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Abstract

The energy of a smoothly parameterized knot $\gamma(t)$ is defined as

$$\int_0^{2\pi}\int_0^{2\pi}\left\{\frac{1}{\|\gamma(s)-\gamma(t)\|^2}-\frac{1}{(D(\gamma(s),\gamma(t)))^2}\right\}\left\|\frac{d\gamma}{ds}\right\|\left\|\frac{d\gamma}{dt}\right\|dsdt$$

where $D(\gamma(s), \gamma(t))$ is the arc length between the two points $\gamma(s)$ and $\gamma(t)$ on the curve. Simple calculus based arguments are used to locate critical values of the energy functional for torus knots. Explicitly the curves given parametrically by $\sigma_{(a,b)}(t) = \left(\frac{\cos(2a\pi t)}{\sqrt{2}-\sin(2b\pi t)}, \frac{\sin(2a\pi t)}{\sqrt{2}-\sin(2b\pi t)}, \frac{\cos(2b\pi t)}{\sqrt{2}-\sin(2b\pi t)}\right)$ are critical points of the energy functional whenever a and b are relatively prime.

Introduction

One of the earliest appearances of knots in physics occured in 1867 when Lord Kelvin put forward the idea that atoms were vortex tubes. More recently, knot theory is being studied as a possible means of quantizing gravity (Baez and Munian 1994). An early attempt to define the energy of a knot was done by Fukuhara (1988). This was based on the usual $\frac{1}{r}$ potential of electrostatics. Unfortunately, this potential doesn't give a finite energy when passing to a continuous charge distribution. It also isn't strong enough to prevent various parts of the knot from touching. More recently O'Hara (1991, 1992) has described a number of renormalized "energies" for knots. Freedman et al., have shown one of these energy functionals to be both scale and conformally invariant. They then used this to show that the least energy configuration over all knot types is the round circle. This energy functional for a smoothly parameterised knot $\gamma(t)$ is defined as

$$(1)\quad \int_0^{2\pi}\int_0^{2\pi}\left\{\frac{1}{\|\gamma(s)-\gamma(t)\|^2}-\frac{1}{(D(\gamma(s),\gamma(t)))^2}\right\}\left\|\frac{d\gamma}{ds}\right\|\left\|\frac{d\gamma}{dt}\right\|dsdt$$

where $D(\gamma(s),\gamma(t))$ is the arc length between the two points $\gamma(s)$ and $\gamma(t)$ on the curve. The terms $\left\|\frac{d\gamma}{ds}\right\|$ and $\left\|\frac{d\gamma}{ds}\right\|$ should be thought of as the charges in a small bit of arc length assuming there is a uniform charge distribution of one on the curve. The second term is the renormalization factor while the 2^{nd} power is used instead of the first to provide for a stronger potential.

The problem of minimizing the energy of various knot types is amenable to the basic calculus tools available to any good undergraduate. Most of the basic calculations in this paper were done by the second author during his senior year as part of a SILO undergraduate research grant. The simplest knots after the round circle are the torus knots. The first to study the energy of torus knots were Kim and Kusner (1993). In this paper we show that the (a, b)-torus knot given parametrically by $\sigma(t) =$ $\left(\frac{\cos(2a\pi t)}{\sqrt{2}-\sin(2b\pi t)}, \frac{\sin(2a\pi t)}{\sqrt{2}-\sin(2b\pi t)}, \frac{\cos(2b\pi t)}{\sqrt{2}-\sin(2b\pi t)}\right)$ is a critical point of the energy functional (1). This is done by recognizing that $\sigma(t)$ is the sterographic projection of the curve $\gamma(t) = (\cos(2at\pi), \sin(2at\pi), \cos(2bt\pi), \sin(2bt\pi))$ in R^4. This curve lies on S^3. Since the energy functional (1) is conformally invariant, if $\gamma(t)$ is a critical point of (1) its stereographic projection $\sigma(t)$ is a critical point of (1). $\gamma(t) = (\cos(2at\pi), \sin(2at\pi), \cos(2bt\pi), \sin(2bt\pi))$ is shown to be a critical point of the energy functional (1) by approximating it as a discrete set of charged points $\{p_k\} = \{(\cos(2ak\pi/n), \sin(2bk\pi/n), \sin(2bk\pi/n))\}_{k=1}^{n}$ connected by straight line edges and then letting $n \to \infty$. Since any C^1 curve can be approximated using a polygonal path, any C^1 curve near $\gamma(t)$ can be approximated by perturbing the points p_k. It is known that the energy functional (1) is continuous for $C^{1,1}$ curves (corollary 6.3, Freedman et al., 1994). This result can be extended to piecewise $C^{1,1}$ curves, which the various piecewise linear curves used in approximating $\gamma(t)$ are, as is $\gamma(t)$. Since all our curves have constant speed, to show that the first variation of the energy is zero at $\gamma(t)$, it suffices to show that the gradiant of the discrete model of the energy is zero in the limit as $n \to \infty$.

Calculations

In (1) the analytic potential assumes a uniform charge denisty of one. The analogue for the discrete case has the charge at each point equial to 1/2 of the sum of the lengths of the two edges which meet at that point. The energy in the discrete case is obtained as the double sum

$$(2)\qquad \sum_{s\neq t=1}^{n} \frac{q_s q_t}{\|p_s - p_t\|^2} - \frac{q_s q_t}{D(p_s, p_t)^2}$$

where p_s and p_t are the positions of two of the charged points, q_s and q_t their charges, and $D(p_s, p_t)$ is the length of the shortest edge path between the two points p_s and p_t.

Since the configuration of the points $\{p_k\}_{k=1}^{n}$ is symmetric, if it can be shown that the four partials associated to varying the single point (1,0,1,0) are all zero in the limit as $n \to \infty$, then all the partials associated to varying any point will also be zero in the limit, and so the gradient will be zero. Symmetry shows that the partials in the $y = x_2$ and $w = x_4$ directions are zero, since $\sin\left(\frac{2(n-l)\pi}{n}\right) = \sin\left(\frac{2l\pi}{n}\right)$. The choice of the points p_k also shows that the calculation of the derivative in the $z = x_3$ direction is identical to the calculation in the $x = x_1$ direction with the roles of a and b reversed. To show that the gradient is zero in the limit as $n \to \infty$, it suffices to vary only the point (1,0,1,0) in the x-direction and see that in the limit this partial goes to zero.

Note: for antipodal points on the circle there are two shortest connecting paths on the circle. So when modeling the circle by a discrete number of points, it is best to use an odd number of points to avoid having to choose a shortest path.

To begin the calculation of the energy for the discrete model the following simplifying notations are used

r_k is the distance between the points (x,0.1.0) and p_k $(1 \leq k \leq n-1)$

$r_{k,j}$ is the distance from p_k and p_j $(1 \leq k \neq j \leq n-1)$

q is the distance from two adjacent points if neither is (x,0,1,0).

q is chosen for this last distance because the charge at all but the points p_1, p_{n-1}, and (x,0,1,0) is q. The charge at (x,0,1,0) is r_1 while the charge at both p_1, and p_{n-1} is $1/2\,(r_1 + q)$.

Since the charges are different at different points, the calculation of the energy is broken down into the following cases: the potential between the point (x,0,1,0) and the points $\{p_k\}_{k=2}^{n-2}$; the potential between (x,0,1,0) and both p_1, and p_{n-1}; the potential between both p_1, and p_{n-1} and the points $\{p_k\}_{k=2}^{n-2}$; the potential between p_1, and p_{n-1}; and finally the potential between pairs of the points $\{p_k\}_{k=2}^{n-2}$. The various parts of the energy are given as follows. The part of the energy associated to the charges at the point (x,0,1,0) and charges at the points $\{p_k\}_{k=2}^{n-2}$ is given by

$$2\sum_{k=2}^{\frac{n-1}{2}} \left(\frac{r_1 q}{r_k^2} - \frac{r_1 q}{(r_1 + (k-1)q)^2}\right) + 2\sum_{k=\frac{n+1}{2}}^{n-2} \left(\frac{r_1 q}{r_k^2} - \frac{r_1 q}{(r_1 + (n-k-1)q)^2}\right).$$

The 2 multiplying each of the sums is there because the sum in equation (2) has each pair of points appearing twice. Using the symmetry of p_k and p_{n-k} this can be simplified further to

$$(3)\qquad 4\sum_{k=2}^{\frac{n-1}{2}} \left(\frac{r_1 q}{r_k^2} - \frac{r_1 q}{(r_1 + (k-1)q)^2}\right).$$

The potential associated with adjacent points in the model does not contribute to the energy since the distance between adjacent points and the distance along the curve between them is the same, and thus the renormalization factor cancels the $\frac{1}{r^2}$ potential. Because of this there is no contribution to the potential from the points p_1 and p_{n-1} with the point (x,0,1,0).

Symmetry shows the potential between the points p_1 and p_{n-1} and the points $\{p_k\}_{k=}^{n-2}$ is twice that between the points p_1 and the points $\{p_k\}_{k=2}^{n-2}$. The potential between the points p_1 and the points $\{p_k\}_{k=2}^{n-2}$ breaks into two cases depending on whether the shortest arc length path goes through the point (x,0,1,0) or not. It doesn't for the points $\{p_k\}_{k=2}^{(n+1)/2}$ and does for the points $\{p_k\}_{k=(n+3)/2}^{n-2}$. So the potential associated with the points p_1 and p_{n-1} and the points $\{p_k\}_{k=2}^{n-2}$ is

$$(4)\quad 4\sum_{k=3}^{\frac{n+1}{2}} \left(\frac{\frac{1}{2}(r_1+q)q}{r_{1,k}^2} - \frac{\frac{1}{2}(r_1+q)q}{(k-1)^2 q^2}\right) + 4\sum_{k=\frac{n+3}{2}}^{n-2} \left(\frac{\frac{1}{2}(r_1+q)q}{r_{1,k}^2} - \frac{\frac{1}{2}(r_1+q)q}{(2r_1 + (n-1-k)q)^2}\right).$$

Here the sum starts at $k = 3$ because p_1 is adjacent to p_2 and the 4 appears because each pair of points is counted twice in sum (2) and then this is doubled to take into account the potential with the point p_{n-1}.

The potential between p_1 and p_{n-1} is

$$(5)\qquad 2\left(\frac{\frac{1}{4}(r_1+q)^2}{r_{1,n-1}^2} - \frac{\frac{1}{4}(r_1+q)^2}{(2r_1)^2}\right).$$

Again the 2 appears to take into account that the pairs of points appear twice in equation (2).

The potential between a pair of the points $\{p_k\}_{k=2}^{n-2}$ depends on whether the shortest are length path goes through the point (x,0,1,0) or not. Looking at only the points p_k, $2 \leq k \leq (n-1)/2$, the points for which the shortest path doesn't pass through (x,0,1,0) are p_j for $2 \leq j \leq \frac{(n-1)}{2} + k$. Thus for the point p_k, $2 \leq k \leq (n-1)/2$, the energy associated with it and the p_j, $j \neq 1, n-1$ or (x,0,1,0) is

$$2\sum_{j\neq k, j=2}^{\frac{n-1}{2}+k} \left(\frac{q^2}{r_{k,j}^2} - \frac{q^2}{(j-k)^2 q^2}\right) + 2\sum_{j=\frac{n+1}{2}+k}^{n-2} \left(\frac{q^2}{r_{k,j}^2} - \frac{q^2}{(2r_1 + (n+k-j-2)q)^2}\right).$$

When summing up over all the p_k, $2 \leq k \leq (n-2)$, one must be careful not to double count the points. So in the case of the first sum above, one should restrict oneself to looking at only those j with $k < j < \frac{(n-2)}{2} + k$. Also the largest value of k for which there are points in the second sum is $\frac{n-5}{2}$ (this is because all potentials with $(x,0,1,0)$ and p_{n-1} have already been taken into account). Summing the potential associated to the points p_k, $2 \leq k \leq (n-1)/2$ with the points p_k, $2 \leq k \leq (n-2)$ gets all the potential between pairs of the points p_k, $2 \leq k \leq (n-2)$, except for that associated with pairs of points in $(n+1)/2 \leq k \leq (n-2)$. Taking these into account and summarizing all of this gives

(6)
$$2\sum_{k=2}^{\frac{n-1}{2}} \sum_{j=k+1}^{\frac{n-1}{2}+k} \left(\frac{q^2}{r_{k,j}^2} - \frac{q^2}{(j-k)^2 q^2}\right) + 2\sum_{k=2}^{\frac{n-5}{2}} \sum_{j=\frac{n+1}{2}+k}^{n-2} \left(\frac{q^2}{r_{k,j}^2} - \frac{q^2}{(2r_1 + (n+k-j-2)q)^2}\right) + 2\sum_{k=\frac{n+1}{2}}^{n-2} \sum_{j=k+2}^{n-2} \left(\frac{q^2}{r_{k,j}^2} - \frac{q^2}{(j-k)^2 q^2}\right).$$

Since x is only found in r_k, the only terms for which the partial derivative of the energy with respect to x will be non-zero are

$$4\sum_{k=2}^{\frac{n-1}{2}} \left(\frac{r_1 q}{r_k^2} - \frac{r_1 q}{(r_1+(k-1)q)^2}\right) + 4\sum_{k=3}^{\frac{n+1}{2}} \left(\frac{\frac{1}{2}(r_1+q)q}{r_{1,k}^2} - \frac{\frac{1}{2}(r_1+q)q}{(k-1)^2 q^2}\right) + 4\sum_{k=\frac{n+3}{2}}^{n-2} \left(\frac{\frac{1}{2}(r_1+q)q}{r_{1,k}^2} - \frac{\frac{1}{2}(r_1+q)q}{(2r_1+(n-1-k)q)^2}\right) + 2\left(\frac{\frac{1}{4}(r_1+q)^2}{r_{1,n-1}^2} - \frac{\frac{1}{4}(r_1+q)^2}{(2r_1)^2}\right)$$
$$-2\sum_{k=2}^{\frac{n-5}{2}} \sum_{j=\frac{n+1}{2}+k}^{n-2} \left(\frac{q^2}{(2r_1+(n+k-j-2)q)^2}\right).$$

The derivative of this is

(7)
$$4\sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_1 q \cdot r_k^2 - r_1 q \cdot 2r_k dr_k}{r_k^4} - \frac{dr_1 q \cdot (r_1+(k-1)q)^2 - r_1 q \cdot 2(r_1+(k-1)q)dr_1}{(r_1+(k-1)q)^4}\right)$$
$$+2\sum_{k=2}^{\frac{n+1}{2}} \left(\frac{dr_1 q}{r_{1,k}^2} - \frac{dr_1 q}{(k-1)^2 q^2}\right)$$
$$+2\sum_{k=\frac{n+3}{2}}^{n-2} \left(\frac{dr_1 q}{r_{1,k}^2} - \frac{dr_1 q \cdot (2r_1+(n-1-k)q)^2 - (r_1+q)q \cdot 2(2r_1+(n-1-k)q)2dr_1}{(2r_1+(n-1-k)q)^4}\right)$$
$$+\frac{1}{2}\left(\frac{2(r_1+q)dr_1}{r_{1,n-1}^2} - \frac{2(r_1+q)dr_1 \cdot (2r_1)^2 - (r_1+q)^2 \cdot 2(2r_1)2dr_1}{(2r_1)^4}\right)$$
$$-2\sum_{k=2}^{\frac{n-5}{2}} \sum_{j=\frac{n+1}{2}+k}^{n-2} \frac{-2q^2 \cdot 2dr_1}{(2r_1+(n+k-j-2)q)^3}.$$

Reorganizing this gives

(8)
$$\frac{\partial E}{\partial x} = 4q \cdot \sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_1}{r_k^2} - \frac{2r_1 \cdot dr_k}{r_k^3} + \frac{2r_1 \cdot dr_1}{(r_1+(k-1)q)^3} - \frac{dr_1}{(r_1+(k-1)q)^2}\right)$$
$$+2q \cdot \sum_{k=3}^{\frac{n+1}{2}} \left(\frac{dr_1}{r_{k,1}^2} - \frac{dr_1}{(k-1)^2 q^2}\right)$$
$$+2q \cdot \sum_{k=\frac{n+3}{2}}^{n-2} \left(\frac{dr_1}{r_{k,1}^2} + \frac{4(r_1+q) \cdot dr_1}{(2r_1+(n-1-k)q)^3} - \frac{dr_1}{(2r_1+(n-1-k)q)^2}\right)$$
$$+\frac{(r_1+q)dr_1}{r_2^2} + \frac{2(r_1+q)^2 dr_1}{(2q)^3} - \frac{(r_1+q)dr_1}{(2q)^2} + 8q^2 \cdot \sum_{k=2}^{\frac{n-5}{2}} \sum_{j=\frac{n+1}{2}+k}^{n-2} \frac{dr_1}{(2r_1+(n+k-j-2)q)^3}.$$

When evaluating this derivative at $x = 1$, note that
(a) $r_1|_{x=1} = q$, and (b) $r_{j,k}|_{x=1} = r_{j-k}|_{x=1}$,
Using these to simplify (8) gives

(9)
$$\frac{\partial E}{\partial x} = 4q \cdot \sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_1}{r_k^2} - \frac{2q \cdot dr_k}{r_k^3} + \frac{2q \cdot dr_1}{(kq)^3} - \frac{dr_1}{(kq)^2}\right)$$
$$+2q \cdot \sum_{k=3}^{\frac{n+1}{2}} \left(\frac{dr_1}{r_{k-1}^2} - \frac{dr_1}{(k-1)^2 q^2}\right) + 2q \cdot \sum_{k=\frac{n+3}{2}}^{n-2} \left(\frac{dr_1}{r_{k-1}^2} + \frac{8q \cdot dr_1}{((n+1-k)q)^3} - \frac{dr_1}{((n+1-k)q)^2}\right)$$
$$+\frac{(2q)dr_1}{r_2^2} + \frac{dr_1}{(2q)} + 8q^2 \cdot \sum_{k=2}^{\frac{n-5}{2}} \sum_{j=\frac{n+1}{2}+k}^{n-2} \frac{dr_1}{((n+k-j)q)^3}.$$

The symmetry of the problem gives that
(c) $r_{n-k}|_{x=1} = rk|_{x=1}$, and (d) $dr_k|_{x=1} = dr_{n-k}|_{x=1}$.
These can be used to reindex the sums giving

(10)
$$\frac{\partial E}{\partial x} = 4q \cdot \sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_1}{r_k^2} - \frac{2q \cdot dr_k}{r_k^3} + \frac{2q \cdot dr_1}{(kq)^3} - \frac{dr_1}{(kq)^2}\right)$$
$$+2q \cdot \sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_1}{r_k^2} - \frac{dr_1}{k^2 q^2}\right) + 2q \cdot \sum_{k=3}^{\frac{n-1}{2}} \left(\frac{dr_1}{r_k^2} + \frac{8q \cdot dr_1}{(kq)^3} - \frac{dr_1}{(kq)^2}\right)$$
$$+\frac{2q \cdot dr_1}{r_2^2} + \frac{dr_1}{2q} + 8 \cdot \sum_{k=2}^{\frac{n-5}{2}} \sum_{j=k+2}^{\frac{n-1}{2}} \frac{dr_1}{k^3 q}.$$

Recognizing that $8 \cdot \sum_{k=2}^{\frac{n-5}{2}} \sum_{j=k+2}^{\frac{n-1}{2}} \frac{dr_1}{k^3 q} = 8 \cdot \sum_{k=4}^{\frac{n-1}{2}} \frac{(k-3)dr_1}{k^3 q}$ and reorganizing to combine like terms gives

(11)
$$\frac{\partial E}{\partial x} = 6q \cdot \sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_1}{r_k^2}\right) + \frac{2q \cdot dr_1}{r_2^2} + 2q \cdot \sum_{k=3}^{\frac{n-1}{2}} \left(\frac{dr_1}{r_k^2}\right) - 8q^2 \cdot \sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_k}{r_k^3}\right)$$
$$-6 \cdot \sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_1}{k^2 q}\right) + \frac{dr_1}{2q} - 2 \cdot \sum_{k=3}^{\frac{n-1}{2}} \left(\frac{dr_1}{k^2 q}\right) + 8 \cdot \sum_{k=4}^{\frac{n-1}{2}} \left(\frac{dr_1}{k^2 q}\right)$$
$$+8 \cdot \sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_1}{k^3 q}\right) + 16 \cdot \sum_{k=3}^{\frac{n-1}{2}} \left(\frac{dr_1}{k^3 q}\right) - 24 \cdot \sum_{k=4}^{\frac{n-1}{2}} \left(\frac{dr_1}{k^3 q}\right),$$

which upon starting all the terms in the sums at the same place becomes

(12)
$$\frac{\partial E}{\partial x} = 8q \cdot \sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_1}{r_k^2}\right) - 8q^2 \cdot \sum_{k=2}^{\frac{n-1}{2}} \left(\frac{dr_k}{r_k^3}\right)$$

$$-6\frac{dr_1}{4q}-8\frac{dr_1}{9q}+\frac{dr_1}{2q}+8\cdot\frac{dr_1}{8q}+24\cdot\frac{\cdot dr_1}{27q}.$$

This reduces to

(13) $$\frac{\partial E}{\partial x}=8q\cdot\sum_{k=2}^{\frac{n-1}{2}}\left(\frac{r_k\cdot dr_1-q\cdot dr_k}{r_k^3}\right).$$

Evaluating at $x = 1$ and using

$$r_k|_{x=1}=\sqrt{\left(\cos\left(\tfrac{2ak\pi}{n}\right)-1\right)^2+\sin^2\left(\tfrac{2ak\pi}{n}\right)+\left(\cos\left(\tfrac{2bk\pi}{n}\right)-1\right)^2+\sin^2\left(\tfrac{2bk\pi}{n}\right)}$$
$$=2\sqrt{\sin^2\left(\tfrac{ak\pi}{n}\right)+\sin^2\left(\tfrac{bk\pi}{n}\right)},$$
$$dr_k|_{x=1}=\frac{2\sin^2\left(\frac{ak\pi}{n}\right)}{r_k},\text{ and}$$
$$r_1|_{x=1}=q$$

(13) becomes

(14)
$$\frac{\partial E}{\partial x}=4\sum_{k=2}^{\frac{n-1}{2}}\left(\frac{\sin^2\left(\frac{bk\pi}{n}\right)\cdot\sin^2\left(\frac{a\pi}{n}\right)-\sin^2\left(\frac{b\pi}{n}\right)\cdot\sin^2\left(\frac{ka\pi}{n}\right)}{\left(\sin^2\left(\frac{ak\pi}{n}\right)+\sin^2\left(\frac{bk\pi}{n}\right)\right)^2}\right).$$

It remains to show in the limit as $n\to\infty$ that this is zero. This is done by breaking the sum (14) into two parts. If $a < b$ split the sum for $k \le n/2b$ and $k > n/2b$, if $a<b$ split the sum between $k \le n/2a$ and $k > n/2a$.

Assuming $a < b$, and since $\sin(x) > 2x/\pi$ for $0 < x < \pi/2$, we have that

(15)
$$\sum_{k=2}^{[[\frac{n}{2b}]]}\frac{\sin^2\left(\frac{bk\pi}{n}\right)\sin^2\left(\frac{a\pi}{n}\right)-\sin^2\left(\frac{ak\pi}{n}\right)\sin^2\left(\frac{b\pi}{n}\right)}{\left(\sin^2\left(\frac{ak\pi}{n}\right)+\sin^2\left(\frac{bk\pi}{n}\right)\right)^2}$$
$$<\sum_{k=2}^{[[\frac{n}{2b}]]}\frac{\sin^2\left(\frac{bk\pi}{n}\right)\sin^2\left(\frac{a\pi}{n}\right)-\sin^2\left(\frac{ak\pi}{n}\right)\sin^2\left(\frac{b\pi}{n}\right)}{\left(\left(\frac{2ak}{n}\right)^2+\left(\frac{2bk}{n}\right)^2\right)^2}$$
$$<\sum_{k=2}^{[[n/2b]]}\frac{\sin^2\left(\frac{bk\pi}{n}\right)\sin^2\left(\frac{a\pi}{n}\right)-\sin^2\left(\frac{ak\pi}{n}\right)\sin^2\left(\frac{b\pi}{n}\right)}{\left(\frac{2ak}{n}\right)^4+\left(\frac{2bk}{n}\right)^4}$$
$$=n^4\sum_{k=2}^{[[n/2b]]}\frac{\sin^2\left(\frac{bk\pi}{n}\right)\sin^2\left(\frac{a\pi}{n}\right)-\sin^2\left(\frac{ak\pi}{n}\right)\sin^2\left(\frac{b\pi}{n}\right)}{16\,(a^4+b^4)k^4}.$$

Using the Taylor expansion for $\sin(x)$ in (15) gives

(16)
$$\frac{n^4}{16(a^4+b^4)}\sum_{k=2}^{[[n/2b]]}\frac{\left(\frac{bk\pi}{n}-\frac{1}{6}\left(\frac{bk\pi}{n}\right)^3+\cdots\right)^2\left(\frac{a\pi}{n}-\frac{1}{6}\left(\frac{a\pi}{n}\right)^3+\cdots\right)^2-\left(\frac{ak\pi}{n}-\frac{1}{6}\left(\frac{ak\pi}{n}\right)^3+\cdots\right)^2\left(\frac{b\pi}{n}-\frac{1}{6}\left(\frac{b\pi}{n}\right)^3+\cdots\right)^2}{k^4}$$
$$=\frac{n^4}{16(a^4+b^4)}\sum_{k=2}^{[[n/2b]]}\frac{\left(\left(\frac{bk\pi}{n}\right)^2-\frac{1}{3}\left(\frac{bk\pi}{n}\right)^4+\cdots\right)\left(\left(\frac{a\pi}{n}\right)^2-\frac{1}{3}\left(\frac{a\pi}{n}\right)^4+\cdots\right)-\left(\left(\frac{ak\pi}{n}\right)^2-\frac{1}{3}\left(\frac{ak\pi}{n}\right)^4+\cdots\right)\left(\left(\frac{b\pi}{n}\right)^2-\frac{1}{3}\left(\frac{b\pi}{n}\right)^4+\cdots\right)}{k^4}$$
$$=\frac{n^4}{16(a^4+b^4)}\sum_{k=2}^{[[n/2b]]}\frac{(a^2k^2b^4+a^4k^4b^2-b^2k^2a^4-b^4k^4a^2)\frac{\pi^6}{3n^6}+\text{higher order in }1/n}{k^4}$$
$$=\frac{\pi^6}{48\,(a^4+b^4)\,n^2}\sum_{k=2}^{[[n/2b]]}\frac{(a^2k^2b^4+a^4k^4b^2-b^2k^2a^4-b^4k^4a^2)+\cdots}{k^4}.$$

Since the terms inside the sum are all bounded, the sum itself is on the order of $n/2b$, so (16) is of order $1/n$, thus its limit is zero, that is

(17)
$$\lim_{n\to\infty}\sum_{k=2}^{[[n/2b]]}\frac{\sin^2\left(\frac{bk\pi}{n}\right)\sin^2\left(\frac{a\pi}{n}\right)-\sin^2\left(\frac{ak\pi}{n}\right)\sin^2\left(\frac{b\pi}{n}\right)}{\left(\sin^2\left(\frac{ak\pi}{n}\right)+\sin^2\left(\frac{bk\pi}{n}\right)\right)^2}=0.$$

For $k > n/2b$, since a and b are relatively prime, the denominator for

(18)
$$\sum_{k=[[n/2b]]+1}^{(n-1)/2}\frac{\sin^2\left(\frac{bk\pi}{n}\right)\sin^2\left(\frac{a\pi}{n}\right)-\sin^2\left(\frac{ak\pi}{n}\right)\sin^2\left(\frac{b\pi}{n}\right)}{\left(\sin^2\left(\frac{ak\pi}{n}\right)+\sin^2\left(\frac{bk\pi}{n}\right)\right)^2}$$

is bounded away from zero, so

(19)
$$\lim_{n\to\infty}\sum_{k=[[n/2b]]+1}^{\frac{(n-1)}{2}}\frac{\sin^2\left(\frac{bk\pi}{n}\right)\sin^2\left(\frac{a\pi}{n}\right)}{\left(\sin^2\left(\frac{2ak\pi}{n}\right)+\sin^2\left(\frac{bk\pi}{n}\right)\right)^2}\le\lim_{n\to\infty}\sum_{k=[[n/2b]]+1}^{\frac{(n-1)}{2}}\frac{\sin^2\left(\frac{bk\pi}{n}\right)\sin^2\left(\frac{a\pi}{n}\right)}{C}\le\lim_{n\to\infty}\frac{1}{C}\sum_{k=[[n/2b]]+1}^{\frac{(n-1)}{2}}$$
$$\sin^2\left(\frac{a\pi}{n}\right)\le\lim_{n\to\infty}\sum_{k=[[n/2b]]+1}^{\frac{(n-1)}{2}}\left(\frac{a\pi}{n}\right)^2\le\lim_{n\to\infty}\left(\frac{a\pi}{n}\right)^2\left(\frac{n-1}{2}\right)=0.$$

Similarly

(20)
$$\lim_{n\to\infty}\sum_{k=[[n/2b]]+1}^{\frac{(n-1)}{2}}\frac{\sin^2\left(\frac{ak\pi}{n}\right)\sin^2\left(\frac{b\pi}{n}\right)}{\left(\sin^2\left(\frac{ak\pi}{n}\right)+\sin^2\left(\frac{bk\pi}{n}\right)\right)^2}=0,$$

so (19) and (20) combine to give

(21)
$$\lim_{n\to\infty}\sum_{k=[[n/2b]]+1}^{\frac{(n-1)}{2}}\frac{\sin^2\left(\frac{bk\pi}{n}\right)\sin^2\left(\frac{a\pi}{n}\right)-\sin^2\left(\frac{ak\pi}{n}\right)\sin^2\left(\frac{b\pi}{n}\right)}{\left(\sin^2\left(\frac{ak\pi}{n}\right)+\sin^2\left(\frac{bk\pi}{n}\right)\right)^2}=0.$$

(17) and (21) combine to give

(22)
$$\lim_{n\to\infty}\frac{\partial E}{\partial x}=\lim_{n\to\infty}\sum_{k=2}^{\frac{(n-1)}{2}}\frac{\sin^2\left(\frac{bk\pi}{n}\right)\sin^2\left(\frac{a\pi}{n}\right)-\sin^2\left(\frac{ak\pi}{n}\right)\sin^2\left(\frac{b\pi}{n}\right)}{\left(\sin^2\left(\frac{ak\pi}{n}\right)+\sin^2\left(\frac{bk\pi}{n}\right)\right)^2}=0.$$

(22) together with the symmetry of our discrete

model shows, in the limit as the number of points in the model goes to infinity, that all the partials of the energy for this configuration are zero. Since the energy functional is continuous at $\gamma(t)$, the first variation of the energy for the curve $\gamma(t) = (\cos(2at\pi), \sin(2at\pi), \cos(2bt\pi), \sin(2bt\pi))$ is zero. Thus this curve is a critical point of the energy. Projecting this curve to R^3 gives the curve

$$\sigma(t) = \left(\frac{\cos(2a\pi t)}{\sqrt{2}\text{-}\sin(2b\pi t)}, \frac{\sin(2a\pi t)}{\sqrt{2}\text{-}\sin(2b\pi t)}, \frac{\cos(2b\pi t)}{\sqrt{2}\text{-}\sin(2b\pi t)} \right)$$

Using the conformal property of the energy functional (1) shows that the curve $\sigma(t)$ is a critical point of the energy.

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