty(bD) \to L^\infty(bD_\varepsilon)$ is bounded for all $\varepsilon > 0$ with constant independent of ε. That is,

   (b) $\|Tf\|_{L^\infty(bD_\varepsilon)} \leq C_1 \|f\|_{L^\infty(bD)}$ for all $\varepsilon > 0$

   because $bD_\varepsilon \subset D$ for all $\varepsilon > 0$.

2. $T : L^1(bD, d\sigma) \to L^1(bD_\varepsilon, d\sigma_\varepsilon)$ is bounded for all $0 < \varepsilon \ll 1$ that is

   (a) $\| Tf \|_{L^1(bD_\varepsilon)} \leq C_{\circ,\varepsilon} \| f \|_{L^1(bD)}$ for all $\varepsilon > 0$

   (b) $C_\circ := \sup_{\varepsilon > 0} C_{\circ,\varepsilon} < 0$

Then the Reisz-Thorin [4] theorem in variable spaces grants

$\forall\ 1 < p < \infty \qquad T : L^p(bD) \to L^p(bD_\varepsilon)$ is bounded uniformly in $\varepsilon > 0$, with

$$
\begin{aligned}
\| Tf \|_{L^p(bD_\varepsilon)} &\leq C_\circ^{1-t} \cdot C_1^t \| f \|_{L^p(bD)} \\
&= C_\circ^{\frac{1}{p}} \cdot C_1^{1-\frac{1}{p}} \| f \|_{L^\infty(bD)}
\end{aligned}
$$

where $\frac{1}{p} = 1 - t$ and $0 < t < 1$.

Now the conclusion above along with 2.(b) automatically gives us

$$\sup_{\varepsilon > 0} \|Tf\|_{L^p(bD_\varepsilon, d\sigma_\varepsilon)} \leq C_\circ^{1-t} \cdot C_1^t \|f\|_{L^p(bD, d\sigma)},$$

are defined. Thus, we are left to show 1.(a), 2.(a), and 2.(b) above for the operator

T defined by equation (21). To this end, for $z \in D$, we have:

$$\begin{aligned}
|Tg(z)| &= \left| \frac{1}{(2\pi i)^n} \int_{\omega \in bD} g(\omega)\{\gamma(\omega) - \gamma(z)\} \frac{\partial\lambda(\omega) \wedge (\overline{\partial}\partial(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n} \right| \\
&\leq \frac{1}{(2\pi)^n} \int_{\omega \in bD} |g(\omega)| \left| \frac{\gamma(\omega) - \gamma(z)}{\langle \partial\lambda(\omega), \omega - z\rangle^n} \right| d\sigma(\omega) \\
&\leq \frac{1}{(2\pi i)^n} \int_{\omega \in bD} \|g(\omega)\|_{L^\infty(bD)} \left| \frac{\gamma(\omega) - \gamma(z)}{\langle \partial\lambda(\omega), \omega - z\rangle^n} \right| d\sigma(\omega) \\
&= \frac{\|g\|_{L^\infty(bD)}}{(2\pi)^n} \int_{\omega \in bD} \frac{|\gamma(\omega) - \gamma(z)|}{|\langle \partial\lambda(\omega), \omega - z\rangle|^n} d\sigma(\omega) \\
&\leq \frac{\|g\|_{L^\infty(bD)}}{(2\pi)^n} \int_{\omega \in bD} \frac{C|\omega - z|}{|\langle \partial\lambda(\omega), \omega - z\rangle|^n} d\sigma(\omega) \\
&\leq C_{n,\gamma} \|g\|_{L^\infty(bD)} \int_{\omega \in bD} \frac{|\omega - z|}{C|\omega - z|^{2n}} d\sigma(\omega) \\
&= C_{n,\gamma,D} \|g\|_{L^\infty(bD)} \int_{\omega \in bD} |\omega - z|^{2n-1} d\sigma(\omega)
\end{aligned}$$

Converting to polar coordinates centered at z we have

$$\begin{aligned}
&C_{n,\gamma,D} \|g\|_{L^\infty(bD)} \int_{\omega \in bD} |\omega - z|^{2n-1} d\sigma(\omega) \\
= \; &C_{n,\gamma} \|g\|_{L^\infty(bD)} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{c(z)}^{\tilde{c}} r^{1-2n} \cdot r^{2n-1} dr d\theta_1 \cdots d\theta_{2n}
\end{aligned}$$

where $0 < c(z) := \operatorname{dist}(z, bD) \leq |z - \omega| \leq \operatorname{diam}(bD) =: \tilde{c}$.

Notice that

1. $0 < c(z) < \tilde{c}$ for all $z \in D$

2. As $z \to bD$, $c(z) \to 0$ and so $(c(z), \tilde{c}) \subset (0, \tilde{c})$ for all $z \in D$.

Thus

$$\begin{aligned} & C_{n,\gamma}\|g\|_{L^\infty(bD)} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{c(z)}^{\tilde{c}} r^{1-2n} \cdot r^{2n-1} dr d\theta_1 \cdots d\theta_{2n} \\ \leq \;& C_{n,\gamma,D}\|g\|_{L^\infty(bD)} \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{c(z)}^{\tilde{c}} 1 dr d\theta_1 \cdots \theta_{2n} \\ \leq \;& C_{n,\gamma,D}\|g\|_{L^\infty(bD)} \end{aligned}$$

We have shown that for all $z \in D$

$$|Tg(z)| \;\leq\; \mathbf{C}\|g\|_{L^\infty(bD)}$$

and so

$$\sup_{z \in D} |Tg(z)| \;\leq\; \mathbf{C}\|g\|_{L^\infty(bD)}.$$

That is,

$$\|Tg\|_{L^\infty(D)} \;\leq\; \mathbf{C}\|g\|_{L^\infty(bD)} \tag{22}$$

We have shown part of the hypothesis of the Reisz-Thorin theorem holds for T and so we have left to show the rest of the hyothesis. That is, we need to show

$$\int_{z \in bD_\varepsilon} |Tg(z)| d\sigma_\varepsilon(z) \;\leq\; C_\circ \|g\|_{L^1(bD)}$$

Now

$$
\begin{aligned}
&\int\limits_{z\in bD_\varepsilon} |Tg(z)|d\sigma_\varepsilon(z)\\
= &\int\limits_{z\in bD_\varepsilon} \frac{1}{2\pi i}\int_{\omega\in bD} g^+(\omega)\{\gamma(\omega)-\gamma(z)\}\ j^*\left(\frac{\partial\lambda(\omega)\wedge(\bar\partial\partial\lambda(\omega))^{n-1}}{\langle\partial\lambda(\omega),\omega-z\rangle^n}\right)d\sigma_\varepsilon(z)\\
\leq\ & C_n\int\limits_{z\in bD_\varepsilon}\left(\int\limits_{\omega\in bD}|g^+(\omega)|\cdot\left|\frac{\gamma(\omega)-\gamma(z)}{\langle\partial\lambda(\omega),\omega-z\rangle^n}\right|d\sigma(\omega)\right)d\sigma_\varepsilon
\end{aligned}
$$

Now, $g \in L^1(bD)$ by hypothesis of the Reisz-Thorin theorem. We now consider for $\omega \in bD$ and $z \in bD_\varepsilon$-fixed,

$$
\begin{aligned}
\left|\frac{\gamma(\omega)-\gamma(z)}{\langle\partial\lambda(\omega),\omega-z\rangle^n}\right| &= \frac{|\gamma(\omega)-\gamma(z)|}{|\langle\partial\lambda(\omega),\omega-z\rangle|^n}\\
&\leq \frac{C|\omega-z|}{|\langle\partial\lambda(\omega),\omega-z\rangle|^n}\\
&\leq C_{n,\gamma,D}\frac{|\omega-z|}{|\omega-z|^{2n}}
\end{aligned}
$$

as $\gamma \in Lip(\mathbb{C}^n)$ and D is strongly convex. So we have

$$
\begin{aligned}
&\int\limits_{z\in bD_\varepsilon} |Tg(z)|d\sigma_\varepsilon(z)\\
&\qquad\leq \int\limits_{z\in bD_\varepsilon}\int\limits_{\omega\in bD}|g(\omega)|\cdot\left|\frac{\gamma(\omega)-\gamma(z)}{\langle\partial\lambda(\omega),\omega-z\rangle^n}\right|d\sigma(\omega)d\sigma_\varepsilon(z)\\
&\qquad\leq C_\gamma\int\limits_{z\in bD_\varepsilon}\int\limits_{\omega\in bD}|g(\omega)|\cdot|\omega-z|^{1-2n}d\sigma(\omega)d\sigma_\varepsilon(z)
\end{aligned}
$$

Note that for fixed $\varepsilon > 0$, $|\omega - z|^{1-2n} \leq C\varepsilon^{1-2n}$ for all $\omega \in bD$ and for all $z \in bD_\varepsilon$.

By Fubini's Theorem, we have

$$\begin{aligned}
&C_\gamma \int\limits_{z\in bD_\varepsilon} \int\limits_{\omega\in bD} |g(\omega)| \cdot |\omega - z|^{1-2n} d\sigma(\omega) d\sigma_\varepsilon(z) \\
&= C_\gamma \int\limits_{\omega\in bD} |g(\omega)| \left[\int\limits_{z\in bD_\varepsilon} |\omega - z|^{1-2n} d\sigma_\varepsilon(z) \right] d\sigma(\omega)
\end{aligned}$$

<u>*Claim.*</u>

$$\int\limits_{z\in bD_\varepsilon} |\omega - z|^{1-2n} d\sigma_\varepsilon(z) < C_{n,D} \text{ for all } \omega \in bD \text{ and for all } \varepsilon \ll 1$$

Converting to polar coordinates centered at z, that is, $r = |\omega - z|$, we have

$$\begin{aligned}
&\int\limits_{z\in bD_\varepsilon} |\omega - z|^{1-2n} d\sigma_\varepsilon(z) \\
&= \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{\varepsilon=\text{dist}(\omega, bD_\varepsilon)}^{\tilde{c}} r^{1-2n} \cdot r^{2n-1} dr d\theta_1 \cdots d\theta_{2n} \\
&= \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{\varepsilon=\text{dist}(\omega, bD_\varepsilon)}^{\tilde{c}} 1 dr d\theta_1 \cdots d\theta_{2n} \\
&= C_n r\Big|_\varepsilon^{\tilde{c}} = C_n \cdot (\tilde{c}_D - \varepsilon) < C_n \cdot \tilde{c}_D
\end{aligned}$$

for all $\varepsilon > 0$. So our claim holds. Putting all of this together we have

$$\begin{aligned}
\int\limits_{z\in bD_\varepsilon} |Tg(z)| d\sigma_\varepsilon(z) &< C_\gamma \cdot C_{n,D} \int\limits_{\omega\in bD} |g(\omega)| d\sigma(\omega) \\
&=: C_{\gamma,n,D} \|g\|_{L^1(bD)}.
\end{aligned}$$

□

Theorem 3.4. *Suppose $f \in H^p(D)$, $1 \leq p \leq \infty$, and suppose that γ is defined and satisfies a Lipschitz condition on $\mathbb{C}^n$, that is*

$$|\gamma(z) - \gamma(\zeta)| \;\leq\; C|z - \zeta| \;\text{ for all } \zeta, z \;\in \mathbb{C}^n$$

then the function defined by

$$F(z) = \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega)\gamma(\omega)\; j^*\left(\frac{\partial\lambda(\omega) \wedge (\bar{\partial}\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}\right), z \in D$$

belongs to $H^p(D)$.

Proof. Let $z \in D$. Then define

$$F(z) := \frac{1}{(2\pi i)^n} \int_{\omega \in bD} f^+(\omega)\gamma(\omega)\; j^*\left(\frac{\partial\lambda(\omega) \wedge (\bar{\partial}\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}\right), z \in D$$

By Theorem 3.2 we have that

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega)\; j^*\left(\frac{\partial\lambda(\omega) \wedge (\bar{\partial}\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}\right)$$

So we can write:

$$\begin{aligned}
\gamma(z)f(z) &= \frac{\gamma(z)}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega)\; j^*\left(\frac{\partial\lambda(\omega) \wedge (\bar{\partial}\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}\right) \\
&= \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega)\gamma(z)\; j^*\left(\frac{\partial\lambda(\omega) \wedge (\bar{\partial}\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}\right)
\end{aligned}$$

Then, by Theorem 3.2 we have:

$$\begin{aligned}
F(z) &= \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega)\gamma(\omega)\ j^*\left(\frac{\partial\lambda(\omega) \wedge (\bar\partial\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}\right) \\
&= \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega)\gamma(\omega)\ j^*\left(\frac{\partial\lambda(\omega) \wedge (\bar\partial\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}\right) - f(z)\gamma(z) + f(z)\gamma(z) \\
&= \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega)\gamma(\omega)\ j^*\left(\frac{\partial\lambda(\omega) \wedge (\bar\partial\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}\right) + \\
&\quad -\frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega)\gamma(z)\ j^*\left(\frac{\partial\lambda(\omega) \wedge (\bar\partial\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}\right) + f(z)\gamma(z) \\
&= \frac{1}{(2\pi i)^n} \int\limits_{\omega \in bD} f^+(\omega)\{\gamma(\omega) - \gamma(z)\}\ j^*\left(\frac{\partial\lambda(\omega) \wedge (\bar\partial\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}\right) + f(z)\gamma(z)
\end{aligned}$$

Define the operator $T = T_\gamma$ by

$$Tg(z) := \frac{1}{(2\pi i)^n} \int_{\omega \in bD} g^+(\omega)\{\gamma(\omega) - \gamma(z)\} \frac{\partial\lambda(\omega) \wedge (\bar\partial\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}$$

With these notations we have

$$F(z) = Tf^+(z) + f(z)\gamma(z) \quad \text{for } z \in D.$$

We need to show:

1. $F \in \mathcal{O}(D)$

2. $\sup\limits_{\varepsilon > 0} \int\limits_{bD_\varepsilon} |F|^p d\sigma_\varepsilon < \infty \quad \forall\ 1 \le p \le \infty$

Notice that as D is strongly convex, $\dfrac{\partial\lambda(\omega) \wedge (\bar\partial\partial\lambda(\omega))^{n-1}}{\langle \partial\lambda(\omega), \omega - z\rangle^n}$ is holomorphic in D, and so $F \in \mathcal{O}(D)$. Thus, we need only prove the second item above. Now

$$\begin{aligned} \|F\|^p_{L^p(bD_\varepsilon)} &= \|Tf^+(z) + f(z)\gamma(z)\|^p_{L^p(bD_\varepsilon)} \\ &\leq 2^{p-1}\left(\|Tf^+\|^p_{L^p(bD_\varepsilon)} + \|f\gamma\|^p_{L^p(bD_\varepsilon)}\right) \end{aligned}$$

and

$$\|f\gamma\|^p_{L^p(bD_\varepsilon)} = \int\limits_{\omega \in bD_\varepsilon} |f(\omega)\gamma(\omega)|^p d\sigma - \varepsilon(\omega) \tag{23}$$

$$\leq \int\limits_{\omega \in bD_\varepsilon} |f(\omega)|^p |\gamma(\omega)|^p d\sigma_\varepsilon(\omega) \tag{24}$$

But γ is Lipschitz on $\mathbb{C}^n$, $D \subset\subset \mathbb{C}^n$ and since $bD_\varepsilon \subset \overline{D}$ for all $\varepsilon > 0$ we have

$$\begin{aligned} &\int\limits_{\omega \in bD_\varepsilon} |f(\omega)|^p |\gamma(\omega)|^p d\sigma_\varepsilon(\omega) \\ \leq\ & M_D \int\limits_{\omega \in bD_\varepsilon} |f(\omega)|^p d\sigma_\varepsilon(\omega) \end{aligned}$$

where $M_D := \max\limits_{\omega \in \overline{D}} |\gamma(\omega)|$. But,

$$M_D \int\limits_{\omega \in bD_\varepsilon} |f(\omega)|^p d\sigma_\varepsilon(\omega) \leq \sup_{\varepsilon > 0} M_D \int\limits_{\omega \in bD_\varepsilon} |f(\omega)|^p d\sigma_\varepsilon(\omega) < \infty$$

by the hypothesis that $f \in H^p(D)$. Now we need only consider

$$\|Tf^+\|^p_{l^p(bD_\varepsilon)} = \int\limits_{\omega \in bD_\varepsilon} |Tf^+(\omega)|^p d\sigma_\varepsilon(\omega)$$

By equation (21) along with $F(z) = Tf^+(z) + f(z)\gamma(z)$, we have that $F \in H^P(D)$.

□

Corollary 3.5. *Suppose $f \in H^p(D)$ and $U = (U_1, ..., U_q)$ is a finite open cover of bD. Then there exists $f_1 ... f_q \in H^p(D)$ such that*

$$f = f_1 + \cdots + f_q.$$

Furthermore, for all $j = 1, ..., q$ there is an open neighborhood of $bD \backslash U_j$, $W(bD \backslash U_j)$ such that $f_j \in \mathcal{O}(W(bD \backslash U_j))$.

Proof. Let γ_j be a smooth partition of unity subordinated to the covering U_j, $j = 1, ..., q$ and satisfies $\gamma_j(\omega) = \gamma_j(|\omega|)$.

<u>Claim 1:</u> For all fixed j, $\gamma_j \in Lip(\mathbb{C}^n)$

<u>proof</u>

For ω, ω' in U_j we need to show that

$$|\gamma_j(\omega) - \gamma_j(\omega')| \ < \ C_{\gamma_j}|\omega - \omega'| \ \text{ for all } \omega \in \mathbb{C}^n$$

Now

$$\gamma_j(\omega) = \gamma_j(\omega') + \int\limits_{\zeta \in [\omega, \omega']} (\triangledown \gamma_j(\zeta)) \cdot d\sigma(\zeta)$$

For $\zeta = s\omega + (1 - s)\omega'$, $s \in [0, 1]$ we have that $d\sigma(\zeta) = (\omega - \omega')ds$ and so

$$\begin{aligned} & \gamma_j(\omega') + \int\limits_{\zeta \in [\omega, \omega']} (\triangledown \gamma_j(\zeta)) \cdot d\sigma(\zeta) \\ = \ & \gamma_j + \int\limits_0^1 (\triangledown \gamma_j(s\omega + (1 - s)\omega')) \cdot (\omega - \omega')\, ds \end{aligned}$$

So we get that

$$|\gamma_j(\omega) - \gamma_j(\omega')| \leq \int_0^1 |\bigtriangledown\gamma_j(s\omega + (1-s)\omega') \cdot (\omega - \omega')| \, ds$$

By the Cauchy Schwartz inequality we have

$$\begin{aligned} & \int_0^1 |\bigtriangledown\gamma_j(s\omega + (1-s)\omega') \cdot (\omega - \omega')| \, ds \\ \leq & \int_0^1 |\bigtriangledown\gamma_j(s\omega + (1-s)\omega')| \cdot |(\omega - \omega')| \, ds \\ = & \; |\omega - \omega'| \cdot \int_0^1 |\bigtriangledown\gamma_j(s\omega + (1-s)\omega')| \, ds \end{aligned}$$

But $\gamma_j \in C_\circ^\infty(\mathbb{C}^n)$ and therefore $\bigtriangledown\gamma_j \in L^\infty(\mathbb{C}^n)$. Putting this all together we see

$$|\gamma_j(\omega) - \gamma_j(\omega')| \leq |\omega - \omega'| \, C_{\gamma_j} \cdot \int_0^1 1 ds = C_{\gamma_j} \, |\omega - \omega'|$$

Our claim is proved. By Theorem 3.4 we have

$$f_j(z) := \frac{1}{2\pi i} \int_{\omega \in bD} f^+(\omega)\gamma_j(\omega) j^* \left(\frac{\partial\lambda(\omega) \wedge \left(\overline{\partial}\partial\lambda\right)^{n-1}(\omega)}{\langle \partial\lambda(\omega), \omega - z \rangle} \right)$$

where $Supp\gamma_j \subset U_j = U_j(\omega_j)$. In the sequel we will use the short-hand notation:

$$L(\omega, z) = j^* \left(\frac{\partial\lambda(\omega) \wedge \left(\overline{\partial}\partial\lambda\right)^{n-1}(\omega)}{\langle \partial\lambda(\omega), \omega - z \rangle} \right).$$

<u>Claim 2:</u> $f_j \in \mathcal{O}\left(U\left(bD \backslash U_j\right)\right)$

<u>proof</u>

Since $\gamma_j \in C_\circ^\infty(U_j)$ there exists $U_j' \subset\subset U_j$ such that, in fact:

$$f_j(z) = \frac{1}{2\pi i} \int_{\omega \in bD \cap U_j'} f^+(\omega)\gamma_j(\omega) L(\omega, z).$$

Now $L(\omega, z)$ fails to be holomorphic only if either:

1. $\partial\lambda(\omega) = 0$

2. $\omega \neq z$ and $\langle \partial\lambda(\omega), \omega - z\rangle = 0$ or

3. $\omega = z$.

Now we know that:

1. $\partial\lambda(\omega) \neq 0$ for all $\omega \in bD$ and so by continuity we must also have that for all $\omega \in U(bD)$, $\partial\lambda(\omega) \neq 0$.

2. $|\langle \partial\lambda(\omega), \omega - z\rangle| \geq c|\omega - z|$ for all $\omega \in bD$ and for all $z \in \overline{D}$ by the strong convexity of D, so $\langle \partial\lambda(\omega), \omega - z\rangle \neq 0$ when $\omega \neq z$.

3. By hypothesis, i.e. $z \in U(bD\backslash U_j)$ along with

$$f_j(z) = \frac{1}{2\pi i} \int\limits_{\omega \in bD \cap U_j'} f^+(\omega)\gamma_j(\omega)L(\omega, z).$$

   We have that $(\omega, z) \in U((bD\backslash U_j') \times U(bD\backslash U_j)$ and $(bD \cap U_j') \cap (bD\backslash U_j) = \emptyset$. That is, $|z - \omega| > dist(U_j, U_j') = c_j > 0$.

So, for $z \in U(bD\backslash U_j)$, $L(\omega, z) \neq 0$. Thus, $L(\omega, z) \in \mathcal{O}_z(U(bD\backslash U_J))$ and our claim is proved. $\square$

4

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5 Appendix

Theorem 5.1 (Chen,Reisz-Thorin Theorem)**.** *Let* (X, μ) *and* (Y, ν) *be two measure spaces and* p_0, p_1, q_0, q_1 *be numbers in* $[1, \infty]$*. If* T *is of type* (p_i, q_i) *with* (p_i, q_i)*-norm* M_i*,* $i = 0, 1$*, then* T *is of type* (p_t, q_t)

$$\|Tf\|_{L^{q_t}} \leq M_0^{1-t} M_1^t \|f\|_{L^{p_t}},$$

provided

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

with $0 < t < 1$.