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# INTRACTABILITY AND UNDECIDABILITY IN SMALL SETS OF WANG TILES

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**Abstract**

*Imagine a never-ending checkerboard, red and black squares alternating forever in every direction. Now close your eyes, wait for a second, and open them again. There is still the checkerboard, but is it different? Has somebody moved the checkerboard over two squares? Four squares? One million squares? It still looks the same. This is the nature of periodic tilings. Wang tiles are squares, much like the red and black ones used on a checkerboard, except Wang tiles have colors on their edges instead of on the whole square. Also, Wang tiles can only be put edge-to-edge with each other where these colors are the same. So what's so special about Wang tiles? If you cover the infinite plane with certain sets of Wang tiles, close your eyes, and open them again, you will always be able to tell if it has changed. In these sorts of tilings, there is always something that does not quite overlap when moved any amount in any direction. This is the nature of aperiodic tilings. The smallest known such set of Wang tiles has thirteen tiles. This paper computationally explores sets of six, seven, and eight Wang tiles, looking for the same aperiodic structure.*

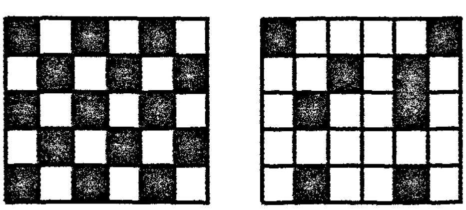

*Figure 1. Periodic and non-periodic*

To humans, the most important foundation for order is the establishment of patterns. The recognition of patterns is one of the traits that set us apart from the animals. In effect, the fact that we can do mathematics sets us a step above the animal kingdom, for mathematics is the science of patterns. Take, for example, the two patterns shown above (Figure 1). The pattern on the left is highly recognizable as a checkerboard pattern, and we are comfortable with its familiar pattern, even if it were to extend infinitely. However, the pattern on the right looks somewhat alien and strange because your brain is not instantly able to find a relationship between the squares. In the branch of mathematics known as tilings, we call the pattern on the left a periodic pattern, because we can find an infinite set of translations that leave the tiling unchanged. For example, in the case of the checkerboard, we could move the infinite tiling two squares in any of four directions, and an observer who had not seen the movement would not be able to tell that anything had changed. However, if an infinite extension of the random tiling on the right were moved any distance in any direction, an observer would be able to tell that the tiling had changed, even if they had not seen the actual translation. These sorts of tilings are called non-periodic.

What happens when an infinite extension is not even possible? If I give you a set of tiles, what we call a 'protoset,' could you even tell me if you can use the tiles to tile infinitely, cover the entire Euclidean plane? This problem is known in mathematics as the Domino Problem. Certainly, if I gave you the black and white squares used in the checkerboard (Figure 1), you could tile those forever (I imagine a 1950's diner with an endless floor). This means that the Domino Problem is decidable for certain sets, that is, given certain protosets can easily show that the tiles can cover the plane forever. However, if this is the case for all sets, then each would have a compact fundamental domain, a finite, closed, continuous piece of the plane that contains a single copy of the most basic building block of the infinitely repeating pattern. In the case of the checkerboard, an example of a fundamental domain would be a black square next to a white square, and this pattern covers the infinite plane. However, in 1966, R. Berger gave a protoset of over twenty thousand tiles that admitted a tiling of the plane, yet had no compact fundamental domain, thus, proving that for a general protoset, there could be no algorithm to decide whether the given protoset admits a tiling of the infinite plane. Berger's protoset was the first aperiodic protoset, a set of tiles that does admit a tiling of the plane, yet only gives a non-periodic structure.

In the field of aperiodic tilings, the search continues for aperiodic sets of manageable size, since Berger's twenty-thousand-tile set was unwieldy even for a computer. Currently, there are different aperiodic protosets having just two tiles by Penrose, Amman, and Goodman-Strauss, but the search for such sets has never been approached systematically. In this paper, what we are looking for is a small set of a certain type of tiles that can only tile non-periodically. For this purpose, we have designed a computer algorithm to take a given set of these certain types of tiles, and in a sense, categorize it. Protosets can be classified into three behaviors: non-tiling, periodic, and aperiodic. The first two categories are decidable under the Domino Problem, that is, we can write a computer algorithm to model the tiles and have an outcome that the protoset either does not tile the infinite plane or has a compact fundamental domain. The third category will, however, confuse the computer algorithm, as it will tile forever, yet the computer will not be able to recognize any distinguishable pattern. A slight variation on the second possibility would be a periodic tiling, but the size of the fundamental domain has a combinatorial complexity vastly more complicated than one would expect given the combinatorial complexity of the protoset itself. Although these fall under the periodic category, they are interesting in that they also might be able to confuse the computer. These types of tilings are called intractable.

The method for construction of aperiodic tilings dealt with in this article is known as Wang tiles. They are unit squares, similar to the ones covering the checkerboard, with the twist that Wang tiles have their edges colored and can only touch other Wang tiles where the colors match. These are the types of tiles that were used in the original aperiodic set of over twenty thousand tiles. Thinking in terms of colored edges, it is very easy to extend the matching rules of Wang tiles to let two edges meet not only where the two colors are the same, such as two red edges meeting, but also where one color is a primary constituent of another, such as red and purple (red + blue) edges meeting. Henceforth, the original definition will be referred to as strict Wang tiles, while the extended definition will be referred to as loose Wang tiles. This extended definition allows for greater flexibility in tiling the plane since it allows for lack of transitivity between tiles. In strict Wang tiles, a red edge could only meet a red edge, so if tiles A, B, and C had red edges, any of those edges could join with either of the other two. However, in loose Wang tiles, tile B might have an orange edge and tile C might have a purple edge, so although tile A's red edge could meet both of them, B and C could not connect at those two edges.

The first and foremost goal of this research is to explore the structure of small sets of loose Wang tiles with a new algorithm for categorizing protosets. We hope to find a small set that confuses the computer algorithm and could therefore be classified as intractable. From a theory by Robinson [2] saying that no aperiodic set has fewer than four tiles, and knowing that the smallest known aperiodic set of Wang tiles has thirteen tiles [3], the algorithm in this paper was applied to protosets of six, seven, and eight loose Wang tiles.

The first problem in computer handling of protosets and tilings are computer representation of the protosets. The easiest way to do this for small protosets is a modification of traditional adjacency matrices. If a protoset has $n$ tiles, then we will have two $nxn$ Boolean matrices, LRmatrix and UDmatrix, which represent the possible left/right and up/down adjacencies respectively. That way, tile $x$ can be on the left of tile $y$ if LRmatrix[$x$][$y$]=1. Notice that this also means tile $y$ can be on the right of tile $x$. Similarly, we say tile $x$ can go above tile $y$ if UDmatrix[$x$][$y$]=1.

Once we have the matrix representation for a protoset, we must find a way to categorize it. A logical approach to determining if a tiling is periodic or not is to search for a fundamental domain. Now, if we look at an infinitely repeating pattern, we can find vectors $(a,b)$ and $(c,d)$ such that every tile is invariant when translated $a$ tiles to the right and $b$ tiles down ($c$ and $d$ respectively). From algebra, we know that the greatest common divisor (gcd) of two numbers is also the smallest positive linear combination of those numbers, so if we were to take gcd($a,c$), that would be the smallest horizontal translation possible where those vectors could give an invariant. Furthermore, we can deduce from algebra that the fundamental domain will have an area of determinant[ $(a,b)$ $(c,d)$ ]=($ad$-$bc$), so we can conclude that the smallest vertical translation possible that could leave the tiling invariant would be that area divided by the corresponding horizontal translation, or ($ad$-$bc$)/gcd($a,c$). So instead of using a protoset to tile the plane and then look for patterns, we can generate fundamental domains using vectors, and see if our protoset can tile the fundamental domain with the rule that the fundamental domain must be able to go next to itself. That means the tiles on the right of the fundamental domain must be able to be adjacent to the tiles on the left of the domain, and likewise for top and bottom. If a protoset successfully tiles our generated fundamental domain, then it has an infinitely repeating pattern with which it can tile the infinite plane, and can therefore be categorized as a periodic protoset. If the protoset does not tile any fundamental domain up to a certain size, then the protoset will be tested to see if it tiles a finite square of the plane. If the protoset cannot even tile the finite square, it certainly cannot tile the infinite plane, and the protoset is categorized as non-tileable. However, if it does tile the finite square, then the protoset has effectively confused the computer since the algorithm can extract no patterns from the tiling. These types of protosets are set aside for human analysis. Only after the human has proved anything about the protoset can it be classified as aperiodic or intractably periodic.

Each of the protoset sizes, six, seven, and eight tiles, was tested for two weeks of computer time, and the results were as we expected to a large degree. As stated earlier, even when we narrow our consideration of matrices around the 25% adjacency pivot, there are still so many that we are only able to scratch the surface. From previous knowledge of aperiodic tilings, we can generalize that if there is one protoset of a certain size of loose Wang tiles that forces a non-periodic tiling, then it is probably not the only one of that size. We would probably expect there to be around ten such protosets, if there are any at all, so for size six protosets we would have 10 in $2^{72}$ chances (about one in four thousand million million million) of randomly stumbling upon an intractable or aperiodic protoset. For this reason, our results, which do not even begin to draw near those numbers, are largely what we expect. In two weeks of computer time for each size, the following numbers were generated:

| Protoset sizes | # non-tileaable protosets | # periodic protosets |
|---|---|---|
| 6 tiles | 11,889,226 | 8,590,793 |
| 7 tiles | 5,038, 438 | 4,654,499 |
| 8 tiles | 4,214,480 | 5,627,316 |

The computer found no interesting (aperiodic or intractable) protosets, and this was expected. What also was expected was that the ratio of non-tileable protosets to periodic protosets would decrease for larger sizes since having more tiles allows a greater possibility of tiling the plane.

In conclusion, although we did not uncover any interesting protosets, we did not fully expect to do so. It was merely a hope that the implementation of this categorization algorithm for sets of loose Wang tiles would reveal a previously unknown interesting set. However, we have demonstrated that it is likely that Robinson's theorem can be extended to sets of at least eight Wang tiles, i.e., that at least eight tiles are required for an aperiodic set. The actual lower bound remains an open question.

**References**

**[1] Goodman-Strauss, C. "Open Questions in Tilings" [online]. January 2000.**

**[2] Grünbaum, B. and G.C. Shephard. *Tilings and Patterns*. New York: W.H. Freeman and Company, 1987.**

**[3] Kari, J. "A small aperiodic set of Wang tiles." *Discrete Mathematics*. 1996, vol. 160, s. 259-264.**

**Faculty comments**

Janet Woodland, Mr. DeLisse's mentor, made the following comments in her letter os support for publication of his work:

> I met Adam Delisse in the first semester of his first year at the University of Arkansas, when he was enrolled in my Discrete Mathematics class, and his

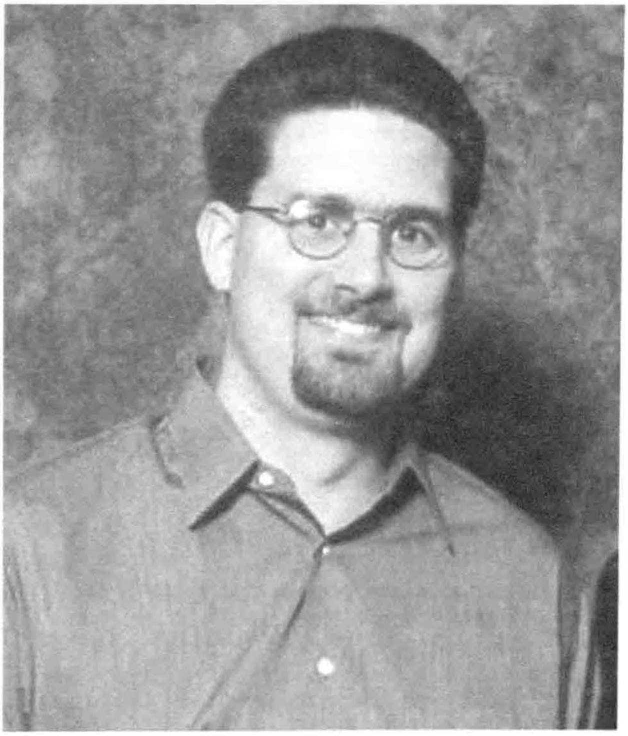

*Figure 2. Adam Delisse*

> academic abilities were immediately evident. This course is a "cornerstone" for the mathematics major, addressing such fundamental concepts as logic, set theory, and combinatorics. Adam was an energetic and alert participant in the class, expressing his comprehension of the course material with unusual clarity. He was particularly talented in recognizing mathematical patterns, and anticipating natural extensions of the course material, and curious about its relationships with other mathematical topics. I was also impressed by the questions Adam asked, in addition to how well he answered mine. His genuine curiosity and attention to detail enabled him to produce some of the most elegant mathematical arguments I have seen from such a young student, and ever since then he has showed the persistence and creativity needed to solve difficult problems.
>
> Though that was the last time Adam took a class with me, we have remained in contact through his continued presence in the department, and his involvement in Pi Mu Epsilon (our undergraduate mathematics society). He has always been one of our best students, and in fact, one of the College's best students, since he is about to complete an Honors degree and has been the recipient of many fellowships including the Sturgis and the Goldwater, and he has presented his work at the regional meeting of the Mathematical Association of America.
>
> This article summarizes the work more fully described in Adam's honors thesis, and combines his chief areas

of interest, mathematics and computer science. The original question Adam addresses was posed by my colleague Dr. Chaim Goodman-Strauss (among the experts in this field), and I have taken on the role of mentor in his absence. The thesis describes Adam's long-term investigation of a certain category of tilings of the plane, with deeper underlying issues such as decidability and intractability. (Kurt Gödel proved in the early 1930s that certain questions *cannot* be answered - that certain statements cannot be *disproved).* The question of how a given set of tiles can cover the plane, and whether or not a given set *must* tile aperiodically, is a major topic of current research in this area of mathematics.

A member of Mr. Delisse's honors thesis committee, Suzanne McCray, Associate Director of Honors Studies in Fulbright College, had this to say about him:

> Adam DeLisse, a senior Sturgis Fellow and mathematics major, shines both in and outside his field of study. He chose the Fulbright College of Arts and Sciences Honors Scholars Program, a demanding four-year curriculum pursued by only two percent of the students at the University of Arkansas. Adam also opted to take the Honors Humanities Roots of Culture series of courses. The average ACT score for students who opt for this option is 32 (1400 SAT equivalent). In the third semester, he was the absolute star of the class. The reading load was heavy, and the research project was demanding, Adam's performance was exemplary. His papers were always interesting. He is a thorough researcher and is skeptical when it comes to historical bromides. He always wants to know if the data really supports standard assumptions. There are certainly easier humanities courses to take that will satisfy the requirements, and students in the sciences often opt for them-not Adam. Not only did he choose our most ambitious core curriculum, he also began taking upper-division courses in mathematics as a freshman and is now taking graduate level courses. We expected Adam to do well when we recruited him for our program. We were delighted, when he chose to accept the Sturgis Fellowship. He has been a wonderful member of the Honors community. The Honors students elected him by an overwhelming vote to be the student representative from the sciences on the University Honors Council.
>
> Adam's research abilities are well documented. His research mentor, Professor Chaim Goodman Strauss, is a tough taskmaster, but you could not tell that when talking to him about Adam whom he Praises highly and at length. For two years in a row Adam received a science Information Liaison Office Undergraduate research Fellowship for his work with Strauss on aperiodic tilings. According to Strauss, Adam's presentation of his material at the Mathematical Association of America was very professional. A previous research project resulted in a newsletter publication for the Society of Actuaries: Time to Dig Out the Old Dividend Discount Model?" Last year Adam received the nationally competitive Barry Goldwater Scholarship for his outstanding achievements in mathematics and for his commitment to research.
>
> My degrees are in English, and the world of mathematics has always been a mysterious one to me. That is why I approached being on Adam DeLisse's honors thesis committee with some trepidation. Wang tiles meant very little to me. The only comprehensible tile to me was on a floor or in a quilt. But I have had long, interesting conversations with Adam on this topic, and I am genuinely happy that I have served on this committee. On several occasions in talking with Adam, mathematical lights have come on for me.
>
> What Adam is doing is remarkable. As I understand it, no one has approached these tile patterns (or more importantly the possibility of non-patterns) this systematically before. Tile studies are relatively new, originating in the 1960's. According to Adam, scholars have used mathematical theories to prove that nonpatterns do exit with thirteen or more tiles. Adam DeLisse's goal is to demonstrate through the use of computers that even fewer tiles will produce non-patterns. Through his work~ he has been able to conclude that eight or fewer tiles will produce patterns to infinity. Intellectually the project is extremely interesting and one day will likely have important practical applications. Finally, Adam DeLisse's work is both interesting and readable.

One of Mr. Delisse's mathematics instructors, Loredana Lanzani, also had high praise for his work, saying:

> Adam was a student in my differential equations course during the Fall of 1997, which happened to be my very first semester at the University of Arkansas. It was clear from day one that Adam would define the top of the class. During that semester I often compared my teaching experience at the University of Arkansas with my very fresh memories from Purdue University, where I had had extensive contacts with many science or engineering majors. None of the students I had known in Purdue could even remotely compare to Adam in terms of mathematical ability and rigor, intuition, enthusiasm and curiosity.
>
> Not one lecture went by without Adam being with me or, more often, ahead of me in the presentation. He showed equal enthusiasm both for the theoretical aspects of the subject (in fact, I could tell by his remarks that Adam was consistently able to pin down the details that I had left out in the proofs) and for the many applications to Physics, Statistics and Engineering that we studied. It goes without saying that Adam's written work was exceptional and he

ended up with the best score in the class. (I should add that, later on, I realized that in this class the number of talented students was unusually large). By the end of the semester it was clear to me that Adam would produce an excellent senior thesis in any branch of mathematics.

My expectations have been met beyond my wildest hopes. First of all, the thesis topic, a tiling problem, is a wonderful blend of geometry and combinatorics and perfectly suits Adam's choices for his major (Mathematics) and minor (Computer Science). I was also impressed by the large body of information, both in terms of mathematics and computer programming, that he had to master in order to test the theory on a concrete set of computer simulations. Last but not least, when reading his thesis I was very much impressed by Adam's ability to explain such complex work in a clear, precise and yet entertaining and compelling manner. His frequent comparisons with familiar patterns from everyday life (the checkerboard and the 1950's diner with an endless floor are the first examples I can remember) make sure that non-specialist readers have close at hand very pertinent and concrete examples that they can relate to to help them keep track of the main ideas in his work.