Inquiry: The University of Arkansas Undergraduate Research Journal

Volume 1 Article 5

Fall 2000

# An Analysis of the Theory of Functions of One Real Variable

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# AN ANALYSIS OF THE THEORY OF FUNCTIONS OF ONE REAL VARIABLE

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**Abstract:**

*Few undergraduates are aware that the Riemann integral taught in introductory calculus courses has only limited application—essentially this integral can be used only to integrate continuous functions over intervals. The necessity to integrate a broader class of functions over a wider range of sets that arises in many applications motivates the theory of abstract integration and functional analysis. The founder of this theory was the French mathematician Henri Lebesgue, who in 1902 defined the "Lebesgue measure" of subsets of the real line. The purpose of this project is to elucidate the theory of abstract measure spaces and of important spaces of functions (a critical example of which are Banach spaces), and extend the application of this theory. Developing the tools for doing so has been the focus of my advisor Professor Dmitry Khavinson and me over the past three years.*

*The primary goal of the thesis is to make this highly formal and abstract material accessible to an undergraduate having only a year of coursework in advanced calculus. These concepts are typically introduced at the graduate level, but the ideas require only a familiarity with the analytic style of proof learned as an undergraduate. It would be advantageous to expose advanced undergraduates to this material since these ideas form the foundation for how mathematical research is done at the professional level. The addition of interesting and practical examples (which are scarce in the standard graduate texts) will help to make the concepts more familiar and down-to-earth.*

*The motivation for a new theory of integration came from the Riemann integral's apparent inability to operate on functions that fail to be continuous. For example, the Riemann integral of the function that assigns the value 1 to rational numbers and 0 to irrational numbers can be evaluated over the interval [0, 1] with equally valid justification to be 0 or 1. This is because the definition of the Riemann integral depends on partitioning the* domain *of the function to be integrated, and finding the maximum and minimum values of the function over each partition. The Lebesgue integral, on the other hand, partitions the range of the function to be integrated and then considers the length of the pre-image of each partition as well as the maximum and minimum values of the function of the partition. The utility of this change of perspective arises when we refine what is meant by "length" in the aforementioned pre-image. The Riemann integral requires that the domain consist of intervals of real numbers (where length makes sense), while the Lebesgue integral can be used with a much broader class of sets. Lebesgue modified the notion of length by defining the* measure *of a set E to be the smallest possible total length of all collections of intervals that cover E. Using this ingenious method, Lebesgue constructed a theory of integration which forms the most useful example of all general integration theories. The theory has important applications in many areas of science and engineering as well as probability and statistics.*

*Jason Reed and Dmitry Khavinson*

*Our approach to the subject has emphasized theory developed in H.L. Royden's classic text,* Real Analysis. *My project has included analysis of each concept in the text, and I have developed for each major subject a collection of problems solved and applications of major theorems that were explored. The result has been comprehension of many of the foundational ideas in the field. We have used a number of supplemental texts to gain depth of understanding where Royden's text provides only a survey, such as the Riesz Representation theorem, and to extend important ideas, such as the consideration of complex-valued (in addition to real-valued) measures.*

*The synthesis has been a comprehensive paper which describes the theoretical directions the research has taken, the major results and theorems with proof, and applications and examples which are worked out in detail. The final record of my research will be divided into the following six sections: Lebesgue measure, Lebesgue integral, relationship between differentiation and Lebesgue integration, Banach space theory, abstract measure theory, and general integration theory. The analysis encompasses discussion of the main ideas (what it means for a set function to be a measure, how an integral can be defined in a coherent way with respect to a measure, when the derivative of an integral of a function is the function itself, different ideas about what it means for a sequence of functions to converge to a function, what are the properties of Banach spaces and why they are useful, etc.), as well as important ideas and theorems that interrelate these concepts (i.e., when we can interchange the limit of a sequence of functions and the integral, how we can represent a bounded linear functional, the structure of certain spaces of integrable functions).*

## The Paper:

The Riemann integral taught in introductory calculus courses has limited application—essentially this integral can be used only to integrate continuous functions over intervals.[1] Even bounded functions such as the characteristic function of the rational numbers on the interval [0, 1] have a Riemann integral whose value can be equally well justified to be 0 or 1. A new theory of integration was called for, and the French mathematician Henri Lebesgue invented it in the early twentieth century. Instead of partitioning the *domain* of the function to be integrated into a number of intervals of fixed length and then sampling it on each partition, Lebesgue's idea was to partition the *range* of the function and consider the length of the corresponding preimages. These preimages, however, were no longer necessarily as simple as intervals (sets whose length is easily defined), and so a new notion of the length of a set was needed. Lebesgue thus defined the "measure" of a set E to be the infimum of the total lengths of all open covers of E. From this springboard he defined a class of "measurable sets" of the real line, over which measurable functions could be integrated. Lebesgue's generalization permitted vastly more functions to be integrated; and, as E. B. Van Vleck wrote shortly after its conception in 1916, "This new integral of Lebesgue is proving itself a wonderful tool. I might compare it with a modern Krupp gun, so easily does it penetrate barriers which before were impregnable." The purpose of the present project is to explicate the theory of Lebesgue measure and the Lebesgue integral as well as abstract measure theory.

The approach to the subject has emphasized the development of the theory of functions of one real variable in H.L. Royden's classic text, *Real Analysis*. The project has included analysis of each chapter in the text, and for each subject a collection of solved problems and applications of major theorems which were explored. The project has covered many of the major ideas and theorems that form the foundation of the field of what is presently known as "real analysis," a subject typically learned in a one-year graduate course in mathematics. Supplemental texts were used to gain understanding of major parts of the theory that are only surveyed in Royden's text (such as the Riesz Representation theorem) and to extend various important ideas (such as the consideration of complex-valued in addition to real-valued measures).

The synthesis of the project has been a comprehensive paper which describes the theoretical concepts, the major theorems with proof, and applications and examples which are worked out in detail. The final record of my research will be divided into the following five sections: Lebesgue measure, the Lebesgue integral, the relationship between differentiation and Lebesgue integration, Banach space theory, and abstract measure and integration theory. The development of Lebesgue measure includes an axiomatic introduction and analysis of its salient properties (analyzing its vital axioms: countable additivity, translation invariance, and determination that the calculated "measure" of an interval agrees with the established notion of length), the more general notion of outer measure, measurable sets and the construction of a nonmeasurable set of the real line,[2] measurable functions, and J. E. Littlewood's famous "three principles." Littlewood's principles are the most instrumental notions that form the foundation of functional analysis. Various incarnations of his principles are realized in Egoroff's theorem, Lusin's theorem, the possibility of approximating measurable sets with open sets, and the action of finding a sequence of continuous functions which can uniformly approximate any measurable function. Finally, we demonstrate what it means for a sequence of measurable functions to converge in measure to a function $f$. This is a notion that is weaker than uniform and even pointwise convergence but has important applications in functional analysis.

Considering Lebesgue's theory of integration, we first review the formal definition of the Riemann integral, and illustrate its limitations. The presentation of the Lebesgue integral then proceeds by construction in successively more complicated settings. At the beginning of this process, we base the definition of the integral on the integral of a *simple function*, which is a function that assumes only finitely many values on some finite sequence of measurable sets. Its integral is simply the sum ofthe measures of the sets on which the function is defined times the value of the function on these sets. We proceed to defne the integral of a bounded function $f$ on a set of finite measure to be the supremum of the integrals of all simple functions which do not exceed $f$. We then naturally extend the definition to the integral of any non-negative function $g$, defining the integral of $g$ to be the supremum of the integrals of $f$ for all bounded measurable functions which do not exceed $g$. This definition enables us to establish the essential convergence theorems: Fatou's lemma, the monotone convergence theorem, and Lebesgue's dominated convergence theorem. These convergence theorems express the integral of a function $f$ in terms of the limit of a sequence of integrals of functions $f_n$, when the functions $f_n$ converge to $f$ for each point in their domain (except perhaps for a set of measure zero). Fatou's lemma requires only pointwise convergence but in return concludes only that the limit of the integrals of $f_n$ is less than or equal to the integral of $f$.

The hypothesis of the monotone convergence theorem is that the functions $f_n$ be increasing to $f$, then asserts that the limit of the sequence of integrals of $f_n$ is equal to the integral of $f$. Lebesgue's dominated convergence theorem requires that each $f_n$ be bounded above and below by a fixed integrable function $h$ and concludes equality of the limit of integrals of $f_n$ and the integral of $f$. These theorems provide powerful tools with which we can determine the integrability of a function and also open the door on the subject of functional analysis. There, we explore the absolute continuity of the integral and approximation of integrable functions by sequences of simple functions, step functions, or continuous functions.

Our consideration of differentiation and integration explores in which settings differentiation and integration are indeed "inverse" operations. The conclusion of this consideration is encapsulated in a result known as the *Fundamental Theorem of Calculus.* We first analyze the application of the derivative to measurable functions and Vitali's covering theorem. This requires introducing the concept of the four derivates of a measurable function, which must be equal for the derivative of the function to exist. Then we define what it means for a function to be of bounded variation and the relationship of bounded variation to absolute continuity. Absolute continuity, which is stronger even than uniform continuity, is the key with which we establish the Fundamental Theorem of Calculus. It so happens that we can relax the restrictions placed on $f$ in order to be able to recover $f$ by differentiating its integral; but in order to have the converse—to have the integral of the derivative of $f$ be equal to $f$ itself—$f$ must be absolutely continuous. We finally consider the properties enjoyed by convex functions (a special class of measurable functions) and prove Jensen's inequality.

The theory of Banach spaces, of which the $p$-integrable measurable functions provide the most salient example, is of vital importance in order to develop a coherent theory of abstract measure spaces. We will cover in detail these so-called $L^p$ spaces, prove the Holder and Minkowski inequalities, and explore the role played by completeness in analyzing the convergence of functions in Banach spaces. In addition, we introduce bounded linear functionals. The Riesz representation theorem is the most important result in this area, allowing us in $L^p$ to find a representation via integration of each bounded linear functional against a function in its dual space, $L^q$.

Abstract measure and integration theory and functional analysis are the culmination of this project. We distill the most vital properties of Lebesgue measure, and with these define what it means for an arbitrary set function to be a measure. We thus concentrate on "measurable spaces" (an abstract space along with a measure and a collection of measurable sets) and consider which properties of Lebesgue measure remain valid. We then look at a number of general measures that arise in this setting: signed measures (which can take both positive and negative values), complex measures, and product measures. We further consider the decomposition of a measure into disjoint parts (one absolutely continuous and the other singular) and the extension of a measure from an algebra of sets to a æ-algebra. Analyzing the corresponding extension of integration, we prove the general convergence theorems, which allow not only a sequence of functions to vary but also a sequence of measures. We conclude with the Radon-Nikodym theorem, which reveals that each measure can be represented as an integral of a given function against a measure with respect to which the original measure is absolutely continuous.. The result gives rise to the Radon-Nikodym derivative, an important function representing a ratio of sorts between the two measures considered.

[1] *More precisely, a bounded function f is Riemann integrable on [a, b] if and only if the set of points at which f is discontinuous has measure zero.*
[2] *This is a surprisingly difficult task! The fact that this is so seems to justify the usefulness of Lebesgue's definition of what constitutes a "measurable set."*

### Faculty Comments:

**Dr. Dmitry Khavinson**, Mr. Reed's mentor, made these comments about the work:

> I have known Jason for five years, since he came to the University of Arkansas as a freshman, holding a prestigious Sturgis Fellowship. He has been my student ever since, first in the three-semester Honors Calculus course and, during his third year, in a directed reading course (studying the Principles of Mathematical Analysis by Walter Rudin). He has continued his reading course with me through his fourth year, studying measure and integration theory in Royden's fundamental text, *Real Analysis*. This year he is reading Kai Lai Chung's book, *A Course in Probability Theory*, whichforms the basis for his senior thesis.

The thesis consists of exposition some of the most "treacherous" places in the Lebesgue theory of integration—some difficult problems used in graduate courses in Real Analysis and Probability. This work may not be highly original in regard to its scientific merit but is highly so in relation to pedagogy and exposition, and I strongly support this project.

In all my calculus classes he was without a doubt the best student; and, over all, I would not hesitate to place him among the three best undergraduate students I have ever encountered during my twenty years of teaching, both here at the University of Arkansas, and at the University of Alabama, the University of Michigan, and Brown University. Jason is a solid person, a hard worker, and a stable learner. What I find to be his most important quality is that he has a tremendous amount of curiosity for mathematics and science, in general.

Jason has received a number of impressive awards during his undergraduate career. The most prestigious and competitive award that I want to mention here is the Barry Goldwater Scholarship for 1998-99 that allowed him to participate in the Penn State Mathematics Advanced Study Semester with 25 other students selected from throughout the United States. Lately Reed has been actively working on his senior project and has given a number of excellent presentations at our seminars and also at various gatherings of undergraduate students around the country on a rich variety of topics. Also, in 1999 he was awarded an undergraduate fellowship at the Department of Energy at Argonne National Laboratory. In short, Jason Reed is as good a student and a beginning scholar as one can possibly hope for.

Professor **John Akeroyd**, also of the Mathematics Department, echoed Professor Khavinson's comments:

Jason was a student of mine in a course in discrete mathematics and in a two-semester course in mathematical analysis. In all of these courses, Jason defined the top of the class. Though I had a number of bright students in these classes (in fact, more so than ever before), no one was quite able to match Jason's combination of enthusiasm and ability. During my lectures, I was ever conscious that Jason was with or ahead of me in the presentation. His questions and comments in class were insightful and helped other students learn the subject. His test and homework scores were always excellent. I further got to know Jason in his capacity as Pi Mu Epsilon president; I am one of two faculty advisors for this math club at the University of Arkansas. Jason's enthusiasm for mathematics has made my job easy. He organizes talks each semester, asking both students and faculty to participate. He also represents this math club at national meetings and carefully prepares an interesting talk for each of these occasions.

In recent years he has been studying graduate-level mathematics under the guidance of various mathematicians in our department. One of the subjects to which he has been effectively devoting time is Analysis; Professor D. Khavinson has been his supervising instructor in this area. Having a firm grasp of Analysis is an essential first step in understanding many branches of mathematics. Frequently I have had the opportunity to discuss this subject with Jason and witness his development. He is well on his way to gaining fluency in this subject and shows the ability and enthusiasm not only to master it but to make contributions.

Jason has definite goals, and he pursues them with great determination and paradoxically with a joyful and outgoing manner. Among his goals for the 1999-2000 academic year is to continue his studies in Analysis; Dr. Khavinson is on leave, serving with the NSF for a year or two, and I have agreed to fillin. Our goal is to finish readings in a *Real Analysis* text written by H. L. Royden (a very standard text in the field), gain a firm grasp of the fundamentals of Functional Analysis (primarily using well-known texts of W. Rudin) and then proceed into the subject of Mathematical Probability, which is a very natural path to follow after an exposure to Real Analysis. We will budget our time carefully and maintain a rigorous pace. I am confident that Jason is up to the task and indeed will surpass my expectations as he always does; we have already made great strides this year and (if anything) are ahead of schedule. He pushes himself and the instructor with much fervor -- such is his enthusiasm for learning. Jason is very easy to communicate with and indeed is an excellent public speaker. The discipline and enthusiasm that he brings to each of his efforts has been an inspiration to other students. I can think of no student in my years here whom I can recommend as highly as Jason. His innate ability and the maturity and determination with which he pursues his goals are quite remarkable.

**Serge Tabachnikov**, another of Mr. Reed's professors, had this to say:

I know Jason very well: four times he pursued independent studies with me (each project was a semester long; we spent about two hours weekly discussing the subject). The topics included differential geometry and topology of plane curves, "quantum" knot invariants, and differential geometry of smooth and polyhedral surfaces. All these projects went far beyond the standard curriculum and were quite challenging for an undergraduate student,. Each of the studies resulted in an essay, written by Jason, and talks, given by him at regional and national conferences. Jason also has the distinctive qualities of a leader. He is the President of the local Pi Mu Epsilon Mathematical Society. It is due to his enthusiasm and organization skills that the Society remains the center of attraction to all undergraduates at UARK who are interested in mathematics.