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The Szego Kernel of Certain Polynomial Models, and Heat Kernel Estimates for Schrodinger Operators with Reverse Holder Potentials

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Abstract

We present two different results on operator kernels, each in the context of its relationship to a class of CR manifolds

$$M = \left\{(z, w_1, \ldots, w_n) \in \mathbb{C}^{n+1} \;:\; \operatorname{Im} w_i = \phi_i(\operatorname{Re} z)\right\}$$

where $n \leq 2$ and $\phi_i(x)$ is subharmonic for $i = 1, \ldots, n$. Such models have proven useful for studying canonical operators such as the Szegő projection on weakly pseudoconvex domains of finite type in $\mathbb{C}^2$, and may play a similar role in work on higher codimension CR manifolds in $\mathbb{C}^3$.

Our study in Part II concerns the Szegő kernel on M for which the ϕ_i are subharmonic nonharmonic polynomials. We wish to develop, for $n = 2$, an approach based on [36], Nagel's estimation of the Szegő kernel through an explicit integral formula when $n = 1$. After a careful review of his methods and the related control geometry, we write out the analogous integral formula in codimension two. For the "degenerate" case of $M \subset \mathbb{C}^3$ with $\phi_2(x) = a\,\phi_1(x)$ for $a \in \mathbb{R}$, we prove a simple relationship between the Szegő kernel on M and the kernel on the codimension one CR manifold defined by $\phi_1(x)$.

Part III of the dissertation considers only $n = 1$. Identifying M with $\mathbb{C} \times \mathbb{R}$ with coordinates (x, y, t) and taking a partial Fourier transform in the y and t directions, $\bar{\partial}_b$ on $L^2(M)$ is transformed to a two parameter family of differential operators $\bar{D}_{\eta\tau} = \partial_x - \eta + \phi_1'\tau$ on $L^2(\mathbb{R})$. For $\tau > 0$ we study $\bar{D}_{\eta\tau} D_{\eta\tau}$ and $D_{\eta\tau}\bar{D}_{\eta\tau}$ as real Schrödinger operators on $L^2(\mathbb{R})$. Using Auscher and Ben Ali's work [4] on reverse Hölder potentials, we obtain new upper bounds on the heat kernels associated to these operators for a large class of $\phi_1(x)$. In fact, our estimates apply to the heat kernel of any Schrödinger operator on $L^2(\mathbb{R}^n)$ whose potential satisfies a reverse Hölder inequality. For Schrödinger operators with potentials in the supremum reverse Hölder class, we also prove heat kernel lower bounds derived from van den Berg's estimates [55] on the Dirichlet Laplacian.

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Dedication

To Jian.

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Part I

Preliminaries

Chapter 1

Background on CR Manifolds

In Chapter 2 we will survey the contents of the dissertation. Here we prepare by giving some background on CR manifolds, in particular those arising as smooth submanifolds of $\mathbb{C}^n$. Such $M \subset \mathbb{C}^n$ are characterized by the complex subbundle of $\mathbb{C}T(M)$ which is invariant under the ambient complex structure. Study of the involutive properties of this subbundle then leads naturally to the key notion of pseudoconvexity. We conclude the chapter by defining the tangential Cauchy-Riemann complex $\bar{\partial}_b$ on a CR manifold; the action of $\bar{\partial}_b$ on functions motivates everything that follows in Parts II and III.

1.1 Embedded CR manifolds

Our discussion throughout this chapter primarily follows the notes in Peloso [44], with Boggess [10] as a reference. First, the definition of a CR manifold does not require an ambient complex structure.

Definition 1.1. *Let M be a smooth manifold of real dimension $2n + k$ where $n, k \geq 1$. We say that M is a CR manifold of CR dimension n and CR codimension k if there exists a complex subbundle $\mathcal{L}$ of the complexified tangent bundle $\mathbb{C}T(M)$ (that is, $T(M) \otimes_{\mathbb{R}} \mathbb{C}$) such that three conditions hold:*

1. *$\text{rank}_{\mathbb{C}}\, \mathcal{L} = n$.*

2. *$\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$.*

3. *$\mathcal{L}$ is involutive (that is, closed under the Lie bracket of its sections.)*

The complex subbundle $\mathcal{L}$ is called the CR structure of M; and $(M, \mathcal{L})$ is called a CR manifold of type (n, k). A manifold of type $(n, 1)$ is also said to be of hypersurface type.

The theory may then be based on just the intrinsic structure $\mathcal{L}$ and a Hermitian metric assumed to exist on $\mathbb{C}T(M)$. We will, however, focus on CR manifolds which arise more concretely in an ambient space $\mathbb{C}^n$. Let us recall the the complex structure in this space.

Definition 1.2. *Let $\mathbb{C}^n$ be identified with $\mathbb{R}^{2n}$ via the map $(z_1, \ldots, z_n) \mapsto (x_1, y_1, \ldots, x_n, y_n)$. Then for any point $p \in \mathbb{C}^n$ the tangent space $T_p(\mathbb{C}^n)$ is spanned by*

$$\left(\frac{\partial}{\partial x_1}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \cdots \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_n}\right)_p$$

Define an $\mathbb{R}$-linear map J from $T_p(\mathbb{C}^n)$ onto itself by

$$J\left(\frac{\partial}{\partial x_j}\right)_p = \left(\frac{\partial}{\partial y_j}\right)_p, J\left(\frac{\partial}{\partial y_j}\right)_p = -\left(\frac{\partial}{\partial x_j}\right)_p$$

for all $j = 1, \ldots, n$. Evidently $J^2 = -1$, and J is called the complex structure on $\mathbb{C}^n$.

The complex structure J extends to the whole complexified tangent space $\mathbb{C}T_p(\mathbb{C}^n)$ by setting $J(v \otimes \alpha) = (Jv) \otimes \alpha$. Now for a real submanifold $M \subset \mathbb{C}^n$ of real dimension m, we focus on the part of $T_p(M)$ that is invariant under the complex structure on $\mathbb{C}^n$.

Definition 1.3. *Given a point $p \in M$, the complex tangent space of M at p is the vector space*

$$H_p(M) = T_p(M) \cap J(T_p(M))$$

And the totally real part of $T_p(M)$ is the quotient space

$$X_p(M) = T_p(M)/H_p(M)$$

It is an important fact that the dimensions of $H_p(M)$ and $X_p(M)$ need not be constant for all points in M. But when they are, we have the following definition.

Definition 1.4. *If $M \subset \mathbb{C}^n$ is a smooth real submanifold and* $\dim_{\mathbb{R}} H_p(M)$ *is independent of p, then M is called an embedded CR manifold. If the real codimension of M coincides with the CR codimension, M is called generic.*

To justify why such M are CR manifolds in the sense of Definition 1.1, consider that at each $p \in M$, the complexified complex tangent space $\mathbb{C}H_p$ is the orthogonal sum of the $+i$- and $-i$-eigenspaces of $J|_{\mathbb{C}H_p(M)}$. These are denoted by $H_p^{1,0}(M)$ and $H_p^{0,1}(M)$, respectively; and because $\dim_{\mathbb{R}} H_p(M)$ is constant, we may define associated complex subbundles of $\mathbb{C}T(M)$.

Definition 1.5. *The vector bundle $\mathbb{C}H(M) = \cup_{p\in M}\mathbb{C}H_p(M)$ is called the complex tangent bundle of M; the vector bundle $H^{1,0}(M) = \cup_{p\in M}H_p^{1,0}(M)$ is called the holomorphic tangent bundle of M; and the vector bundle $H^{0,1}(M) = \cup_{p\in M}H_P^{0,1}(M)$ is called the antiholomorphic tangent bundle of M.*

Now take $\mathcal{L}$ in Definition 1.1 to be $H^{1,0}(M)$; and assume the basic condition

$$0 < \operatorname{rank}_{\mathbb{C}} H^{1,0}(M) < \frac{1}{2}\dim_R M.$$

For condition (1) in Definition 1.1 we just let the CR dimension be $\operatorname{rank}_{\mathbb{C}} H^{1,0}(M)$. Next, since $\overline{H^{1,0}(M)} = H^{0,1}(M)$, and these are eigenspaces of a linear map relative to different eigenvalues, condition (2) is also satisfied. Finally we have $H^{1,0}(M) = T^{1,0}(\mathbb{C}^n) \cap \mathbb{C}T(M)$, the intersection of two involutive vector bundles. Hence condition (3) is also satisfied, and M is indeed a CR manifold of type $(\operatorname{rank}_{\mathbb{C}} H^{1,0}(M), \dim_{\mathbb{R}} M - 2\operatorname{rank}_{\mathbb{C}} H^{1,0}(M))$.

1.2 Description in local coordinates and the Levi form

The embedded CR manifolds in this dissertation will all be generic, and defined globally by smooth functions $\rho_1, \ldots, \rho_k$. That is,

$$M = \{z \in \mathbb{C}^n \colon \rho_1(z) = \cdots = \rho_k(z) = 0\}$$

with $d\rho_1 \wedge \cdots \wedge d\rho_m \neq 0$ everywhere. Let us note a useful characterization of the holomorphic and antiholomorphic tangent bundles in this case.

Lemma 1.6. *Let $M \subset \mathbb{C}^d$ be an embedded CR manifold of type (n, k) defined by $\rho_1, \ldots, \rho_k$. Then a global holomorphic vector field $V = \sum_{j=1}^d v_j \partial_{z_j}$ belongs to $H^{1,0}(M)$ if and only if*

$$V(\rho_l) = \sum_{j=1}^d v_j \partial_{z_j}\rho_l \equiv 0 \qquad (l = 1, \ldots, k)$$

Similarly an antiholomorphic vector field belongs to $H^{0,1}(M)$ if and only if it annihilates all the defining functions of M. It follows that the complex dimension of $H^{1,0}(M)$ and $H^{0,1}(M)$ are both $d - k$. □

Now, we have emphasized that $H^{1,0}(M)$ is involutive. But in general this is not true for $\mathbb{C}H(M) = H^{1,0}(M) \oplus H^{0,1}(M)$. (Indeed, if $\mathbb{C}H(M)$ is involutive, then M can be locally foliated by a complex manifold whose complex tangent bundle is $\mathbb{C}H(M)$, see [44].) The Levi form defined below measures "how much" the complex tangent space fails to be involutive at each point in an embedded CR manifold M.

Definition 1.7. *For each point $p \in M$ define the the projection map*

$$\pi_p \colon \mathbb{C}T_p(M) \mapsto \mathbb{C}T_p(M)/H_p(M)$$

Then the Levi form at p is the map

$$\mathbf{L}_p \colon H_p^{1,0}(M) \times H_p^{1,0}(M) \mapsto \mathbb{C}T_p(M)/H_p(M)$$

defined as

$$\mathbf{L}_p(Z_p, W_p) = \frac{i}{2}\pi_p\left([Z_p, \overline{W}_p]\right)$$

For a generic embedded CR manifold $M \subset \mathbb{C}^d$ defined by $\rho_1, \ldots, \rho_k$ with $|\nabla \rho_j| = 1$ on M, the action of the Levi form on two vectors $Z_p = \sum_{j=1}^d a_j \partial_{z_j}$ and $W_p = \sum_{j=1}^d b_j \partial_{z_j}$ is given in coordinates by

$$\mathbf{L}_p(Z_p, W_p) = \sum_{l=1}^{k} \left(\sum_{i,j=1}^{d} \frac{\partial^2 \rho_l(p)}{\partial_{z_i} \partial_{\bar{z}_j}} a_i \bar{b}_j \right) (J \nabla \rho_l(p)) \tag{1.1}$$

We will not be directly concerned with the Levi form in what follows, but need it to state a geometric notion that is central to analysis in $\mathbb{C}^n$.

Definition 1.8. *For a CR manifold M of hypersurface type, we say that M is pseudoconvex (strongly psuedoconvex) at a point $p \in M$ if the Levi form $\mathbf{L}_p$ is positive semidefinite (positive definite) at this point; that is, if*

$$\sum_{i,j=1}^{d} \frac{\partial^2 \rho(p)}{\partial_{z_i} \partial_{\bar{z}_j}} a_i \bar{a}_j \geq (>)\ 0 \qquad \text{for all} \quad \sum_{j=1}^{d} a_j \partial_{z_j} \in H_p^{1,0}(M)$$

If this condition holds uniformly in p, then M is pseudoconvex (strongly pseudoconvex.)

When pseudoconvex M is the boundary of a domain $\Omega \subset \mathbb{C}^n$, we say accordingly that Ω is pseudoconvex. Such domains are the natural domains of definition for holomorphic functions in $\mathbb{C}^n$; see Range [47].

1.3 The Cauchy-Riemann complex $\bar{\partial}_b$

Now let $M \subset \mathbb{C}^d$ as above be equipped with a Hermitian metric on $\mathbb{C}T(M)$ so that for each $p \in M$, $H_p^{1,0}(M)$ and $H_p^{0,1}(M)$ are orthogonal. Writing $N_p(M)$ for the orthogonal complement of $H_p(M)$ in $\mathbb{C}T_p(M)$, we then have

$$\mathbb{C}T(M) = H^{1,0}(M) \oplus H^{0,1}(M) \oplus N(M)$$

and the pointwise metric on $\mathbb{C}T(M)$ induces a pointwise dual metric on the space of 1-forms on M; that is, on $\mathbb{C}T^*M$. Let $\{L_1, \dots, L_n\}$ be a basis for the smooth sections of $H^{1,0}(M)$, $\{\bar{L}_1, \dots, \bar{L}_n\}$ be a basis for the smooth sections of $H^{0,1}(M)$, and finally $\{T_1, \dots, T_k\}$ be a basis for the smooth sections of $N(M)$. The tangential Cauchy-Rieman complex is then defined relative to a dual basis of 1-forms

$$\{\omega_1, \dots, \omega_n, \bar{\omega}_1, \dots, \bar{\omega}_n, \tau_1, \dots, \tau_k\}$$

where the metric on $\mathbb{C}T^*(M)$ is extended to the exterior algebra of forms in such a way that

$$\{\omega^I \wedge \tau^K \wedge \bar{\omega}^J : |I| + |K| = p, |J| = q, |K| = r\}$$

is an orthonormal basis over all increasing multi-indices I, J, K such that $p, q \leq n$ and $r \leq k$.

For a function $f \in C^\infty(M)$, which is the only case that concerns us, $\bar{\partial}_b f$ is the $(0,1)$-form on M given by

$$\langle \bar{\partial}_b f, \bar{L} \rangle = \bar{L}(f)$$

for any smooth section $\bar{L}$ of $H^{0,1}(M)$. This definition extends to a $(0,q)$-form ϕ by

$$\begin{aligned}\langle \bar{\partial}_b \phi, (\bar{L}_1, \dots, \bar{L}_{q+1}) \rangle = \frac{1}{q+1} \Bigg\{ & \sum_{j=1}^{q+1} (-1)^{j+1} \bar{L}_j \langle \phi, (\bar{L}_1, \dots, \widehat{\bar{L}_j}, \dots, \bar{L}_{q+1}) \rangle \\ & + \sum_{i<j} (-1)^{i+j} \langle \phi, ([\bar{L}_i, \bar{L}_j], \bar{L}_1, \dots, \widehat{\bar{L}_i}, \dots, \widehat{\bar{L}_j}, \dots, \bar{L}_{q+1}) \rangle \Bigg\}\end{aligned}$$

It can be proven that $\bar{\partial}_b^2 = 0$, and because $H^{0,1}(M)$ is involutive the following sequence is a complex.

$$0 \longrightarrow \Lambda^{p,0}(M) \xrightarrow{\bar{\partial}_b} \Lambda^{p,1}(M) \xrightarrow{\bar{\partial}_b} \cdots \xrightarrow{\bar{\partial}_b} \Lambda^{p,n-1}(M) \longrightarrow 0.$$

Chapter 2

Overview of the Dissertation

2.1 The Szegő kernel on polynomial models

Given a domain $\Omega \subset \mathbb{C}^n$, the Bergman projection onto the nullspace of $\bar{\partial}$ in $L^2(\Omega)$ has fundamental connections to the geometry of $\partial\Omega$ and the regularity of the $\bar{\partial}$ solution operator in Ω. In Chapter 3 we recall the analogous projection for a CR manifold $M \subset \mathbb{C}^n$, the Szegő projection onto the nullspace of $\bar{\partial}_b$ in $L^2(M)$. When M is of hypersurface type, the theory of the Szegő projection often rivals in completeness the theory of the Bergman projection. But difficulties can arise for M with codimension greater than one. For example, very little is known about the the singularities of the distribution kernel of the Szegő projection on *any* codimension two $M \subset \mathbb{C}^3$ that is not a quadratic submanifold.

We contrast this especially with the case that $M = \partial\Omega$ is the boundary of a finite type domain $\Omega \subset \mathbb{C}^2$. There is an intrinsic "control metric" on M determined by the commutation properties of the real and imaginary parts of its Levi vector field. The Szegő kernel on M is then fully understood as a singular integral kernel with respect to the family of balls in the control metric.

$$|S(u,v)| \lesssim |B(u, d(u,v))|^{-1}$$

Nagel et al. proved this in [39], making crucial use of the prior work of Nagel et al. in [43] that gave a tractable description of the control geometry on $\partial\Omega$. We review the control geometry and their work in the remainder of Chapter 3.

Nagel had discovered an early clue pointing toward the conclusions of [39] when as in [36] he computed an explicit integral formula for the Szegő kernel on finite type domain boundaries

$$M_\phi = \{(z,w) \in \mathbb{C}^2 \colon \operatorname{Im} w = \phi(\operatorname{Re} z)\}$$

where ϕ is a subharmonic nonharmonic polynomial on $\mathbb{R}$. His formula follows by identifying M_ϕ with $\mathbb{C} \times \mathbb{R}$ and using a partial Fourier transform $\mathcal{F}$ and multiplication operator H_Ψ to write the Szegő projection P_S in terms of a similar projection P onto the nullspace of $\frac{\partial}{\partial x}$ in an appropriate weighted space $L^2(\mathbb{R}, d\omega)$.

$$P_S\, g = \mathcal{F}^{-1} H_{\Psi^{-1}} P H_{\Psi} \mathcal{F}\, g$$

Estimates on the integral formula then provide direct insight into the Szegő kernel's relationship to the control geometry on M_ϕ. It is natural to ask what clues might be found in higher codimension by using an analogous integral formula for the Szegő kernel on models

$$M_{\phi,\psi} = \{(z, w_1, w_2) \in \mathbb{C}^3 \colon \operatorname{Im} w_1 = \phi(\operatorname{Re} z) \text{ and } \operatorname{Im} w_2 = \psi(\operatorname{Re} z)\} \tag{2.1}$$

with ϕ and ψ subharmonic nonharmonic polynomials. Essentially Part II of the dissertation consists of our struggle to find such clues.

The obvious first step is to understand the details of Nagel's work on codimension one models. After our review of the control geometry on $\partial\Omega$ in Chapter 3, we move in Chapter 4 to retrace Nagel's steps and estimate the Szegő kernel on M_ϕ through the integral formula

$$\int_0^\infty e^{-2\pi\tau[(\phi(x)+\phi(q))+i(v-t)]} \int_{\mathbb{R}} \frac{e^{2\pi\eta[x+q+i(y-r)]}}{\int_{\mathbb{R}} e^{4\pi[\eta\varsigma-\tau\phi(\varsigma)]}\, d\varsigma}\, d\eta\, d\tau \tag{2.2}$$

Primarily we fill in certain details that are omitted in the literature. The heart of the estimate involves scaling the coefficients of the convex polynomial $\tau\phi(\varsigma)$ so that they belong to a compact set. This characterizes the inner integral as the Fourier transform of a compact class of Schwartz functions.

Chapter 5 contains our work on models $M_{\phi,\psi}$ given by (2.1). We write out the details of extending Nagel's work to obtain an integral formula for the codimension two Szegő kernel. Compared to (2.2) we have on $M_{\phi,\psi}$ with $\deg\psi > \deg\phi$ the formula

$$\int_0^\infty e^{-2\pi\tau[\psi(x)+\psi(q)+i(v-t)]} \int_{\mathbb{R}} e^{-2\pi\sigma[\phi(x)+\phi(q)+i(u-s)]} \int_{\mathbb{R}} \frac{e^{2\pi\eta[x+qi(y-r)]}\, d\eta\, d\sigma\, d\tau}{\int_{\mathbb{R}} e^{4\pi[\eta\varsigma-(\sigma\phi(\varsigma)+\tau\psi(\varsigma))]}\, d\varsigma} \tag{2.3}$$

At first glance the formulas (2.2) and (2.3) look quite similar. Indeed, in Section 5.4 we show that for the "degenerate" case $\psi = a\phi$ for $a \in \mathbb{R}$, the codimension two kernel is just (2.2) times a δ distribution. But the non-degenerate formula (2.3) actually presents a serious new technical problem. The polynomial $\sigma\phi(\varsigma) + \tau\psi(\varsigma)$ in the exponent of the integrand of the denominator integral need not be convex on $\mathbb{R}$ when $\sigma < 0$. This prevents us from using the scaling which was so effective for estimating (2.2); and even for the kernel on M_{x^2,x^4}, it is both unclear how to proceed and hard to say what role, if any, the control geometry plays.

Let us now address our apparently inconsistent emphasis throughout Part II on the control geometry and Nagel's estimates. After all, the goal of the integral formula approach is to get insight into phenomena which are likely very *different* than their codimension one analogues. But even though we took only a very small step in this direction, it was still necessary to master the technical details missing in [36] before we could begin.

2.2 Schrödinger operators with A_∞ potentials

Schrödinger operators of the form $H = -\Delta + V$, defined for example on $L^2(\mathbb{R}^n)$, are of tremendous interest to physicists. Thousands of papers have been devoted to just the study of quantal anharmonic oscillators; that is, H with potential $V(x) = x^2 + \lambda x^{2m}$ on $L^2(\mathbb{R})$, see [32]. In Chapter 6 we give a several complex variables motivation for studying a large class of H on $L^2(\mathbb{R})$. When a manifold M_ϕ from Part II is identified with $\mathbb{C} \times \mathbb{R}$, the Kohn Laplacian on $L^2(M_\phi)$ goes over, under a partial Fourier transform, to a two-parameter family of Schrödinger operators $-\Delta + V_{\eta\tau}$ on $L^2(\mathbb{R})$. Use of the Fourier transform in this general setting is well known from the work of authors such as Christ ([14]) and Raich ([45]).

In the remainder of Part III we then focus on proving upper and lower bounds on heat kernels $p(x, y, t)$ that satisfy

$$\begin{cases} (\partial_t + H)p(\cdot, y, t) = 0 & \text{on } \mathbb{R}^n \times (0, \infty) \\ \lim_{t\to 0} p(\cdot, y, t) = \delta(\cdot - y) & \text{in } L^2(\mathbb{R}^n) \end{cases} \tag{2.4}$$

where $H = -\Delta + V$ is defined on $L^2(\mathbb{R}^n)$ with $n \geq 1$, and $V \geq 0$ belongs to an appropriate reverse Hölder class. The reverse Hölder class RH_q, $1 < q \leq \infty$, consists of potentials in $L^q_{\text{loc}}(\mathbb{R}^n)$ which for all cubes Z satisfy a uniform estimate

$$\left(\frac{1}{|Z|}\int_Z V^q \, dx\right)^{1/q} \lesssim \frac{1}{|Z|}\int_Z V \, dx$$

(If $q = \infty$, the left hand side is the essential supremum over Q.) A key technical point is that the class $A_\infty = \cup_{q>1} RH_q$ is exactly the class of Muckenhoupt potentials. It includes the singular power weights $|x|^{-\alpha}$ when $\alpha < n$, as well as potentials with non-uniform end behavior such as $V(x_1, x_2, x_3) = (x_1 x_2 x_3)^2$.

Now with just this condition on V, it is not even immediate why H is well-defined on $L^2(\mathbb{R}^n)$; much less that there exists a heat kernel satisfying (2.4). Hence in Chapter 7 we review how H is given meaning through its densely-defined quadratic form

$$Q(u,v) = \int_\Omega \nabla u \cdot \nabla v + Vuv\, dx$$

We show Q meets the Buerling-Deny conditions, and hence extends to a positive self-adjoint H on $L^2(\mathbb{R}^n)$. The Dirichlet form machinery also implies H generates a strongly continuous semigroup e^{-Ht} on $L^2(\mathbb{R}^n)$. By leveraging the theory of the classical Laplacian, we furthermore demonstrate e^{-Ht} to be a contraction from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ for all $t > 0$. The existence of the heat kernel then follows by a classical theorem in functional analysis.

This construction also yields a pointwise Gaussian bound on the heat kernel, holding for any nonnegative $V \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$p(x,y,t) \leq (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$$

So when studying upper bounds on $p(x,y,t)$, the goal is to obtain extra-Gaussian decay in terms of V, for as large a class of V as possible. Most notable is Davies' work [16] that applies to H with a continuous potential diverging at infinity. But to our knowledge there are not currently published upper bounds which hold under just the assumption that $V \in A_\infty$ for all $n \geq 1$. The strongest result in this direction seems to come from Shen's work on Schrödinger operators with reverse Hölder potentials in [50]. Namely, in [31] Kurata built on Shen's work to prove with $n \geq 2$ and $V \in RH_q$ with $q \geq n/2$, that

$$p(x,y,t) \leq \frac{c_0}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \exp\left\{-c_1(1 + m_V(x)^2 t)^{1/(2(k_0+1))}\right\} \tag{2.5}$$

were $m_V(x)$ is a size function measuring the effective growth of V near $x \in \mathbb{R}^n$.

But in Chapter 8 we reuse Kurata's argument to prove a heat kernel upper bound that does apply for all $n \geq 1$ and $V \in A_\infty$. Let us preview this argument at a high level. First, one notes for fixed $y \in \mathbb{R}^n$ that $p(\cdot, y, t)$ is a weak subsolution to the heat equation $(\partial_t - \Delta)u = 0$ in any cylinder $Q_{2r} \subset \mathbb{R}^n \times (0, \infty)$. So Moser's work in [34] gives

$$\sup_{Q_{r/2}} p \leq \left(\frac{C}{r^{n+2}} \iint_{Q_{2r/3}} p^2 \, dx \, dt \right)^{1/2} \tag{2.6}$$

Next one follows Shen's ideas from [50], incorporating V's effect into the right-hand side by iterating Caccioppoli and Fefferman-Phong inequalities over a sequence of cylinders $\{Q_{r_j}\}_{j=0}^k$ that increase from $Q_{2r/3}$ to Q_r.

Then a Caccioppoli inequality

$$\iint_{Q_{r_j}} |\nabla p|^2 + V p^2 \, dx \, dt \leq \frac{C_k}{r^2} \iint_{Q_{r_{j+1}}} p^2 \, dx \, dt \tag{2.7}$$

is easily obtained from the fact $p(\cdot, y, t)$ is a weak solution to $(\partial_t + H)u = 0$. On the other hand, to get a viable Fefferman-Phong inequality, one must effectively reverse (2.7) for balls, proving uniformly in B_{r_j} that

$$\frac{\omega_V(B_{r_j})}{r^2} \int_{B_{r_j}} p^2 \, dx \leq C \int_{B_{r_j}} |\nabla p|^2 + V p^2 \, dx \tag{2.8}$$

for some weight $\omega_V(B_{r_j})$. This is far more subtle. But if it is accomplished, one easily iterates successive applications of (2.8) and (2.7) over the sequence $\{Q_{r_j}\}_{j=0}^k$ and converts Moser's estimate (2.6) into

$$\sup_{Q_{r/2}} p \leq (C_k \cdot \omega_V(B_{r_j})^{-1})^{k/2} \left(\frac{C}{r^{n+2}} \iint_{Q_r} p^2 \, dx \, dt \right)^{1/2}$$

And as long as the growth of C_k in k is "not too fast", this—combined with the standard Gaussian bound on p—implies exponential decay of the heat kernel in $\omega_V(B_{r_j})$.

Shen proved a Fefferman-Phong inequality essentially of the form

$$\int_{B_r} |\nabla p|^2 + V \, |p|^2 \, dx \gtrsim \int_{B_r} m_V(x)^2 |p|^2 \, dx$$

which holds as long as $n \geq 2$ and $V \in RH_q$ with $q \geq n/2$. Kurata used this inequality for (2.8) in the preceding argument; evidently Shen's restrictions on n and RH_q explain his as well. We simply take advantage of more recent work by Auscher and Ben Ali in [4]. They proved for $V \in A_\infty$ with $n \geq 1$ that any $u \in H^1(\mathbb{R}^n)$ is subject on cubes Z_r to a uniform estimate

$$\frac{m_\beta(r^2 \mathrm{av}_{Z_r} V)}{r^2} \int_{Z_r} |u|^2 \, dx \lesssim \int_{Z_r} |\nabla u|^2 + V |u|^2 \, dx \tag{2.9}$$

The function $m_\beta(x)$ is x for $x \le 1$, but for $x \ge 1$ it is x^β with $0 < \beta < 1$. Using (2.9) in place of (2.8), we can then prove by the preceding argument

$$p(x, y, t) \le \frac{c_0}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \exp\left\{-c_1 m_\beta(t \operatorname{av}_{Z_{\sqrt{t}}(x)} V)^{1/2}\right\} \tag{2.10}$$

In the case that V is a power weight $|x|^\alpha$ with $\alpha > 0$, we obtain effectively the same order of decay as did Kurata when $x = y$. However, sharp estimates such as Sikora's in [51] show this is not the right order of decay.

We also prove in Chapter 9 a lower bound for the heat kernel of H with potential in the supremum reverse Hölder class, RH_∞. For a uniform $0 < \kappa < 1$ we have

$$p(x, y, t) \ge \begin{cases} \frac{c_0}{t^{n/2}} \exp\{-c_1 t \operatorname{av}_{Z_{\sqrt{t}}(x)} V\} & |x - y| < \kappa\sqrt{t} \\ \frac{c_0}{t^{n/2}} e^{-c_3 \frac{|x-y|^2}{t}} \exp\{-c_1 t (c_2^{\frac{|x-y|^2}{t}} \operatorname{av}_{Z_{t/|x-y|}(x)} V)\} & |x - y| \ge \kappa\sqrt{t} \end{cases}$$

These bounds are an application of van den Berg's estimates in [55] on the heat kernel of the Dirichlet Laplacian. The tools needed are a maximum principle for weak solutions of parabolic Schrödinger equations, the semigroup property of $p(x, y, t)$, and a lemma on doubling measures. Notice our on-diagonal lower bound is satisfactory when compared to (2.10), while the off-diagonal estimates are weaker. A possible way to improve the off-diagonal estimates would be transplanting into the context of the parabolic Schrödinger equation ideas from Aizenman and Simon's proof in [1] of their Harnack inequality for Schrödinger operators.

Part II

The Szegő Kernel on Polynomial Models

Chapter 3

The Szegő Kernel and Control Geometry in $\mathbb{C}^2$

In this chapter we introduce the Szegő projection onto the nullspace of $\bar{\partial}_b$, and begin to develop background for our study of the Szegő kernel on codimension two polynomial models in $\mathbb{C}^3$. Our first goal is to understand Nagel's estimates for the Szegő kernel on domain boundaries of the form

$$M_\phi = \{(z,w) \in \mathbb{C}^2 \colon \operatorname{Im} w = \phi(\operatorname{Re} z)\} \tag{3.1}$$

where ϕ is a subharmonic nonharmonic polynomial. In this setting the Szegő kernel is given by an integral formula and estimates may be made directly; we hope to reuse or modify some of his techniques. But Nagel's estimates are explained by the control geometry associated to a domain of finite type in $\mathbb{C}^2$. Hence we take some time to recall this geometry, and only return our focus to his integral formula in Chapter 4.

3.1 Background on the Szegő projection

Let us set the stage by mentioning the Szegő projection's $\bar{\partial}$ analogue, the Bergman projection. Fix a smoothly bounded domain $\Omega \subset \mathbb{C}^n$. Write

$$A^2(\Omega) = \{f \in L^2(\Omega) \colon \ f \text{ is holomorphic in } \Omega\}.$$

Then the Bergman projection P_B is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$, and (see [30]) is given by integration against a unique kernel $B(z,\zeta)$, namely,

$$P_B[f] = \int_\Omega B(z,\zeta) f(\zeta)\, d\zeta$$

This kernel plays a fundamental role in analysis on $\mathbb{C}^n$.

Bergman proved in [7] that a biholomorphic mapping between two domains is an isometry relative to Hermitian metrics defined in terms of the Bergman kernels of the domains. Fefferman highlighted the importance of this metric invariant in [17] when he combined it with an on-diagonal asymptotic expansion of the Bergman kernel to prove smoothness up to the boundary of biholomorphic mappings between smooth strictly pseudoconvex domains. Estimates on the

Bergman kernel are also valuable because they determine the mapping properties of the Bergman projection, which are of inherent interest. For example, a result of Straube and Boas in [9] states that regularity of the Bergman projection is equivalent to regularity of the $\bar{\partial}$-Neumann operator on bounded smooth pseudoconvex domains—and this holds at all form levels.

Since the projection onto the nullspace of $\bar{\partial}$ is of such importance, it certainly interesting by analogy to study the same for $\bar{\partial}_b$.

Definition 3.1. *Let M be a CR manifold. Define $H^2(M)$ as*

$$H^2(M) = \{f \in L^2(M) \colon\ \bar{\partial}_b f = 0 \text{ as a distribution}\}$$

Then the Szegő projection for M is the orthogonal projection from $L^2(M)$ onto $H^2(M)$.

There has been considerable progress on the Szegő projection at the first level of Fefferman's hierarchy in [19]; that is, in understanding C^∞ regularity of the projection operator. A major result in this direction was Shaw and Wang's work in [49] on regularity of the Szegő projection operator in nonisotropic Hölder and Sobolev spaces when M is compact and satisfies condition $Y(1)$. For smooth compact M of hypersurface type in $\mathbb{C}^2$, Christ has even proved pointwise bounds on the Szegő kernel in [13]; and these come with a corresponding theory of singular integrals which yields L^p boundedness of the Szegő projection operator. Again for $M \subset \mathbb{C}^n$ of hypersurface type, in [24] Harrington, Peloso, and Raich have very recently established that regularity of the complex Green operator is equivalent to regularity of the Szegő projection at all form levels; note the close analogy with Boas and Straube's classical result relating the $\bar{\partial}$-Neumann operator and Bergman projection.

Nonetheless, the final level of Fefferman's hierarchy—derivation of sharp mapping properties from a theory of singular integrals—appears in higher codimension to be far out of reach. There is no theory of singular integrals generally appropriate for the distribution kernel of the Szegő projection operator. Indeed, the Szegő kernel in higher codimension may be singular off the diagonal, which precludes a standard Calderón-Zygmund type framework (see Stein's concluding remarks in [54]). Nagel et al. did very notably surmount this difficulty in the case of quadratic surfaces of higher codimension, developing a theory of "flag kernels" for product spaces in [38].

But even for a codimension two model such as

$$M_{\phi,\psi} = \{(z, w_1, w_2) \in \mathbb{C}^3 \colon \operatorname{Im} w_1 = \phi(\operatorname{Re} z) \text{ and } \operatorname{Im} w_2 = \psi(\operatorname{Re} z)\}$$

where ϕ and ψ are subharmonic nonharmonic polynomials, there are no concrete estimates for the Szegő kernel on M. (See Halfpap's comments in [23].) Our original goal in this part of the dissertation was to repair this absence. In any case, we proceed by recalling some well-known results in codimension one; in particular, when M is a domain boundary.

3.2 The Szegő kernel on domain boundaries

Suppose that $\Omega \subset\subset \mathbb{C}^n$ has C^1 boundary, and let $M = \partial\Omega$. Denote by $A^2(\bar{\Omega})$ the algebra of functions holomorphic on Ω and continuous on $\bar{\Omega}$. Now $H^2(M)$ is the $L^2(\partial\Omega)$ closure of $A^2(\bar{\Omega})$ restricted to the boundary (see again [30]). Indeed, functions $f \in H^2(M)$ are a.e. the boundary values of their Poisson integrals $P[f]$, which allows the Szegő kernel for M to be simply characterized in terms of Hilbert space representatives of the continuous functionals

$$\Psi_z \colon f \to P[f](z), \qquad f \in H^2(M)$$

Specifically, if $k_z(\zeta)$ is the $H^2(M)$ Hilbert space representative of Ψ_z, then the Szegő kernel is given by $S(z, \zeta) = \overline{k_z(\zeta)}$. The Szegő projection on $L^2(M)$ thus acts as the integral operator

$$f \mapsto \int_{\partial\Omega} f(\zeta) S(z, \zeta)\, d\sigma(\zeta)$$

In particular, both the Bergman and the Szegő projections fix vectors from $A^2(\bar{\Omega})$ as restricted to their respective domains.

Hence we see the geometric considerations inherent to $\bar{\partial}$ and the Bergman projection (e.g. pseudoconvexity) must also be central for the Szegő projection on a domain boundary. Beginning with Kohn's insights in [28], a tremendous amount of work has been done on the precise connection between boundary geometry and the Szegő projection for domains $\Omega \subset\subset \mathbb{C}^2$. Of special significance for us is the work of Nagel et al. [39], wherein they gave a complete account of the Szegö and Bergman kernels on smoothly bounded pseudoconvex domains $\Omega \subset \mathbb{C}^2$ that satisfy a finite type hypothesis. Let us recall the meaning of a (global) finite type hypothesis.

Definition 3.2. *Suppose* Ω *is as above with a global tangential antiholomorphic vector field* $\bar{L} = X_1 + iX_2$. *Fix a smooth vector field* T *on* $\partial\Omega$ *so that* $\{X_1, X_2, T\}$ *everywhere spans the real tangent space to* $\partial\Omega$*; define for each* k*-tuple* $(i_1, \ldots, i_k)$ *with* $i_j \in \{1, 2\}$ *a corresponding smooth function* $\lambda_{i_1,\ldots,i_k}$ *on* $\partial\Omega$ *by requiring* $\lambda_{i_1,\ldots,i_k}$ *be the* T*-component of the commutator* $[X_{i_k}, [\ldots[X_{i_2}, X_{i_1}]\ldots]]$ *in the* $\{X_1, X_2, T\}$ *basis. For* $l \geq 2$*, let* Λ_l *on* $\partial\Omega$ *be the sum*

$$\Lambda_l(p) = \left[\sum_{(i_1,\ldots,i_k)} |\lambda_{i_1,\ldots,i_k}(p)|^2\right]^{1/2}, \qquad 2 \leq k \leq l$$

Then Ω *is of finite commutator type* m *if for every point* $p \in \partial\Omega$ *there is* $n \leq m$ *such that*

$$\Lambda_2(p) = \cdots = \Lambda_{n-1}(p) = 0 \qquad \text{but} \qquad \Lambda_n(p) \neq 0$$

That is, Ω is finite type if it has a tangential antiholomorphic vector field whose real and imaginary parts X_1 and X_2, along with a finite number of their iterated commutators, span the real tangent space at every point of the boundary. (This vector field need not be globally defined for Nagel et al.'s work, but will always be for the domains that concern us.) For finite type Ω one also defines, using the notation of Definition 3.2, the "higher Levi-invariant" Λ on the boundary by

$$\Lambda(p, \delta) = \sum_{j=2}^{m} \Lambda_j(p)\delta^j$$

To state precisely the significance of this geometry for the Szegő kernel, we must know how the vector fields $\{X_1, X_2\}$ induce a *control metric* on $\partial\Omega$. In [42] Nagel and Stein summarized the idea elegantly: "The distance in the control metric between two points is the infimum of the times required to flow from one point to the other along absolutely continuous curves whose tangent, almost everywhere, is a bounded linear combination of the vector fields X_1 and X_2." We will spell this out exactly in the next section.

But for now, Nagel et al.'s result on the Szegő kernel was that for finite type pseudoconvex domains in $\mathbb{C}^2$, the Szegő kernel on $\partial\Omega \times \partial\Omega$ is a singular integral kernel with respect to the balls given by the control metric. That is,

$$S(p, q) \lesssim |B(p, d(p, q))|^{-1} \qquad p, q \in \partial\Omega \tag{3.2}$$

where $d(p, q)$ is the control distance between p and q, and the indicated ball is in the control geometry. Recall at the beginning of this chapter we said Nagel provided an elementary path to study the Szegő kernel on domain boundaries of the form

$$M_\phi = \{(z, w) \in \mathbb{C}^2 \colon \operatorname{Im} w = \phi(\operatorname{Re} z)\}$$

These are clearly boundaries of finite type domains, so in fact (3.2) applies. This motivates us to understand the control geometry in some detail. We proceed with this task, taking the Heisenberg group in $\mathbb{C}^2$ as a concrete example.

3.3 Control metrics and the case of the Heisenberg group

One might wonder if the control metric as summarized above is even finite between every two points of $\partial\Omega$, since X_1 and X_2 cannot span the tangent space. But as W. Chow first proved in [12], one does obtain a finite metric from the precise construction. Even more interesting is how the metric reflects the commutation properties of X_1 and X_2. Essentially, the greater the length of the iterated commutator required to reach a direction in the tangent space, the greater the "cost" to move in that direction. Nagel, Stein, and Wainger made this explicit in an equivalent formulation of the control metric on the boundary of a domain of finite type m. (We will use this formulation as our definition of the control metric; see [43] for more historical context and the proof of equivalence.)

Theorem 3.3 (Nagel, Stein, Wainger)**.** *Using the notation of Definition 3.2, let $\{Y_1, \ldots, Y_q\}$ be some enumeration of the vector fields X_1, X_2, and all their iterated commutators of length less than or equal m. Define the "degree" of each vector field Y_j by*

$$d(Y_j) = \text{ length of the iterated commutator that forms } Y_j$$

Now let the distance between $p, q \in \partial\Omega$ be the infimum of $\delta > 0$ such that there is an absolutely continuous map $\gamma : [0, 1] \mapsto \partial\Omega$ with $\gamma(0) = p$, $\gamma(1) = q$ so that for almost all $t \in (0, 1)$

$$\gamma'(t) = \sum_{j=1}^{q} a_j(t) Y_j(\gamma(t)), \qquad |a_j(t)| < \delta^{d(Y_j)}$$

This distance function defines a metric equivalent to the control metric on $\partial\Omega$. □

Let us make two comments. First, taking $\delta < 1$, Theorem 3.3 shows how the metric "controls costs" by increasingly penalizing higher-order commutators. Second, the restriction to a three-dimensional manifold $\partial\Omega$ is obviously artificial; and Nagel et al.'s work in [43] on metrics constructed from vector fields actually holds in great generality. But apart from an easy extension for codimension two models in Chapter 5, Theorem 3.3 is all we will need, so we have chosen simplicity over generality. To make things even more concrete, it is valuable to see how this geometry plays out on the boundary of Siegel upper half space in $\mathbb{C}^2$, namely the Heisenberg group $\mathbb{H}_1$, where

$$\mathbb{H}_1 = \{(z,w) \in \mathbb{C}^2 \ : \ \operatorname{Im} w = |z|^2\}$$

Our treatment of the control geometry on the Heisenberg group interleaves abridged versions of Nagel's exposition from both [36] and [37].

The control metric on $\mathbb{H}_1$

First we consider the antiholomorphic vector field $\bar{L}$ on $\mathbb{C}^2$ given by

$$\bar{L} = \frac{\partial}{\partial \bar{z}} - 2iz\frac{\partial}{\partial \bar{w}}$$

This is clearly tangential to the upper half space defined by $\rho(z,w) = |z|^2 - \operatorname{Im} w$. Restricting $\bar{L}$ to $\mathbb{H}_1$ and using coordinates $z = x + iy$ and $w = t + iv$, we have

$$\begin{aligned} 2\bar{L} &= 2\left(\frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial t}\right) \\ &= \left(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\right) + i\left(\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}\right) \\ &= X_1 + iX_2 \end{aligned}$$

Furthermore the commutator of the real and imaginary parts of $\bar{L}$ is

$$[X_1, X_2] = 4\frac{\partial}{\partial t} = 4T$$

And since $[T, X_1] = 0 = [T, X_2]$, all further commutators vanish.

The higher Levi-invariant at each $(x,y,t) \in \mathbb{H}_1$ is therefore

$$\Lambda\left((x,y,t),\delta\right) = \Lambda_2\left((x,y,t),\delta\right) = 4\delta^2$$

In particular, $\mathbb{H}_1$ is of finite type of degree two and has a finite control metric. We now obtain a much more explicit description of this control metric and the geometry that results when it is combined with surface measure on $\mathbb{H}_1$. Take, as in Theorem 3.3, an absolutely continuous mapping $\gamma(r)\colon [0,1] \mapsto \mathbb{H}_1$. Suppose its (x, y, t) components are $(\phi(r), \eta(r), \tau(r))$, respectively. Then wherever $\gamma'(r)$ exists, the tangent vector is given by

$$\gamma'(r) = \phi'(r)X_1|_{\gamma(r)} + \eta'(r)X_2|_{\gamma(r)} + [\tau'(r) - 2(\phi'(r)\eta(r) - \eta'(r)\phi(r))]\, T$$

Since $\mathrm{d}(T) = 2$, Theorem 3.3 says the distance between points $p, q \in \mathbb{H}_1$ is less than $\delta > 0$ if and only if they are connected by such γ where for almost every $r \in (0,1)$

$$|\phi'(r)| < \delta \quad \text{and} \quad |\eta'(r)| < \delta \quad \text{and} \quad |\tau'(s) - 2(\phi'(r)\eta(r) - \eta'(r)\phi(r))| < \delta^2$$

With this observation and a connection of points $p = (x, y, t)$ and $q = (u, v, s)$ by curves of the form $\gamma(r) = (u + r(x-u), v + r(y-v), s + r(t-s))$, we can prove the following proposition (see [37] for complete details.)

Proposition 3.4 (Nagel, Stein, Wainger)**.** *Define a pseudometric $d\left((x,y,t),(u,v,s)\right)$ on $\mathbb{H}_1$ by*

$$d\left((x,y,t),(u,v,s)\right) = \sup\left\{|x-u|, |y-v|, |t-s+2(yu-xv)|^{1/2}\right\}$$

Then d is equivalent to the control metric ρ on $\mathbb{H}_1$. □

Notice that writing $z = x + iy$ and $w = u + iv$, one has $yu - xv = \mathrm{Im}\, \bar{w}z$. So in particular the control metric requires

$$|t - s + 2\mathrm{Im}\, \bar{w}z| < \delta^2 = \Lambda(p, \delta)$$

if (x, y, t) and (u, v, s) are to be closer than δ. We will see that this relationship, in which the control metric restricts movement in a "twist" of the totally real direction through the higher Levi-invariant, generalizes to polynomial models such as (3.1).

Size of the control balls on $\mathbb{H}_1$

To make sense of the volume of control balls on $\mathbb{H}_1$, we view $\mathbb{H}_1$ as an embedded manifold in $\mathbb{C}^2$ and recall the following fact (see [36].) For any domain $\Omega \in \mathbb{C}^2$ with polynomial defining function $\rho(\zeta) : \mathbb{C}^2 \mapsto \mathbb{R}$

$$\rho(\zeta) = \sum a_{\alpha\beta}\zeta^\alpha\bar{\zeta}^\beta,$$

the polarization $R : \mathbb{C}^2 \times \mathbb{C}^2 \mapsto \mathbb{C}$ given by

$$R(\zeta, \chi) = \sum a_{\alpha\beta}\zeta^\alpha\bar{\chi}^\beta$$

defines by $R(\cdot, \chi) = 0$, for $\chi \in \partial\Omega$, a holomorphic hypersurface tangent to $\partial\Omega$ at χ. In particular,

$$\{\zeta \in \mathbb{C}^2 \; : \; R(\zeta, \chi) = 0\}$$

is the complex tangent space to $\partial\Omega$ at χ.

Now fix a point $p = (z_0, w_0)$ in $\mathbb{H}_1$. From the preceding comments

$$T_{\mathbb{C},p} = \{(z, w) \in \mathbb{C}^2 \; : \; (w - \bar{w}_0 - 2iz\bar{z}_0) = 0\}$$

is the complex tangent space to $\mathbb{H}_1$ at p. Let $q = (z, w)$ also be in $\mathbb{H}_1$. According to Proposition 3.4, if q in the control ball $B(p, \delta)$ then

$$|z - z_0| < \delta \quad \text{and} \quad |\mathrm{Re}\,(w - w_0) + 2(\mathrm{Im}\,z \cdot \mathrm{Re}\,z_0 - \mathrm{Re}\,z \cdot \mathrm{Im}\,z_0)| < \delta^2$$

Let us see what these inequalities imply for the distance from q to $T_{\mathbb{C},p}$. This distance is essentially $|w - \bar{w}_0 - 2iz\bar{z}_0|$, and satisfies

$$\begin{aligned}
|w - \bar{w}_0 - 2iz\bar{z}_0| &= |\mathrm{Re}\,(w - \bar{w}_0) + i(|z|^2 + |z_0|^2 - 2z\bar{z}_0)| \\
&= \left|\mathrm{Re}\,(w - \bar{w}_0 - 2iz_0\bar{z}) + i|z - z_0|^2\right| \\
&\approx |z - z_0|^2 + |\mathrm{Re}\,(w - w_0 - 2iz_0\bar{z})| \\
&= |z - z_0|^2 + |\mathrm{Re}\,(w - w_0) + 2(\mathrm{Im}\,z \cdot \mathrm{Re}\,z_0 - \mathrm{Re}\,z \cdot \mathrm{Im}\,z_0)| \\
&\lesssim \delta^2
\end{aligned}$$

We conclude that balls on $\mathbb{H}_1$ in the control metric are equivalent to "squashed cubes" of side length δ in the complex directions X_1 and X_2, but side length δ^2 in the orthogonal real direction T. The next proposition states that what we have just demonstrated also generalizes to the polynomial models we study in Chapter 4.

Proposition 3.5 (Nagel, Stein, Wainger). *For all $(x, y, t) \in \mathbb{H}_1$ and $\delta > 0$,*

$$|B((x,y,t),\delta)| \approx \delta \cdot \delta \cdot \delta^2 \approx \delta^2 \Lambda(p,\delta)$$

where $B((x,y,t),\delta)$ is the control ball centered at (x,y,t) of radius δ. $\square$

The Szegő kernel on $\mathbb{H}_1$

For the sake of completeness, let us check that the Szegő kernel on $\mathbb{H}_1$ has the promised relationship to the control geometry. Our global tangential antiholomorphic vector field was

$$\bar{L} = \frac{\partial}{\partial \bar{z}} - 2iz\frac{\partial}{\partial t}$$

And using our same notation, $\{L, \bar{L}, T\}$ is a basis for the smooth sections of the complexified tangent bundle $T_{\mathbb{C}}(\mathbb{H}_1)$. In particular, $\bar{L}$ is a basis of the conjugate complex subbundle of $T_{\mathbb{C}}(\mathbb{H}_1)$ and

$$\begin{aligned} H^2(\mathbb{H}_1) &= \{f \in L^2(\mathbb{H}_1) \ : \ \bar{\partial}_b f \equiv 0 \text{ as a distribution}\} \\ &= \{f \in L^2(\mathbb{H}_1) \ : \ \bar{L} f \equiv 0 \text{ as a distribution}\} \end{aligned}$$

So the Szegő projection is the orthogonal projection from $L^2(\mathbb{H}_1)$ onto this subspace.

Building on Folland and Stein's work in [21] on harmonic analysis on the Heisenberg group, Nagel and Stein provide a calculation of the Szegő kernel of $\mathbb{H}_n$ for all dimensions $n \geq 1$ in [40]. When $n = 1$ and coordinates are $z = x + iy$ and $w = u + iv$ this kernel is

$$S((z,t),(w,s)) = \frac{1}{\pi^2 \left[(t - s + 2(yu - xv)) + i|z - w|^2\right]^2}$$

Define $\delta = \rho((z,t),(w,s))$ to be the distance between (z,t) and (w,s) in the control metric. Recalling Propositions 3.4 and 3.5, it follows that the Szegő kernel is indeed a singular integral kernel with respect to the control balls on $\mathbb{H}_1$.

Proposition 3.6. *Let (z,t) and (w,s) be points in $\mathbb{H}_1$. The Szegő kernel satisfies*

$$|S((z,t),(w,s))| \lesssim \inf\{|t - s + 2Im\,\bar{w}z|^{-2}, |z - w|^{-4}\} \lesssim |B((z,t),\delta)|^{-1}$$

where $B((z,t),\delta)$ is the control ball centered at (z,t) which just includes the point (w,s). $\square$

3.4 Exponential balls

Let us summarize what we have observed about the balls in the control geometry on $\mathbb{H}_1$, in the most general terms.

1. A control ball $B((z_0, w_0), \delta) \subset \mathbb{H}_1$ has Euclidean size in the complex directions, so it contains (z, w) only if $|z - z_0| < \delta$.

2. The Euclidean size of $B((z_0, w_0), \delta)$ in the orthogonal real direction is restricted by $\Lambda((z_0, w_0), \delta)$; and for R a polarization of ρ, this size is effectively captured by $|R((z, w), (z_0, w_0))|$ given that $|z - z_0| < \delta$.

So to directly generalize from $\mathbb{H}_1$ to boundaries of finite type domains we should have

Theorem 3.7 (Nagel, Stein, Wainger). *Given a domain of finite type*

$$\Omega = \{(z, w) \in \mathbb{C}^2 \ : \ \rho(z, w) < 0\},$$

the control ball $B((z_0, w_0), \delta) \subset \partial\Omega$ is equivalent, uniformly in (z_0, w_0) and δ, to

$$B_P((z_0, w_0), \delta) = \{(z, w) \in \partial\Omega\colon |z - z_0| < \delta;\ |R((z, w), (z_0, w_0)| < \Lambda((z_0, w_0), \delta)\} \tag{3.3}$$

where R is a polarization of ρ. □

This characterization of the control geometry is very useful on polynomial models. The general proof, however, is extremely technical. We now indicate where the technical difficulty lies, as this will lead us to a final key theorem of Nagel et al., one which characterizes control balls as the images of exponential maps. The ball (3.3) only directly uses information about the commutators of X_1 and X_2 at its center, where $\Lambda((z_0, w_0), \delta)$ measures the size of their T-components. The control metric, on the other hand, determines the distance between (z_0, w_0) and some other point $(z, w) \in \partial\Omega$ by the sizes of the coefficients of commutators

$$\{X_1, X_2, \cdots, [X_{i_k}, [\ldots [X_{i_2}, X_{i_1}] \ldots]], \cdots\}$$

as they form tangents of all absolutely continuous curves connecting (z_0, w_0) and (z, w). So the control ball uses information about the commutators essentially everywhere! This asymmetry presents a serious obstacle to proving the balls given by (3.3) are equivalent to the control balls on $\partial\Omega$.

On $\mathbb{H}_1$, where $[X_1, X_2] \approx T$, the obstacle vanishes. The machinery needed to overcome the difficulties in the general case require, among other techniques, the Taylor series expansion of exponential maps in canonical coordinates, and the Baker-Campbell-Hausdorff formula. See [43] for details on the following key theorem.

Theorem 3.8 (Nagel, Stein, Wainger)**.** *Identify $\partial\Omega$ with $\mathbb{C} \times \mathbb{R}$. Define the "exponential ball" of radius $\delta > 0$ centered at $(x, y, t) \in \partial\Omega$ to be the image of the box in $\mathbb{R}^3$*

$$\left\{(\alpha_1, \alpha_2, \tau) \in \mathbb{R}^3 \ : \ |\alpha_i| < \delta \quad (i = 1, 2) \quad \textit{and} \quad |\tau| < \Lambda((x, y, t), \delta)\right\}$$

under the exponential mapping $\exp\left[\alpha_1 X_1 + \alpha_2 X_2 + \tau T\right](x, y, t) \colon \mathbb{R}^3 \mapsto \partial\Omega$. *This family of exponential balls is equivalent to the balls defined by the control metric.* □

Recall that the mapping $\exp\left[\alpha_1 X_1 + \alpha_2 X_2 + \tau T\right](x, y, t)$ in the theorem denotes the point obtained by flowing for unit time along the integral curve of $\alpha_1 X_1 + \alpha_2 X_2 + \tau T$ from an initial point (x, y, t). Also recall that this exponential mapping, viewed as a diffeomorphism from

$$\left\{(\alpha_1, \alpha_2, \tau) \in \mathbb{R}^3 \ : \ |\alpha_i| < \delta \quad (i = 1, 2) \quad \text{and} \quad |\tau| < \Lambda((x, y, t), \delta)\right\}$$

onto the exponential ball, has Jacobian the absolute value of $\det[X_1, X_2, T]$; this fact will be key for us in the next chapter as we turn to polynomial models.

Chapter 4

Nagel's Integral Formula for Polynomial Models

We are now ready to investigate, in detail, an elementary method introduced by Nagel for estimating the Szegő kernel $S\left((x,y,t),(q,r,s)\right)$ on the boundary of a domain

$$\Omega_\phi = \left\{(z,w) \in \mathbb{C}^2 \ : \ \phi(\operatorname{Re} z) - \operatorname{Im} w < 0\right\}$$

where $\phi(x)$ is a subharmonic nonharmonic polynomial of degree m. This is a pseudoconvex domain of finite type m, and as discussed in Chapter 3 the general work of Nagel et al. applies to give on $\partial\Omega_\phi \times \partial\Omega_\phi$ that

$$S\left((x,y,t),(q,r,s)\right) \lesssim |B((x,y,t),\delta)|^{-1} \tag{4.1}$$

where δ is the distance between (x,y,t) and (q,r,s) in the control metric and $B((x,y,t),\delta)$ denotes the control ball. But our purpose in this chapter is to recover the upper bound directly from an integral formula for the Szegő kernel. Complete details are not readily available in the literature and we provide them here, especially with the hope they might prove useful for work on codimension two polynomial models in Chapter 5.

4.1 Control geometry on polynomial models

As before we denote the boundary of Ω_ϕ by $M_\phi \subset \mathbb{C}^2$. This is the embedded three-dimensional CR manifold defined by

$$\rho(z,w) = \phi(\operatorname{Re} z) - \operatorname{Im} w$$

and one readily checks that

$$\bar{D} = \frac{\partial}{\partial \bar{z}} - i\phi' \frac{\partial}{\partial \bar{w}}$$

is a global tangential antiholomorphic vector field. Now identify M_ϕ with $\mathbb{C}\times\mathbb{R}$ using coordinates (x,y,t). The restriction of $\bar{D}$ to M_ϕ is, under the identification,

$$\bar{D} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\left[\frac{\partial}{\partial y} - \phi' \frac{\partial}{\partial t}\right] = \frac{1}{2}(X_1 + iX_2)$$

Thus $\bar{L} = 2\bar{D}$ is a basis for the conjugate complex subbundle of $T_{\mathbb{C}}(M_\phi)$; and the Szegő projection on M_ϕ is onto the space

$$H^2(M_\phi) = \{f \in L^2(M_\phi) \; : \; \bar{L}f = 0 \text{ as a distribution}\}$$

So we want tractable descriptions of the control geometry on M_ϕ generated by the vector fields $X_1 = \mathrm{Re}, \bar{L}$ and $X_2 = \mathrm{Im}\, \bar{L}$, namely

$$X_1 = \frac{\partial}{\partial x} \qquad \text{and} \qquad X_2 = \frac{\partial}{\partial y} - \phi'(x)\frac{\partial}{\partial t}$$

We will broadly follow Nagel in [36], describing the control metric by an equivalent pseudometric that is implied by Theorem 3.7; and then describing the volume of the control balls in terms of the equivalent family of exponential balls from Theorem 3.8.

A pseudometric equivalent to the control metric

The first fundamental object is the higher Levi-invariant $\Lambda((x, y, t), \delta)$ on M_ϕ. It is clear that all possibly non-zero commutators of X_1 and X_2 have the form

$$Y_k = [\overbrace{X_1, [X_1, \cdots, [X_1}^{k-1 \text{ times}}, X_2] \cdots]] = -\phi^{(k)}(x)\frac{\partial}{\partial t} \qquad (2 \le k \le m)$$

Hence (see Definition 3.3) we have

$$\Lambda((x, y, t), \delta) = \Lambda(x, \delta) = \sum_{k=2}^{m} |\phi^{(k)}(x)|\delta^k$$

Now we use the characterization in (3.3) of control balls in terms of a polarization of $\rho(z, w)$.

In our case $\rho(z, w)$ is a polynomial, explicitly

$$\rho(z, w) = \phi\left(\frac{1}{2}(z + \bar{z})\right) - \frac{1}{2i}(w - \bar{w})$$

Let us abuse notation slightly and denote by $\phi\left(\frac{1}{2}(\zeta + \bar{\chi})\right)$ the complex polynomial on $\mathbb{C}^2$ that restricts to our original $\phi : \mathbb{R} \mapsto \mathbb{R}$ on the diagonal $\chi = \zeta$. Then a polarization of ρ is given by

$$R((z, w), (z_0, w_0)) = w - \bar{w}_0 - 2i\,\phi\left(\frac{1}{2}(z + \bar{z}_0)\right)$$

Or, fixing $\bar{z_0}$ and considering the right-hand term as a polynomial in z, the Taylor expansion about $z = z_0$ yields

$$R((z,w),(z_0,w_0)) = w - \bar{w_0} - 2i\sum_{k=0}^{m}\frac{\phi^{(k)}(\operatorname{Re} z_0)}{2^k k!}(z-z_0)^k$$

Now take two points in M_ϕ, say $(z,w) = ((q,r),(v,\phi(q)))$ and $(z_0,w_0) = ((x,y),(t,\phi(x)))$. Theorem 3.7 says that (z,w) being of distance less than δ from (z_0,w_0) in the control metric is equivalent to the pair of inequalities

$$|z - z_0| \lesssim \delta \quad \Leftrightarrow \quad |q-x| \lesssim \delta \text{ and } |r-y| \lesssim \delta$$

and

$$\left| w - \bar{w_0} - 2i\textstyle\sum_{k=0}^{m}\frac{\phi^{(k)}(x)}{2^k k!}(z-z_0)^k \right| \lesssim \Lambda(x,\delta)$$

Of course the second inequality is also equivalent to $\Lambda(x,\delta)$ dominating the size of both the real and imaginary parts of the left-hand term. The real part of this term relates a "twist" in the totally real direction, which we denote in these coordinates by $T_{x,y}^{q,r}$ as below

$$v - t + 2\mathrm{Im}\sum_{k=1}^{m}\frac{\phi^{(k)}(x)}{2^k k!}\big(q-x+i(r-y)\big)^k = v - t + T_{x,y}^{q,r}$$

And the imaginary part of $w - \bar{w_0} - 2i\sum_{k=0}^{m}\frac{\phi^{(k)}(x)}{2^k k!}(z-z_0)^k$ is

$$\begin{aligned}
\phi(q) + \phi(x) - 2&\left(\phi(x) + \frac{1}{2}\phi'(x)(q-x) + \operatorname{Re}\sum_{k=2}^{m}\frac{\phi^{(k)}(x)}{2^k k!}\big(q-x+i(r-y)\big)^k\right) \\
&= \phi(q) - \phi(x) - \phi'(x)(q-x) - 2\mathrm{Re}\sum_{k=2}^{m}\frac{\phi^{(k)}(x)}{2^k k!}\big(q-x+i(r-y)\big)^k \\
&= \sum_{k=2}^{m}\frac{\phi^{(k)}(x)}{k!}(q-x)^k - 2\mathrm{Re}\sum_{k=2}^{m}\frac{\phi^{(k)}(x)}{2^k k!}\big(q-x+i(r-y)\big)^k
\end{aligned}$$

But if $|x-q| < \delta$ and $|y-r| < \delta$ then both summations are clearly dominated by $\Lambda(x,\delta)$.

We conclude that asserting the control distance between (x,y,t) and (q,r,v) is less than δ is equivalent to asserting

$$|q-x| \lesssim \delta \text{ and } |r-y| \lesssim \delta \text{ and } |v-t+T_{x,y}^{q,r}| \lesssim \Lambda(x,\delta)$$

where the implied constants are independent of all coordinates. For $x \in \mathbb{R}$ fixed, let us write the inverse of $\Lambda(x,\cdot)$ as $\mu(x,\cdot)$. Then the control metric is globally equivalent to the pseudometric

$$d((x,y,t),(q,r,v)) = \sup\{|x-q|, |y-r|, \mu(x,|v-t+T_{x,y}^{q,r}|)\} \tag{4.2}$$

The proof in Section 4.3 of the Szegő kernel estimate (4.1) will be broken into cases, one for each regime of this pseudometric.

Volume of the exponential balls

Writing $T = \frac{\partial}{\partial t}$, it is clear that $\{X_1, X_2, T\}$ form a global basis for the tangent space at every point of M_ϕ. Fix a point $(x, y, t) \in M_\phi$. For our tangent basis, the exponential ball from Theorem 3.8 that centered at (x, y, t), of radius $\delta > 0$, is the image of the box

$$Z_{\delta,x} = \{(\alpha_1, \alpha_2, \tau) \in \mathbb{R}^3 \colon |\alpha_i| < \delta \ (i = 1, 2);\ \text{and}\ |\tau| < \Lambda(x, \delta)\}$$

under the mapping $(\alpha_1, \alpha_2, \tau) \mapsto \exp[\alpha_1 X_1 + \alpha_2 X_2 + \tau T](x, y, t)$. But the Jacobian of this mapping is

$$|\det[X_1, X_2, T]| = \text{abs} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \phi'(x) & 1 \end{vmatrix} = 1$$

Hence the volume of the exponential ball is given by the volume of the box in $Z_{\delta,x} \subset \mathbb{R}^3$. In particular, the volume of the control ball $B((x, y, t), \delta) \subset M_\phi$ is

$$|B((x, y, t), \delta)| \approx \delta^2 \Lambda(x, \delta) \tag{4.3}$$

with constants independent of (x, y, t) and δ.

Final geometric preliminaries

Given the fundamental nature of $\Lambda(x, \delta) = \sum_{k=2}^{m} |\phi^{(k)}(x)|\delta^k$ and its inverse $\mu(x, \lambda)$, we will need two further observations on these objects. First,

$$\Lambda(x, \delta) \approx \Lambda(q, \delta) \quad \text{whenever} \quad \delta \geq |x - q| \tag{4.4}$$

with constants independent of x and q. To see this we simply write the coefficients of $\Lambda(x, \delta)$ in Taylor expansions about $x = q$.

$$\Lambda(x, \delta) = \sum_{k=2}^{m} |\phi^{(k)}(x)|\delta^k = \sum_{k=2}^{m} \left| \sum_{l=k}^{m} \frac{\phi^{(l)}(q)}{(l-k)!}(x - q)^{l-k} \right| \delta^k \leq \sum_{l=2}^{m} \sum_{k=2}^{l} |\phi^{(l)}(q)|\delta^l \lesssim \Lambda(q, \delta)$$

Second, using $\approx$ to denote uniform comparability in x and λ, we have the following characterization of the size of the inverse $\mu(x,\lambda)$:

$$\lambda = \sum_{k=2}^{m} |\phi^{(k)}(x)|\mu(x,\lambda)^k \approx \sup_{2\le k\le m} |\phi^{(k)}(x)|\,\mu(x,\lambda)^k$$

So that in fact

$$\sup_{2\le k\le m} \frac{|\phi^{(k)}(x)|}{\lambda}\,\mu(x,\lambda)^k \approx 1$$

This yields

$$\mu(x,\lambda) \approx \inf_{2\le k\le m} \lambda^{1/k}\,|\phi^{(k)}(x)|^{-1/k}$$

And similarly

$$\mu(x,\lambda)^{-1} \approx \sup_{2\le k\le m} \lambda^{-1/k}\,|\phi^{(k)}(x)|^{1/k} \approx \sum_{k=2}^{m} \left|\lambda^{-1}\phi^{(k)}(x)\right|^{1/k} \tag{4.5}$$

One last geometric detail is relevant. In [36] Nagel works with smooth versions of the higher Levi-invariant and its inverse, namely

$$\Xi(x,\delta) = \sum_{k=2}^{m} |\phi^{(k)}(x)|^2\delta^{2k}$$

and its inverse $\nu(x,\xi)$. The definitions of these objects makes it clear that

$$\Xi(x,\delta) \approx \Lambda(x,\delta)^2 \quad \text{and} \quad \nu(x,\xi) \approx \mu(x,\sqrt{\xi})$$

More details are available in [42].

4.2 Derivation of the integral formula

The point in [36] is that projection onto $H^2(M_\phi)$ is similar to projection onto the constants in an appropriate weighted space $L^2(\mathbb{R}, d\omega)$. Let us see how this works, following Nagel's exposition. Because the variable coefficients in

$$\bar{L} = \frac{\partial}{\partial x} + i\left[\frac{\partial}{\partial y} - \phi'\frac{\partial}{\partial t}\right]$$

depend only on x, we take a partial Fourier transform in the y and t variables. Given $g = g(x,y,t) \in L^2(M_\phi)$, identifying M_ϕ with $\mathbb{R}^3$, define

$$\mathcal{F}g = \hat{g}(x,\eta,\tau) = \iint_{\mathbb{R}^2} e^{-2\pi i(y\eta+t\tau)} g(x,y,t)\,dy\,dt$$

so that

$$g(x,y,t) = \mathcal{F}^{-1}\hat{g} = \iint_{\mathbb{R}^2} e^{2\pi i(y\eta + t\tau)}\hat{g}(x,\eta,\tau)\,d\eta\,d\tau$$

$\mathcal{F}$ is an isometry of $L^2(M_\phi)$, and on the transform side

$$\hat{\bar{L}} = \mathcal{F}\bar{L} = \frac{d}{dx} - 2\pi\eta + 2\pi\phi'(x)\tau$$

Using the notation

$$\Psi = \Psi(x,\eta,\tau) = e^{-2\pi(x\eta - \phi(x)\tau)} \quad \text{and} \quad M_\Psi[f] = \Psi \cdot f$$

We can then express $\bar{L}$ as an interleaving of $\frac{d}{dx}$ by multiplication operators and Fourier transforms.

$$\bar{L}g = \mathcal{F}^{-1}M_{\Psi^{-1}}\frac{\partial}{\partial x}M_\Psi\mathcal{F}g$$

The key point is that

$$M_\Psi \colon L^2(\mathbb{R}^3, dx\,d\eta\,d\tau) \mapsto L^2(\mathbb{R}^3, e^{4\pi(x\eta - \phi(x)\tau)}dx\,d\eta\,d\tau)$$

and

$$M_{\Psi^{-1}} \colon L^2(\mathbb{R}^3, e^{4\pi(x\eta - \phi(x)\tau)}dx\,d\eta\,d\tau) \mapsto L^2(\mathbb{R}^3, dx\,d\eta\,d\tau)$$

are isometries. Thus $\bar{L}$ on $L^2(M_\phi)$ is similar to $\frac{\partial}{\partial x}$ acting on functions which satisfy

$$\iiint_{\mathbb{R}^3} |g(x,\eta,\tau)|^2 e^{4\pi(x\eta - \phi(x)\tau)}\,dx\,d\eta\,d\tau < +\infty$$

Now the kernel of $\frac{\partial}{\partial x}$ consists of functions $g(\eta,\tau)$ such that

$$\iint_{\mathbb{R}^2} |g(\eta,\tau)|^2 \left[\int_{\mathbb{R}} e^{4\pi(x\eta - \phi(x)\tau)}\,dx\right] d\eta\,d\tau < +\infty$$

Certainly the support of such functions must be in the set

$$\Sigma = \left\{(\eta,\tau) \in \mathbb{R}^2 \,:\, C_{\eta,\tau} = \int_{\mathbb{R}} e^{4\pi(x\eta - \phi(x)\tau)}\,dx < +\infty\right\}$$

Hence the projection operator P onto the kernel of $\frac{\partial}{\partial x}$ in $L^2(\mathbb{R}^3, e^{4\pi(x\eta - \phi(x)\tau)}dx\,d\eta\,d\tau)$ must vanish for $(\eta,\tau) \notin \Sigma$. On the other hand, if $(\eta,\tau) \in \Sigma$, then P acts by projecting a function $g(x,\eta,\tau) =$

$g_{\eta,\tau}(x)$ onto the space of constants in $L^2(\mathbb{R}, e^{4\pi(x\eta-\phi(x)\tau)}\,dx)$. Let us denote this projection by $P_{\eta,\tau}$; it is given by the integral operator

$$\begin{aligned} P_{\eta,\tau}g(x) &= \frac{\langle g, 1\rangle}{\langle 1, 1\rangle} \\ &= \int_{\mathbb{R}} g(q)\, C_{\eta,\tau}^{-1}\, e^{4\pi(q\eta-\phi(q)\tau)}\, dq \end{aligned}$$

So set

$$K_{\eta,\tau}(x,q) = \begin{cases} C_{\eta,\tau}^{-1}\, e^{4\pi(q\eta-\phi(q)\tau)} & \text{if } (\eta,\tau) \in \Sigma \\ 0 & \text{otherwise} \end{cases}$$

Using this kernel, we may extend the above integral formula for $P_{\eta,\tau}$ to all (η,τ)

$$P_{\eta,\tau}g(x) = \int_{\mathbb{R}} g(q) K_{\eta,\tau}(x,q)\, dq$$

And writing the Szegő projection operator P_S on $L^2(\mathcal{M}_\phi)$ as

$$P_S\, g = \mathcal{F}^{-1} M_{\Psi^{-1}} P M_{\Psi} \mathcal{F}\, g$$

It then follows

$$P_S f(x,y,t) = \iiint_{\mathbb{R}^3} f(q,r,v) S\big((x,y,t);(q,r,v)\big)\, dq\, dr\, dv$$

where the kernel obtained from a careful combination of like terms is

$$\begin{aligned} S((x,y,t),(q,r,v)) &= \iint_{\mathbb{R}^2} e^{2\pi i[(y-r)\eta+(t-v)\tau]} e^{2\pi[(x-q)\eta-(\phi(x)-\phi(q))\tau]} K_{\eta,\tau}(x,q)\, d\eta\, d\tau \\ &= \iint_{\Sigma} C_{\eta,\tau}^{-1}\, e^{-2\pi\tau[(\phi(x)+\phi(q))+i(v-t)]}\, d\eta\, d\tau \end{aligned}$$

Since $\phi(x)$ is a subharmonic nonharmonic polynomial, this can be simplified further. Indeed, $\Sigma = \{(\eta,\tau) \in \mathbb{R}^2 \,:\, \tau > 0\}$ and we conclude

$$S((x,y,t),(q,r,v)) = \int_0^\infty e^{-2\pi\tau[(\phi(x)+\phi(q))+i(v-t)]} \int_{\mathbb{R}} \frac{e^{2\pi\eta i(y-r)}}{\int_{\mathbb{R}} e^{4\pi[\eta(\varsigma-\frac{x+q}{2})-\tau\phi(\varsigma)]}\, d\varsigma}\, d\eta\, d\tau \tag{4.6}$$

where we have expanded $C_{\eta,\tau}^{-1}$ as the denominator integral, and moved the $e^{2\pi\eta(x+q)}$ factor from numerator to denominator. Now we will prove estimate (4.1); that is,

$$|S((x,y,t),(q,r,v))| \lesssim \delta^{-2}\Lambda(x,\delta)^{-1}$$

where $\delta = d((x,y,t),(q,r,v))$ is the distance between (x,y,t) and (q,r,v) in the pseudometric equivalent to the control metric given in (3.3). Several technical lemmas will be used; their proofs appear in Section 4.4.

4.3 Szegő kernel estimates from Nagel's formula

The length of the formula for $S((x,y,t),(q,r,v))$ makes it useful to abbreviate some of the arguments. We will write $a_0 = x_q, a_1 = \phi(x)+\phi(q), b_0 = y-r$, and $b_1 = v-t$. So the kernel is

$$\int_0^\infty e^{-2\pi\tau[a_1+ib_1]} \int_{\mathbb{R}} \frac{e^{2\pi i(\eta\cdot b_0)}}{\int_{\mathbb{R}} e^{4\pi[\eta(\varsigma-\frac{a_0}{2})-\tau\phi(\varsigma)]}\,d\varsigma}\,d\eta\,d\tau$$

Shifting $\varsigma \mapsto \varsigma + \frac{a_0}{2}$, this reads

$$\int_0^\infty e^{-2\pi\tau[a_1+ib_1]} \int_{\mathbb{R}} \frac{e^{2\pi i(\eta\cdot b_0)}}{\int_{\mathbb{R}} e^{4\pi[\eta\varsigma-\tau\phi(\varsigma+\frac{a_0}{2})]}\,d\varsigma}\,d\eta\,d\tau$$

Note that when we write $\phi(\varsigma+\frac{a_0}{2})$ in a Taylor expansion about $\varsigma = 0$, the integral in the denominator is

$$\int_{\mathbb{R}} e^{4\pi[\eta\varsigma-\sum_{k=0}^m \tau\frac{\phi^{(k)}(\frac{a_0}{2})}{k!}\varsigma^k]}\,d\varsigma = \int_{\mathbb{R}} e^{4\pi[-\tau\phi(\frac{a_0}{2})+(\eta-\tau\phi'(\frac{a_0}{2}))\varsigma-\sum_{k=2}^m \tau\frac{\phi^{(k)}(\frac{a_0}{2})}{k!}\varsigma^k]}\,d\varsigma$$

So shifting $\eta \mapsto \eta + \tau\phi'(\frac{a_0}{2})$ and collecting like terms, the kernel is

$$\int_0^\infty e^{-2\pi\tau[(a_1-2\phi(\frac{a_0}{2})+i(b_1-b_0\phi'(\frac{a_0}{2}))]} \int_{\mathbb{R}} \frac{e^{2\pi i(\eta\cdot b_0)}}{\int_{\mathbb{R}} e^{4\pi[\eta\varsigma-\sum_{k=2}^m \tau\frac{\phi^{(k)}(\frac{a_0}{2})}{k!}\varsigma^k]}\,d\varsigma}\,d\eta\,d\tau$$

The crucial step follows. Define a scaling $\breve{\mu} = \breve{\mu}(a_0,\tau)$ by the requirement

$$\sum_{k=2}^m \left|\tau\frac{\phi^{(k)}(\frac{a_0}{2})}{k!}\breve{\mu}^k\right|^2 = 1$$

The relationship to $\mu(x,\cdot)$ as defined in Subsection 4.1 is key; $\breve{\mu}$ here is essentially

$$\breve{\mu}(\frac{x+q}{2},\tau) \approx \nu(\frac{x+q}{2},\tau^{-2}) \approx \mu(\frac{x+q}{2},\tau^{-1})$$

in that notation. Now writing $c_k = \tau\frac{\phi^{(k)}(\frac{a_0}{2})}{k!}\breve{\mu}^k$ for $k = 2,\ldots,m$ and scaling $\varsigma \mapsto \breve{\mu}\varsigma$ and $\eta \mapsto \breve{\mu}^{-1}\eta$, we have

$$\int_0^\infty e^{-2\pi\tau[(a_1-2\phi(\frac{a_0}{2})+i(b_1-b_0\phi'(\frac{a_0}{2}))]}\breve{\mu}^{-2} \int_{\mathbb{R}} \frac{e^{2\pi i(\eta\cdot b_0\breve{\mu}^{-1})}}{\int_{\mathbb{R}} e^{4\pi[\eta\varsigma-\sum_{k=2}^m c_k\varsigma^k]}\,d\varsigma}\,d\eta\,d\tau$$

Because $\phi(x)$ is convex, so is the mapping $\varsigma \mapsto \sum_{k=2}^m c_k\varsigma^k$ for any $\vec{c} = (c_2,\ldots,c_m)$ obtained in this way. Let us write

$$\mathcal{C} = \left\{ (c_2,\ldots,c_m) \in \mathbb{R}^{m-1} \colon \sum_{k=2}^m |c_k|^2 = 1 \text{ and } \varsigma \mapsto \sum_{k=2}^m c_k\varsigma^k \text{ is convex} \right\}$$

And for $\vec{c} \in \mathcal{C}$, define

$$\theta_{\vec{c}}(\eta) = \left\{ \int_{\mathbb{R}} e^{4\pi[\eta\varsigma - \sum_{k=2}^{m} c_k \varsigma^k]} \, d\varsigma \right\}^{-1}$$

The set of all such functions—call it $\mathcal{S}_{\mathcal{C}}$—is a compact class of Schwartz functions, which we prove in Lemma 4.2 by establishing the Schwartz seminorms $\sup_{y\in\mathbb{R}} |y|^n |\hat{\theta}_{\vec{c}}^{(j)}(y)|$ change continuously with the choice of $\vec{c}$.

All told, the preceding transformations have led us from (4.6) to

$$S((x,y,t),(q,r,v)) = \int_0^\infty e^{-2\pi\tau[(a_1 - 2\phi(\frac{a_0}{2})) + i(b_1 - b_0\phi'(\frac{a_0}{2}))]} \breve{\mu}^{-2} \hat{\theta}_{\vec{c}}\big(b_0 \breve{\mu}^{-1}\big) \, d\tau \tag{4.7}$$

We now prove the estimate (4.1) in each of the three regimes of the pseudometric in (4.2).

The $|x-q|$ regime

First suppose that $d((x,y,t),(q,r,v)) = |x-q|$. By compactness of $\mathcal{S}_{\mathcal{C}}$, the expression $\hat{\theta}_{\vec{c}}\big(b_0\breve{\mu}^{-1}\big)$ in (4.7) is bounded uniformly in $\vec{c}$. Also note that by the estimate in (4.5) and the fact $\breve{\mu}(\frac{x+q}{2}, \tau) \approx \mu(\frac{x+q}{2}, \tau^{-1})$ it follows

$$\breve{\mu}^{-1} \approx \sum_{k=2}^{m} \left| \phi^{(k)}(\frac{a_0}{2})\tau \right|^{1/k}$$

Let us now write $\Delta_\phi(x,q) = \phi(x) + \phi(q) - 2\phi(\frac{x+q}{2})$. Bringing absolute values inside the kernel formula (4.7), we obtain

$$\begin{aligned}
|S\big((x,y,t);(q,r,v)\big)| &\lesssim \sum_{k=2}^{m} |\phi^{(k)}(\frac{x+q}{2})|^{2/k} \int_0^\infty \tau^{2/k} e^{-2\pi\tau\Delta_\phi(x,q)} \, d\tau \\
&\lesssim \Delta_\phi(x,q)^{-1} \sum_{k=2}^{m} \left| \frac{\phi^{(k)}(\frac{x+q}{2})}{\Delta_\phi(x,q)} \right|^{2/k} \int_0^\infty \tau^{2/k} e^{-2\pi\tau} \, d\tau \\
&\lesssim \Delta_\phi(x,q)^{-1} \sum_{k=2}^{m} \left| \frac{\phi^{(k)}(\frac{x+q}{2})}{\Delta_\phi(x,q)} \right|^{2/k}
\end{aligned}$$

It is easy to see that the division by $\Delta_\phi(x,q)$ makes sense. For suppose without loss that $x < q$. Then if $\Delta_\phi(x,q) = 0$ we have

$$\phi(q) - \phi(\frac{x+q}{2}) = \phi(\frac{x+q}{2}) - \phi(x)$$

and in particular the slope of the secant line from $(x,\phi(x))$ to $(\frac{x+q}{2}, \phi(\frac{x+q}{2}))$ is the same as the slope of the secant line from $(\frac{x+q}{2}, \phi(\frac{x+q}{2}))$ to $(q,\phi(q))$. But the slopes of secant lines to the graph of a convex function are increasing (see e.g. [48]). So ϕ is linear on $(\frac{x+q}{2}, q)$—but this contradicts our hypothesis that ϕ is a nonharmonic polynomial.

In fact the convexity of ϕ implies much more, namely that

$$\Delta_\phi(x,q)) \sim \Lambda(\frac{x+q}{2}, |x-q|)$$

This is proved in Lemma 4.1 below. We then need only observe that

$$\left|\frac{\phi^{(k)}(\frac{x+q}{2})}{\Lambda(\frac{x+q}{2}, |x-q|)}\right| \lesssim |x-q|^{-k}$$

And because $\Lambda(\frac{x+q}{2}, |x-q|) \approx \Lambda(x, |x-q|)$ it follows from the preceding chain of inequalities

$$|S\big((x,y,t);(q,r,v)\big)| \lesssim |x-q|^{-2}\Lambda(x,|x-q|)^{-1}$$

The $|y-r|$ regime

Now we assume that $d((x.y,t),(q,r,v)) = |y-r|$. Write

$$S(a_0,b_0,a_1,b_1) = \frac{1}{b_0}\int_0^\infty e^{-2\pi\tau[\Delta_\phi + i(b_1 - b_0\phi'(\frac{a_0}{2}))]}\breve{\mu}^{-1}\cdot b_0\breve{\mu}^{-1}\hat{\theta}_{\vec{c}}(b_0\breve{\mu}^{-1})\,d\tau \tag{4.8}$$

For each $\vec{c}\in\mathcal{C}$ and $n\in\mathbb{N}$ we have $C_{\vec{c},n}$ such that

$$|y|^n\,|\hat{\theta}_{\vec{c}}(y)| \le C_{\vec{c},n}|y|^{-n} \qquad \text{whenever} \quad y\neq 0 \tag{4.9}$$

This is true in particular for $n=1$; as well as for n large enough that

$$\int_{\Lambda(x,|y-r|)^{-1}}^\infty \breve{\mu}^{n-1}\,d\tau$$

converges. (Since the slowest possible decay of $\breve{\mu}$ is essentially $\tau^{-1/\deg\phi}$, we may just take $n = \deg\phi + 2$.) For either $n=1$ or $n=\deg\phi+2$, compactness of $\mathcal{S}_C$ yields $C>0$ so that (4.9) holds uniformly in $\vec{c}$ with C in place of $C_{\vec{c},n}$.

So now we split (4.8) into two integrals, and bring absolute values inside to obtain with this $C>0$ and $n=\deg\phi+2$ the inequality

$$\begin{aligned}|S\big((x,y,t);(q,r,v)\big)| &\le \frac{C}{|y-r|}\left[\int_0^{\Lambda(x,|y-r|)^{-1}}\breve{\mu}^{-1}\,d\tau + \int_{\Lambda(x,|y-r|)^{-1}}^\infty \frac{\breve{\mu}^{-1}}{(|y-r|\breve{\mu}^{-1})^n}\,d\tau\right]\\ &= \frac{C}{|y-r|}(A+B)\end{aligned}$$

The key observation is that

$$\breve{\mu}(\frac{x+q}{2}, \Lambda(x, |y-r|)^{-1}) \approx \mu(\frac{x+q}{2}, \Lambda(x, |y-r|)) \approx |y-r|$$

because $|y-r| > |x-q|$ by choice of regime, and $\mu(x, \cdot)$ preserves comparability. Thus

$$|A| \lesssim |y-r|^{-1}\Lambda(x, |y-r|)^{-1}$$

since $\breve{\mu}^{-1}$ is increasing.

Furthermore, by the proof of Lemma 4.3 we have $-\breve{\mu}' > \frac{\breve{\mu}}{\tau}$. Hence the pointwise equality

$$\frac{\partial}{\partial \tau}[\tau \breve{\mu}^{n-1}] = (n-1)\tau\breve{\mu}^{n-2}\breve{\mu}' + \breve{\mu}^{n-1}$$

and the fact $n \geq 4$ imply the inequality

$$\begin{aligned}\left|\int_{\Lambda(x,|y-r|)^{-1}}^{\infty} \frac{\partial}{\partial \tau}[\tau\breve{\mu}^{n-1}]\, d\tau\right| &= (n-1)\int_{\Lambda(x,|y-r|)^{-1}}^{\infty} \tau\breve{\mu}^{n-2}(-\breve{\mu}')\, d\tau + \int_{\Lambda(x,|y-r|)^{-1}}^{\infty} \breve{\mu}^{n-1}\, d\tau \\ &\geq \int_{\Lambda(x,|y-r|)^{-1}}^{\infty} \breve{\mu}^{n-1}\, d\tau\end{aligned}$$

We conclude

$$\begin{aligned}|y-r|^n B &= \int_{\Lambda(x,|y-r|)^{-1}}^{\infty} \breve{\mu}^{n-1}\, d\tau \\ &\leq \left|\int_{\Lambda(x,|y-r|)^{-1}}^{\infty} \frac{\partial}{\partial \tau}[\tau\breve{\mu}^{n-1}]\, d\tau\right| \\ &\lesssim \Lambda(x, |y-r|)^{-1}|y-r|^{n-1}\end{aligned}$$

Combining the inequalities for A and B yield the desired estimate of the Szegő kernel.

The $\mu(x, |v-t+T^{q,r}_{x,y}|)$ regime

In the final case, we need to observe that our pseudometric in (4.2) is, for any $n \in \mathbb{N}$, equivalent to the pseudometric

$$d_n((x,y,t),(q,r,v)) = \sup\{n\,|x-q|, n\,|y-r|, \mu(x, |v-t+T^{q,r}_{x,y}|)\}$$

So it suffices to prove this section's desired estimate

$$|S((x,y,t),(q,r,v))| \lesssim \mu(x, |v-t+T^{q,r}_{x,y}|)^{-2}\,|v-t+T^{q,r}_{x,y}|^{-1}$$

under the assumption $d_n((x,y,t),(q,r,v)) = \mu(x, |v-t+T^{q,r}_{x,y}|)$ for some n that is independent of all coordinates.

The term $|t-v+T^{q,r}_{x,y}|$ does not appear explicitly in (4.7), but nonetheless

$$\begin{aligned}
|v-t+T^{q,r}_{x,y}| &= |v-t+2\mathrm{Im}\sum_{k=1}^{m}\frac{\phi^{(k)}(x)}{2^k k!}\big(q-x+i(r-y)\big)^k| \\
&\leq |v-t+\phi'(x)(r-y)| + 2\left|\mathrm{Im}\sum_{k=2}^{m}\frac{\phi^{(k)}(x)}{2^k k!}(q-x+i(r-y))^k\right|
\end{aligned}$$

And when we expand all $\phi^{(j)}$ about $\frac{x+q}{2}$ we obtain

$$\begin{aligned}
&\leq |v-t+\phi'(\frac{x+q}{2})(r-y)| + \left|\sum_{k=1}^{m-1}\frac{\phi^{(k+1)}(\frac{x+q}{2})}{2^k k!}(x-q)^k(r-y)\right| \\
&\quad + 2\left|\mathrm{Im}\sum_{k=2}^{m}\frac{\phi^{(k)}(x)}{2^k k!}(q-x+i(r-y))^k\right| \\
&\leq |v-t+\phi'(\frac{x+q}{2})(r-y)| + (N-1)\cdot\Lambda(x,\max\{|x-q|,|y-r|\})
\end{aligned}$$

for some $N \in \mathbb{N}$ independent of all coordinates. But in the $\mu(x, |v-t+T^{q,r}_{x,y}|)$ regime of the pseudometric $d_n((x,y,t),(q,r,v))$, for n sufficiently large, we have

$$N\cdot\Lambda(x,\max\{|x-q|,|y-r|\}) \leq \Lambda(x,\max\{n\cdot|x-q|, n\cdot|y-r|\}),$$

Combined with the preceding inequality this says

$$|v-t+T^{q,r}_{x,y}| \lesssim |v-t+\phi'(\frac{x+q}{2})(r-y)|$$

when $d_n((x,y,t),(q,r,v)) = \mu(x,|v-t+T^{q,r}_{x,y}|)$. We assume this holds for n as described.

Now $|v-t+\phi'(\frac{x+q}{2})(r-y)| = |b_1 - \phi'(\frac{a_0}{2})b_0|$ is exactly the size of the oscillation in the τ-integral in (4.7). For brevity, set

$$\lambda = |v-t+\phi'(\frac{x+q}{2})(r-y)|$$

and still use our abbreviation $\Delta_\phi = \Delta_\phi(x,q) = a_1 - 2\phi(\frac{a_0}{2})$. So we wish to understand the size of

$$S(a_0,b_0,a_1,b_1) = \int_0^\infty e^{-2\pi\tau[\Delta_\phi + i\lambda]}\breve{\mu}^{-2}\hat{\theta}_{\vec{c}}(b_0\breve{\mu}^{-1})\,d\tau$$

in terms of the size of its oscillation. Observe that the oscillation in the kernel $S(a_0, b_0, a_1, b_1)$ is beneficial only when $\tau > \lambda^{-1}$. So choose $\zeta(x) \in C_c^\infty(\mathbb{R})$ such that $\zeta \equiv 1$ on $[-1, 1]$, $\text{supp}(\zeta) = [-2, 2]$, and all of the derivatives of ζ are uniformly bounded. Now

$$
\begin{aligned}
S(a_0, b_0, a_1, b_1) &= \int_0^\infty e^{-2\pi\tau[\Lambda_\phi + i\lambda]} \breve{\mu}^{-2} \hat{\theta}_{\vec{c}}(b_0 \breve{\mu}^{-1})\, d\tau \\
&= \int_0^{2/\lambda} \zeta(\tau\lambda) e^{-2\pi\tau[\Delta_\phi + i\lambda]} \breve{\mu}^{-2} \hat{\theta}_{\vec{c}}(b_0 \breve{\mu}^{-1})\, d\tau \\
&\qquad + \int_{1/\lambda}^\infty (1 - \zeta(\tau\lambda)) e^{-2\pi\tau[\Delta_\phi + i\lambda]} \breve{\mu}^{-2} \hat{\theta}_{\vec{c}}(b_0 \breve{\mu}^{-1})\, d\tau \\
&= A + B
\end{aligned}
$$

For A, bringing absolute values inside the integral yields

$$
|A| \leq \int_0^{2/\lambda} \breve{\mu}^{-2}\, d\tau \leq 2\lambda^{-1} \breve{\mu}(\frac{a_0}{2}, 2\lambda^{-1})^{-2} \lesssim \lambda^{-1} \mu(\frac{a_0}{2}, \lambda)^{-2}
$$

For B, integrating by parts three times gives

$$
\frac{-1}{8\pi^3[\Delta_\phi + i\lambda]^3} \cdot \int_{1/\lambda}^\infty e^{-2\pi\tau[\Delta_\phi + i\lambda]} \frac{d^3}{d\tau^3} \big[(1 - \zeta(\tau\lambda)) \breve{\mu}^{-2} \hat{\theta}_{\vec{c}}(b_0 \breve{\mu}^{-1})\big]\, d\tau
$$

Simply by the product rule, we see the integral in B is the sum of two different pieces; one in which $(1-\zeta(\tau\lambda))$ gets no derivatives, and another in which the compact support of ζ's derivatives take effect.

The first piece is

$$
B_1 = \sum_{l=0}^{3} \int_{1/\lambda}^\infty (1 - \zeta(\tau\lambda)) e^{-2\pi\tau[\Delta_\phi + i\lambda]} \frac{d^l}{d\tau^l} \big[\breve{\mu}^{-2}\big] \frac{d^{3-l}}{d\tau^{3-l}} \big[\hat{\theta}_{\vec{c}}(b_0 \breve{\mu}^{-1})\big]\, d\tau
$$

By Lemma 4.3, for $1 \leq l \leq 3$ we have

$$
\left| \frac{d^l}{d\tau^l} \big[\breve{\mu}^{-1}\big] \right| \lesssim \frac{\breve{\mu}^{-1}}{\tau^l} \qquad \text{so that} \qquad \left| \frac{d^l}{d\tau^l} \big[\breve{\mu}^{-2}\big] \right|. \lesssim \frac{\breve{\mu}^{-2}}{\tau^l}
$$

and it is Corollary 4.4 that

$$
\left| \frac{d^{3-l}}{d\tau^{3-l}} \big[\hat{\theta}_{\vec{c}}(b_0 \breve{\mu}^{-1})\big] \right| \lesssim \tau^{l-3}
$$

Bringing absolute values inside the integral in B_1, we thus obtain

$$
\begin{aligned}
|B_1| &\lesssim \int_{1/\lambda}^\infty \tau^{-3} \breve{\mu}^{-2}\, d\tau \\
&\lesssim \left| \int_{1/\lambda}^\infty \frac{d}{d\tau} [\tau^{-2} \breve{\mu}^{-2}]\, d\tau \right| \\
&\lesssim \lambda^2 \mu(\frac{x+q}{2}, \lambda)^{-2}
\end{aligned}
$$

where the first integral on the right-hand side converges because the fastest growth of $\breve{\mu}^{-1}$ is essentially $\tau^{1/2}$; and the second inequality follows as in the argument for the $|y-r|$ regime.

The second piece of B, where the compact support of ζ's deratives comes into play, is

$$B_2 = \sum_{l=1}^{3} \int_{1/\lambda}^{2/\lambda} \lambda^l \zeta^{(l)}(\tau\lambda) e^{-2\pi\tau[(a_1 - 2\phi(\frac{a_0}{2})) + i\lambda]} \frac{d^{3-l}}{d\tau^{3-l}} \left[\breve{\mu}^{-2} \hat{\theta}_{\vec{c}}(b_0 \breve{\mu}^{-1})\right] d\tau$$

Given a typical term of B_2, bringing absolute values inside the integral and using the above arguments, it follows

$$|B_{2,l}| \lesssim \lambda^l \int_{1/\lambda}^{2/\lambda} \tau^{l-3} \breve{\mu}^{-2}\, d\tau \lesssim \lambda^l \left| \int_{1/\lambda}^{2/\lambda} \frac{d}{d\tau}[\tau^{l-2}\breve{\mu}^{-2}]\, d\tau \right| \lesssim \lambda^2 \mu(\frac{x+q}{2}, \lambda)^{-2}$$

So in conclusion

$$\begin{aligned} |B| &\leq \frac{|B_1| + |B_2|}{8\pi^3 |\Delta_\phi + i\lambda|^3} \\ &\lesssim \lambda^{-3}(|B_1| + |B_2|) \\ &\lesssim \lambda^{-1} \mu(\frac{x+q}{2}, \lambda)^{-2} \\ &\lesssim \mu(x, |v - t + T_{x,y}^{q,r}|)^{-2}\, |v - t + T_{x,y}^{q,r}|^{-1} \end{aligned}$$

where we have again used the facts that μ preserves comparability and is increasing. This completes the main proof.

4.4 Proof of Lemmas 4.1–4.3

We now establish the technical lemmas used in the preceding argument. First we have the characterization of $\Delta_\phi(x,q)$ that underlies work in the $|x-q|$ regime.

Lemma 4.1. *Let $\phi(x)$ be a subharmonic nonharmonic polynomial of degree $m \geq 2$. Then the two quantities*

$$\Delta_\phi(x,q) = \phi(x) + \phi(q) - 2\phi(\frac{x+q}{2}) \quad \text{and} \quad \Lambda(\frac{x+q}{2}, |x-q|) = \sum_{k=2}^{m} \left| \phi^{(k)}(\frac{x+q}{2}) \right| |x-q|^k$$

are equivalent; that is, there exists $C > 0$ so that

$$C^{-1}\Delta_\phi(x,q) \leq \Lambda(\frac{x+q}{2}, |x-q|) \leq C\Delta_\phi(x,q), \qquad \forall (x,q) \in \mathbb{R}^2$$

Proof. Using the coordinates $(\gamma, \kappa) = (\frac{x+q}{2}, x-q)$ and expanding ϕ about $\frac{x+q}{2}$ gives

$$\Delta_\phi(x,q) = \sum_{l=1}^{m/2} c_{2l}\, \phi^{(2l)}(\gamma)|\kappa|^{2l}$$

with c_{2l} of course independent of γ, κ for $l = 1, \ldots, m/2$. And by definition

$$\Lambda(\frac{x+q}{2}, |x-q|) = \sum_{k=2}^{m} |\phi^{(k)}(\gamma)|\, |\kappa|^k$$

From these two sums $\Delta_\phi(x,q) \lesssim \Lambda(\frac{x+q}{2}, |x-q|)$ is clear. We now consider the reverse bound.

First, for $|\gamma| > M_0$ sufficiently large, $\phi^{(2l)}(\gamma) \approx |\gamma|^{m-k}$. So uniformly in large $|\gamma|$, we have

$$\sum_{l=1}^{m/2} c_{2l}\, \phi^{(2l)}(\gamma)|\kappa|^{2l} \gtrsim \sum_{l=1}^{m/2} |\gamma|^{m-2l}\, |\kappa|^{2l}$$

and

$$\sum_{k=2}^{m} |\phi^{(k)}(\gamma)|\, |\kappa|^k \lesssim \sum_{k=2}^{m} |\gamma|^{m-k}\, |\kappa|^k$$

If k is odd, the k-th term in $\sum_{k=2}^{m} |\gamma|^{m-k}\, |\kappa|^k$ is dominated by either the $\frac{k-1}{2}$-th or $\frac{k+1}{2}$-th term of $\sum_{l=1}^{m/2} |\gamma|^{m-2l}\, |\kappa|^{2l}$, depending on the relative sizes of $|\gamma|$ and $|\kappa|$. Thus the reverse bound holds on $\{(\gamma, \kappa) \in \mathbb{R}^2 \colon |\gamma| > M_0, \kappa \in \mathbb{R}\}$.

Next, if $|\gamma| \leq M_0$, there is a uniform upper bound on the size of the derivatives of ϕ. Taking $|\kappa| > M_1$ large enough, both sums are similar to $|\kappa|^m$ because $\phi^{(m)} > 0$, and $\Delta_\phi(x,q)$ also dominates $\Lambda(\frac{x+q}{2}, |x-q|)$ on $\{(\gamma, \kappa) \in \mathbb{R}^2 \colon |\gamma| \leq M_0, |\kappa| > M_1\}$. The situation is less clear when $|\kappa|$ is small, for the low order derivatives of ϕ may vanish. So write the zeros of ϕ'' as $\{\gamma_1, \ldots, \gamma_q\}$, assuming without loss that they lie within $[-M_0, M_0]$, and enclose each zero in an arbitrarily small neighborhood N_p which will be shrunk as necessary.

Fix some zero, say γ_p. There is a smallest k, say k_p, for which $\phi^{(k_p)}(\gamma_p)$ does not vanish. Convexity of ϕ implies k_p is even and $\phi^{(k_p)}(\gamma_p) > 0$. Pick any d where $2 \leq d < k_0$. Then

$$\phi^{(d)}(\gamma) = \sum_{j=k_0}^{m} \frac{\phi^{(j)}(\gamma_p)}{(j-d)!}(\gamma - \gamma_p)^{j-d}.$$

And by shrinking N_p enough we may conclude for $\gamma \in N_p$

$$|\phi^{(d)}(\gamma)| \leq \sum_{j=k_p}^{m} \frac{|\phi^{(j)}(\gamma_p)|}{(j-d)!}|\gamma - \gamma_p|^{j-d} \lesssim |\gamma - \gamma_p|^{k_p - d}$$

where the implied constant is independent of p because there are only finitely many zeros. On the other hand, taking $|\kappa| < M_2$ sufficiently small, we also have in N_p that

$$\sum_{j=k_p}^{m} |\phi^j(\gamma)|\,|\kappa|^j \lesssim |\kappa|^{k_p}$$

It follows directly

$$\sum_{k=2}^{m} |\phi^{(k)}(\gamma)|\,|\kappa|^k \lesssim \left(|\kappa|^{k_p} + \sum_{k=2}^{k_p-1} |\gamma-\gamma_p|^{k_p-k}\,|\kappa|^k\right) \tag{4.10}$$

inside $R_p = \{(\gamma,\kappa) \in \mathbb{R}^2 \colon \gamma \in N_p, |\kappa| < M_2\}$.

Now look again at some $\phi^{(2d)}(\gamma)$ in N_p where $1 \le d < k_0/2$:

$$\phi^{(2d)}(\gamma) = \sum_{j=k_0}^{m} \frac{\phi^{(j)}(\gamma_p)}{(j-2d)!}(\gamma-\gamma_p)^{j-2d}.$$

Near γ_p it is clear that this sum is dominated by the term with $|\gamma-\gamma_p|^{j-2d}$. Hence we also have in R_p that

$$\sum_{l=1}^{m/2} c_{2l}\,\phi^{(2l)}(\gamma)|\kappa|^{2l} \gtrsim \left(|\kappa|^{k_p} + \sum_{l=1}^{(k_p-2)/2} |\gamma-\gamma_p|^{k_p-2l}\,|\kappa|^{2l}\right) \tag{4.11}$$

Comparing the summations on the right-hand sides of (4.10) and (4.11), we once again observe the k-th term in $\sum_{k=2}^{k_p-1} |\gamma-\gamma_p|^{k_p-k}\,|\kappa|^k$ is dominated by either the $\frac{k-1}{2}$-th or $\frac{k+1}{2}$-th term of $\sum_{l=1}^{(k_p-2)/2} |\gamma-\gamma_p|^{k_p-2l}\,|\kappa|^{2l}$, depending on the relative sizes of κ and $|\gamma-\gamma_p|$. Combining (4.10) and (4.11), we thus obtain the reverse bound of the lemma on the union of the R_p.

For $\gamma \in [-M_0, M_0] \setminus \{N_1 \cup \cdots \cup N_q\}$, on the other hand, $\phi''(\gamma)$ *is* positive and bounded away from zero. Shrinking M_2 as necessary, this implies both sums are similar to $|\kappa|^2$ in

$$\{(\gamma,\kappa) \in \mathbb{R}^2 \colon |\gamma| \in [-M_0, M_0] \setminus \{N_1 \cup \cdots \cup n_q\}, |\kappa| < M_2\}$$

Thus we obtain the reverse bound on all of

$$\{(\gamma,\kappa) \in \mathbb{R}^2 \colon |\gamma| \le M_0, |\kappa| < M_2\}$$

On the remaining compact set $\{(\gamma,\kappa) \in \mathbb{R}^2 \colon \gamma \le M_0,\, M_2 \le |\kappa| \le M_1\}$, both $\Delta_\phi(x,q)$ and $\Lambda(\frac{x+q}{2}, |x-q|)$ are continuous functions bounded away from zero, hence comparable. (Convexity of ϕ ensures $\Delta_\phi(x,q)$ is strictly positive when $x \ne q$; see comments in Section 4.3.) This establishes the reverse bound uniformly on $\mathbb{R}^2$. □

Compactness of $\mathcal{S}_{\mathcal{C}}$ in the Schwartz space

Next we prove that $\mathcal{S}_{\mathcal{C}} \subset\subset S(\mathbb{R})$. Note that if one naively brings absolute values inside the integral formula before scaling, this essential information is lost.

Lemma 4.2. *For even integer $m \geq 2$, define a mapping on the compact subset of $\mathbb{R}^{m-1}$*

$$\mathcal{C} = \left\{ \vec{c} = (c_2, \ldots, c_m) \in \mathbb{R}^{m-1} \colon \sum_{k=2}^{m} |c_k|^2 = 1 \text{ and } \varsigma \mapsto \sum_{k=2}^{m} c_k \varsigma^k \text{ is convex} \right\}$$

by

$$\vec{c} \mapsto \theta_{\vec{c}}(\eta) = \left\{ \int_{\mathbb{R}} e^{4\pi[\eta\varsigma - \sum_{k=2}^{m} c_k \varsigma^k]} \, d\varsigma \right\}^{-1}$$

This mapping is continuous on $\mathbb{R}^{m-1}$ when composed with any Schwartz seminorm

$$\sup_{\eta \in \mathbb{R}} |\eta|^n \, |\theta_{\vec{c}}^{(j)}(\eta)|, \qquad (n \in \mathbb{N})$$

In particular, its image $\mathcal{S}_{\mathcal{C}}$ is a compact class of Schwartz functions on $\mathbb{R}$.

Proof. For fixed $\vec{c}_0 \in \mathcal{C}$, write $p_0(x) = \sum_{k=2}^{m} c_{0,k} \, x^k$ and $q_0^\eta(x) = \eta x - p_0(x)$. Then

$$\theta_0(\eta) = \left\{ \int_{\mathbb{R}} e^{4\pi q_0^\eta(x)} \, dx \right\}^{-1} = g_0(\eta)^{-1}$$

is a function in $\mathcal{S}_{\mathcal{C}}$. We first examine the concave polynomial $q_0^\eta(x)$. Let $x_0 = x_0(\eta)$ be its global maximum; so $\eta = \sum_{k=2}^{m} c_{0,k} \, k {x_0}^{k-1}$ and

$$q_0^\eta(x_0) = \sum_{k=2}^{m} c_{0,k} \, (k-1) {x_0}^k.$$

It is clear we may choose $M_0 > 0$ so large that for $|\eta| > M_0$, we have $|x_0| \approx |\eta|^{\frac{1}{m-1}}$ and hence $|q_0^\eta(x_0)| \approx |\eta|^{\frac{m}{m-1}}$, where the implied constants are uniform in $\mathcal{C}$.

Now suppose $\{\vec{c}_n\}$ is a sequence in $\mathcal{C}$ converging to $\vec{c}_0$. Let $x_n = x_n(\eta)$ be the corresponding sequence of global maxima of concave polynomials $\eta x - \sum_{k=2}^{m} c_{n,k} x^k$. The key point is that there is a $n_0 \in \mathbb{N}$ sufficiently large that the equivalences

$$|x_n| \approx |\eta|^{\frac{1}{m-1}} \text{ and } |q_n^\eta(x_n)| \approx |\eta|^{\frac{m}{m-1}} \qquad (|\eta| \geq M_0) \tag{4.12}$$

also hold with constants independent of n, whenever $n \geq n_0$. Explicitly, choose $n_0 \in \mathbb{N}$ large enough that $2^{-1}|c_{0,k}| \leq |c_{n,k}| \leq 2|c_{0,k}|$ and $2^{-1}|x_0| \leq |x_n| \leq 2|x_0|$ for $2 \leq k \leq m$, whenever

$n \geq n_0$. Then for $\eta \geq M_0$ large enough, with n past n_0 we have

$$|\eta| \;\leq\; \sum_{k=2}^{m} |c_{n,k}|\, k|x_n|^{k-1} \;\leq\; \sum_{k=2}^{m} 2^k |c_{0,k}|\, k|x_0|^{k-1} \;\lesssim\; |x_n|^{m-1}$$

and

$$|\eta| \;\geq\; 2^{-m} c_{0,m} m|x_0|^{m-1} - \sum_{k=2}^{m-1} 2^k |c_{0,k} k|x_0|^{k-1} \;\gtrsim\; |x_n|^{m-1}$$

Similarly

$$\begin{aligned} |q_n^\eta(x_n)| &= \sum_{k=2}^{m} c_{n,k}\,(k-1){x_n}^k \\ &\geq 2^{-(m+1)} c_{0,m}\,(m-1)|x_0|^m - \sum_{k=2}^{m-1} 2^{k+1}|c_{0,k}|\,(k-1)|x_0|^k \\ &\gtrsim |\eta|^{\frac{m}{m-1}} \end{aligned}$$

and

$$|q_n^\eta(x_n)| \;\leq \sum_{k=2}^{m} 2^{k+1}|c_{0,k}|(k-1)|x_0|^k \;\lesssim\; |\eta|^{\frac{m}{m-1}}$$

We have been so explicit here just to emphasize that (4.12) holds with constants that depend only on x_0 and not at all on n once we are past n_0.

These estimates will be directly applicable to $g_n(\eta)$ when we rewrite $q_n^\eta(x)$ in its Taylor expansion about $x = x_n$ and scale the resulting integral in the fashion of Section 4.3

$$\begin{aligned} g_n(\eta) &= e^{4\pi q_n^\eta(x_n)} \int_{\mathbb{R}} e^{-4\pi \sum_{k=2}^{m} \frac{p_n^{(k)}(x_n)}{k!}(x-x_n)^k}\, dx \\ &= e^{4\pi q_n^\eta(x_n)} \tilde{\mu}_n(x_n) \int_{\mathbb{R}} e^{-\sum_{k=2}^{m} d_{n,k}\, x^k}\, dx \end{aligned}$$

where now $d_{n,k} = 4\pi \frac{p_n^{(k)}(x_n)}{k!} \tilde{\mu}_n^k$ and $\tilde{\mu}_n$ is defined by the requirement $\sum_{k=2}^{m} |d_{n,k}|^2 = 1$. Note that the integral in this expression of $g_n(\eta)$ is bounded and bounded away from zero independent of n, since the map $\vec{d_n} \to \int_{\mathbb{R}} e^{-\sum_{k=2}^{m} d_{n,k}\, x^k}\, dx$ is continuous and each $\vec{d_n}$ belongs to the compact set $\mathcal{C} \subset \mathbb{R}^{m-1}$.

From the preceding estimates on x_0, and the fact $\tilde{\mu}_0 \approx \left\{\sup_{k=2}^{m} |d_{0,k}|^{1/k}\right\}^{-1}$, it is clear that $|\tilde{\mu}_0(x_0)| \approx |\eta|^{\frac{2-m}{2m-2}}$ for $|\eta| > M_0$. Then arguments entirely analogous to those used for (4.12) yield

$$|\tilde{\mu}_n(x_n)| \approx |\eta|^{\frac{2-m}{2m-2}} \qquad (|\eta| > M_0)$$

with constants independent of $n \geq n_0$. It is now clear that e.g.

$$|\theta_n(\eta)| \lesssim e^{-\alpha|\eta|^{\frac{m}{m-1}}} \qquad (|\eta| \geq M_0 \text{ and } n \geq n_0)$$

with $\alpha > 0$ depending only on $\vec{c}_0$. But to prove Schwartz estimates we will need to look more closely at the form of the derivatives $\theta_n^{(j)}(\eta)$.

To this end, note for $j \in \mathbb{N}$

$$\begin{aligned} g_n^{(j)}(\eta) &= \int_{\mathbb{R}} (4\pi x)^j \cdot e^{4\pi q_n^\eta(x)}\, dx \\ &= e^{4\pi q_n^\eta(x_n)} \int_{\mathbb{R}} (4\pi x)^j e^{-4\pi \sum_{k=2}^m \frac{p_n^{(k)}(x_n)}{k!}(x-x_n)^k}\, dx \\ &= e^{4\pi q_n^\eta(x_n)} \tilde{\mu}(x_n)^{j+1} \int_{\mathbb{R}} (4\pi x)^j e^{-\sum_{k=2}^m d_{n,k}\, x^k}\, dx \end{aligned}$$

with the integral again bounded and bounded away from zero. Now a simple induction shows that $\theta_n^{(j)}(\eta)$ is a linear combination of terms of the form of the form

$$\frac{\prod_{p=1}^{l-1} g_n^{(d_p)}(\eta)}{g_n(\eta)^l} \qquad (l \leq j+1 \text{ and } \sum_{p=1}^{l-1} d_p = j)$$

which by the expression for ${g_n}^{(j)}$ are bounded (up to constants) by $e^{-4\pi q_n^\eta(x_n)} \tilde{\mu}_n(x_n)^{j-1}$. In particular,

$$|\theta_n^{(j)}(\eta)| \lesssim e^{-\alpha_j|\eta|^{\frac{m}{m-1}}} \qquad (|\eta| > M_0 \text{ and } n \geq n_0)$$

where $\alpha_j > 0$ depends only on $\vec{c}_0$. This shows that $\mathcal{S}_C$ belongs to $\mathcal{S}(\mathbb{R})$.

In fact this estimate also implies compactness of $\mathcal{S}_C$. For let

$$\|f\|_{j,N} := \sup_{\eta \in \mathbb{R}} |\eta^N f^{(j)}(\eta)|$$

be a Schwartz seminorm. Fix $\epsilon > 0$. From above, there are $n_0 \in \mathbb{N}$, $M_0 > 0$, and $\alpha_j > 0$ so that $|\theta_n^{(j)}(\eta)| \lesssim e^{-\alpha_j|\eta|^{\frac{m}{m-1}}}$ whenever $n \geq n_0$ and $|\eta| \geq M_0$. Now choose $M \geq M_0$ so large that $M^N e^{-\alpha_j M^{\frac{m}{m-1}}} < \epsilon/2$; and $n_1 \geq n_0$ so that uniform convergence on $[-M, M]$ implies

$$\sup_{\eta \in [-M,M]} |\eta^N \theta_n^{(j)}(\eta) - \eta^N \theta_0^{(j)}(\eta)| < \epsilon$$

whenever $n \geq n_1$. Clearly $\|\theta_n - \theta_0\|_{j,N} < \epsilon$ whenever $n \geq n_1$. □

Derivatives of $\breve{\mu}^{-1}(x,\tau)$

Finally we establish some estimates on the τ-derivatives of $\breve{\mu}^{-1}(x,\tau)$ which are needed in both the $|y-r|$ and $\mu(|v-t+T^{q,r}_{x,y}|)$ regimes.

Lemma 4.3. *Given $\phi(x)$ a subharmonic nonharmonic polynomial of degree m, define $\breve{\mu}=\breve{\mu}(x,\tau)$ by the requirement*

$$\sum_{k=2}^{m}\left|\tau\frac{\phi^{(k)}(x)}{k!}\breve{\mu}^{k}\right|^{2}=1$$

Then for $l=1,2,3$

$$\left|\frac{\partial^{l}}{\partial\tau^{l}}[\breve{\mu}^{-1}]\right|\lesssim\frac{\breve{\mu}^{-1}}{\tau^{l}}$$

with implied constants independent of x.

Proof. Starting with the definition of $\breve{\mu}$, writing $c_k=|\frac{\phi^{(k)}(x)}{k!}|^2$, and taking three τ-derivatives, we obtain the equalities

$$\sum_{k=2}^{m}c_k\big(\tau\breve{\mu}^{2k}+k\tau^2\breve{\mu}^{2k-1}\breve{\mu}'\big)=0 \tag{4.13}$$

$$\sum_{k=2}^{m}c_k\big(\breve{\mu}^{2k}+4k\tau^2\breve{\mu}^{2k-1}\breve{\mu}'+k(2k-1)\tau^2\breve{\mu}^{2k-2}(\breve{\mu}')^2+k\tau^2\breve{\mu}^{2k-1}\breve{\mu}''\big)=0 \tag{4.14}$$

$$\begin{aligned}\sum_{k=2}^{m}c_k\big(&6k\breve{\mu}^{2k-1}\breve{\mu}'+6k(2k-1)\tau\breve{\mu}^{2k-2}(\breve{\mu}')^2+6k\tau\breve{\mu}^{2k-1}\breve{\mu}''\\&+3k(2k-1)\tau^2\breve{\mu}^{2k-2}\breve{\mu}'\breve{\mu}''+k(2k-1)(2k-2)\tau^2\breve{\mu}^{2k-3}(\breve{\mu}')^3+k\tau^2\breve{\mu}^{2k-1}\breve{\mu}^{(3)}\big)=0\end{aligned} \tag{4.15}$$

where $\breve{\mu}^{(l)}\equiv\frac{\partial^l}{\partial\tau^l}\breve{\mu}$.

Now rearrange (4.13) as

$$\sum_{k=2}^{m}c_k\tau\breve{\mu}^{2k}=-\breve{\mu}'\sum_{k=2}^{m}k\tau^2\breve{\mu}^{2k-1}$$

Note the left-hand side is τ^{-1}; and multiplying both sides by $\breve{\mu}$ yields

$$\frac{\breve{\mu}}{\tau}=-\breve{\mu}'\sum_{k=2}^{m}kc_k\tau^2\breve{\mu}^{2k}$$

We conclude $\breve{\mu}'\approx-\frac{\breve{\mu}}{\tau}$. (And in particular $-\breve{\mu}'>\frac{\breve{\mu}}{\tau}$.) This yields the $l=1$ estimate of the lemma because

$$\frac{d}{d\tau}[\breve{\mu}^{-1}]=\frac{-\breve{\mu}'}{\breve{\mu}^2}\approx\frac{\breve{\mu}^{-1}}{\tau}$$

The $l = 2$ and $l = 3$ cases follow similarly. That is, rearranging (4.14) and balancing powers of $\breve{\mu}$ implies

$$\left|\frac{\breve{\mu}''}{\breve{\mu}}\right| \lesssim \left|\sum_{k=2}^{m} c_k \breve{\mu}^{2k}\right| + \left|4\tau \frac{\breve{\mu}'}{\breve{\mu}} \sum_{k=2}^{m} k c_k \breve{\mu}^{2k}\right| + \left|\tau^2 \frac{(\breve{\mu}')^2}{\breve{\mu}^2} \sum_{k=2}^{m} k(2k-1) c_k \breve{\mu}^{2k}\right|$$

So $|\breve{\mu}''| \lesssim \frac{\breve{\mu}}{\tau^2}$ from the estimate on $\breve{\mu}'$, and thus

$$\left|\frac{d^2}{d\tau^2}[\breve{\mu}^{-1}]\right| = \left|2(\breve{\mu}')^2 \breve{\mu}^{-3} - \breve{\mu}'' \breve{\mu}^{-2}\right| \lesssim \frac{\breve{\mu}^{-1}}{\tau^2}$$

And rearranging (4.15)

$$\left|\frac{\breve{\mu}^{(3)}}{\breve{\mu}}\right| \lesssim \left|6\frac{\breve{\mu}'}{\breve{\mu}} \sum_{k=2}^{m} k c_k \breve{\mu}^{2k}\right| + \left|6\frac{(\breve{\mu}')^2}{\breve{\mu}^2} \sum_{k=2}^{m} k(2k-1) c_k \breve{\mu}^{2k}\right| + \left|6\tau \frac{\breve{\mu}''}{\breve{\mu}} \sum_{k=2}^{m} k c_k \breve{\mu}^{2k}\right| +$$
$$\left|\tau^2 \frac{(\breve{\mu}')^3}{\breve{\mu}^3} \sum_{k=2}^{m} k(2k-1)(2k-2) c_k \breve{\mu}^{2k}\right| + \left|3\tau^2 \frac{\breve{\mu}'\breve{\mu}''}{\breve{\mu}^2} \sum_{k=2}^{m} 3k(2k-1) c_k \breve{\mu}^{2k}\right|$$

Then $|\breve{\mu}^{(3)}| \lesssim \frac{\breve{\mu}}{\tau^3}$ from the estimates on $\breve{\mu}'$ and $\breve{\mu}''$. Hence

$$\left|\frac{d^3}{d\tau^3}[\breve{\mu}^{-1}]\right| = \left|6\breve{\mu}'\breve{\mu}''\breve{\mu}^{-3} - 6(\breve{\mu}')^3 \breve{\mu}^{-4} - \breve{\mu}^{(3)} \breve{\mu}^{-2}\right| \lesssim \frac{\breve{\mu}^{-1}}{\tau^3}$$

□

Corollary 4.4. *Let $\hat{\theta}_{\vec{c}}$ be a compact class of Schwartz functions on $\mathbb{R}$ indexed by $\vec{c}$, and $b_0 \in \mathbb{R}$. Then with $\breve{\mu}(x, \tau)$ as in Lemma 4.3, we have for $l = 1, 2, 3$*

$$\left|\frac{\partial^l}{\partial \tau^l} \hat{\theta}(b_0 \breve{\mu}^{-1})\right| \lesssim \tau^{-l}$$

with implied constants independent of b_0 and x.

Proof. This is easily seen by writing out the first three τ-derivatives of $\hat{\theta}(b_0\breve{\mu}^{-1})$ and then applying the estimates of Lemma 4.3. □

Chapter 5

Observations and Difficulties in Codimension Two

In this chapter we finally consider the Szegő kernel for CR manifolds of the form

$$M_{\phi,\psi} = \{\, (z, w_1, w_2) \in \mathbb{C}^2 \colon \operatorname{Im} w_1 = \phi(\operatorname{Re} z) \text{ and } \operatorname{Im} w_2 = \psi(\operatorname{Re} z) \,\}$$

where $\phi(x)$ and $\psi(x)$ are convex polynomials of degree at least 2. By reasoning parallel to Nagel's in Chapter 4, we provide an integral formula for the Szegő kernel on $M_{\phi,\psi} \times M_{\phi,\psi}$. When $\phi(x) = \psi(x) = x^2$ this kernel simplifies to a δ function times the Szegő kernel for M_{x^2}; in Section 5.4 we show that a similar relationship holds whenever $\psi(x) = a\phi(x)$ for $a \in \mathbb{R}$. The remainder of the chapter, through special study of M_{x^2,x^4}, indicates the technical difficulties that arise when working with the integral formula on general $M_{\phi,\psi}$. We also discuss the possible significance of the control metric on $M_{\phi,\psi}$.

5.1 Extension of Nagel's integral formula

Suppose $\phi(x)$ and $\psi(x)$ are as above, with maximum degree m. Then $M_{\phi,\psi}$ is defined by

$$\rho_1(z, w) = \phi(\operatorname{Re} z) - \operatorname{Im} w_1 \quad \text{and} \quad \rho_2(z, w) = \psi(\operatorname{Re} z) - \operatorname{Im} w_2$$

And it is easy to check that

$$\bar{D} = \frac{\partial}{\partial \bar{z}} - i\phi' \frac{\partial}{\partial \bar{w}_1} - i\psi' \frac{\partial}{\partial \bar{w}_2}$$

is a global tangential antiholomorphic vector field for $M_{\phi,\psi}$. We now identify $M_{\phi,\psi}$ with $\mathbb{C} \times \mathbb{R}^2$ with coordinates (x, y, s, t). Under the identification $2\bar{D}$ is

$$\bar{L} = \frac{\partial}{\partial x} + i\left[\frac{\partial}{\partial y} - \phi' \frac{\partial}{\partial s} - \psi' \frac{\partial}{\partial t}\right] = (X_1 + iX_2)$$

So the Szegő projection is onto the kernel of $\bar{L}$ in $L^2(M_{\phi,\psi})$.

Now what follows is completely analogous to Section 4.2, but we repeat certain steps to fix notation. For $u(x, y, s, t) \in L^2(\mathbb{R}^4)$ define the partial Fourier transform $\mathcal{F}$ by

$$\mathcal{F}u = \hat{u}(x, \eta, \sigma, \tau) = \iiint_{\mathbb{R}^3} e^{-2\pi i(y\eta + s\sigma + t\tau)} u(x, y, s, t)\, dy\, ds\, dt$$

so that

$$u(x,y,s,t) = \mathcal{F}^{-1}\hat{u} = \iiint_{\mathbb{R}^3} e^{2\pi i(y\eta + s\sigma + t\tau)}\hat{u}(x,\eta,\sigma,\tau)\, d\eta\, d\sigma\, d\tau$$

And writing

$$M_\Psi[g] = \Psi \cdot g \quad \text{with} \quad \Psi = e^{-2\pi(x\eta - \phi(x)\sigma - \psi(x)\tau)}$$

we obtain the formula

$$\bar{L}u = \mathcal{F}^{-1} M_{\Psi^{-1}} \frac{d}{dx} M_\Psi \mathcal{F} u.$$

As before, we can analyze $\bar{L}$ on $M_{\phi,\psi}$ by investigating $\frac{\partial}{\partial x}$ acting on functions that satisfy

$$\iiiint_{\mathbb{R}^4} |g(x,\eta,\sigma,\tau)|^2 e^{4\pi(x\eta - \phi(x)\sigma - \psi(x)\tau)} dx\, d\eta\, d\sigma\, d\tau < +\infty.$$

The kernel of $\frac{\partial}{\partial x}$ here consists of functions $g(\eta,\sigma,\tau)$ such that

$$\iiint_{\mathbb{R}^3} |g(\eta,\sigma,\tau)|^2 \left[\int_{\mathbb{R}} e^{4\pi(x\eta - \phi(x)\sigma - \psi(x)\tau)} dx\right] d\eta\, d\sigma\, d\tau < \infty.$$

Because these functions must vanish when $\int_{\mathbb{R}} e^{4\pi(x\eta - \phi(x)\sigma - \psi(x)\tau)} dx = +\infty$, the projection operator P on the kernel of $\frac{\partial}{\partial x}$ has nontrivial behavior only in the region

$$\Sigma = \left\{ (\eta,\sigma,\tau) \in \mathbb{R}^3 \ : \ \int_{\mathbb{R}} e^{4\pi(x\eta - \phi(x)\sigma - \psi(x)\tau)} dx < +\infty \right\}.$$

When $(\eta,\sigma,\tau) \in \Sigma$, the action of P is to project $g(x,\eta,\sigma,\tau)$ onto the space of constants in $L^2(e^{4\pi(x\eta - \phi(x)\sigma - \psi(x)\tau)} dx)$. Denote this projection operator $P_{\eta,\sigma,\tau}$; writing

$$C_{\eta,\sigma,\tau} = \int_{\mathbb{R}} e^{4\pi(\varsigma\eta - \phi(\varsigma)\sigma - \psi(\varsigma)\tau)} d\varsigma$$

we then have

$$P_{\eta,\sigma,\tau} g(x) = \int_{\mathbb{R}} g(q)\, C_{\eta,\sigma,\tau}^{-1}\, e^{4\pi(q\eta - \phi(q)\sigma - \psi(q)\tau)} dq$$

And setting

$$K_{\eta,\sigma,\tau}(x,q) = \begin{cases} C_{\eta,\sigma,\tau}^{-1}\, e^{4\pi(q\eta - \phi(q)\sigma - \psi(q)\tau)} & \text{if } (\eta,\sigma,\tau) \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

We obtain P as an integral operator on $L^2(e^{4\pi(x\eta - \phi(x)\sigma - \psi(x)\tau)} dx)$ with kernel $K_{\eta,\sigma,\tau}(x,q)$.

Finally, because the Szegő projection operator P_S is given by

$$P_S = \mathcal{F}^{-1} M_{\Psi^{-1}} P M_\Psi \mathcal{F},$$

we conclude that

$$P[f](x,y,s,t) = \iiiint_{\mathbb{R}^4} f(q,r,u,v) S((x,y,s,t),(q,r,u,v))\, dq\, dr\, du\, dv$$

where the Szegő kernel is

$$S((x,y,s,t),(q,r,u,v)) = \iiint_{\Sigma} C^{-1}_{\eta,\sigma,\tau} e^{-2\pi\sigma[(\phi(x)+\phi(q))+i(u-s)]} e^{-2\pi\tau[(\psi(x)+\psi(q))+i(v-t)]} e^{2\pi\eta[(x+q)+i(y-r)]}\, d\eta\, d\sigma\, d\tau \quad (5.1)$$

5.2 A comment on the control metric

Since on a finite type domain boundary in $\mathbb{C}^2$ the Szegő kernel is governed by the control metric, we may naturally ask whether the same holds on a codimension two CR manifold. It is easy to extend the notion of finite type; we simply require that the real and imaginary parts of $\bar{L}$, X_1 and X_2, along with a finite number m of their iterated commutators, span the real tangent space at every point of $M_{\phi,\psi}$. Then the definition of the control metric is the same as in Theorem 3.3, with $\partial\Omega$, with replaced by $M_{\phi,\psi}$.

Definition 5.1. *With notation as above, let $\{Y_1, \ldots, Y_q\}$ be some enumeration of the vector fields X_1, X_2, and all their iterated commutators of length less than or equal m. Define the "degree" of each vector field Y_j by*

$$d(Y_j) = \text{ length of the iterated commutator that forms } Y_j$$

Now let the distance between $p, q \in M_{\phi,\psi}$ be the infimum of $\delta > 0$ such that there is an absolutely continuous map $\gamma : [0,1] \mapsto M_{\phi,\psi}$ with $\gamma(0) = p$, $\gamma(1) = q$ so that for almost all $t \in (0,1)$

$$\gamma'(t) = \sum_{j=1}^{q} a_j(t) Y_j(\gamma(t)), \qquad |a_j(t)| < \delta^{d(Y_j)}$$

This distance function defines the control metric on $M_{\phi,\psi}$.

The great difficulty here lies in formulating a tractable description of the metric. Indeed, even on a domain boundary M_ϕ, a serious amount of work is required to prove the equivalence of the control metric to the pseudometric (4.2) discovered by Nagel et al. (We essentially hid all this work in Chapter 4 by taking as a fact the characterization of control balls in terms of a polarization in Theorem 3.7.) As for the volume of the control balls on $\mathcal{M}_{\phi,\psi}$, the closest thing to (4.3) is another of Nagel et al.'s theorems in [43].

Theorem 5.2 (Nagel, Stein, Wainger). *With $\{Y_1, \ldots, Y_q\}$ as above, the volume of the control ball of radius $\delta > 0$ centered at $(x, y, s, t) \in M_{\phi,\psi}$ is, with uniform constants, given by*

$$|B((x, y, s, t), \delta)| \approx \sum_{(i_1,i_2,i_3)} |\det[Y_{i_1}\ Y_{i_2}\ Y_{i_3}]|\, \delta^{d(Y_{i_1})+d(Y_{i_2})+d(Y_{i_3})} \tag{5.2}$$

where all determinants are evaluated at (x, y, s, t) and (i_1, i_2, i_3) range over all increasing triplets from the set $\{1, \ldots, q\}$. □

For general $\phi(x)$ and $\psi(x)$, it is not necessarily obvious what to make of this expression either. But on M_{x^2,x^4} the formula (5.2) is quite tractable, and we will return to it in Section 5.5. Before doing this, however, we study some particularly simple manifolds which are not of finite type.

5.3 Calculations with quadratic defining functions

The very simplest manifold meeting our description is M_{x^2,x^2}. The crucial set Σ introduced in Section 5.1, namely

$$\Sigma = \left\{ (\eta, \sigma, \tau) \in \mathbb{R}^3 \ :\ C_{\eta,\sigma,\tau} = \int_{\mathbb{R}} e^{4\pi(x\eta - \phi(x)\sigma - \psi(x)\tau)} dx < +\infty \right\}$$

is now seen to be

$$\Sigma = \{ (\eta, \sigma, \tau) \in \mathbb{R}^3 \ :\ \sigma + \tau > 0 \}$$

Where inside Σ we may compute

$$C_{\eta,\sigma,\tau} = \int_{\mathbb{R}} e^{4\pi[x\eta - x^2(\sigma+\tau)]}\, dx = \frac{1}{2} \frac{e^{\frac{\pi\eta^2}{\sigma+\tau}}}{\sqrt{\sigma+\tau}}$$

We now use this formula, and the fact the Fourier transform of a Gaussian is another Gaussian, to compute the Szegő kernel explicitly.

As usual, certain abbreviations will be necessary. Take

$$F_{x,q,s,u}(\sigma) = e^{-2\pi\sigma[(x^2+q^2)+i(u-s)]} \quad \text{and} \quad F_{x,q,t,v}(\tau) = e^{-2\pi\tau[(x^2+q^2)+i(v-t)]}$$

The kernel $S((x, y, s, t), (q, r, u, v))$ as in (5.1) is then

$$2 \int_{\mathbb{R}} F_{x,q,t,v}(\tau) \int_{-\tau}^{\infty} \sqrt{\sigma+\tau} F_{x,q,s,u}(\sigma) \int_{\mathbb{R}} e^{-\frac{\pi\eta^2}{\sigma+\tau}} \cdot e^{2\pi\eta[(x+q)+i(y-r)]}\, d\eta\, d\sigma\, d\tau$$

And the inner integral can again be computed

$$\int_{\mathbb{R}} e^{-\frac{\pi\eta^2}{\sigma+\tau}} \cdot e^{2\pi\eta[(x+q)+i(y-r)]}\, d\eta = \sqrt{\sigma+\tau}\, e^{(\sigma+\tau)\cdot\pi[(x+q)+i(y-r)]^2}$$

Using the additional abbreviation

$$\begin{aligned}
G_{x,q,y,r,t,v}(\tau) &= e^{\tau\pi[(x+q)+i(y-r)]^2} F_{x,q,t,v}(\tau) \\
&= e^{\tau\pi[(x+q)+i(y-r)]^2} \cdot e^{-2\pi\tau[(x^2+q^2)+i(v-t)]} \\
&= e^{-\tau\pi\left((x-q)^2+(y-r)^2+2i[(v-t)-(x+q)(y-r)]\right)}
\end{aligned}$$

and re-expanding $F_{x,q,u,s}(\sigma)$, this leads us to the expression for $\frac{1}{2}S((x,y,s,t),(q,r,u,v))$

$$\begin{aligned}
&\int_{\mathbb{R}} e^{\tau\pi[(x+q)+i(y-r)]^2} F_{x,q,t,v}(\tau) \int_{-\tau}^{\infty} (\sigma+\tau) e^{-\sigma\pi\left(2[(x^2+q^2)+i(u-s)]-[(x+q)+i(y-r)]^2\right)}\, d\sigma\, d\tau \\
&\quad= \int_{\mathbb{R}} \tau G_{x,q,y,r,t,v}(\tau) \int_{-\tau}^{-\infty} e^{-\sigma\pi\left(2[(x^2+q^2)+i(u-s)]-[(x+q)+i(y-r)]^2\right)}\, d\sigma\, d\tau \\
&\qquad+ \int_{\mathbb{R}} G_{x,q,y,r,t,v}(\tau) \int_{-\tau}^{\infty} \sigma e^{-\sigma\pi\left(2[(x^2+q^2)+i(u-s)]-[(x+q)+i(y-r)]^2\right)}\, d\sigma\, d\tau \\
&\quad= \int_{\mathbb{R}} \tau G_{x,q,y,r,t,v}(\tau) \int_{-\tau}^{\infty} e^{-\sigma\pi\left((x-q)^2+(y-r)^2+2i[(u-s)-(x+q)(y-r)]\right)}\, d\sigma\, d\tau \\
&\qquad+ \int_{\mathbb{R}} G_{x,q,y,r,t,v}(\tau) \int_{-\infty}^{\tau} \sigma e^{\sigma\pi\left((x-q)^2+(y-r)^2+2i[(u-s)-(x+q)(y-r)]\right)}\, d\sigma\, d\tau \\
&\quad= A + B
\end{aligned}$$

For both A and B, the inner integrals converge absolutely away from the diagonal and may be computed. First,

$$\begin{aligned}
A &= \left(\pi[(x-q)^2+(y-r)^2] + 2\pi i[(u-s)-(x+q)(y-r)]\right)^{-1} \cdot \\
&\qquad \int_{\mathbb{R}} \tau G_{x,q,y,r,t,v}(\tau) \cdot e^{\tau\pi\left((x-q)^2+(y-r)^2+2i[(u-s)-(x+q)(y-r)]\right)}\, d\tau \\
&= C^{-1}_{x,q,y,r,s,u} \cdot \int_{\mathbb{R}} \tau e^{2\pi i\tau[(u-s)-(v-t)]}\, d\tau
\end{aligned}$$

understood in the sense of tempered distributions. Similarly,

$$B = -C^{-1}_{x,q,y,r,s,u} \cdot \int_{\mathbb{R}} \tau e^{2\pi i\tau[(u-s)-(v-t)]}\, d\tau + C^{-2}_{x,q,y,r,s,u} \cdot \int_{\mathbb{R}} e^{2\pi i\tau[(u-s)-(v-t)]}\, d\tau$$

Hence

$$A + B \;=\; C^{-2}_{x,q,y,r,s,u} \cdot \int_{\mathbb{R}} e^{2\pi i\tau[(u-s)-(v-t)]}\, d\tau \;=\; \delta\big((u-s)-(v-t)\big) \cdot C^{-2}_{x,q,y,r,s,u}$$

in the sense of distributions.

We conclude that

$$S((x,y,s,t),(q,r,u,v)) = \frac{2\delta\big((u-s)-(v-t)\big)}{\big(\pi[(x-q)^2+(y-r)^2]+2\pi i[(s-u)-(x+q)(y-r)]\big)^2}$$

On the other hand, the same sort of calculations yield for M_{x^2} that

$$\begin{aligned} S((x,y,t),(q,r,v)) &= \int_0^\infty e^{-2\pi\tau[(x^2+q^2)+i(v-t)]} \int_{\mathbb{R}} \frac{e^{2\pi\eta[(x+q)+i(y-r)]}}{\int_{\mathbb{R}} e^{4\pi[\eta\varsigma-\tau\varsigma^2]}\,d\varsigma}\,d\eta\,d\tau \\ &= \frac{2}{\big(\pi[(x-q)^2+(y-r)^2]+2\pi i[(u-s)-(x+q)(y-r)]\big)^2} \end{aligned}$$

Evidently the Szegő kernel on M_{x^2,x^2} is just the kernel on M_{x^2} multiplied by an appropriate delta distribution. We next generalize this relationship to any case where $\psi(x)$ is a constant multiple of $\phi(x)$.

5.4 The Szegő kernel on $M_{\phi,\psi}$ not of finite type

With the integral formulas from Sections 4 and 5.1 in hand, we can identify the general relationship between M_ϕ and $M_{\phi,a\phi}$.

Proposition 5.3. *With notation as above, let $S_\phi((x,y,s),(q,r,u))$ denote the Szegő kernel on M_ϕ; for $a\in\mathbb{R}$, let $S_{\phi,a\phi}((x,y,s,t),(q,r,u,v))$ be the Szegő kernel on $M_{\phi,a\phi}$. Then we have*

$$S_{\phi,a\phi}((x,y,s,t),(q,r,u,v)) = \delta\big((t-v)-a(s-u)\big)\cdot S_\phi((x,y,s),(q,r,u)) \tag{5.3}$$

Proof. We know the kernels S_ϕ and $S_{\phi,a\phi}$ are given by appropriate integrals over, respectively, the regions

$$\Sigma_1 = \left\{ (\eta,\mu)\in\mathbb{R}^2 \,:\, C_{\eta,\mu} = \int_{\mathbb{R}} e^{4\pi(\varsigma\eta-\phi(\varsigma)\mu)}\,d\varsigma < +\infty \right\}$$

and

$$\Sigma_2 = \left\{ (\eta,\sigma,\tau)\in\mathbb{R}^3 \,:\, C_{\eta,\sigma,\tau} = \int_{\mathbb{R}} e^{4\pi[\varsigma\eta-\phi(\varsigma)(\sigma+a\tau)]}\,d\varsigma < +\infty \right\}$$

In particular, $\Sigma_1 = \{(\eta,\mu)\in\mathbb{R}^2 : \mu>0\}$ and $\Sigma_2=\{(\eta,\sigma,\tau)\in\mathbb{R}^3 : \sigma+a\tau>0\}$. It is also clear that $C_{\eta,\sigma,\tau} = C_{\eta,\sigma+a\tau}$.

Now recall the abbreviation from Section 5.3

$$F_{x,q,y,r}(\eta) = e^{2\pi\eta[(x+q)+i(y-r)]}$$

Then from Section 4 we know

$$S_\phi((x,y,s),(q,r,u)) = \int_0^\infty e^{-2\pi\mu[(\phi(x)+\phi(q))+i(u-s)]} \int_{\mathbb{R}} C^{-1}_{\eta,\mu} F_{x,q,y,r}(\eta)\, d\eta\, d\mu$$

And we calculate

$$\begin{aligned}
&S_{\phi,a\phi}((x,y,s,t),(q,r,u,v)) \\
&\quad = \iiint_{\Sigma_2} C^{-1}_{\eta,\sigma,\tau} F_{x,q,y,r}(\eta) e^{-2\pi\sigma[(\phi(x)+\phi(q))+i(u-s)]} e^{-2\pi\tau[(a\cdot\phi(x)+a\cdot\phi(q))+i(v-t)]}\, d\eta\, d\sigma\, d\tau \\
&\quad = \iiint_{\Sigma_2} C^{-1}_{\eta,\sigma,\tau} F_{x,q,y,r}(\eta) e^{-2\pi(\sigma+a\tau)[\phi(x)+\phi(q)]} e^{-2\pi i[\sigma(u-s)+\tau(v-t)]}\, d\eta\, d\sigma\, d\tau \\
&\quad = \iiint_{\Sigma_2} C^{-1}_{\eta,\sigma,\tau} F_{x,q,y,r}(\eta) e^{-2\pi i\tau[(v-t)-a(u-s)]} e^{-2\pi(\sigma+a\tau)[(\phi(x)+\phi(q))+i(u-s)]}\, d\eta\, d\sigma\, d\tau \\
&\quad = \int_{\mathbb{R}} e^{2\pi i\tau[(t-v)-a(s-u)]} \int_{-a\tau}^\infty e^{2\pi(\sigma+a\tau)[(\phi(x)+\phi(q))+i(u-s)]} \int_{\mathbb{R}} C^{-1}_{\eta,\sigma+a\tau} F_{x,q,y,r}(\eta)\, d\eta\, d\sigma\, d\tau
\end{aligned}$$

Upon shifting $\sigma \mapsto \sigma - a\tau$, this gives

$$\begin{aligned}
&S_{\phi,a\phi}((x,y,s,t),(q,r,u,v)) \\
&\quad = \int_{\mathbb{R}} e^{2\pi i\tau[(t-v)-a(s-u)]} \int_0^\infty e^{-2\pi\sigma[(\phi(x)+\phi(q))+i(u-s)]} \int_{\mathbb{R}} C^{-1}_{\eta,\sigma} F_{x,q,y,r}(\eta)\, d\eta\, d\sigma\, d\tau \\
&\quad = S_1\big((x,y,s);(q,r,u)\big) \int_{\mathbb{R}} e^{2\pi i\tau[(t-v)-a(s-u)]}\, d\tau
\end{aligned}$$

Which is exactly (5.3) in the sense of distributions. $\square$

Connection to the control geometry

Note that $M_{\phi,a\phi}$ does not satisfy a finite type hypothesis. Indeed, with $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial y} - \phi'\frac{\partial}{\partial s} - a\phi'\frac{\partial}{\partial t}$, every potentially non-zero commutator is of the form

$$Y_k = [\overbrace{X_1, [X_1, \cdots, [X_1}^{k-1\text{ times}}, X_2]\cdots]] \qquad (2 \le k \le m)$$

That is,

$$Y_k = -\phi^{(k)}(x)\frac{\partial}{\partial s} - a\phi^{(k)}(x)\frac{\partial}{\partial t}$$

And evidently $\{X_1, X_2, Y_{k_1}, Y_{k_2}\}$ is limited to the $(1,a)$ subspace of the $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ directions for any choice of k_1, k_2. So the real tangent space is never spanned, at any point of $M_{\phi,a\phi}$, and $\{X_1, X_2\}$ do not generate a finite control metric.

Nonetheless we may talk about the control distance on $M_{\phi,a\phi}$. This distance is, we recall, finite and less than some $\delta > 0$ if and only if there exists an absolutely continuous curve $\gamma : [0,1] \mapsto M_{\phi,a\phi}$ such that

$$\gamma'(\varsigma) = a_0(\varsigma)X_1(\gamma(\varsigma)) + a_1(\varsigma)X_2(\gamma(\varsigma)) + \sum_{k=2}^{m} a_k(\varsigma)Y_k(\gamma(\varsigma))$$

with $|a_0(\varsigma)|, |a_1(\varsigma)| < \delta$ and $|a_k(\varsigma)| < \delta^k$ for almost all $\varsigma \in (0,1)$. Given our previous comments, such a curve $\gamma(\varsigma) = (\gamma_1(\varsigma), \gamma_2(\varsigma), \gamma_3(\varsigma), \gamma_4(\varsigma))$ can only exist if $\gamma_4'(\varsigma) = a\gamma_3'(\varsigma)$ for almost all $\varsigma \in (0,1)$. And then of course

$$\begin{pmatrix} u-s \\ v-t \end{pmatrix} = \int_0^1 \begin{pmatrix} {p_3}'(\varsigma) \\ a{p_3}'(\varsigma) \end{pmatrix} d\varsigma$$

so that $(v-t) = a(u-s)$. Now recall (5.3). We see the Szegő kernel on $M_{\phi,a\phi}$ is a singular distribution supported on exactly the subspace where the control distance on $M_{\phi,a\phi}$ is finite.

5.5 Obstacles from losing convexity

We were motivated in Chapter 4 by our hope that the technical ingredients for Nagel's Szegő kernel estimates on M_ϕ could be could be reused in codimension two. Let us now evaluate this prospect in the general case where $\deg(\phi) \neq \deg(\psi)$. Suppose without loss that $m_0 = \deg(\phi) < \deg(\psi) = m_1$. Then

$$\Sigma = \left\{ (\eta,\sigma,\tau) \in \mathbb{R}^3 : C_{\eta,\sigma,\tau} = \int_{\mathbb{R}} e^{4\pi[x\eta - \phi(x)\sigma - \psi(x)\tau)]}\, dx < \infty \right\}$$

is just the set in $\mathbb{R}^3$ with $\tau > 0$. Writing $a_0 = x+q$, $a_1 = \phi(x)+\phi(q)$, $a_2 = \psi(x)+\psi(q)$, $b_0 = y-r$, $b_1 = u-s$, and $b_2 = v-t$, the expression for the kernel is

$$S\big((x,y,s,t);(q,r,u,v)\big) = \int_0^\infty e^{-2\pi\tau[a_2+ib_2]} \int_{\mathbb{R}} e^{-2\pi\sigma[a_1+ib_1]} \int_{\mathbb{R}} \frac{e^{2\pi\eta[a_0+ib_0]}\, d\eta\, d\sigma\, d\tau}{\int_{\mathbb{R}} e^{4\pi[\eta\varsigma - (\tau\psi(\varsigma)+\sigma\phi(\varsigma))]}\, d\varsigma}$$

Nothing prevents us from beginning in analogy to the methods of Section 4.2. Shifting $\varsigma \mapsto \varsigma + \frac{a_0}{2}$, expanding $\phi(\varsigma + \frac{a_0}{2})$ and $\psi(\varsigma + \frac{a_0}{2})$ about $\varsigma = 0$, and finally shifting $\eta \mapsto \eta + \phi'(\frac{a_0}{2}) + \psi'(\frac{a_0}{2})$ yields a kernel $S((x,y,s,t),(q,r,u,v))$ equal to

$$\int_0^\infty e^{-2\pi\tau[\Delta_\psi + i\lambda_\psi]} \int_{\mathbb{R}} e^{-2\pi\sigma[\Delta_\phi + i\lambda_\phi]} \int_{\mathbb{R}} \frac{e^{2\pi i\eta[b_0]}\, d\eta\, d\sigma\, d\tau}{\int_{\mathbb{R}} e^{4\pi\left[\eta\varsigma - \left(\sigma\sum_{k=2}^{m_0} \frac{\phi^{(k)}(\frac{a_0}{2})}{k!}\varsigma^k + \tau\sum_{j=2}^{m_1} \frac{\psi^{(j)}(\frac{a_0}{2})}{j!}\varsigma^j\right)\right]}\, d\varsigma} \tag{5.4}$$

where in direct analogy to the notation of Section 4.3 we have written

$$\Delta_\phi = \phi(x) + \phi(q) - 2\phi(\frac{x+q}{2}) \qquad \Delta_\psi = \psi(x) + \psi(q) - 2\psi(\frac{x+q}{2})$$

and

$$\lambda_\phi = (u-s) + \phi'(\frac{x+q}{2})(r-y) \qquad \lambda_\psi = (v-t) + \psi'(\frac{x+q}{2})(r-y)$$

So, for example, Lemma 4.1 on $\Delta_\phi(x,q)$ could still be of value. But there is a major complication. The polynomial

$$\varsigma \mapsto \sigma \sum_{k=2}^{m_0} \frac{\phi^{(k)}(\frac{a_0}{2})}{k!}\varsigma^k + \tau \sum_{j=2}^{m_1} \frac{\psi^{(j)}(\frac{a_0}{2})}{j!}\varsigma^j \tag{5.5}$$

in the denominator integral is no longer convex on all of $\mathbb{R}$ when $\sigma < 0$. Hence there is simply no use in scaling the coefficients of (5.5) into a compact set via some $\varsigma \mapsto \breve{\mu}(x,\sigma,\tau)\varsigma$, since the size of this $\breve{\mu}$ would have no predictable relationship to the total size of the denominator integral.

The case $\phi(x) = x^2$ and $\psi(x) = x^4$

At this point we turn to the simplest possible example, namely M_{x^2,x^4}. Supposing the control geometry on M_{x^2,x^4} still has some influence on the size of its Szegő kernel, we first consider the volume of balls in the control metric. The relevant vector fields are

$$Y_1 = X_1 = \tfrac{\partial}{\partial x} \qquad Y_2 = X_2 = \tfrac{\partial}{\partial y} - 2x\tfrac{\partial}{\partial s} - 4x^3\tfrac{\partial}{\partial t}$$
$$Y_3 = [X_1, X_2] = -2\tfrac{\partial}{\partial s} - 12x^2\tfrac{\partial}{\partial t} \qquad Y_4 = [X_1,[X_1,X_2]] = -24x\tfrac{\partial}{\partial t}$$
$$Y_5 = [X_1,[X_1,[X_1,X_2]]] = -24\tfrac{\partial}{\partial t}$$

From Theorem 5.2, we conclude that the volume of a control ball $B((x,y,s,t),\delta)$ is essentially

$$\begin{aligned} |B((x,y,s,t),\delta)| &\approx |\det[Y_1\ Y_2\ Y_3\ Y_4]|\ \delta^7 + |\det[Y_1\ Y_2\ Y_3\ Y_5]|\ \delta^8 \\ &\approx |x|\cdot\delta^7 + \delta^8 \end{aligned} \tag{5.6}$$

Next we observe an analogue to the "twisted distance" in the totally real direction that appeared in Chapter 4. In that case, taking two points (x,y,t) and (q,r,v) in a codimension one model M_ϕ, we may parameterize a path $\gamma\colon [0,1] \mapsto M_\phi$ between them by

$$\gamma(\alpha) = (x,y,t) + \alpha\cdot(q-x, r-y, v-t)$$

So $\gamma'(\alpha) = (q-x, r-y, v-t)$ for all $\alpha \in [0,1]$. Now we need to form this tangent vector as a linear combination of the vector fields $Z_1 = \frac{\partial}{\partial x}$, $Z_2 = \frac{\partial}{\partial y} - \phi'(x)\frac{\partial}{\partial t}$, and all their iterated commutators. Evidently the coefficient of Z_1 in this linear combination must be must be $(q-x)$, and the coefficient of Z_2 must be $(r-y)$. So the remaining commutators in the linear combination must have components that sum to yield $v - t + \phi'(x)(r-y)$ in the totally real direction.

Now, recall that the expression $v - t + \phi'(\frac{x+q}{2})(r-y)$ was equivalent to the twisted distance in the totally real direction, so it may be worthwhile to consider a similar exercise on M_{x^2,x^4}. Connecting (x,y,s,t) and (q,r,u,v) by

$$\gamma(\alpha) = (x,y,s,t) + \alpha \cdot (q-x, r-y, u-s, v-t)$$

it follows that the coefficient of Y_1 above is $(q-x)$, the coefficient of Y_2 is $(r-y)$, the coefficient of Y_3 is $-\frac{1}{2}(u-s+2x(r-y))$; and thus Y_4 and Y_5 must be combined to yield a total component of $v - t + 4x^3(r-y) - 6x^2(u-s+2x(r-y))$ in the second totally real direction. Indeed, this term suggests a particular transformation in the integral formula for the Szegő kernel on M_{x^2,x^4}.

The formula for the kernel on M_{x^2,x^4}, after performing the transformations given up to (5.4) and writing $\mathrm{av} = \frac{x+q}{2}$, is

$$\begin{aligned}
&\int_0^\infty \exp\left\{-2\pi\tau\left[3(\mathrm{av})^2(x-q)^2 + \frac{1}{8}(x-q)^4 + i(v-t+4(\mathrm{av})^3(r-y))\right]\right\} \\
&\quad \cdot \int_{\mathbb{R}} \exp\left\{-2\pi\sigma\left[\frac{1}{2}(x-q)^2 + i(u-s+2(\mathrm{av})(r-y))\right]\right\} \\
&\quad\quad \cdot \int_{\mathbb{R}} \frac{e^{2\pi i\eta(y-r)}\, d\eta\, d\sigma\, d\tau}{\int_{\mathbb{R}} \exp\left\{4\pi\left[\eta\varsigma - \sigma\varsigma^2 - \tau 6(\mathrm{av})^2\varsigma^2 - \tau 4(\mathrm{av})\varsigma^3 - \tau\varsigma^4\right]\right\}\, d\varsigma}
\end{aligned}$$

And now our preceding comments suggest transforming $\sigma \mapsto \sigma - \tau 6(\mathrm{av})^2$, which yields

$$\begin{aligned}
&\int_0^\infty \exp\left\{-2\pi\tau\left[\frac{1}{8}(x-q)^4 + i(v-t+4(\mathrm{av})^3(r-y) - 6(\mathrm{av})^2(u-s+2(\mathrm{av})(r-y)))\right]\right\} \\
&\quad \cdot \int_{\mathbb{R}} \exp\left\{-2\pi\sigma\left[\frac{1}{2}(x-q)^2 + i(u-s+2(\mathrm{av})(r-y))\right]\right\} \\
&\quad\quad \cdot \int_{\mathbb{R}} \frac{e^{2\pi i\eta(y-r)}\, d\eta\, d\sigma\, d\tau}{\int_{\mathbb{R}} \exp\left\{4\pi\left[\eta\varsigma - \sigma\varsigma^2 - \tau 2(\mathrm{av})\varsigma^3 - \tau\varsigma^4\right]\right\}\, d\varsigma}
\end{aligned}$$

Note that this cleanly preserves the nonnegative real part of the exponential in τ, while giving oscillation in the τ integral that is controlled by the analogue to the codimension one twisted distance. But it is still not at all clear how the inverse of the volume in (5.6) might be related to a tangible estimate on this formula.

Part III

Schrödinger Operators With A_∞ Potentials

Chapter 6

Motivation and Related Work

This part of this dissertation primarily concerns the heat kernels of Schrödinger operators $H = -\Delta + V$ on $L^2(\mathbb{R}^n)$, where nonnegative $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies a reverse Hölder inequality. Nonetheless there still exists a connection to CR manifolds similar to those discussed in Part II. For take with subharmonic ϕ the three-dimensional CR manifold

$$M = \{(z, w,) \in \mathbb{C}^2 \colon \ \operatorname{Im} w = \phi(z)\}$$

Then the Kohn-Laplacian on M, under an identification of M with $\mathbb{C} \times \mathbb{R}$ and a partial Fourier transform, goes over to a Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R})$. We motivate the partial Fourier transform by discussing its use by several other authors in a more general setting; and then introduce the reverse Hölder condition we will impose on V.

6.1 The Kohn-Laplacian heat equation

For any CR manifold the Kohn-Laplacian $\Box_b$ is defined by

$$\Box_b = \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*$$

where $\bar{\partial}_b^*$ is the formal adjoint of $\bar{\partial}_b$ in $L^2(M)$. The theory of $\Box_b$ when M is the boundary of a weakly pseudoconvex domain of finite type in $\mathbb{C}^2$ has been developed through the work of many authors, including Kohn [29], Fefferman [20], and Christ [13]. Of special interest to us are Nagel and Stein's results in [41] for the heat operator

$$L = \partial_t + \Box_b$$

Note that since $\Box_b$ acts as $\bar{\partial}_b^* \bar{\partial}_b$ on functions and $\bar{\partial}_b \bar{\partial}_b^*$ on $(0, 1)$-forms, there are actually two operators here. But, as Nagel and Stein point out, the bundle of $(0, 1)$-forms on M may be identified with functions, so the analysis corresponding to $\bar{\partial}_b^* \bar{\partial}_b$ is entirely analogous to that for $\bar{\partial}_b \bar{\partial}_b^*$.

For example, when $\Box_b$ acts on functions, they prove that the semigroup of operators $e^{-\Box_b t}$ on $L^2(M)$ are given by integration against a distribution kernel $H(t, p, q)$, and that for smooth data f the initial value problem

$$\begin{cases} L[F]f(p,t) = 0 & \text{on } M \times (0,\infty) \\ \lim_{t\to 0} F(\cdot,t) = f(\cdot) & \text{in } L^2(M) \end{cases}$$

is solved by $F = e^{-\Box_b t}[f]$. Then a central result in [41] is that for $p \neq q$, the kernel $H(t,p,q)$ satisfies, for any $N \in \mathbb{N}$,

$$H(t,p,q) \leq \frac{C_N}{|B(p,d(p,q))|}\left[\frac{t^N}{t^N + d(p,q)^{2N}}\right]$$

where $d(p,q)$ is the control distance on M from p to q, and the ball in the denominator is also in terms of the control geometry described in Chapter 3. In particular, when $p \neq q$, the Kohn-Laplacian heat kernel decays rapidly in $\frac{d(p,q)^2}{t}$.

6.2 Analysis of $\mathcal{F}[\Box_b]$ by Christ, and later work by Raich

For the particular case of $M \subset \mathbb{C}^2$ mentioned above, namely

$$M = \{(z,w) \in \mathbb{C}^2 \colon \operatorname{Im} w = \phi(z)\}$$

with ϕ subharmonic, several authors have used an alternative approach in analyzing $\Box_b$. Identifying M with $\mathbb{C}\times\mathbb{R}$ using coordinates (z,t), $\bar{\partial}_b$ goes over to the global tangential antiholomorphic vector field

$$\bar{L} = \frac{\partial}{\partial \bar{z}} - i\phi_{\bar{z}}(z)\frac{\partial}{\partial t}$$

Now a partial Fourier transform in the t variable yields $\hat{\bar{L}}_\tau = \partial_{\bar{z}} + \tau\phi_{\bar{z}}$ with L^2-adjoint $\hat{L}_\tau = -\partial_z + \tau\phi_z$. Hence one may study $\mathcal{F}[\Box_b]$ on the transform side in terms of the operators $\hat{\bar{L}}_\tau\hat{L}_\tau$ and $\hat{L}_\tau\hat{\bar{L}}_\tau$ on $L^2(\mathbb{C})$.

In [14], Christ undertook this study under the assumption that $\tau > 0$ and $dV = \tau\Delta\phi$ is a doubling Borel measure whose volume on balls of unit radius is uniformly bounded below. He estimated, for example, the distribution kernel $G(z,\zeta)$ of the solution operator $(\hat{\bar{L}}_\tau\hat{L}_\tau)^{-1}$, obtaining bounds in terms of a smooth Riemannian metric ρ and a function ϱ. The metric ρ derives from partitioning $\mathbb{C}$ into disjoint cubes $\{Q_j\}$ such that $dV(Q_j)$ is essentially constant; and then requiring that ρ measure the diameters of these Q_j as uniformly comparable to 1. The function $\varrho(z)$, on the other hand, specifies radii such that $dV(B(z,\varrho(z)) \sim 1$ uniformly in z. With these objects,

$$|G(z,\zeta)| \leq C \begin{cases} \log(2\varrho(z)/|z-\rho|) & \text{for } |z-\zeta| \leq \varrho(z) \\ e^{-\epsilon\rho(z,\zeta)} & \text{for } |z-\zeta| \geq \varrho(z) \end{cases}$$

where $C > 0$ and $\epsilon > 0$ depend only on the doubling constant of dV. Note that the abstract construction of ρ and ϱ means we "lose track" of the transform parameter τ in these estimates. So there is little hope of recovering information about $\Box_b$ on M by inverting the Fourier transform.

In a series of papers, Raich later surmounted this problem in the case that $\phi(z)$ is a subharmonic nonharmonic polynomial. First for $\tau > 0$ he proved in [45] that the distribution kernel $H_\tau(z,w,s)$ of the semigroup $e^{-s\hat{\bar{L}}_\tau \hat{L}_\tau}$ satisfies

$$\left|\frac{\partial^n}{\partial s^n} Y^\alpha H_\tau(z,w,s)\right| \leq \frac{c_0}{s^{n+\frac{1}{2}|\alpha|+1}} e^{-\frac{|z-w|^2}{32s}} e^{-c_1\frac{s}{\mu(z,1/\tau)^2}} e^{-c_1\frac{s}{\mu(w,1/\tau)^2}}$$

where $Y^{|\alpha|}$ is a product of $|\alpha|$ operators $Y = \hat{\bar{L}}_\tau$ or $\hat{L}_\tau$ when acting in z; or the complex conjugate of these operators when acting in w. The function $\mu(\cdot,\delta)$ is the inverse of the higher Levi-invariant from the control geometry on the polynomial model M, as described in Chapter 4. Then in [46] Raich observed that $\hat{\bar{L}}_\tau \hat{L}_\tau$ with $\tau < 0$ is equivalent to $\hat{L}_\tau \hat{\bar{L}}_\tau$ with $\tau > 0$, and proved compatible estimates on the distribution kernel $\tilde{H}_\tau(z,w,s)$ of the semigroup $e^{-s\hat{L}_\tau\hat{\bar{L}}_\tau}$. A representative estimate is

$$\left|Y^\alpha \tilde{H}_\tau(z,w,s)\right| \leq c_{|\alpha|} e^{-c\frac{|z-w|^2}{s}} \max\left\{\frac{e^{-c\frac{s}{\mu(z,1/\tau)^2}} e^{-c\frac{s}{\mu(w,1/\tau)^2}}}{s^{1+\frac{1}{2}|\alpha|}}, \frac{e^{-c\frac{|z-w|}{\mu(z,1/\tau)^2}} e^{-c\frac{|z-w|}{\mu(w,1/\tau)^2}}}{\mu(w,1/\tau)^{2+|\alpha|}}\right\}$$

with notation analogous to that in the $H_\tau(z,w,s)$ estimate. Finally in [11], joint work with Boggess, he combined these estimates with a characterization of functions whose Fourier transforms have exponential decay to obtain Gaussian decay for the $\Box_b$ heat kernel $\mathcal{H}(\alpha,\beta,s)$ on M, along with its derivatives in L and $\bar{L}$.

$$|X_\alpha^J X_\beta^{J'} \mathcal{H}(\alpha,\beta,s)| \leq C\frac{e^{-c\frac{d(\alpha,\beta)^2}{s}}}{d(\alpha,\beta)^{|J|+|J'|}|B(\alpha,d(\alpha,\beta))|} \tag{6.1}$$

The geometry here is again the control geometry on M, and e.g. X_α^J denotes a product of $|J|$ operators, each either L or $\bar{L}$, acting in the α coordinate of $\mathcal{H}$. This estimate improves the rapid decay of Nagel and Stein, and the development of (6.1) even suggests a strategy for obtaining Gaussian decay of the $\Box_b$ heat kernel on finite type M.

6.3 The case $\phi(z) = \phi(x)$ and a reverse Hölder condition

As in Part II we will assume that $\phi(z) = \phi(x)$. Let us see how this simplifies analysis of $\Box_b$ under the Fourier transform. First, if for general $\phi(z)$ we write out $\hat{\bar{L}}_\tau \hat{L}_\tau$ explicitly, we obtain

$$4\hat{\bar{L}}_\tau \hat{L}_\tau = -\Delta + \tau\Delta\phi + \tau^2|\nabla\phi|^2 + 2i\tau\left(\phi_x\partial_y - \phi_y\partial_x\right)$$

Both Christ and Raich point out that the first order terms in this family of operators are a source of tremendous complications. But under our assumption, $\bar{\partial}_b$ is identified with global tangential vector field

$$\bar{L} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\left[\frac{\partial}{\partial y} - \phi'\frac{\partial}{\partial t}\right]$$

which is now translation invariant in both t and y. As in Chapter 4, this suggests analyzing, for example, $\bar{\partial}_b\bar{\partial}_b^*$ under a partial Fourier transform in both the y and t directions. With dual variables η and τ respectively, we are now studying on the transform side a two-parameter family of operators, given (up to a constant multiple) by

$$\hat{\bar{L}}_{\eta\tau}\hat{L}_{\eta\tau} = -\Delta + \phi''\tau + (\eta - \tau\phi')^2$$

That is, we have a two-parameter family of Schrödinger-type operators on $L^2(\mathbb{R})$.

There are at least two natural directions to take in the study of these operators. We could assume that the underlying CR manifold M is the boundary of a weakly pseudoconvex domain of finite type, giving good smoothness properties to the zero order terms of $\hat{\bar{L}}_{\eta\tau}\hat{L}_{\eta\tau}$. Then we would hope to follow one or more of Raich's footsteps, first estimating derivatives of the heat kernel of the semigroup $e^{-s\hat{\bar{L}}_{\eta\tau}\hat{L}_{\eta\tau}}$ while retaining explicit control of the transform parameters η and τ. The second direction is to make assumptions on ϕ that ensure only mild regularity of the zero order term, and obtain non-sharp estimates on the transform side. This is what we will do. Precisely, we will require that the function

$$V_{\eta\tau}(x) = \phi''(x)\tau + (\eta - \tau\phi')^2$$

satisfy a reverse Hölder inequality. Let us recall what this means.

Definition 6.1. *For $1 < q \leq \infty$, nonnegative $V \in L^q_{loc}(\mathbb{R}^n)$ belongs to the reverse Hölder class RH_q if there exists $C > 0$ such that for all cubes Q of $\mathbb{R}^n$,*

$$\left(\frac{1}{|Q|}\int_Q V^q\,dx\right)^{1/q} \leq \frac{C}{|Q|}\int_Q V\,dx$$

where for $q = \infty$ the left hand side is the ess sup over Q. We write $A_\infty = \cup_{q>1} RH_q$.

We have stated the definition for $\mathbb{R}^n$, $n \geq 1$, because our estimates will actually hold in this generality. In particular, for $H = -\Delta + V$ on $L^2(\mathbb{R}^n)$, we will obtain upper bounds on the heat kernel of H with $V \in A_\infty$; and lower bounds on the heat kernel of H with $V \in RH_\infty$.

Typical examples of potentials in A_∞ are the power weights $|x|^\alpha$ with $-n < \alpha$. The A_∞ class was originally discovered by B. Muckenhoupt in connection to his work in [35] on the Hardy-Littlewood maximal function; it is exactly the class of weights ω such that the Hardy-Littlewood maximal function is a bounded operator on $L^p(d\omega)$ for some $1 \leq p < \infty$. (See García-Cuerva and Rubio De Francia [22] or Stein [53] for the precise relationship between the RH_q and A_p classes.) We will use Muckenhoupt's characterization of A_∞ when considering the sharpness of our upper bounds in the case that V is a power weight. It will also be important for us to note that A_∞ weights are doubling measures; Stein's exposition of this fact in [53] includes examples of doubling measures which are not A_∞ weights.

Chapter 7

Real Schrödinger Operators on $L^2(\mathbb{R}^n)$

In this chapter we give useful background for our work, recalling the quadratic form definition of a Schrödinger operator $H = -\Delta + V$ on $L^2(\Omega)$ when $\Omega \subseteq \mathbb{R}^n$ and $V \in L^1_{\text{loc}}(\Omega)$ is nonnegative. We review why H generates a strongly continuous symmetric semigroup e^{-Ht} on $L^2(\mathbb{R}^n)$, and recall the existence of its heat kernel. We also indicate some classical sharp estimates on Schrödinger heat kernels, noting that stronger assumptions than just $V \in A_\infty$ are required for these estimates. (Detailed review of existing work with A_∞ potentials is postponed to Chapter 8, since there we will build on it directly to obtain our own upper bounds.) At the end of this chapter we compute a heat kernel explicitly for later use in sharpness considerations.

7.1 Definition of $-\Delta + V$ with $V \geq 0$ in L^1_{loc}

Our aim is simply to recall the key results from Dirichlet form theory which motivate defining $H = -\Delta + V$ through its quadratic form. After showing that this form meets the conditions required of a Dirichlet form, we obtain a host of good properties for e^{-Ht}. Our primary source is [16] by Davies. Although the results we prove in Chapters 8 and 9 are all for Schrödinger operators on $L^2(\mathbb{R}^n)$, it will also be useful for us to know that, for smoothly bounded $\Omega \subset\subset \mathbb{R}^n$, we may define H on $L^2(\Omega)$ with Dirichlet boundary conditions and obtain the same good properties for the heat kernel. Hence the results that follow are given for H on $L^2(\Omega)$.

The Buerling-Deny conditions and symmetric Markov semigroups

In their paper [8] Beurling and Deny introduce the concept of a Dirichlet form Q on a measure space (X, μ). Such a Q is a positive symmetric bilinear form defined on a dense subspace $D \subset L^2(X, \mu)$ which meets the following two conditions.

1. D under the inner product $\langle u, v\rangle_Q = \langle u, v\rangle_{L^2(X,\mu)} + Q(u, v)$ is a real Hilbert space.

2. For all $u \in D$, $v = \min(u_+, 1)$ also belongs to D, and $Q(v) := \langle v, v\rangle_Q \leq Q(u)$.

We will always take (X, μ) to be (Ω, dx); and in this case there are numerous equivalent formulations for the Beurling-Deny conditions, as we will note below.

It is Theorem 1.2.1 in [16] that any positive symmetric bilinear Q which satisfies condition (1) uniquely extends to a positive self-adjoint operator H on all of $L^2(\Omega)$. Then if Q also satisfies condition (2), the operator e^{-Ht} defined on $L^2(\Omega)$ through the functional calculus is known as a symmetric Markov semigroup, and one has the following.

Theorem 7.1. *If e^{-Ht} is a symmetric Markov semigroup on $L^2(\Omega)$ then $L^1 \cap L^\infty$ is invariant under e^{-Ht}, and e^{-Ht} may be extended from $L^1 \cap L^\infty$ to a positive one-parameter contraction semigroup $T_p(t)$ on L^p for all $1 \le p \le \infty$. These semigroups are strongly continuous if $1 \le p < \infty$, and $T_p(t)f = T_q(t)f$ when $f \in L^p \cap L^q$.* □

This is Theorem 1.4.1 in [16], and will yield in particular the crucial property that e^{-Ht} with $H = -\Delta + V$ is strongly continuous on $L^2(\Omega)$—once we have verified that the quadratic form defining $-\Delta + V$ is a Dirichlet form, of course.

Checking the Buerling-Deny conditions

The quadratic form corresponding to $-\Delta + V$ is

$$Q(u, v) = \int_\Omega \nabla u \cdot \nabla v + V uv \, dx$$

well-defined on the domain

$$D = \{f \in L^2(\Omega) \colon \nabla f \text{ and } V^{1/2} f \in L^2(\Omega)\}$$

Evidently Q is symmetric, $Q(u) \ge 0$, and D is dense in $L^2(\Omega)$. To check Buerling-Deny condition (1) we need to verify that D is a real Hilbert space under the norm

$$\|f\|_D = \left(\|f\|_2^2 + \|\nabla f\|_2^2 + \|V^{1/2} f\|_2^2\right)^{1/2}$$

But we may just note

$$\|f\|_D^2 = \|f\|_{H^1(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega, dV^{1/2})}^2$$

and since both Sobolev and weighted L^2 spaces are well known to be Hilbert spaces, the right-hand norms all satisfy the parallelogram law, and $\|f\|_D$ does as well. Hence Q extends uniquely to a positive self-adjoint operator H on $L^2(\Omega)$.

Checking the second Buerling-Deny condition is more involved. We use an equivalent formulation of the condition that is implied by Theorems 1.3.2 and 1.3.3 in [16], namely

Theorem 7.2. *A closed positive symmetric bilinear form Q on $L^2(\Omega)$ is a Dirichlet form if the semigroup e^{-Ht} generated by its associated operator H is positivity-preserving and a contraction on $L^\infty(\Omega)$ for all $t \geq 0$.* $\square$

Let us write $H_0 = -\Delta$ as the Laplacian on Ω, with Dirichlet boundary conditions if $\Omega \subset\subset \mathbb{R}^n$. If $\Omega = \mathbb{R}^n$, then from the classical theory of the heat equation,

$$e^{-H_0 t} = \left((4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}} \right) * f$$

so we have the norm inequalities

$$\|e^{-H_0 t}\|_{\infty,1} \leq (4\pi t)^{-n/2} \quad \text{and} \quad \|e^{-H_0 t}\|_{\infty,\infty} \leq 1 \tag{7.1}$$

And e^{-Ht} is a contraction on L^∞. Then it follows as in Theorem 2.1.6 of [16] that $e^{-H_0 t}$ for $\Omega \subset\subset \mathbb{R}^n$ satisfies the same inequalities. Now we may apply the Trotter product formula for self-adjoint contraction semigroups (see [26]), that is

$$e^{-Ht} f = \lim_{n\to\infty} (e^{-H_0 t/n} e^{-Vt/n})^n f \qquad f \in L^2(\Omega),\ \forall t \geq 0 \tag{7.2}$$

Hence e^{-Ht} is the limit of positivity-preserving contractions on L^∞, and by Theorems 1.2.2 and 1.2.3 in [16] possesses the same properties itself. We conclude by Theorem 7.2 that e^{-Ht} is a symmetric Markov semigroup. It is interesting to note that because $C_c^\infty(\Omega)$ is dense in D in the norm $\|\cdot\|_D$ (proved in Theorem 1.8.1 in [16]), it follows per Spina's comments in [52] that we would actually get the same operator by defining $H : L^2(\Omega) \mapsto \mathcal{S}'(\Omega)$ in the sense of distributions. The power of the Dirichlet theory is highlighted by the distance the latter definition would seem to leave us from the conclusions of Theorem 7.1.

7.2 Existence of the heat kernel and known estimates

The existence of the heat kernel now follows from the classical result (see e.g. [2]) that a bounded operator from $L^1(\Omega)$ to $L^\infty(\Omega)$ is given almost everywhere by integration against a kernel in $L^\infty(\Omega \times \Omega)$. For, letting $\|A\|_{q,p}$ be the operator norm of $A\colon L^p(\Omega) \mapsto L^q(\Omega)$, we have by taking adjoints that

$$\|e^{-Ht}\|_{\infty,1} \leq \|e^{-Ht/2}\|_{\infty,2} \cdot \|e^{-Ht/2}\|_{2,1} = \|e^{-Ht/2}\|^2_{\infty,2} \tag{7.3}$$

And observe that (7.2) implies $\|e^{-Ht}\|_{\infty,2} \leq \|e^{-H_0}t\|_{\infty,2}$. Furthermore, by the Riesz-Thorin interpolation theorem and (7.1) we know since $V \geq 0$ that

$$\|e^{-H_0t}\|_{\infty,2} \leq \|e^{-H_0t}\|^{1/2}_{\infty,1} \cdot \|e^{-H_0t}\|^{1/2}_{\infty,\infty} \leq (4\pi t)^{-n/4}$$

which combined with (7.3) gives boundedness of e^{-Ht} from L^1 to L^∞ for every $t > 0$. Hence

$$e^{-Ht}f = \int_\Omega p(x,y,t)f(y)\,dy \qquad f \in L^2(\Omega),\ t \geq 0$$

We say $p(x,y,t)$ is the heat kernel associated to H.

In this generality we cannot actually conclude that $p(\cdot,y,t)$ is a classical solution to the equation $(\partial_t + H)u = 0$ in $\Omega \times (0,\infty)$. This would mean that on every cylinder in $\Omega \times (0,\infty)$ of the form

$$Q_r(x_0,t_0) = B(x_0,r) \times I_{t_0,r} = B(x_0,r) \times (t_0 - r^2, t_0)$$

we would have $p(\cdot,y,t) \in C^{2,1}(Q_r(x_0,t_0))$ and $-\Delta p(\cdot,y,t) = \partial_t p(\cdot,y,t)$ inside $Q_r(x_0,t_0)$. (For a such a regularity result when $V \in C^\alpha_{\mathrm{loc}}(\Omega)$ satisfies a Hölder continuity hypothesis, see [33].) Instead we can only say that $p(\cdot,y,t)$ is a weak solution of $(\partial_t + H)u = 0$ on every such cylinder, in the sense defined below. This crucial conclusion follows because e^{-Ht} is a strongly continuous semigroup on $L^2(\Omega)$; for a proof see [5] by J.M. Ball.

Definition 7.3. *A real-valued function $u(x,t)$ is a weak solution to $(\partial_t + H)u = 0$ in $Q_r(x_0,t_0)$ if $u \in L^\infty(L^2(B(x_0,r)); I_{t_0,r}) \cap L^2(H^1(B(x_0,r)); I_{t_0,r})$ satisfies*

$$\begin{aligned}\int_{B(x_0,r)} u(x,t)\phi(x,t)\,dx &- \int_{t_0-r^2}^t \int_{B(x_0,r)} u(x,s)\partial_s\phi(x,s)\,dx\,ds \\ &+ \int_{t_0-r^2}^t \int_{B(x_0,r)} \big(\nabla u(x,s)\cdot\nabla\phi(x,s) + V(x)u(x,s)\phi(x,s)\big)\,dx\,ds = 0\end{aligned} \tag{7.4}$$

for $t_0 - r^2 < t \leq t_0$ *and for every* $\phi(x,s) \in \mathcal{C}$, *where*

$$\mathcal{C} = \{\phi \in L^2(H^1(B(x_0,r)); I_{t_0,r}) \text{ and } \partial_s\phi \in L^2(L^2(B(x_0,r); I_{t_0,r}); \phi(x, r_0 - r^2)) = 0\}$$

Notable estimates on heat kernels

Given existence of the heat kernel for $H = -\Delta + V$, it follows directly from (7.2), taking f to be a sequence of test functions converging to a point mass, that we have a pointwise Gaussian bound

$$p(x,y,t) \leq (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$$

So the basic theme in work on upper bounds for $p(x,y,t)$ is finding extra-Gaussian decay in terms of V, for as large a class of potentials as possible. Significant here are Davies' applications of his theory of logarithmic Sobolev inequalities for second order partial differential operators. For H as above with continuous potential diverging to infinity as $|x| \to \infty$, he proves in [16] that

$$p(x,y,t) \leq c(t)\phi(x)\phi(y) \tag{7.5}$$

where ϕ is the L^2-normalized ground state of H, and $c(t)$ has an explicit description as $t \to 0$. Davies has also proven, by extension of the same techniques, far more complicated upper bounds holding even for potentials with local singularities—at least for H defined with Dirichlet conditions on smoothly bounded domains in $\mathbb{R}^n$, $n \geq 3$.

For general V we clearly cannot have a global Gaussian lower bound on $p(x,y,t)$. A precise answer to the question of when such a bound exists was obtained in [57] by Zhang and Zhao for potentials which belong to a local Kato class. They proved lower bounds of the form

$$p(x,y,t) \geq \begin{cases} \frac{c_1}{t^{n/2}} e^{-c_2 K_{V^+}(t)} & |x-y|^2 \leq t \\ \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}[1 + K_{V^+}(\frac{t^2}{|x-y|^2})]} & |x-y|^2 \geq t \end{cases} \tag{7.6}$$

where

$$K_V(t) = \sup_x \int_0^t \int_{\mathbb{R}^n} \frac{1}{(t-s)^{n/2}} e^{-c\frac{|x-y|^2}{t-s}} |V(y)|\, dy\, ds$$

A direct implication for nonnegative V is that $p(x,y,t)$ has a global Gaussian lower bound if and only if V is Green-bound; i.e. its convolution with the fundamental solution of the Laplacian is globally bounded.

Note that neither (7.5) nor (7.6) applies to the situation we motivated in Section 6.3, when H is defined on $L^2(\mathbb{R})$ with $V \in A_\infty$. Potentials in A_∞ may have local singularities; and they need not behave uniformly at infinity—consider $V(x) = x^2(1 + \sin(|x|^{1/2}))$, for example. However, Shen's work in [50] on Schrödinger operators with A_∞ potentials provides an excellent entry point for our analysis, in particular as it was applied by K. Kurata in [31] to heat kernels of Schrödinger operators. Both of these authors' work actually pertains to a larger class of Schrödinger operators with non-vanishing magnetic potentials such as

$$H_M = (i^{-1}\nabla - a(x))^2 + V(x)$$

But in Chapter 8 we will only treat the case $H = -\Delta + V$. Let us first pause to calculate a heat kernel explicitly; this will afford us some direct perspective of the sharpness of the bounds we obtain.

7.3 A calculation for quadratic potentials on $\mathbb{R}$

Take $V(x) = \sum_{i=0}^{2} a_i x^i$ with $a_2 > 0$; we will compute the heat kernel $p(x, y, t)$ associated to $H = -\Delta + V$ on $L^2(\mathbb{R})$. Note that it suffices to treat the case $a_0 = 0$, because

$$\left[\partial_t - \Delta + (a_2 x^2 + a_1 x + a_0)\right] u = e^{-a_0 t} \left[\partial_t - \Delta + (a_2 x^2 + a_1 x)\right] e^{a_0 t} u. \tag{7.7}$$

So if $p_0(x, y, t)$ is the heat kernel for the operator with potential $V(x) = \sum_{i=1}^{2} a_i x^i$, then $e^{-a_0 t} p_0(x, y, t)$ is directly checked to be the heat kernel when $V(x) = \sum_{i=0}^{2} a_i x^i$.

Several approaches to the calculation are possible. Interpreting $p(x, y, t)$ as the transition probability of a system from state x to state y in time t, the kernel is determined by a certain path integral of the Lagrangian given by $(-\Delta + V)$. For quadratic V this path integral can then be computed using the van Vleck determinant (see [56] for details.) Another possibility is to begin with the Mehler kernel of the harmonic oscillator (see again [16]) and study the behavior of this kernel under appropriate scalings and translations of the harmonic oscillator. Our method, requiring rather less theory than either of the above, is to take from [6] an ansatz proposed by Beals:

$$p(x, y, t) = \phi(t) \exp\left\{-\frac{1}{2}\left(\alpha(t)x^2 + \gamma(t)y^2 + 2\beta(t)xy\right)\right\} \exp\left\{-\mu(t)x - \nu(t)y\right\}.$$

We then simply attempt to enforce on this ansatz the two conditions

$$\begin{cases} (\partial_t + H)p(\cdot, y, t) = 0 & \text{on } \mathbb{R} \times (0, \infty) \\ \lim_{t \to 0^+} p = \delta(x - y) & \text{in } L^1(\mathbb{R}) \end{cases} \tag{7.8}$$

The differential condition in (7.8) yields a system of six ODE's in t.

$$\alpha' = -2\alpha^2 + 2a_2 \tag{7.9}$$

$$\beta' = -2\alpha\beta \tag{7.10}$$

$$\gamma' = -2\beta^2 \tag{7.11}$$

$$\mu' = a_1 - 2\mu\alpha \tag{7.12}$$

$$\nu' = -2\mu\beta \tag{7.13}$$

$$\phi'/\phi = -\alpha + \mu^2 \tag{7.14}$$

These equations are solvable, in order, by elementary methods. The constants of integration must be chosen according to the second condition in (7.8). For $\beta(t)$ this is actually easy to do, but in solving the remaining equations we set constants of integration to zero and identify their true values only after the functional form of $p(x, y, t)$ is known.

First, separating variables in (7.9) gives

$$\alpha(t) = \sqrt{a_2} \coth 2\sqrt{a_2}t$$

Then we have with $C \in \mathbb{R}$ arbitrary

$$\beta(t) = C \operatorname{csch} 2\sqrt{a_2}t$$

Which implies

$$\gamma(t) = \frac{C^2}{\sqrt{a_2}} \coth 2\sqrt{a_2}t$$

Given (7.8), we expect the exponentials in the ansatz to tend to a constant as $t \to 0$ when $x = y$. This makes it natural to set $C = -\sqrt{a_2}$, the value which satisfies

$$\sqrt{a_2} + \frac{C^2}{\sqrt{a_2}} + 2C = 0$$

(In any case this choice is justified by our later calculations.) It then follows

$$\mu(t) = \frac{a_1}{2\sqrt{a_2}} \coth 2\sqrt{a_2}t \quad \text{and} \quad \nu(t) = \frac{-a_1}{2\sqrt{a_2}} \operatorname{csch} 2\sqrt{a_2}t$$

And finally for some volume constant $C > 0$

$$\phi(t) = C\,(\operatorname{csch} 2\sqrt{a_2}t)^{1/2}\ e^{\frac{a_1^2}{4a_2}t} \exp\left\{-\frac{{a_1}^2}{(2\sqrt{a_2})^3}\coth 2\sqrt{a_2}t\right\}$$

Now the singularity condition of (7.8) is equivalent to two properties. First,

$$\lim_{t\to 0} p(x, y, t) = 0$$

whenever $x \neq y$; and, second,

$$\lim_{t\to 0} \int_{\mathbb{R}} p(x, y, t)\, dy = 1$$

for any $x \in \mathbb{R}$. For suppose both these properties hold. Then given $\epsilon > 0$, the Gaussian is an integrable function on $\mathbb{R} \setminus B_\epsilon(x)$ that dominates $p(x, y, t)$. It follows by the Lebesgue dominated convergence theorem that

$$\lim_{t\to 0} \int_{\mathbb{R}\setminus B_\epsilon(x)} p(x, y, t)\, dy = 0$$

for any $\epsilon > 0$, so $\lim_{t\to 0} p = \delta(x - y)$ in $L^1(\mathbb{R})$.

To ensure the first property, we write the candidate kernel $p_a(x, y, t)$ in terms of $(x - y)$ wherever possible and obtain

$$\begin{aligned} p_a(x, y, t) = C\,(\operatorname{csch} 2\sqrt{a_2}t)^{1/2}\ & e^{\frac{a_1^2}{4a_2}t} \exp\left\{-\frac{{a_1}^2}{(2\sqrt{a_2})^3}\coth 2\sqrt{a_2}t\right\} \\ &\cdot \exp\left\{-\frac{\sqrt{a_2}}{2}\Big((x-y)^2 \operatorname{csch} 2\sqrt{a_2}t + (x^2+y^2)(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\Big)\right\} \\ &\cdot \exp\left\{-\frac{a_1}{2\sqrt{a_2}}\Big((x-y)\operatorname{csch} 2\sqrt{a_2}t + x(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\Big)\right\} \end{aligned}$$

Note that the difference $(\coth - \operatorname{csch})(t)$ converges to 0 as $t \to 0$. Hence, writing $\epsilon = (y - x)$, we essentially have

$$\begin{aligned} p_a(x, y, t) = C f(x, y, t) \cdot (\operatorname{csch} 2\sqrt{a_2}t)^{1/2} \\ \cdot \exp\left\{-\frac{1}{2}\operatorname{csch} 2\sqrt{a_2}t\Big(\sqrt{a_2}\,\epsilon^2 - \frac{a_1}{\sqrt{a_2}}\,\epsilon + \frac{a_1^2}{4(\sqrt{a_2})^3}\cosh 2\sqrt{a_2}t\Big)\right\} \end{aligned}$$

where $\lim_{t\to 0^+} f = 1$ independent of x and y. Completing the square in ϵ, it follows

$$\begin{aligned} p_a(x,y,t) = C f(x,y,t) \cdot (\operatorname{csch} 2\sqrt{a_2}t)^{1/2} \\ \cdot \exp\left\{-\frac{1}{2}\operatorname{csch} 2\sqrt{a_2}t\left(\sqrt{a_2}\,(\epsilon - \frac{a_1}{2a_2})^2\right)\right\} \\ \cdot \exp\left\{-\frac{1}{2}\operatorname{csch} 2\sqrt{a_2}t\left(\frac{{a_1}^2}{4(\sqrt{a_2})^3}(\cosh 2\sqrt{a_2}t - 1)\right)\right\} \end{aligned}$$

So our first candidate for the kernel concentrates its mass at $y = (x + a_1/2a_2)$ as $t \to 0$, instead of the desired $y = x$. We correct this by shifting $y \mapsto (y + a_1/2a_2)$ and obtain

$$\begin{aligned} p_b(x,y,t) = C\,(\operatorname{csch} 2\sqrt{a_2}t)^{1/2}\ e^{\frac{a_1^2}{4a_2}t} \exp\left\{-\frac{{a_1}^2}{4(\sqrt{a_2})^3}(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\right\} \\ \cdot \exp\left\{-\frac{\sqrt{a_2}}{2}\left((x-y)^2 \operatorname{csch} 2\sqrt{a_2}t + (x^2+y^2)(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\right)\right\} \\ \cdot \exp\left\{-\frac{a_1}{2\sqrt{a_2}}\left((x+y)(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\right)\right\} \end{aligned}$$

It remains to fix the volume constant $C > 0$ so that $\int_{\mathbb{R}} p(x,y,t)\,dy \to 1$ as $t \to 0$, for any $x \in \mathbb{R}$. Integrating in y (which is simply a matter of integrating a Gaussian) we see

$$\begin{aligned} \int_{\mathbb{R}} p_b(x,y,t)\,dy = C\left(\frac{2\pi \operatorname{csch} 2\sqrt{a_2}t}{\sqrt{a_2}\coth 2\sqrt{a_2}t}\right)^{1/2} e^{\frac{a_1^2}{4a_2}t} \\ \cdot \exp\left\{-\frac{{a_1}^2}{8(\sqrt{a_2})^3}(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t \operatorname{sech} 2\sqrt{a_2}t)\right\} \\ \cdot \exp\left\{-\frac{a_1}{2\sqrt{a_2}}\,x(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t \operatorname{sech} 2\sqrt{a_2}t)\right\} \\ \cdot \exp\left\{-\frac{\sqrt{a_2}}{2}\,x^2(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t \operatorname{sech} 2\sqrt{a_2}t)\right\} \end{aligned}$$

And from l'Hôpitals rule it is clear that $\lim_{t\to 0} \int_{\mathbb{R}} p_b(x,y,t)\,dy = C\,(2\pi/\sqrt{a_2})^{1/2})$; so we choose $C = (\sqrt{a_2}/2\pi)^{1/2}$. We conclude that in this setting the heat kernel is

$$\begin{aligned} p(x,y,t) = \left(\frac{\sqrt{a_2}\operatorname{csch} 2\sqrt{a_2}t}{2\pi}\right)^{1/2} e^{\left(\frac{a_1^2}{4a_2} - a_0\right)t} \\ \cdot \exp\left\{-\frac{{a_1}^2}{4(\sqrt{a_2})^3}(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\right\} \\ \cdot \exp\left\{-\frac{\sqrt{a_2}}{2}\left((x-y)^2 \operatorname{csch} 2\sqrt{a_2}t + (x^2+y^2)(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\right)\right\} \\ \cdot \exp\left\{-\frac{a_1}{2\sqrt{a_2}}\left((x+y)(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\right)\right\} \end{aligned}$$

The symmetry in x and y was of course guaranteed by the theory of Section 7.2.

Chapter 8

Upper Bounds for A_∞ Potentials

We now continue our project to estimate the heat kernel of the operator $H = -\Delta + V$ on $L^2(\mathbb{R})$ when $V \in A_\infty$. Our result is an upper bound that applies to the heat kernels of Schrödinger operators with reverse Hölder potentials on any $L^2(\mathbb{R}^n)$, and gives extra-Gaussian decay in terms of a sub-linear function m_β of the time-scaled average of V over cubes.

$$p(x, y, t) \leq \frac{c_0}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \exp\left\{-c_1 m_\beta(t \operatorname{av}_{Z_{\sqrt{t}}(x)} V)^{1/2}\right\} \tag{8.1}$$

We obtain the estimate by combining an iteration argument used in Kurata [31] with a Fefferman-Phong inequality proved by Auscher and Ben Ali in [4]. The results of these authors are covered in Sections 8.1 and 8.2; the proof in Section 8.3 is then a matter of verifying that Kurata's machinery still runs with an different motor.

8.1 Ingredients of Kurata's estimates

In [31], Kurata draws on ideas of Shen to estimate the heat kernel of H on $L^2(\mathbb{R}^n)$ when $n \geq 2$ and $V \in RH_q$ with $q \geq n/2$. He obtains upper bounds of the form

$$p(x, y, t) \leq \frac{c_0}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \exp\left\{-c_1(1 + m_V(x)^2 t)^{1/(2(k_0+1))}\right\} \tag{8.2}$$

where $m_V(x)$ is a function measuring the effective growth of V near x. His argument uses three inequalities: local boundedness of weak solutions to the equation $(\partial_t + H)u = 0$; a Caccioppoli-type inequality; and a Fefferman-Phong inequality. When one considers the information flow in these inequalities from the right perspective, an iteration argument combining them becomes highly intuitive. In this section we carefully introduce each inequality as it appears in [31]. This illuminates how the argument works, and in particular shows why Kurata's results do not apply to $H = -\Delta + V$ on $L^2(\mathbb{R})$ with $V \in A_\infty$.

Local boundedness of weak solutions

The starting point of the argument, per Section 7.2, is that for $y \in \mathbb{R}^n$ fixed, the heat kernel $p(\cdot, y, t)$ is a weak solution to $(\partial_t + H)u = 0$, in any cylinder $Q_r(x_0, t_0)$ with $0 < r < \sqrt{t_0}$. The key result is then

Theorem 8.1 (Moser)**.** *Let $u \geq 0$ be a weak solution of $(\partial_t + H)u = 0$ in $Q_{2r}(x_0, t_0)$. There exists $C > 0$, depending only on $n \geq 1$, such that*

$$\sup_{Q_{r/2}(x_0,t_0)} |u(x,t)| \leq \left(\frac{C}{r^{n+2}} \iint_{Q_{2r/3}(x_0,t_0)} |u|^2 \, dx \, dt \right)^{1/2} \tag{8.3}$$

Sketch of proof. We give a precise reference because this fact is so basic for us. Suppose $Q_{2r}(x_0, t_0) = B_2(0) \times (0, 4)$. Note that because $V \geq 0$, u is a weak subsolution of $(\partial_t - \Delta)u = 0$ in $Q_2(0, 4)$. So subsolution estimates for the heat equation apply. In particular, a slight modification in the geometry of Moser's Theorem 1 in [34] (see especially pp. 124-125) establishes

$$\sup_{Q_{1/2}(0,4)} |u(x,t)| \leq \left(C \iint_{Q_{2/3}(0,4)} |u|^2 \, dx \, dt \right)^{1/2}$$

Translation invariance of the heat equation then implies the lemma with $r = 1$, and the result for arbitrary $r > 0$ follows from invariance of the heat equation under the scaling $x \to rx$, $t \to r^2 t$. $\square$

A Caccioppoli-type inequality

Roughly speaking, a Caccioppoli inequality bounds the local energy of a weak solution to an elliptic or parabolic equation by its L^2 norm over a slightly larger set. (See the notes by J. Hutchinson in [25].) The following standard result is Kurata's Lemma 3, with the proof repeated here for completeness.

Lemma 8.2. *Fix $\sigma \in (0, 1)$. If u is a weak solution to $(\partial_t + H)u = 0$ in $Q_{2r}(x_0, t_0)$, then there exists $C > 0$ such that*

$$\sup_{t_0-(\sigma r)^2 \leq t \leq t_0} \int_{B(x_0,\sigma r)} |u(x,t)|^2 \, dx \; + \iint_{Q_{\sigma r}(x_0,t_0)} \left(|\nabla u|^2 + V\, |u|^2 \right) dx \, ds \leq \frac{C}{(1-\sigma)^2 r^2} \iint_{Q_r(x_0,t_0)} |u|^2 \, dx \, dt$$

The constant C depends only on $n \geq 1$.

Proof. First, the argument in §9 of [3] allows us to assume that u has a strong derivative $\partial_t u \in L^2(Q_{2r}(x_0,t_0))$. Now choose nonnegative cutoff functions $\chi(x) \in C_0^\infty(B(x_0,r))$ and $\eta(s) \in C^\infty(\mathbb{R})$, bounded above by 1 and satisfying

- $\chi(x) \equiv 1$ on $B(x_0,\sigma r)$, $|\nabla\chi(x)| \le \dfrac{C}{(1-\sigma)r}$
- $\eta(s) \equiv 0$ for $s \le t_0 - r^2$, $\eta(s) \equiv 1$ for $s \ge t_0 - (\sigma r)^2$, $|\eta'(s)| \le \dfrac{C}{(1-\sigma)r^2}$

Fix $t \in [t_0 - (\sigma r)^2, t_0]$. Note that the test function $\eta^2(s)\chi^2(x)u(x,s)$ belongs to the class $\mathcal{C}$ specified in Definition 7.3. Hence we may use this function for $\phi(x,s)$ in (7.4). This yields, since $\eta(t) = 1$,

$$\int_{B(x_0,r)} u^2\chi^2\,dx - \int_{t_0-r^2}^{t}\int_{B(x_0,r)} (u^2(2\eta\,\eta')\chi^2 + (u\,\partial_s u)\eta^2\chi^2)\,dx\,ds \\ + \int_{t_0-r^2}^{t}\int_{B(x_0,r)} ((\nabla u\cdot\nabla\chi^2)\eta^2 u + |\nabla u|^2\eta^2\chi^2 + Vu^2\eta^2\chi^2)\,dx\,ds = 0. \quad (8.4)$$

Note that the second integral in (8.4) may be written as

$$\int_{t_0-r^2}^{t}\int_{B(x_0,r)} \frac{1}{2}\partial_s(u^2\eta^2\chi^2)\,dx\,ds + \int_{t_0-r^2}^{t}\int_{B(x_0,r)} (u^2(\eta\,\eta')\chi^2\,dx\,ds.$$

And by the bounded convergence theorem we may interchange integration and differentiation in the first term above, so because $\eta(t_0 - r^2) = 0$ it follows

$$\int_{t_0-r^2}^{t}\int_{B(x_0,r)} \frac{1}{2}\partial_s(u^2\eta^2\chi^2)\,dx\,ds = \frac{1}{2}\int_{B(x_0,r)} u^2\chi^2\,dx$$

Substituting these observations into (8.4), we obtain

$$\frac{1}{2}\int_{B(x_0,r)} u^2\chi^2\,dx + \int_{t_0-r^2}^{t}\int_{B(x_0,r)} \left(|\nabla u|^2\eta^2\chi^2 + Vu^2\eta^2\chi^2\right)dx\,ds \\ = \int_{t_0-r^2}^{t}\int_{B(x_0,r)} \left(u^2\chi^2\eta\,\eta' - (\nabla u\cdot\nabla\chi^2)\,\eta^2 u\right)dx\,ds \quad (8.5)$$

Since t was arbitrary and $V \ge 0$, we next conclude

$$\sup_{t_0-(\sigma r)^2\le t\le t_0} \frac{1}{2}\int_{B(x_0,r)} u^2\chi^2\,dx \le \iint_{Q_r(x_0,t_0)} \left(u^2|\eta'| + |\nabla u|\chi\,\eta^2\,|u||\nabla\chi|\right)dx\,ds$$

In the second term of the righthand integral we apply Cauchy's inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$, using $a = |\nabla u|\chi$, $b = |u||\nabla\chi|$. It follows

$$\sup_{t_0-(\sigma r)^2 \le t \le t_0} \frac{1}{2}\int_{B(x_0,r)} u^2\chi^2\,dx \le \iint_{Q_r(x_0,t_0)} u^2|\eta'|\,dx\,ds\ + \\ \frac{1}{2}\iint_{Q_r(x_0,t_0)} \chi^2\eta^2|\nabla u|^2\,dx\,ds + \frac{1}{2}\iint_{Q_r(x_0,t_0)} |u|^2|\nabla\chi|^2\eta^2\,dx\,ds$$

And from the bounds on $|\nabla\chi|$, η, and $|\eta'|$,

$$\sup_{t_0-(\sigma r)^2 \le t \le t_0} \int_{B(x_0,r)} u^2\chi^2\,dx \\ \le \frac{C}{(1-\sigma)^2r^2}\iint_{Q_r(x_0,t_0)} u^2\,dx\,ds\ + \iint_{Q_r(x_0,t_0)} \chi^2\eta^2|\nabla u|^2\,dx\,ds \quad (8.6)$$

To complete the proof we make another application of (8.5), this time with $t = t_0$. The positivity of the leftmost integral and the bounds on χ, η, and $|\eta'|$ yield

$$\iint_{Q_r(x_0,t_0)} \left(|\nabla u|^2\chi^2\,\eta^2 + Vu^2\chi^2\eta^2\right)dx\,ds \\ \le \frac{C}{(1-\sigma)r^2}\iint_{Q_r(x_0,t_0)} u^2\,dx\,ds + \iint_{Q_r(x_0,t_0)} |\nabla u|\chi\,\eta^2\,|u||\nabla\chi|\,dx\,ds$$

And the above use of Cauchy's inequality gives, after rearranging, absorbing terms, and possibly increasing C,

$$\iint_{Q_r(x_0,t_0)} \chi^2\eta^2|\nabla u|^2\,dx\,ds + \iint_{Q_r(x_0,t_0)} Vu^2\chi^2\eta^2\,dx\,ds \\ \le \frac{C}{(1-\sigma)^2r^2}\iint_{Q_r(x_0,t_0)} u^2\,dx\,ds \quad (8.7)$$

Then we combine (8.6) and (8.7) to obtain

$$\sup_{t_0-(\sigma r)^2 \le t \le t_0} \int_{B(x_0,r)} u^2\chi^2\,dx + \iint_{Q_r(x_0,t_0)} \chi^2\eta^2|\nabla u|^2\,dx\,ds \\ + \iint_{Q_r(x_0,t_0)} Vu^2\chi^2\eta^2\,dx\,ds \le \frac{C}{(1-\sigma)^2r^2}\iint_{Q_r(x_0,t_0)} u^2\,dx\,ds.$$

Restricting the lefthand integrals to where the cutoff functions are unity yields the lemma. We note that actually (8.7) is all that is required for the proof of Theorem 8.4. $\square$

A Fefferman-Phong inequality

In [18], Fefferman's exposition of his joint work with Phong on the uncertainty principle, there is a "Main Lemma" for polynomial potentials V on $\mathbb{R}^n$. Namely, for any cube $Z \subset \mathbb{R}^n$ of sidelength $R > 0$, as long as $\mathrm{av}_Z V \geq R^{-2}$ it holds

$$\int_Z |\nabla u|^2 + V|u|^2 \, dx \gtrsim R^{-2} \int_Z |u|^2 \, dx \tag{8.8}$$

for reasonable functions $u(x)$, where the implied constant depends only on n and the degree of V. Such inequalities, which locally bound a weighted L^2 norm of a function by its energy, are now commonly referred to as Fefferman-Phong inequalities.

Fundamental for Kurata is Shen's work in generalizing (8.8) to reverse Hölder potentials $V \in RH_q$ on $\mathbb{R}^n$ when $n \geq 2$ and $q > n/2$. In [50] Shen defined the function

$$\frac{1}{m_V(x)} = \sup\left\{ r > 0 \ : \ \frac{r^2}{|B(x,r)|} \int_{B(x,r)} V \, dy \leq 1 \right\}$$

And then proved that for any $u \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\nabla u|^2 + V\,|u|^2 \, dx \gtrsim \int_{\mathbb{R}^n} m_V(x)^2 |u|^2 \, dx \tag{8.9}$$

This is the final ingredient for Kurata in [31]; and in fact the limited range for n and RH_q in the definition of $m_V(x)$ is the only reason that Kurata's results do not include the case we set out to study in Section 6.3, with $n = 1$ and $V \in A_\infty$.

Structure of the iteration

Assume u is a nonnegative weak solution of $(\partial_t + H)u = 0$, and appropriate versions of the preceding inequalities hold. Then we have the following steps toward an upper bound on u.

1. By Moser's work, u is locally bounded by its L^2 norm in a cylinder.

2. This L^2 norm is dominated by the energy of u times a V-dependent weight, by the Fefferman-Phong inequality.

3. And this energy is itself dominated by the solution's L^2 norm in a just larger cylinder, by the Caccioppoli inequality.

Since the last L^2 norm in (3) is now dominated as in (2), the final two steps may be iterated ad infinitum. Each iteration extends the L^2 norm in our upper bound to a slightly larger cylinder, and picks up another factor of a V-dependent weight. Now remember that if $u(\cdot, t)$ is the heat kernel $p(\cdot, y, t)$, we already have a Gaussian bound on u, and hence control over the final L^2 norm when our iteration ends. So it makes sense that this process could yield V-dependent extra-Gaussian decay of the heat kernel; we will see the details momentarily in Section 8.3.

8.2 Auscher and Ben Ali's work on A_∞ potentials

In their wide-ranging work [4], Auscher and Ben Ali improved some of Shen's results on L^p estimates for Schrödinger operators. A key step was their development of the following Fefferman-Phong inequality.

Theorem 8.3 (Auscher, Ben Ali). *Suppose $V \in A_\infty$. Then there are constants $C > 0$ and $\beta \in (0, 1)$, depending only on $n \geq 1$ and the A_∞ constant of V, such that for any cube $Z = Z_r(x_0)$ and $u \in C^1(\mathbb{R}^n)$ one has*

$$\int_Z |\nabla u|^2 + V\,|u|^2\,dx \geq C\frac{m_\beta(r^2\,av_Z V)}{r^2}\int_Z |u|^2\,dx \tag{8.10}$$

where $m_\beta(x) = x$ for $x \leq 1$ and $m_\beta(x) = x^\beta$ for $x \geq 1$. In particular, if $V \in A_p$, then one may take $\beta = \frac{2}{2+n(p-1)}$.

As an aside, we note the inequality is stated for $u \in C^1(\mathbb{R}^n)$, but a look at the proof shows all that is required is that there exist $\nabla u \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that

$$u(y) - u(x) = \int_0^1 \nabla u(x + \alpha(y - x)) \cdot (y - x)\,d\alpha \qquad x, y \in Q$$

holds almost everywhere. Since $u \in H^1(\mathbb{R}^n)$ can be approximated in any $L^1_{\text{loc}}(Q)$ by smooth functions whose derivatives also converge to ∇u in $L^1_{\text{loc}}(Q)$, Theorem 8.3 holds just as well for weakly differentiable u. (Indeed, all of Auscher and Ben Ali's own applications in [4] are to functions differentiable in the weak sense.) This comment is important for us because we will wish to apply the inequality to the heat kernel $p(\cdot, y, t)$ which is a priori only weakly differentiable.

Compared to (8.9), the size function $m_V(x)$ has been replaced in (8.10) by an explicit average of V over cubes (foreshadowing the difference in how (8.1) and (8.2) specify extra-Gaussian decay.) In any case, if V is a nonnegative polynomial, Theorem 8.3 strengthens the original Fefferman-Phong inequality when $R^2 \mathrm{av}_Z V > 1$, since (8.8) essentially requires $\beta = 0$. When one becomes aware of Auscher and Ben Ali's work, it is natural to check whether Kurata's arguments can be modified to use (8.10).

8.3 Proof of the upper bound

We now prove the main result of this chapter.

Theorem 8.4. *If $V \in A_\infty$, the heat kernel of the Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^n)$ satisfies*

$$p(x, y, t) \leq \frac{c_0}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \exp\left\{ -c_1 m_\beta (t \, av_{Z_{\sqrt{t}}(x)} V)^{1/2} \right\} \tag{8.11}$$

where m_β is as in Theorem 8.3, and $c_i > 0$ for $i = 0, 1, 2$.

Proof. We will fix $y \in \mathbb{R}^n$ and work locally on the cylinder $Q_r(x, t)$, where $r = \sqrt{t/8}$. Write $u(\cdot, s) = p(\cdot, y, s)$ so that u is a weak solution to $(\partial_s + H)u = 0$ in $Q_{2r}(x, t)$. As described in Section 2.2, the idea is improve Moser's estimate (8.1) on u by iterating the Caccioppoli and Fefferman-Phong inequalities over an increasing sequence of cylinders that starts with $Q_{2/3r}(x, t)$. So choose $k \in \mathbb{N}$; we construct an iteration of length k by defining

$$\rho_j = \frac{2}{3} + \left(\frac{j-1}{k} \right) \frac{1}{3} \qquad \text{for} \quad j = 1, 2, \ldots, k+1$$

These $\rho_1, \ldots, \rho_{k+1}$ are a sequence of k scaling factors increasing from $\rho_1 = 2/3$ to $\rho_{k+1} = 1$. For each $j = 2, \ldots, k+1$, also define nonnegative cutoff functions $\chi_j(z) \in C_0^\infty(B(x, \rho_j r))$ and $\eta_j(s) \in C^\infty(\mathbb{R})$, bounded by 1 and satisfying

- $\chi_j \equiv 1$ on $B(x, \rho_{j-1} r)$, $|\nabla \chi_j| \leq \dfrac{Ck}{r}$
- $\eta_j \equiv 0$ for $t \leq t_0 - (\rho_j r)^2$, $\eta_j \equiv 1$ for $t \geq t_0 - (\rho_{j-1} r)^2$, $|\eta_j'| \leq \dfrac{Ck}{r^2}$.

Note in particular that $\operatorname{supp} \chi_j \eta_j \subset B(x, r) \times [t_0 - r^2, \infty)$.

The first step is to see how the Caccioppoli inequality in Lemma 8.2 applies on a given cylinder $Q_{\rho_{j+1} r}(x, t)$, where $j = 1, \ldots, k$. We take the radius r in Lemma 8.2 to be our $\rho_{j+1} r$;

and set $\sigma = \frac{\rho_j}{\rho_{j+1}}$. Substituting these values into (8.7), we obtain

$$\iint_{Q_{\rho_{j+1}r}(x,t)} \left(|\nabla u|^2 \chi_{j+1}^2 \eta_{j+1}^2 + V u^2 \chi_{j+1}^2 \eta_{j+1}^2 \right) dz\, ds \le \frac{Ck^2}{r^2} \iint_{Q_{\rho_{j+1}r}(x,t)} |u|^2\, dz\, ds$$

But from Cauchy's inequality

$$|\nabla(\eta_{j+1} u \chi_{j+1})|^2 \le 2|\nabla u|^2 \eta_{j+1}^2 \chi_{j+1}^2 + 2|u|^2 |\nabla \chi_{j+1}|^2 \eta_{j+1}^2.$$

So using the bounds on $|\nabla \chi_{j+1}|$ and η_{j+1} and increasing C as necessary, we also have

$$\begin{aligned} \int_{t-(\rho_{j+1}r)^2}^{t} \int_{B_{\rho_{j+1}r}(x)} &\left(|\nabla(\eta_{j+1} u \chi_{j+1})|^2 + V|u|^2 \chi_{j+1}^2 \eta_{j+1}^2 \right) dz\, ds \\ &\le \frac{Ck^2}{r^2} \iint_{Q_{\rho_{j+1}r}(x,t)} |u|^2\, dz\, ds. \quad (8.12) \end{aligned}$$

And note we may apply the Fefferman-Phong inequality 8.10 to the integral in the space directions on the left-hand side of this inequality. We do this on a cube containing the support of χ_{j+1}, namely $Z_{2r}(x)$.

$$\begin{aligned} \int_{B_{\rho_{j+1}r}(x)} &\left(|\nabla(\eta_{j+1} u \chi_{j+1})|^2 + V|u|^2 \chi_{j+1}^2 \eta_{j+1}^2 \right) dz \\ &= \int_{Z_{2r}(x)} \left(|\nabla(\eta_{j+1} u \chi_{j+1})|^2 + V|u|^2 \chi_{j+1}^2 \eta_{j+1}^2 \right) dz \\ &\ge \frac{C}{r^2} m_\beta(r^2 \operatorname{av}_{Z_{2r}(x)} V) \int_{B_{\rho_{j+1}r}(x)} |\eta_{j+1} u \chi_{j+1}|^2\, dz. \end{aligned}$$

Combined with (8.12), this implies

$$\int_{t-(\rho_{j+1}r)^2}^{t} \frac{m_\beta(r^2 \operatorname{av}_{Z_{2r}(x)} V)}{r^2} \int_{B_{\rho_{j+1}r}(x)} |\eta_{j+1} u \chi_{j+1}|^2\, dz\, ds \le \frac{Ck^2}{r^2} \iint_{Q_{\rho_{j+1}r}(x,t)} |u|^2\, dz\, ds,$$

and hence

$$\iint_{Q_{\rho_{j+1}r}(x,t)} |\eta_{j+1} u \chi_{j+1}|^2\, dz\, ds \le \frac{Ck^2}{m_\beta(r^2 \operatorname{av}_{Z_{2r}(x)} V)} \iint_{Q_{\rho_{j+1}r}(x,t)} |u|^2\, dz\, ds.$$

We can now write precisely how the iteration relates the L^2 norm of u at the the ρ_j scaling to the ρ_{j+1} scaling.

$$\begin{aligned} \iint_{Q_{\rho_j r}(x,t)} |u|^2\, dz\, ds &\le \iint_{Q_{\rho_{j+1}r}(x,t)} |\eta_{j+1} u \chi_{j+1}|^2\, dz\, ds \\ &\le \frac{Ck^2}{m_\beta(r^2 \operatorname{av}_{Z_{2r}(x)} V)} \iint_{Q_{\rho_{j+1}r}(x,t)} |u|^2\, dz\, ds \end{aligned}$$

In particular, the $j = 1$ case of this inequality is

$$\iint_{Q_{2r/3}(x,t)} |u|^2\,dz\,ds \leq \frac{Ck^2}{m_\beta(r^2 \operatorname{av}_{Z_{2r}(x)} V)} \iint_{Q_{\rho_2 r}(x,t)} |u|^2\,dz\,ds$$

So an iteration of k steps yields

$$\iint_{Q_{2r/3}(x,t)} |u|^2\,dz\,ds \leq \frac{C^k k^{2k}}{m_\beta(r^2 \operatorname{av}_{Z_{2r}(x)} V)^k} \iint_{Q_r(x,t)} |u|^2\,dz\,ds$$

And substituting this inequality into Moser's estimate (8.3), we obtain

$$\sup_{(z,s)\in Q_{r/2}(x,t)} |u| \lesssim \frac{C^{k/2} k^k}{m_\beta(r^2 \operatorname{av}_{Z_{2r}(x)} V)^{k/2}} \left(\frac{1}{r^{n+2}} \iint_{Q_r(x,t)} |u|^2\,dz\,ds \right)^{1/2} \tag{8.13}$$

with the suppressed constant independent of k.

It remains to exploit the fact that k is arbitrary in the above inequality. From Stirling's formula we know that

$$k^k \sim \frac{e^k k!}{\sqrt{2\pi k}} \qquad \text{as} \qquad k \to \infty,$$

so we may replace k^k in (8.13) with $e^k k!$. Then multiplying through in (8.13) by $\epsilon^k/k!$ for $\epsilon > 0$ to be chosen sufficiently small, we obtain inequalities of the form

$$\sup_{(z,s)\in Q_{r/2}(x,t)} |u| \cdot \frac{(\epsilon\, m_\beta(r^2 \operatorname{av}_{Z_{2r}(x)} V)^{1/2})^k}{k!} \lesssim (\epsilon C e)^k \left(\frac{1}{r^{n+2}} \iint_{Q_r(x,t)} |u|^2\,dz\,ds \right)^{1/2}.$$

Summing over all $k \in \mathbb{N}$, it follows for $\epsilon < (Ce)^{-1}$ that

$$\sup_{(z,s)\in Q_{r/2}(x,t)} |u| \leq c_0 \exp\left\{-c_1 m_\beta(r^2 \operatorname{av}_{Z_{2r}(x)} V)^{1/2}\right\} \left(\frac{1}{r^{n+2}} \iint_{Q_r(x,t)} |u|^2\,dz\,ds \right)^{1/2},$$

and we have

$$p(x,y,t) \leq c_0 \exp\left\{-c_1 m_\beta(t \operatorname{av}_{Z_{\sqrt{t/2}}(x)} V)^{1/2}\right\} \left(\frac{1}{t^{(n+2)/2}} \iint_{Q_{\sqrt{t/8}}(x,t)} |p|^2\,dz\,ds \right)^{1/2}, \tag{8.14}$$

recalling the meaning of u and r.

Because A_∞ potentials are doubling, in (8.14) we may replace the average of V over $Z_{\sqrt{t/2}}(x)$ with its average over $Z_{\sqrt{t}}(x)$, scaling c_1 appropriately. Now as the final step we incorporate the Gaussian bound on the heat kernel

$$p(x,y,t) \lesssim t^{-n/2} \exp(-|x-y|^2/4t) \tag{8.15}$$

In particular, if $|x-y| \approx \sqrt{t}$, we lose nothing by using $p \lesssim t^{-n/2}$ inside the integral in (8.14). Because $|Q_r| \approx r^{n+2}$, this gives

$$p(x,y,t) \leq \frac{c_0}{t^{n/2}} \exp\left\{-c_1 m_\beta (t \operatorname{av}_{Z_{\sqrt{t}}(x)} V)^{1/2}\right\} \tag{8.16}$$

On the other hand, if $\sqrt{t} \ll |x-y|$ then $p \lesssim t^{-n/2}$ is a very poor estimate. We would be better off just using (8.15) directly. So the upper bound of the theorem is a compromise that follows from writing

$$p(x,y,t) = p(x,y,t)^{1/2} \cdot p(x,y,t)^{1/2}$$

and then applying (8.15) to the first term in the product, (8.16) to the second. $\square$

8.4 Sharpness considerations

In this section we make some observations about the sharpness of Theorem 8.4 in two cases. First, when $V(x) = \sum_{i=0}^{2} a_i x^i$ is a nonnegative polynomial on $\mathbb{R}$, we compare our upper bounds to the explicit kernel calculated in Section 7.3. Second, when $V(x) = |x|^\alpha$ ($\alpha > 0$) is a power weight on $\mathbb{R}^n$, we compare our on-diagonal upper bounds to Kurata's bounds, as well as to A. Sikora's estimates in [51]. The observations in this section deal only with orders of decay in the space and time variables, and not in any way with the constants involved.

The Mehler kernel

When $V(x) = \sum_{i=0}^{2} a_i x^i$ we have computed

$$\begin{aligned} p(x,y,t) = &\left(\frac{\sqrt{a_2}\operatorname{csch} 2\sqrt{a_2}t}{2\pi}\right)^{1/2} e^{\left(\frac{a_1^2}{4a_2}-a_0\right)t} \\ &\cdot \exp\left\{-\frac{{a_1}^2}{4(\sqrt{a_2})^3}(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\right\} \\ &\cdot \exp\left\{-\frac{\sqrt{a_2}}{2}\Big((x-y)^2 \operatorname{csch} 2\sqrt{a_2}t + (x^2+y^2)(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\Big)\right\} \\ &\cdot \exp\left\{-\frac{a_1}{2\sqrt{a_2}}\Big((x+y)(\coth 2\sqrt{a_2}t - \operatorname{csch} 2\sqrt{a_2}t)\Big)\right\} \end{aligned}$$

Now recall the following asymptotics

- $\operatorname{csch}(t) \sim t^{-1}$ as $t \to 0^+$ and $\operatorname{csch}(t) \sim e^{-t}$ as $t \to +\infty$
- $(\coth(t) - \operatorname{csch}(t)) \sim t$ as $t \to 0^+$ and $(\coth(t) - \operatorname{csch}(t)) \sim 1$ as $t \to +\infty$

So a sharp upper bound should essentially be

$$p(x,y,t) \lesssim \begin{cases} t^{-1/2} e^{-c_0 \frac{|x-y|^2}{t}} \exp\left\{-c_1 t(x^2+y^2)\right\} & t \le 1 \\ e^{-c_2 t} \exp\left\{-c_3(x^2+y^2)\right\} & t > 1 \end{cases} \tag{8.17}$$

Let us rewrite our upper bound (8.1) to include an extra-Gaussian decay term in y by again using the symmetry of $p(x,y,t)$ in the space variables.

$$\begin{aligned} p(x,y,t) &= p(x,y,t)^{1/2} \cdot p(y,x,t)^{1/2} \\ &\lesssim t^{-1/2} e^{-c_1 \frac{|x-y|^2}{t}} \exp\left\{-c_2\left[m_\beta(t\,\mathrm{av}_{Z_{\sqrt{t}}(x)} V)^{1/2} + m_\beta(t\,\mathrm{av}_{Z_{\sqrt{t}}(y)} V)^{1/2}\right]\right\} \end{aligned} \tag{8.18}$$

Taking $n=1$ and using $V(x)$ as above

$$\mathrm{av}_{Z_{\sqrt{t}}(x)} V = \frac{1}{\sqrt{t}} \int_{x-\frac{1}{2}\sqrt{t}}^{x+\frac{1}{2}\sqrt{t}} V(z)\,dz = a_2\left(x^2 + \frac{t}{12}\right) + a_1 x + a_0.$$

Thus we see that (8.18) is no sharper than a bound of

$$p(x,y,t) \lesssim t^{-1/2} e^{-c_1 \frac{|x-y|^2}{t}} \exp\left\{-c_2\left[m_\beta(t^{1/2}|x|+t) + m_\beta(t^{1/2}|y|+t)\right]\right\}.$$

When all of the three terms $\{\sqrt{t}, |x|, |y|\}$ are small, the Gaussian factor will essentially determine the size of both the above, and of (8.17). But when $t>1$ and say $|x|$ is the dominant term, our upper bound can be no sharper than $\exp\left\{-c|x|^{2\beta}\right\}$, while (8.17) will have decay of the order $\exp\{-cx^2\}$. Hence the presence of the sublinear function m_β guarantees that sharp decay is not attained. (Although if $V(x)$ is a strictly positive polynomial, β may be taken arbitrarily close to 1 since positive polynomials belong to RH_∞; see [31] and our comments on β in the next section.)

Power weights

We need to recall that in Auscher and Ben Ali's Fefferman-Phong inequality (8.10), one may take $\beta = \frac{2}{2+n(p-1)}$ for any p such that V belongs to the Muckenhoupt class A_p. This is useful when combined with the standard fact that $|x|^\alpha \in A_p$ if and only if $-n < \alpha < n(p-1)$; which may be proved with some effort by elementary estimates, or in a few lines by taking advantage of a factorization theorem for A_p weights, see [15]. In any case, for $V(x) = |x|^\alpha$ ($\alpha > 0$), our upper bound (8.18) has $\beta < \frac{2}{\alpha+2}$. Beyond assuming that V is a power weight, we now also focus on the diagonal $x = y$. Kurata's results in this setting were, due to simplified expressions for $m_V(x)$ with power weights,

$$p(x,x,t) \lesssim t^{-n/2} \exp\left\{-c_0 \left(t + t|x|^\alpha\right)^{\frac{1}{\alpha+2}}\right\} \tag{8.19}$$

while Sikora proved in [51] the stronger bounds

$$p(x,x,t) \lesssim \begin{cases} t^{-n/2} \exp\left\{-c_0 t|x|^\alpha\right\} & t \leq (1+|x|)^{1-\alpha/2} \\ e^{-c_1 t} \exp\left\{-c_2 |x|^{1+\alpha/2}\right\} & t > (1+|x|)^{1-\alpha/2} \end{cases} \tag{8.20}$$

in the case the potential $V \gtrsim |x|^\alpha$ on all of $\mathbb{R}^n$.

Applying to (8.18) the inequality

$$\mathrm{av}_{Z_{\sqrt{t}}(x)} |z|^\alpha \, dz \lesssim t^{\alpha/2} + |x|^\alpha$$

we see that our estimate can be no sharper than

$$p(x,x,t) \lesssim t^{-n/2} \exp\left\{-c_0 \left(t^{1/2} + \left(t|x|^\alpha\right)^{\frac{1}{\alpha+2}}\right)\right\}$$

In conclusion, we obtained essentially the same order of decay as did Kurata. The sharp estimates (8.20), on the other hand, provide a measure of how the iteration argument "fails to recognize" well-behaved potentials. It would be very interesting to see a single estimate which both applies to A_∞ potentials, and specializes to Sikora's bounds in the case of nonsingular power weights.

Chapter 9

Lower Bounds for RH_∞ Potentials

Since our heat kernel upper bounds in the previous chapter are not sharp, it is impossible to obtain general lower bounds of the same form as (8.1). But if $V \in RH_\infty$, we prove in this chapter that for fixed $0 < \kappa < 1$ there is a lower bound

$$p(x,y,t) \geq \begin{cases} \frac{c_0}{t^{n/2}} \exp\{-c_1 t \operatorname{av}_{Z_{\sqrt{t}}(x)} V\} & |x-y| < \kappa\sqrt{t} \\ \frac{c_0}{t^{n/2}} e^{-c_3 \frac{|x-y|^2}{t}} \exp\big\{-c_1 t (c_2^{\frac{|x-y|^2}{t}} \operatorname{av}_{Z_{t/|x-y|}(x)} V)\big\} & |x-y| \geq \kappa\sqrt{t} \end{cases} \tag{9.1}$$

For on-diagonal bounds (that is, when $|x-y| \lesssim \sqrt{t}$) this is actually a satisfactory companion to (8.1). The idea of the proof is to establish a bridge between $p(x,y,t)$ and the heat kernel of an appropriate Dirichlet Laplacian, where van den Berg's results in [55] can be applied. We do this in Section 9.3 using the semigroup property of $p(x,y,t)$ and a parabolic maximum principle; but first recall the form of van den Berg's estimates.

9.1 Estimates on the Dirichlet heat kernel

The statement of van den Berg's results differs somewhat for the $n=1$ and $n \geq 2$ cases. In the latter case we need the following definition.

Definition 9.1. *Fix an open set $D \subset \mathbb{R}^n$, where $n \geq 2$. Given $\epsilon > 0$, let D_ϵ be the points in D at least distance ϵ from the boundary; and let $d_\epsilon(x,y)$ for $x, y \in D$ be the infimum of lengths of arcs in D_ϵ with endpoints x and y. When $d_\epsilon(x,y) < \infty$, let γ_ϵ be a minimal geodesic from x to y and define*

$$\alpha(\gamma_\epsilon) = \int_{s:\, \gamma_\epsilon(s) \in D_\epsilon} \left| \frac{d^2 \gamma_\epsilon(s)}{ds^2} \right| ds$$

In practice we will only choose D to be a ball, so that D_ϵ is also a ball and

$$\gamma_\epsilon(s) = x + \frac{s}{|y-x|}(y-x)$$

Hence $\alpha(\gamma_\epsilon)$ will always vanish in our applications of the following theorem.

Theorem 9.2 (van den Berg). *Suppose D is an open set in $\mathbb{R}^n$ with $n \geq 2$. Given $\epsilon > 0$, $0 < \delta \leq \epsilon$, $x \in D$, $y \in D$ such $d_\epsilon(x,y) < \infty$, it holds for all $t > 0$*

$$\Gamma_D(x,y,t) \geq \frac{C}{t^{n/2}} e^{-\frac{\pi^2 n^2 t}{4\epsilon^2}} \exp\left\{ -\frac{d_\epsilon(x,y)^2\big(1 + 2\delta\alpha(\gamma_\epsilon) d_\epsilon(x,y)\big)}{4t} \right\}$$

where Γ_D is the heat kernel of $-\Delta$ on D with Dirichlet boundary conditions and $C < 1$ is a positive constant depending only on n. $\square$

When $n = 1$, van den Berg obtained a lower bound on Γ_D from ingenious use of a special function identity and the eigenfunction expansion of the Dirichlet heat kernel on an interval. Namely,

Proposition 9.3 (van den Berg). *Suppose $D \subset \mathbb{R}$ is an interval, and for some $x < y$ and $\epsilon > 0$ we have $(x - \epsilon, y + \epsilon) \subset D$. Then for all $t > 0$*

$$\Gamma_D(x,y,t) \geq \frac{C}{t^{1/2}} e^{-\frac{|x-y|^2}{4t}} (1 - 2e^{-\frac{\epsilon^2}{t}})$$

where Γ_D is the heat kernel of $-\Delta$ on D with Dirichlet boundary conditions and $C < 1$ is a positive constant depending only on n.

9.2 Some technical points

The above results for the Dirichlet Laplacian are by far the most sophisticated element of our proof of (9.1). But some other introductory remarks are also helpful. The proof has two parts; in the first, we establish lower bounds "on the diagonal"—that is, on a set where $|x - y|$ is uniformly small compared to $\sqrt{t}$. For this step a maximum principle for weak solutions of parabolic equations is key. Then in the second part of the proof, we move off the diagonal by combining the on-diagonal bounds with the fact that e^{-Ht} is a semigroup and a lemma of Christ's that applies to doubling measures. Let us give a little more detail on these ideas.

A maximum principle

Our bridge between the heat kernel of $H = -\Delta + V$ and van den Berg's results is the following well-known maximum principle. Note that we need the boundedness of V implied by membership in RH_∞.

Theorem 9.4. *Suppose $V \geq 0$ is bounded in a cylinder $Q = Q_r(x_0, t_0)$ and $u \in C(\overline{Q})$ is a weak solution of $(\partial_t + H)u = 0$ in Q. Then*

$$\sup_Q u \leq \sup_{\partial Q} u_+ \qquad \text{and} \qquad \inf_Q u \geq \inf_{\partial Q} u_-$$

If u is only a weak supersolution of the same equation in Q, then we still have the second conclusion. □

Since we cannot assume u is a classical solution in Q, the proof is rather involved and best accomplished through functional analytic machinery. Details may be found in [27]; see [52] for other applications of Theorem 9.4 to Schrödinger heat kernels.

The semigroup property

Lemma 9.5. *Let $p(x, y, t)$ be the heat kernel of a Schrödinger operator H on $L^2(\mathbb{R}^n)$ with locally integrable nonnegative potential. Then*

$$p(x, y, t + s) = \int_{\mathbb{R}^n} p(x, z, t)\, p(z, y, s)\, dz$$

for all $x, y \in \mathbb{R}^n$ and $s, t > 0$. □

Here we have simply restated from Section 7.2 that e^{-Ht} is a semigroup; but in the form in which this fact appears in our proof of (9.1). For an off-diagonal estimate of $p(x, y, t)$, we will invoke Lemma 9.5 repeatedly to write $p(x, y, t)$ as an iterated integral of many "copies" of itself at earlier times. Our on-diagonal bounds will then apply to these copies when they are restricted to appropriately small regions in space.

A lemma for doubling measures

The following is a lemma of Christ's in [14] which we will need to compare the averages of V over two cubes whose centers are some distance from each other.

Lemma 9.6. *For any doubling measure ω on $\mathbb{R}^n$, there exist positive $C < \infty$ and $\epsilon < 1$ such that for any cubes $Z' \subset Z$*

$$\int_{Z'} d\omega \leq C \left(\frac{|Z'|}{|Z|} \right)^{\epsilon} \int_Z d\omega$$

where e.g. $|Z|$ denotes the Euclidean measure of Z. □

9.3 Proof of the lower bound

We are now ready to obtain heat kernel lower bounds for Schrödinger operators with potentials in the reverse Hölder supremum class.

Theorem 9.7. *If $V \in RH_\infty$, the heat kernel of the Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^n)$ satisfies with $0 < \kappa < 1$ fixed*

$$p(x, y, t) \geq \begin{cases} \frac{c_0}{t^{n/2}} \exp\{-c_1 t \, av_{Z_{\sqrt{t}}(x)} V\} & |x-y| < \kappa\sqrt{t} \\ \frac{c_0}{t^{n/2}} e^{-c_3 \frac{|x-y|^2}{t}} \exp\{-c_1 t (c_2^{\frac{|x-y|^2}{t}} \, av_{Z_{t/|x-y|}(x)} V)\} & |x-y| \geq \kappa\sqrt{t} \end{cases} \tag{9.2}$$

where $c_i > 0$ for $i = 0, 1, 2, 3$.

Proof. First suppose $|x - y| < \frac{1}{8}\sqrt{t}$. We consider the ball $B = B_{\sqrt{t}}(x)$. Let H_B be the restriction of the operator H to B with Dirichlet boundary conditions; and let $p_B(x, y, t)$ be the associated heat kernel. Note that $u(\cdot, t) = p(\cdot, y, t) - p_B(\cdot, y, t)$ is a weak solution to $(\partial_t + H)u = 0$ on $B \times (0, \infty)$, for any $y \in B$. And because $p_B(\cdot, y, t)$ vanishes on ∂B, u is nonnegative on the boundary, implying by the maximum principle that

$$p(x, y, t) \geq p_B(x, y, t) \qquad \text{in } B \times B \times (0, \infty) \tag{9.3}$$

since the choice of $y \in B$ was arbitrary.

Now we use again the hypothesis that $V \in RH_\infty$. With $C > 0$ the RH_∞ constant of V, we have for $M = C\mathrm{av}_{Z_{2\sqrt{t}}(x)} V$ that $V \leq M$ in B. Let H_D^M be the operator $(-\Delta + M)$ restricted to B with Dirichlet boundary conditions. As in (7.7), its heat kernel is just $e^{-Mt}\Gamma_D$, where Γ_D is the heat kernel of the Dirichlet Laplacian on B. Now for $y \in B$ set

$$w(x, t) = p_B(x, y, t) - e^{-Mt}\,\Gamma_D(x, y, t)$$

Then $w \equiv 0$ on $\partial B \times (0, \infty)$, and inside B we have for any $t > 0$

$$(\partial_t + H_D^M)w = (\partial_t - \Delta + M)p_B = (M - V)p_B \geq 0$$

So w is a supersolution of $(\partial_t + H_B^M)$ in the cylinder $Q = B \times (0, \infty)$, vanishing on the boundary, and by the maximum principle satisfies

$$p_B(x, y, t) \geq e^{-Mt}\,\Gamma_D(x, y, t) \qquad \text{in } B \times B \times (0, \infty)$$

since $y \in B$ was arbitrary.

Applying either Proposition 9.3 or Theorem 9.2 with $\epsilon = \frac{7}{8}\sqrt{t}$, we obtain from the preceding inequality and (9.3) that

$$p(x, y, t) \geq \frac{c_0}{t^{n/2}} \exp\left\{-c_1 t \operatorname{av}_{Z_{2\sqrt{t}}(x)} V\right\} \tag{9.4}$$

where $c_0 < 1$ is a positive constant depending only on n, and c_1 is just the RH_∞ constant of V. Because V is doubling, we may also increase c_1 and replace the cube $Z_{2\sqrt{t}}(x)$ with $Z_{\sqrt{t}}(x)$. These on-diagonal bounds conclude the first part of the proof. Off-diagonal bounds will follow via an argument similar to that used by Zhang and Zhao in [57]. So take $|x - y| \geq \frac{1}{8}\sqrt{t}$.

We begin by considering the line segment from x to y given by

$$l(s) = x + s\frac{y - x}{|y - x|}, \qquad s \in [0, |y - x|]$$

We will partition this segment by a sequence of $M + 1$ points $\{x_i\}_{i=0}^{M}$; the sequence is determined by the requirement that $|x_i - x_{i+1}| = \frac{|y-x|}{M}$, where M is the smallest integer satisfying

$$\frac{|y - x|}{M} < \frac{1}{16}\sqrt{\frac{t}{M}} \Leftrightarrow \frac{256|y - x|^2}{t} < M \tag{9.5}$$

Now directly from Lemma 9.5 we have

$$p(x, y, t) = \int_{\mathbb{R}^n} p(x, z_1, t/M) p(z_1, y, (M - 1)t/M)\, dz_1$$

And applying the semigroup property in the same way to the right-most integrand, $(M - 1)$ times, we get an iterated integral

$$p(x, y, t) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} p(x, z_1, t/M) \cdots p(z_{M-1}, y, t/M)\, dz_1 \cdots dz_{M-1}$$

Upon restricting each dz_i integral to $Z_i = Z_{\sigma\sqrt{t/M}}(x_i)$ with $0 < \sigma < 1$ such that

$$z_i \in Z_i \text{ and } z_{i+1} \in Z_{i+1} \Rightarrow |z_i - z_{i+1}| < \frac{1}{8}\sqrt{t/M}$$

we then obtain

$$p(x, y, t) \geq \int_{Z_1} \cdots \int_{Z_{M-1}} p(x, z_1, t/M) \cdots p(z_{M-1}, y, t/M)\, dz_1 \cdots dz_{M-1} \tag{9.6}$$

And now our on-diagonal lower bounds apply to each term in the integrand.

That is, we have for each $i = 0, \dots, M-1$ that

$$p(z_i, z_{i+1}, t/M) \geq c_0 \left(\frac{M}{t}\right)^{n/2} \exp\left\{-c_1 \frac{t}{M} \operatorname{av}_{Z_{\sigma\sqrt{t/M}}(z_i)} V\right\}$$

To assimilate these into a single lower bound for $p(x, y, t)$, we use Lemma 9.6. In particular we see that

$$\begin{aligned} \int_{Z_{\sigma\sqrt{t/M}}(z_i)} V &\leq C \left(\frac{1}{2^n}\right)^{\epsilon} \int_{Z_{2\sigma\sqrt{t/M}}(x_i)} V \\ &\leq C \int_{Z_i} V \end{aligned}$$

And iterating this inequality up to M times (if $i = M-1$) we may even conclude

$$\int_{Z_{\sigma\sqrt{t/M}}(z_i)} V \leq C^M \int_{Z_0} V = C^M \int_{Z_{\sigma\sqrt{t/M}}(x)} V$$

So in fact we have a lower bound, uniform in i, of

$$p(z_i, z_{i+1}, t/M) \geq c_0 \left(\frac{M}{t}\right)^{n/2} \exp\left\{-c_1 \frac{t}{M} C^M \operatorname{av}_{Z_{\sigma\sqrt{t/M}}(x)} V\right\}$$

It now just remains to apply this to each term in the integrand of (9.6).

This yields

$$\begin{aligned} p(x, y, t) &\geq \prod_{i=0}^{M-1} c_0 \left(\frac{M}{t}\right)^{n/2} \exp\left\{-c_1 \frac{t}{M} C^M \operatorname{av}_{Z_{\sigma\sqrt{t/M}}(x)} V\right\} \cdot \prod_{i=1}^{M-1} |Z_{\sigma\sqrt{t/M}}(x_i)| \\ &\geq \frac{\sigma^{-1}}{t^{n/2}} M^{n/2} (\sigma c_0)^M \exp\left\{-c_1 t \big(C^M \operatorname{av}_{Z_{\sigma\sqrt{t/M}}(x)} V\big)\right\} \end{aligned}$$

Because $c_0 < 1$ and $\sigma < 1$, the factor $M^{n/2}(\sigma c_0)^M$ gives exponential decay in M; and by (9.5), M is comparable to $\frac{|x-y|^2}{t}$. Increasing constants as necessary and using Christ's lemma to replace M with $\frac{|x-y|^2}{t}$ yields (9.2) with $\kappa = 1/8$. $\square$

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