7-2015

Isometries of Besov Type Spaces among Composition Operators

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Mathematics

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July 2015
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Abstract

Let $B_{p,\alpha}$ for $p > 1$ and $\alpha > -1$ be the Besov type space of holomorphic functions on the unit disk $\mathbb{D}$. Given φ, a holomorphic self map of $\mathbb{D}$, we show the composition operator C_φ is an isometry on $B_{p,\alpha}$ if and only if the weighted composition operator $W_{\varphi',\varphi}$, is an isometry on the weighted Bergman space A^p_α. We then characterize isometries among composition operators in $B_{p,\alpha}$ in terms of their Nevanlinna type counting function. Finally, we find that the only isometries among composition operators on $B_{p,\alpha}$, except on $B_{2,0}$, are induced by rotations. This extends known results by Martin, Vukotic and by Allen, Heller and Pons on certain Besov spaces.

Acknowledgments

I would like to thank my advisor, Dr. Maria Tjani, for the patient guidance, encouragement and advice she has provided throughout my time as her student. I have been blessed to have an advisor who cared so much about me and my work. I want to thank the rest of my committee, Dr. John Akeroyd and Dr. Daniel Luecking, for their time and commitment. I would also like to thank Dr. Lance Miller and Dr. Andy Raich for their guidance and support while I applied for jobs and prepared for interviews.

I must express my gratitude to my mother, Charla Behan, and grandma, Pat Rhodes for their continued support and encouragement. Thank you to Jay and Julie St. Clair, Keith and Sonya Wilkins, Les and Dian Jordan, and Scott and Jill Fields for making me part of your family and treating me as your daughter. Thank you Mindy Grusing, Kristy Wisdom McPeak and Lindy Roberts for your guidance and for helping me keep things in perspective. Thank you Taekwondo family for keeping me grounded and entertained. I would not have made it this far without ALL of you.

Much thanks to Mary Powers and staff at the Mathematics department at the University of Arkansas. Special thanks to the University of Arkansas Mathematics department and SREB for funding my education.

Finally, I would like to thank all of my students for challenging me and motivating me to become a better teacher and mathematician.

Dedication

I dedicate my dissertation to Afton Clevenger and my father, Keith Wilkins.

Afton Jade I am so proud of the woman you have become. You are kind, beautiful, brilliant and talented! I look forward to seeing what God has planed for your life. The skies the limit, kid.

Dad, thank you for showing me how my heavenly Father views me. Thank you for being my rock that I can always count on.

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1 Introduction

Let φ be a holomorphic self map of the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ and let $f \in H(\mathbb{D})$, the space of holomorphic functions in $\mathbb{D}$. Then the *composition operator*, C_φ, is the operator that maps $f \mapsto f \circ \varphi$, that is

$$C_\varphi f = f \circ \varphi.$$

Composition operators were originally studied in the context of the Hardy space, H^2. The Hardy space is the space of functions of square summable power series coefficients. In 1925 Littlewood proved a subordination principle. In terms of composition operators the subordination principle implied that composition operators on the Hardy space are bounded. Research on composition operators on spaces of holomorphic functions is fairly recent, dating back to the mid 1960's. A driving force in the study of a concrete operator on spaces of holomorphic functions is seeing how properties of the symbol affects properties of the operator such as boundedness, compactness, closed-range and isometries.

Let X be a normed linear space with norm $\|\cdot\|_X$. An operator T on X is an *isometry* if for each $f \in X$,

$$\|Tf\|_X = \|f\|_X.$$

Stefan Banach was the first to study isometries on specific Banach spaces such as on $C(D)$, the space of continuous functions on a compact metric space D, see [3].

General isometries have been studied on H^∞, the space of bounded holomorphic functions on $\mathbb{D}$, the weighted Bergman spaces, the Bloch space, and Besov spaces (see [5], [9], [13], [14], [15]). Isometries among composition operators have been studied on H^2, H^∞, the Dirichlet space, the Bloch space, the space of BMOA and on Besov spaces (see [1], [4], [6], [16]-[20], [29]).

This thesis focuses on the isometries on the Besov type spaces. Given $p > 1$ and $\alpha > -1$, the *Besov type space*, $B_{p,\alpha}$, is the space of holomomorphic functions such that

$$\|f\|_{B_{p,\alpha}} = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha dA(z) < \infty.$$

This is equivalent to f' belonging on the *weighted Bergman space* A^p_α. The Dirichlet space, the Besov spaces, and the weighted Bergman spaces are all Besov type spaces.

In Chapter 2 we discuss the weighted Bergman spaces, the Besov type spaces, the Hardy space, the Bloch type spaces and BMOA. These are all Banach spaces of holomorphic functions. We give properties of these spaces that we will need in later chapters.

Composition operators are not always bounded on $B_{p,\alpha}$. Given $p > 1$, $\alpha > -1$ and $w \in \mathbb{D}$, the Nevanlinna type *counting function* for $B_{p,\alpha}$ is

$$N_{p,\alpha}(w, \varphi) = \sum_{\varphi(z)=w} |\varphi'(z)|^{p-2}(1-|z|^2)^\alpha$$

where it is understood that if w is not in the range of φ, then $N_{p,\alpha}(w, \varphi) = 0$. It has been used to determine bounded, compact and closed range composition operators, see [25].

In chapter 3 we recall a non univalent change of variables formula involving the counting function for $B_{p,\alpha}$, that is similar to the well known change of variables formula involving the Nevanlinna function and is essential to this work, see Proposition 3.2. We also recall the definition, a property of Besov type Carleson measures and a characterization of bounded composition operators on $B_{p,\alpha}$ using Carleson measures, see Theorems 3.1 and 3.2. We finally see that composition operators are always bounded on the Bloch space, in fact they are contractions if $\varphi(0) = 0$.

Chapter 4 consists of three sections. In section 4.1 we discuss what is known on general isometries on $C(D)$, on H^∞, on weighted Bergman spaces, on Bloch type spaces and on Besov spaces. In section 4.2 we discuss what is known on isometries among composition operators on H^2, on Bloch type spaces, on BMOA and on Besov spaces. Motivated by Allen, Heller and Pons paper [1], we classify isometric composition operators acting on Besov type spaces.

In all known cases of isometries among composition operators, if C_φ is an isometry then $\varphi(0) = 0$.

Our first result in section 4.3 is Theorem 4.1, where we unify the proofs of all known results and extend them to any Banach space satisfying some very general conditions.

Theorem 4.1 *Let X be a Banach space of holomorphic functions containing the constant functions and $Aut(\mathbb{D})$. Moreover assume that for all $f \in X$, $||f||_X = |f(0)| + ||f||_{sX}$ and for any constant c, $||f + c||_{sX} = ||f||_{sX}$. Then C_φ is an isometry on X if and only if $\varphi(0) = 0$ and for all $f \in X$*

$$||f \circ \varphi||_{sX} = ||f||_{sX}.$$

We conclude that on all such Banach spaces, if φ is a unit disk automorphism then the only isometries are the rotations, see Proposition 4.1.

The *weighted composition operator*, $W_{\psi,\varphi}$ is defined by $W_{\psi,\varphi}f = \psi f \circ \varphi$ for $\psi \in H(\mathbb{D})$, and φ a holomorphic self map of $\mathbb{D}$. In Theorem 4.2 we show that under general conditions on a Banach space, whose norm is determined from a seminorm, we can use equality of the seminorm to determine isometries in just operators that are unimodular constant multiples of composition operators.

Theorem 4.2 *Let X be a Banach space of holomorphic functions containing the constant functions and $Aut(\mathbb{D})$. Assume that for all $f \in X$, $\|f\|_X = |f(0)| + \|f\|_{sX}$ and $||f + c||_{sX} = ||f||_{sX}$, for each constant c. Moreover, assume that $W_{\psi,\varphi}$ is an isometry on X. Then $||W_{\psi,\varphi}f||_{sX} = ||f||_{sX}$ for all $f \in X$ if and only if $\psi(z) = \psi(0)$ for all $z \in \mathbb{D}$, $|\psi(0)| = 1$ and $\varphi(0) = 0$.*

In Theorem 4.3 we show that C_φ is an isometry on $B_{p,\alpha}$ if and only if $\varphi(0) = 0$ and $W_{\varphi',\varphi}$ is an isometry on the weighted Bergman space A^p_α. As a consequence of Kolaski's theorem, see Corollary 4.3 we have that isometries are full maps, that is the Lesbesgue measure area of $\mathbb{D}\backslash\varphi(\mathbb{D})$ is 0.

The Dirichlet space is a Besov type space with $p = 2$ and $\alpha = 0$. Martin and Vukotic in [17] characterized isometries on the Dirichlet space $\mathcal{D}$ and showed that a composition operators is an isometry on $\mathcal{D}$ if and only if φ is a univalent full map that fixes the origin. This is equivalent to $\varphi(0) = 0$ and $N_{2,0}(w, \varphi) = 1$ almost everywhere.

In Theorems 4.6 and 4.7 we characterize isometries in $B_{p,\alpha}$ in terms of their Nevanlinna type counting function.

Theorem 4.6, Theorem 4.7 *Let $p > 1$, $\alpha > -1$. Then C_φ is an isometry on $B_{p,\alpha}$ if and only if $\varphi(0) = 0$ and for almost every $w \in \mathbb{D}$, $N_{p,\alpha}(w, \varphi) = (1 - |w|^2)^\alpha$.*

In Theorems 4.4, 4.5 and in Proposition 4.5, we partially solve the problem of isometries on $B_{p,\alpha}$. In Theorems 4.8 and 4.9 we find all isometries among composition operators on $B_{p,\alpha}$.

Theorem 4.8, Theorem 4.9 *If $p > 1$, $\alpha > -1$, except $p = 2$, $\alpha = 0$, then C_φ is an isometry on $B_{p,\alpha}$ if and only if φ is a rotation.*

Our last result is the corollary below.

Corollary 4.7 *Let $p > 1$, $\alpha > -1$, except $p = 2$, $\alpha = 0$. Then, $\varphi(0) = 0$ and for almost every $w \in \mathbb{D}$, $N_{p,\alpha}(w, \varphi) = (1 - |w|^2)^\alpha$ if and only if φ is a rotation.*

If A and B are two quantities that depend on a holomorphic function f on $\mathbb{D}$, we say that A is *equivalent* to B and write $A \asymp B$ if there exists constants $c_1, c_2 > 0$ such that

$$c_1 A \leq B \leq c_2 A.$$

We say that a complex valued function $h(w)$, $w \in \mathbb{D}$ is little o of 1 and write $o(1)$, if as $w \to 0$, $h(w) \to 0$.

2 Banach Spaces of Holomorphic Functions

Let $H(\mathbb{D})$ denote the space of holomorphic functions on $\mathbb{D}$. It is a complete metric space with the topology it inherits from $C(\mathbb{D})$, see [7, Chapter VII Corollary 2.3]; that is, every Cauchy sequence in $H(\mathbb{D})$, converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function on $\mathbb{D}$. A *Banach space* is a normed linear space that is complete with respect to the metric defined by its norm. In this chapter we define and give properties of Banach spaces of holomorphic functions that we will need in later chapters.

Definition 2.1 *A Möbius Transformation is a function of the form* $e^{i\theta}\alpha_\lambda(z)$ *where*

$$\alpha_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$$

for $\lambda \in \mathbb{D}$ *and* $z \in \mathbb{D}$.

These are the conformal automorphisms of $\mathbb{D}$ denoted by $Aut(\mathbb{D})$. Two simple calculations show that $\alpha_\lambda^{-1} = \alpha_\lambda$ and

$$|\alpha_\lambda'(z)| = \frac{1 - |\lambda|^2}{|1 - \bar{\lambda}z|^2}.$$

Also,

$$1 - |\alpha_\lambda(z)|^2 = \frac{(1 - |\lambda|^2)(1 - |z|^2)}{|1 - \bar{\lambda}z|^2} = |\alpha_\lambda'(z)|(1 - |z|^2). \tag{1}$$

Let A denote area measure on $\mathbb{D}$ normalized by the condition $A(\mathbb{D}) = 1$.

Definition 2.2 *Let* $\alpha > -1$. *Define a positive Borel measure* dA_α *on* $\mathbb{D}$ *by* $dA_\alpha(z) = (1 - |z|^2)^\alpha dA(z)$.

Definition 2.3 *Let* $\alpha > -1$, $p \geq 1$. *The Bergman space,* A^p_α, *consist of all holomorphic* *functions on $\mathbb{D}$ such that*

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right)^{\frac{1}{p}} < \infty.$$

Theorem 2.1 *Suppose $f \in A^p_\alpha$, where $p \geq 1$ and $\alpha > -1$. Then*

$$|f(z)| \leq \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}$$

for all $z \in \mathbb{D}$.

Proof: Let $f \in A^p_\alpha$, fix $p \geq 1$ and $\alpha > -1$. Using polar coordinates we see that

$$\begin{aligned} \|f\|_{A^p_\alpha} &= \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) \\ &= \int_0^1 \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p (1-r^2)^\alpha r d\theta \right\} \frac{dr}{\pi}. \end{aligned}$$

Hardy's Convexity Theorem (Theorem 1.5 from [12]) states that integrals of the form

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

are increasing functions of r for $0 \leq r < 1$. Therefore for each $0 \leq r < 1$

$$|f(0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

Multiply both sides above with $(\alpha+1)r(1-r^2)^\alpha$, and integrate with respect to r to get

$$\begin{aligned} (\alpha+1) \int_0^1 |f(0)|^p r(1-r^2)^\alpha dr &\leq \frac{\alpha+1}{2\pi} \int_0^1 \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p (1-r^2) r\theta \right\} dr \\ &= \frac{1}{2} \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA_\alpha(z). \end{aligned}$$

Hence,

$$|f(0)| \leq \|f\|_{A^p_\alpha} \tag{2}$$

and the conclusion is proved if $z = 0$.

Next fix $z_0 \in \mathbb{D} \setminus \{0\}$ and define

$$F(z) = f \circ \alpha_{z_0}(z) \cdot \left[\frac{1-|z_0|^2}{(1-z\bar{z}_0)^2}\right]^{\frac{2+\alpha}{p}}.$$

Then by (1)

$$\begin{aligned}
\|F\|^p_{A^p_\alpha} &= (\alpha+1)\int_{\mathbb{D}} |F(z)|^p (1-|z|^2)^\alpha dA(z) \\
&= (\alpha+1)\int_{\mathbb{D}} |f(\alpha_{z_0}(z))|^p \left|\frac{1-|z_0|^2}{(1-z\bar{z}_0)^2}\right|^{2+\alpha} (1-|z|^2)^\alpha dA(z) \\
&= (\alpha+1)\int_{\mathbb{D}} |f(\alpha_{z_0}(z))|^p \left(\frac{|1-|z_0|^2|(1-|z|^2}{|1-z\bar{z}_0|^2}\right)^\alpha \left|\frac{1-|z_0|^2}{(1-z\bar{z}_0)^2}\right|^2 dA(z) \\
&= (\alpha+1)\int_{\mathbb{D}} |f(\alpha_{z_0}(z))|^p (1-|\alpha_{z_0}(z)|^2)^\alpha |\alpha'_{z_0}(z)|^2 dA(z).
\end{aligned}$$

By making the change of variables $\zeta = \alpha_{z_0}(z)$ we see that

$$\begin{aligned}
\|F\|^p_{A^p_\alpha} &= \int_{\mathbb{D}} |f(\zeta)|^p (1-|\zeta|^2)^\alpha dA_\alpha(\zeta) \\
&= \|f\|^p_{A^p_\alpha}.
\end{aligned}$$

Thus F and f have the same norm. Notice that $F(0) = f(z_0)(1-|z_0|^2)^{\frac{2+\alpha}{p}}$ and from (2) we know that $|F(0)| \leq \|f\|_{A^p_\alpha}$. Therefore,

$$|f(z_0)|(1-|z_0|^2)^{\frac{2+\alpha}{p}} \leq \|f\|^p_{A^p_\alpha}.$$

Hence,

$$|f(z_0)| \leq \frac{\|f\|^p_{A^p_\alpha}}{(1-|z_0|^2)^{\frac{2+\alpha}{p}}}.$$

□

Theorem 2.2 *Let $p \geq 1$ and $\alpha > -1$. Then the Bergman space A^p_α is a Banach space.*

Proof: Let (f_n) be a Cauchy sequence in A^p_α. Then (f_n) is Cauchy in $L^p_\alpha := L^p((1-|z|^2)^\alpha dA(z))$, which is a Banach space, see [22, Theorem 3.11]. Therefore (f_n) converges to some function $g \in L^p_\alpha$, that is $\|f_n - g\|_{L^p_\alpha} \to 0$. By Theorem 2.1 we know that for all $z \in \mathbb{D}$ and all natural numbers m, n,

$$|f_n(z) - f_m(z)| \leq \frac{\|f_n - f_m\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}.$$

For each fixed r and for all $|z| < r$,

$$\frac{1}{(1-|z|^2)^{\frac{2+\alpha}{p}}} < \frac{1}{(1-r^2)^{\frac{2+\alpha}{p}}}.$$

Therefore (f_n) is uniformly Cauchy on compact subsets of $\mathbb{D}$. We conclude that there exist an $f \in H(\mathbb{D})$ such that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. Now since $\|f_n - g\|_{L^p_\alpha} \to 0$ by [22, Theorem 3.12], there exists a subsequence (f_{n_k}) such that $f_{n_k} \to g$ uniformly on compact subsets of $\mathbb{D}$. Thus g is holomorphic and $f = g$. Hence $\|f_n - f\|_{A^p_\alpha} \to 0$ and A^p_α is a Banach space. □

Definition 2.4 *Let $p > 1$ and $\alpha > -1$. The Besov type space $B_{p,\alpha}$ is the space of holomor-*

phic functions f on $\mathbb{D}$ such that

$$||f||^p_{B_{p,\alpha}} = \int_{\mathbb{D}} |f'(z)|^p \,(1-|z|^2)^\alpha \,dA(z) < \infty\,. \tag{3}$$

The following is a norm in $B_{p,\alpha}$:

$$\|f\|_{p,\alpha} = |f(0)| + \left\{ \int_{\mathbb{D}} |f'(z)|^p \,(1-|z|^2)^\alpha \,dA(z) \right\}^{\frac{1}{p}}. \tag{4}$$

Note that $B_{p,p-2}$ is the *Besov space*, $B_{2,0}$ is the *Dirichlet space* with an equivalent norm. Below we describe the growth of a function in $B_{p,\alpha}$. By Proposition 4.27 in Zhu [26] we know that for any holomorphic function $f \in \mathbb{D}$

$$f(z) - f(0) = \frac{1}{\alpha+1} \int_{\mathbb{D}} \frac{(1-|w|^2)^{1+\alpha} f'(w)}{\bar{w}(1-z\bar{w})^{\alpha+2}} dA(w).$$

Then by Hölder's Inequalities

$$\begin{aligned}
|f(z)-f(0)| &\leq \frac{1}{\alpha+1} \int_{\mathbb{D}} \frac{(1-|w|^2)^{1+\alpha}|f'(w)|}{|\bar{w}||1-z\bar{w}|^{\alpha+2}} dA(w) \\
&\leq C \int_{\mathbb{D}} \frac{(1-|w|^2)^{1+\alpha}|f'(w)|}{|1-z\bar{w}|^{\alpha+2}} dA(w) \\
&\leq C \left(\int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^\alpha dA(w) \right)^{\frac{1}{p}} \left(\int_{\mathbb{D}} \left(\frac{1-|w|^2}{|1-z\bar{w}|^{\alpha+2}} \right)^q (1-|w|^2)^\alpha dA(w) \right)^{\frac{1}{q}} \\
&= C\|f\|_{B_{p,\alpha}} \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{q+\alpha}}{|1-z\bar{w}|^{\alpha q+2q}} dA(w) \right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Let $t = q+\alpha$ and $c = \alpha q + q - \alpha - 2$. Then $c = \frac{\alpha - p + 2}{p-1}$.

By Lemma 3.10 in [26] we have the following.

- Assume $p > 1$ and $\alpha = p - 2$. Then $c = 0$ and

$$\begin{aligned} |f(z) - f(0)| &\leq C\|f\|_{p,p-2}\left(\int_{\mathbb{D}} \frac{(1-|w|^z)^t}{|1-z\bar{w}|^{2+t+c}} dA(w)\right)^{\frac{1}{q}} \\ &\leq C\|f\|_{B_{p,p-2}}\left(\log \frac{1}{1-|z|^2}\right)^{\frac{1}{q}}. \end{aligned} \tag{5}$$

- Assume $p > 1$ and $-1 < \alpha < p - 2$. Then $c < 0$ and

$$\left(\int_{\mathbb{D}} \frac{(1-|w|^z)^t}{|1-z\bar{w}|^{2+t+c}} dA(w)\right)^{\frac{1}{q}}$$

is bounded above and below on $\mathbb{D}$. So for all $z \in \mathbb{D}$

$$|f(z) - f(0)| \leq C\|f\|_{B_{p,\alpha}}. \tag{6}$$

- Assume $p > 1$ and $\alpha > p - 2$. Then $c > 0$ and

$$\begin{aligned} |f(z) - f(0)| &\leq C\|f\|_{B_{p,\alpha}}\left(\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2+t+c}p} dA(w)\right)^{\frac{1}{q}} \\ &\leq C\|f\|_{B_{p,\alpha}}\frac{1}{(1-|z|^2)^{\frac{\alpha+2-p}{p}}}. \end{aligned} \tag{7}$$

Proposition 2.1 *Let $p > 1$, $\alpha > -1$. If (f_n), $f \in B_{p,\alpha}$ and $||f_n - f||_{p,\alpha} \to 0$ then f_n converges to f uniformly on compact subsets of $\mathbb{D}$. Moreover, if $-1 < \alpha < p - 2$, $f_n \to f$ uniformly on $\mathbb{D}$.*

Proof: Since $\|f_n - f\|_{p,\alpha} \to 0$ we have, $f_n(0) \to f(0)$ and $\|f_n - f\|_{B_{p,\alpha}} \to 0$. First, let $\alpha = p - 2$. For each fixed $r \in (0, 1)$, for $|z| < r$ and by (5)

$$|f_n(z) - f(z) - f_n(0) + f(0)| \leq C\|f_n - f\|_{B_{p,p-2}}\left(\log \frac{1}{1-|z|^2}\right)^{\frac{1}{q}}$$

$$\leq \quad C\|f_n - f\|_{B_{p,p-2}} \left(\log \frac{1}{1-r^2}\right)^{\frac{1}{q}}.$$

So $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$.

Next for $-1 < a < p-2$. Clearly f_n converges uniformly on $\mathbb{D}$ to f by (6).

Lastly, for $\alpha > p-2$, for each fixed $r \in (0,1)$, for $|z| < r$, and by (7)

$$\begin{aligned} |f_n(z) - f(z) - f_n(0) + f(0)| &\leq C\|f_n - f\|_{B_{p,\alpha}} \frac{1}{(1-|z|^2)^{\frac{\alpha+2-p}{p}}} \\ &\leq C\|f_n - f\|_{B_{p,\alpha}} \frac{1}{(1-r^2)^{\frac{\alpha+2-p}{p}}}. \end{aligned}$$

Thus $f_n - f \to 0$ on compact subsets of $\mathbb{D}$. □

A proof similar to the one in Proposition 2.1 gives the following.

Corollary 2.1 *Let $p > 1$, and $\alpha > -1$, $(f_n) \in B_{p,\alpha}$. If (f_n) is a Cauchy sequence in $B_{p,\alpha}$ then (f_n) is uniformly Cauchy on compact sets.*

Theorem 2.3 *Let $1 < p < \infty$, $\alpha > -1$. Then the Besov type space $B_{p,\alpha}$ is a Banach space.*

Proof: Let (f_n) be a Cauchy sequence in $B_{p,\alpha}$.. Then (f_n') is Cauchy in A^p_α, which is a Banach space by Theorem 2.2. Therefore, f_n' converges to some function $g \in A^p_\alpha$, that is $\|f_n' - g\|_{A^p_\alpha} \to 0$.

Since f_n is Cauchy in $B_{p,\alpha}$, and by Corollary 2.1 f_n is uniformly Cauchy on compact subsets of $\mathbb{D}$, there exists a holomorphic function f on $\mathbb{D}$ such that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. Hence $f_n' \to f'$ uniformly on compact subsets of $\mathbb{D}$.

By [22, Theorem 3.12] and since $\|f_n' - g\|_{A^p_\alpha} \to 0$, there exists a subsequence of f_n' such that $f_n' \to g$ uniformly on compact subsets of $\mathbb{D}$. Therefore f' and g must be the same function. That is $g = f'$ and g is holomorphic. Hence $\|f_n' - f'\|_{A^p_\alpha} \to 0$ and $\|f_n - f\|_{B_{p,\alpha}} \to 0$.

Also from Corollary 2.1, f_n converges to f uniformly on compact sets of $\mathbb{D}$. Choose $\{0\}$ to be the compact subset. Then $|f_n(0) - f(0)| \to 0$. Therefore,

$$\|f_n - f\|_{p,\alpha} = |f_n(0) - f(0)| + \|f_n - f\|_{B_{p,\alpha}} \to 0$$

and $B_{p,\alpha}$ is complete and therefore a Banach space. $\square$

Below is Theorem 4.28 in [26] that gives a characterization of weighted Bergman spaces in terms of the first derivative.

Theorem 2.4 *Suppose $\beta > -1$, $p > 1$ and $f \in H(\mathbb{D})$. Then $f \in A^p_\beta$ if and only if $(1-|z|^2)f'(z) \in L^p(dA_\beta)$. Moreover, if $f \in A^p_\beta$ then*

$$\|f\|_{A^p_\beta} \asymp |f(0)| + \|(1-|z|^2)f'(z)\|_{L^p(dA_\beta)}.$$

Proposition 2.2 *If $p > 1$, $\alpha > p-1$ then $B_{p,\alpha} = A^p_{\alpha-p}$ with equivalent norms; that is, if $f \in B_{p,\alpha}$ then*

$$\|f\|_{B_{p,\alpha}} \asymp \|f\|_{A^p_{\alpha-p}}.$$

Proof: Let $f \in B_{p,\alpha}$ and $\beta = \alpha - p$. Then

$$\begin{aligned} \|f\|^p_{B_{p,\alpha}} &= \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha dA(z) \\ &= \int_{\mathbb{D}} \left[|f'(z)|(1-|z|^2)\right]^p (1-|z|^2)^\beta dA(z). \end{aligned}$$

Therefore, $f \in B_{p,\alpha}$ if and only if $(1-|z|^2)f' \in L^p(dA_\beta)$. Applying Theorem 2.4 we have $f \in B_{p,\alpha}$ if and only if $f \in A^p_{\alpha-p}$, and the equivalency of the norms. $\square$

Theorem 2.5 *For $\alpha > -1$, $p > 1$, $B_{p,\alpha}$ is a Möbius invariant Banach space if and only if $\alpha = p-2$.*

Proof: The space $B_{p,\alpha}$ is Möbius invariant if and only if $\|f \circ \alpha_\lambda\|^p_{B_{p,\alpha}} = \|f\|^p_{B_{p,\alpha}}$ for all $\lambda \in \mathbb{D}$.

Note that

$$\begin{aligned}\|f\circ\alpha_\lambda\|^p_{B_{p,\alpha}} &= \int_{\mathbb{D}} |(f\circ\alpha_\lambda)'(z)|^p(1-|z|^2)^\alpha dA(z)\\ &= \int_{\mathbb{D}} |f'(\alpha_\lambda(z))|^p|\alpha'_\lambda(z)|^p(1-|z|^2)^\alpha dA(z)\end{aligned}$$

Thus, making the change of variables $w=\alpha_\lambda(z)$ and by (1) we have

$$\begin{aligned}\|f\circ\alpha_\lambda\|^p_{B_{p,\alpha}} &= \int_{\mathbb{D}} |f'(w)|^p|\alpha'_\lambda(\alpha_\lambda(w))|^p(1-|\alpha_\lambda(w)|^2|^\alpha|\alpha'_\lambda(w)|^2 dA(w)\\ &= \int_{\mathbb{D}} |f'(w)|^p\frac{1}{|\alpha'_\lambda(w)|^p}(1-|w|^2)^\alpha|\alpha'_\lambda(w)|^\alpha|\alpha'_\lambda(w)|^2 dA(w)\\ &= \int_{\mathbb{D}} |f'(w)|^p(1-|w|^2)^\alpha\frac{|\alpha'_\lambda(w)|^\alpha}{|\alpha'_\lambda(w)|^{p-2}} dA(w).\end{aligned}$$

If $\alpha=p-2$ then it is clear that $B_{p,\alpha}$ is Möbius invariant.

If $\alpha\neq p-2$ and $B_{p,\alpha}$ was Möbius invariant then

$$\int_{\mathbb{D}} |f'(w)|^p(1-|w|^2)^\alpha\left(1-|\alpha'_\lambda(w)|^{\alpha-p+2}\right)dA(w)=0.$$

for all $f\in B_{p,\alpha}$. Apply the above for the function $f(z)=z$ to get

$$\int_{\mathbb{D}}(1-|w|^2)^\alpha|\alpha'_\lambda(w)|^{\alpha-p+2}dA(w)=\int(1-|w|^2)^\alpha dA(w)=\frac{1}{\alpha+1}.$$

Therefore,

$$\int_{\mathbb{D}}\frac{(1-|w|^2)^\alpha}{|1-\bar{\lambda}w|^{2\alpha-2p+4}}dA(w)=\frac{1}{\alpha+1}\frac{1}{(1-|\lambda|^2)^{\alpha-p+2}}.$$

Let $c=\alpha-2p+2$. By examining the three possible cases in Lemma 3.10 in [26], we arrive at a contradiction. Therefore if $\alpha\neq p-2$, $B_{p,\alpha}$ is not a Möbius invariant Banach space. $\square$

Definition 2.5 *The Hardy Space, H^2, is the space of holomorphic functions*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with square-summable power series coefficients, that is

$$\|f\|_{H^2} = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Notice $f \in H^2$ if and only if $f \in B_{2,1}$. That is $H^2 = B_{2,1}$ as sets but they are equipped with different norms.

Theorem 2.6 Littlewood-Paley Identity *For each $f \in H(\mathbb{D})$,*

$$\|f\|_{H^2}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|^2} dA(z).$$

It is known, see for example Theorem 17.11 in [22], that if $f \in H^2$ then the radial boundary values of f,

$$f(\zeta) := \lim_{r \to 1} f(r\zeta)$$

exists almost everywhere. Let $dm = \frac{d\theta}{d\pi}$ denote the normalized Lebesgue measure on the unit circle $\mathbb{T}$. By [22, Theorem 17.1].

$$\|f\|_{H^2}^2 = \int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta) \tag{8}$$

Definition 2.6 *Let $\alpha > 0$. The Bloch type space $\mathcal{B}_\alpha$ is the space of holomorphic functions f on $\mathbb{D}$ such that*

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2)^\alpha < \infty.$$

For each $\alpha > 0$, the space $\mathcal{B}_\alpha$ is a Banach space with norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}_\alpha},$$

see [28, Proposition 1]. The *Bloch space*, denoted by $\mathcal{B}$, is the space $\mathcal{B}_\alpha$ with $\alpha = 1$. It is a Möbius invariant Banach space. That is, if $f \in \mathcal{B}$ then $f \circ \varphi \in \mathcal{B}$ for all Möbius transformations φ. In fact

$$\begin{aligned}
\|f \circ \alpha_\lambda\|_{\mathcal{B}} &= \sup_{z \in \mathbb{D}} |(f \circ \alpha_\lambda)'(z)|(1 - |z|^2) \\
&= \sup_{z \in \mathbb{D}} |f'(\alpha_\lambda(z))||\alpha_\lambda'(z)|(1 - |z|^2) \\
&= \sup_{z \in \mathbb{D}} |f'(\alpha_\lambda(z))|(1 - |\alpha_\lambda(z)|^2) \\
&= \sup_{w \in \mathbb{D}} |f'(w)|(1 - |w|^2) \\
&= \|f\|_{\mathcal{B}}.
\end{aligned}$$

Rubel and Timoney in [21] showed that the Bloch space is the largest Möbius invariant Banach space in the following sense. Let X be a linear space of holomorphic functions with a semi norm $\|\cdot\|_X$ such that $f \circ \varphi \in X$ and $\|f \circ \varphi\|_X = \|f\|_x$ for all $f \in X$ and have a non-zero linear functional L that is decent (that is L extends to a continuous linear functional on the space of holomorphic functions on $\mathbb{D}$). Then X must be a subspace of the Bloch space.

Definition 2.7 *The space BMOA consists of the holomorphic functions $f \in H^2$ such that*

$$\|f\|_* = \sup_{\lambda \in \mathbb{D}} \|f \circ \alpha_\lambda - f(\lambda)\|_2 < \infty.$$

The norm in the space of BMOA is

$$\|f\|_{BMOA} = |f(0)| + \|f\|_*.$$

It is also a Möbius invariant Banach space, see [26, Chapter 9].

3 Composition Operators on Banach spaces of Holomorphic Functions

In this chapter we define and give properties of the composition operator. It is always bounded on H^2 and on $\mathcal{B}$. We recall the Besov type space Carleson measures and the characterization of bounded composition operators on $B_{p,\alpha}$.

Definition 3.1 *Let φ be a holomorphic self map of the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ and let f be a function in a space of holomorphic functions. Then the composition operator, C_φ, is the operator that maps f to $f \circ \varphi$, that is $C_\varphi f = f \circ \varphi$.*

Proposition 3.1 *C_φ is a one to one linear operator on any space of holomorphic functions.*

Proof: Let $f, g \in H(\mathbb{D})$ then

$$\begin{aligned} C_\varphi(f+g) &= (f+g)\circ\varphi \\ &= f\circ\varphi + g\circ\varphi. \end{aligned} \tag{9}$$

And if $a \in \mathbb{C}$

$$\begin{aligned} C_\varphi(af) &= af\circ\varphi \\ &= a(f\circ\varphi) \\ &= a(C_\varphi f) \end{aligned}$$

Thus C_φ is a linear operator. Moreover, if $C_\varphi f = C_\varphi g$ then $f\circ\varphi = g\circ\varphi$ and therefore $f|_{\varphi(\mathbb{D})} = g|_{\varphi(\mathbb{D})}$. Now since φ is holomorphic and $\mathbb{D}$ is open, $\varphi(\mathbb{D})$ is open by the Open Mapping Theorem. Two holomorphic functions that are equal on an open set are equal by the Identity Principle, thus $f \equiv g$ and C_φ is one to one. $\square$

Littlewood in 1925 proved that composition operators on H^2 are bounded.

Theorem (Littlewood's Subordination Principle) *Suppose that φ is a holomorphic self map of $\mathbb{D}$, with $\varphi(0) = 0$. Then for each $f \in H^2$ and $C_\varphi f \in H^2$, $\|C_\varphi f\|_{H^2} \le \|f\|_{H^2}$.*

For each bounded Borel subset E of $\mathbb{T}$ let m_φ denote the pull back measure of m; that is $m_\varphi(E) = m(\varphi^{-1}(E))$. Then for each $f \in H^2$,

$$\int_{\mathbb{T}} |f(\zeta)|^2 dm_\varphi(\zeta) = \int_{\mathbb{T}} |f \circ \varphi(\zeta)|^2 dm(\zeta). \tag{10}$$

Definition 3.2 *Let $p > 1$, $\alpha > -1$ and $w \in \mathbb{D}$. The counting function for $B_{p,\alpha}$ is*

$$N_{p,\alpha}(w, \varphi) = \sum_{\varphi(z)=w} |\varphi'(z)|^{p-2}(1 - |z|^2)^\alpha.$$

If w is not in $\varphi(\mathbb{D})$ then $N_{p,\alpha}(w, \varphi) = 0$.

The proof of the proposition below is similar to the proof of the proposition in section 10.3 in [23].

Proposition 3.2 *If g is a non-negative measurable function on $\mathbb{D}$ and φ is a holomorphic self map of $\mathbb{D}$, then*

$$\int_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^p(1 - |z|^2)^\alpha \, dA(z) = \int_{\mathbb{D}} g(w)N_{p,\alpha}(w, \varphi) \, dA(w).$$

Proof: Let g be a non-negative measurable function on $\mathbb{D}$. Let $Z \subset \mathbb{C}$ be the set where φ' vanishes. That is $Z = \{z \in \mathbb{D} : \varphi'(z) = 0\}$. Then Z is an at most countable set and $\mathbb{D}\backslash Z$ is a finite or countable union of a collection of semi-closed polar rectangles. That is $\mathbb{D}\backslash Z = \bigcup_{n\in A} R_n$.

Notice that for all n, φ is univalent on R_n. Then $\varphi : R_n \to \varphi(R_n)$ is one to one and onto. So the inverse of $\varphi|_{R_n}$ exists, $\psi_n = (\varphi|_{R_n}) : \varphi(R_n) \to R_n$. Now applying the usual change of variables, we have

$$\begin{aligned}
\int_{R_n} g(\varphi(z))|\varphi'(z)|^p(1 - |z|^2)^\alpha \, dA(z) &= \int_{R_n} g(\varphi(z))|\varphi'(z)|^{p-2}(1 - |z|^2)^\alpha|\varphi'(z)|^2 \, dA(z) \\
&= \int_{\varphi(R_n)} g(w)|\varphi'(\psi(w))|^{p-2}(1 - |\psi_n(w)|^2)^\alpha \, dA(w)
\end{aligned}$$

$$= \int_{\mathbb{D}} g(w)\chi_n(w)|\varphi'(\psi(w))|^{p-2}(1-|\psi_n(w)|^2)^\alpha \, dA(w)$$

where χ_n is the characteristic function of $\varphi(R_n)$. Now summing both sides on n we obtain

$$\int_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^p(1-|z|^2)^\alpha \, dA(z) = \int_{\mathbb{D}} g(w) \left\{ \sum_n \chi_n(w)|\varphi'(\psi(w))|^{p-2}(1-|\psi_n(w)|^2)^\alpha \right\} dA(w)$$

For $w \in \varphi(\mathbb{D})\backslash\varphi(Z)$ the points in the inverse image $\varphi^{-1}\{w\}$ have multiplicity one. So the term above in braces is $N_{p,\alpha}(w,\varphi)$ almost everywhere on $\varphi(\mathbb{D})$. Similarly if $w \notin \varphi(\mathbb{D})$ then the term in braces and $N_{p,\alpha}$ are both zero. Thus

$$\int_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^p(1-|z|^2)^\alpha \, dA(z) = \int_{\mathbb{D}} g(w) \sum_{\varphi(z)=w} |\varphi'(z)|^{p-2}(1-|z|^2)^\alpha \, dA(w).$$

$\square$

The counting function is useful in the following change of variables. Let $f \in B_{p,\alpha}$, $p > 1$ and $\alpha > -1$. Then by Proposition 3.2

$$\begin{aligned}
\|C_\varphi f\|^p_{B_{p,\alpha}} &= \int_{\mathbb{D}} |(f\circ\varphi)'(z)|^p(1-|z|^2)^\alpha \, dA(z) \\
&= \int_{\mathbb{D}} |f'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^\alpha \, dA(z) \\
&= \int_{\mathbb{D}} |f'(\varphi(z))|^p|\varphi'(z)|^{p-2}(1-|z|^2)^\alpha|\varphi'(z)|^2 \, dA(z) \\
&= \int_{\mathbb{D}} |f'(w)|^p N_{p,\alpha}(w,\varphi) \, dA(w). \qquad (11)
\end{aligned}$$

The composition operator C_φ is bounded on certain Besov type spaces $B_{p,\alpha}$. But they are not always bounded on all Besov type spaces. Carleson measures have been used to characterize properties of composition operators such as compactness and boundedness in many settings. They can be used to characterize bounded composition operators on $B_{p,\alpha}$, see [25].

Definition 3.3 *Let μ be a positive measure on $\mathbb{D}$. Then μ is a $(B_{p,\alpha}, p)$ Carleson measure* *if there exist a constant* $A > 0$ *such that for all* $f \in B^p_{p,\alpha}$

$$\int_{\mathbb{D}} |f'(w)|^p \, d\mu(w) \leq A \|f\|^p_{p,\alpha}.$$

By Lemma 3.10 (b) in [26]

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^t}{(1-\bar{\lambda}w)^{2+t+c}} \, dA(w) \approx \frac{1}{(1-|\lambda|^2)^c} \tag{12}$$

for $c > 0$ and $t > -1$.

Let $z \in \mathbb{D}$, $\lambda \in \mathbb{D}$, $\delta > \frac{\alpha+2}{p} - 1$ and

$$\beta_{\lambda,\delta}(z) = \frac{(1-|\lambda|^2)^{\delta+1-\frac{\alpha+2}{p}}}{|1-\bar{\lambda}z|^\delta}. \tag{13}$$

Then by (12)

$$\begin{aligned}
\|\beta_{\lambda,\delta}(z)\|^p_{B_{p,\alpha}} &= \int_{\mathbb{D}} |\beta'_{\lambda,\delta}(z)|^p (1-|z|^2)^\alpha \, dA(z) \\
&= \delta^p |\lambda|^p (1-|\lambda|^2)^{\delta p+p-\alpha-2} \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-\bar{\lambda}z)^{\delta p+p}} \, dA(z) \\
&\approx \delta^p |\lambda|^p (1-|\lambda|^2)^{\delta p+p-\alpha-2} \cdot \frac{1}{(1-|\lambda|^2)^{\delta p+p-2-\alpha}} \\
&\leq C. \qquad (14)
\end{aligned}$$

Let $\theta \in [0, 2\pi)$ and $h \in (0,1)$. The *Carleson type set* $S(h,\theta)$ is:

$$S(h,\theta) = \{z \in \mathbb{D} : |z - e^{i\theta}| < h\}.$$

The following theorem characterizes Carleson measures on $B_{p,\alpha}$. It extends and unifies Theorem 13 in [2] to Besov type spaces, and Carleson measures in Hardy and Bergman spaces.

Theorem 3.1 *Let* $p > 1, \alpha > -1$ *and* $\delta > \frac{\alpha+2}{p} - 1$.

Then the following are equivalent

1. *μ is a $(B_{p,\alpha}, p)$ Carleson measure.*

2. *There exists a constant $A > 0$ so that*

$$\mu(S(h,\theta)) \leq Ah^{\alpha+2}$$

   for all $\theta \in [0, 2\pi)$ and for all $h \in (0,1)$.

3. *There exists a constant $B > 0$ so that for all $\lambda \in \mathbb{D}$*

$$\int_{\mathbb{D}} \frac{(1-|\lambda|^2)^{\delta p+p-\alpha-2}}{|1-\bar{\lambda}w|^{\delta p+p}}\, d\mu(w) \leq B$$

Proof: First suppose that *(1)* above holds. Let $\beta_{\lambda,\delta} \in B_{p,\alpha}$ be as in (13). Then, by (14), statement *(3)* of the theorem easily follows. Now suppose that *(3)* holds. Given $h \in (0,1)$ and $\theta \in \mathbb{R}$, let $\lambda = (1-h)e^{i\theta}$. Then for all $w \in S(h,\theta)$

$$\frac{1-|\lambda|^2}{|1-\bar{\lambda}w|^2} \geq \frac{1}{5h}\,.$$

Therefore by *(3)* and since $1 - |\lambda| = h$,

$$\begin{aligned} B &\geq \int_{\mathbb{D}} \frac{(1-|\lambda|^2)^{\delta p+p-\alpha-2}}{|1-\bar{\lambda}w|^{\delta p+p}}\, d\mu(w) \\ &\geq \int_{S(h,\theta)} \left(\frac{1-|\lambda|^2}{|1-\bar{\lambda}w|^2}\right)^{\frac{\delta p+p}{2}} \cdot (1-|\lambda|^2)^{\frac{\delta p+p}{2}-\alpha-2}\, d\mu(w) \\ &\geq \text{const.} \frac{1}{h^{\alpha+2}} \mu(S(h,\theta)) \end{aligned}$$

and *(2)* follows. Finally suppose that *(2)* holds. Let $E_1(z) = \left\{ w \in \mathbb{D} : |z-w| < \frac{1-|z|}{2} \right\}$. Then if $z = |z|e^{i\theta}$, $E_1(z) \subset S(2(1-|z|), \theta)$. Therefore as in the proof of Theorem 13 of [2], if $f \in B_{p,\alpha}$

$$
\begin{aligned}
\int_{\mathbb{D}} |f'(z)|^p \, d\mu(z) &\leq 9 \int_{\mathbb{D}} |f'(w)|^p (1-|w|)^{-2} \mu(S(2(1-|w|,\theta))) \, dA(w) \\
&\leq \int_{\mathbb{D}} |f'(w)|^p \, (1-|w|^2)^{\alpha} \, dA(w)) \, ,
\end{aligned}
$$

and statement *(1)* of the theorem follows. □

By (11) and Theorem 3.1 we obtain the following.

Theorem 3.2 *Let $p > 1$, $\alpha > -1$ and $\delta > \frac{\alpha+2}{p} - 1$. Then the following are equilvalent.*

1. *The composition operator C_φ is a bounded operator on $B_{p,\alpha}$.*

2. *There exists a constant $B > 0$ such that*

$$
\sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1-|\lambda|^2)^{\delta p + p - \alpha - 2}}{|1-\overline{\lambda} w|^{\delta p + p}} \, N_{p,\alpha}(w,\varphi) \, dA(w) \leq B \, .
$$

3. *The measure $N_{p,\alpha}(w,\varphi) \, dA(w)$ is a p-Carleson measure.*

We know that composition operators are not always bounded on $B_{p,\alpha}$. However, composition operators are always bounded on the Bloch space as shown below.

Theorem 3.3 *If $\varphi(0) = 0$ then C_φ is a contraction on $\mathcal{B}$, that is for all $f \in \mathcal{B}$*

$$
\|C_\varphi f\| \leq \|f\|.
$$

Proof: Applying the Schwarz-Pick lemma to line (15) below we have

$$
\begin{aligned}
\|C_\varphi f\| &= |f(\varphi(0))| + \|C_\varphi f\|_{\mathcal{B}} \\
&= |f(0)| + \sup_{z \in \mathbb{D}} |(f \circ \varphi)'(z)| (1-|z|^2)
\end{aligned}
$$

$$
\begin{aligned}
&= |f(0)| + \sup_{z\in\mathbb{D}} |f'(\varphi(z))||\varphi'(z)|(1-|z|^2) \qquad (15)\\
&\leq |f(0)| + \sup_{z\in\mathbb{D}} |f'(\varphi(z))|(1-|\varphi(z)|^2)\\
&\leq |f(0)| + \sup_{w\in\mathbb{D}} |f'(w)|(1-|w|^2)\\
&= \|f\|
\end{aligned}
$$

Thus C_φ is a contraction on $\mathcal{B}$.

$\square$

4 Isometries

4.1 General Isometries

An *isometry* from one metric space to another is a distance preserving map between them. The isometries of the complex plane $\mathbb{C}$ form the *Euclidean group*. They are rotations, translations, reflections, and glide reflections and have the form $e^{i\theta}z + a$, and $e^{i\theta}\overline{z} + a$, $\theta \in \mathbb{R}$ and $a \in \mathbb{C}$.

Let X be normed linear space. A linear operator $T : X \to X$ is an isometry if for all $f \in X$, $||Tf|| = ||f||$.

Let D be a compact metric space. The space of continuous real valued functions defined on D is denoted by $C(D)$. It is a Banach space with the sup norm, that is if $f \in C(D)$ then

$$||f||_\infty = \sup_{t \in D} |f(t)| .$$

Stefan Banach was the first to study isometries on specific Banach spaces such as $C(D)$, l^p and $L^p[0,1]$. In particular Banach gave the following characterization of the onto linear isometries of $C(D)$, see [3], [13].

Theorem (Banach) *Let D be a compact metric space. If T is an onto isometry of $C(D)$, then there exists a real valued function h on D with $|h(t)| = 1$ for all $t \in D$ and φ a homeomorphism of D onto itself such that for all $f \in C(D)$*

$$Tf(t) = h(t)\, f(\varphi(t)) , \quad t \in D.$$

The Banach space of bounded holomorphic functions $\mathbb{D}$ with the sup norm is denoted by H^∞. Below is the characterization of the onto linear isometries of H^∞ given by deLeeuw, Rudin and Wermer.

Theorem (deLeeuw, Rudin, Wermer) *A linear operator T on H^∞ is an onto linear isometry* *if and only if there exists* $\theta \in \mathbb{R}$ *and* $p \in \mathbb{D}$ *such that for all* $f \in H^\infty$

$$T(f)(z) = e^{i\theta}\,(f \circ \alpha_p)(z)\,, \quad t \in D.$$

The following is the characterization that Kolaski gave for the isometries of the weighted Bergman spaces. It is Theorem 1 in [15] p 911.

Theorem (Kolaski) *Let* $1 < p < \infty$, $p \neq 2$ *and* $\alpha > -1$.

1. *If* $T : A^p_\alpha \to A^p_\alpha$ *is a linear isometry and if* $T1$ *is denoted by* ψ, *then there is a holomorphic map* φ *taking* $\mathbb{D}$ *onto a dense subset of* $\mathbb{D}$ *such that*

$$(Tf)(z) = W_{\psi,\varphi} f(z) = \psi(z) \cdot f(\varphi(z)), \tag{16}$$

   for all $f \in A^p_\alpha$. *For every bounded Borel function* h *on* $\mathbb{D}$ *then*

$$\int_{\mathbb{D}} (h \circ \varphi(z))|\psi(z)|^p \, dA_\alpha(z) = \int_{\mathbb{D}} h(z)\, dA_\alpha(z). \tag{17}$$

2. *If* φ *is a holomorphic map of* $\mathbb{D}$ *into* $\mathbb{D}$ *and if* $\psi \in A^p_\alpha$ *satisfies (17) for every continuous function* h *on* $\mathbb{D}$ *then (16) defines an isometry of* A^p_α.

3. *If the linear isometry* T *is onto* A^p_α, *then* $\varphi \in Aut(\mathbb{D})$. *Conversely, if* $\varphi \in Aut(\mathbb{D})$ *and if* $\psi \in A^p_\alpha$ *is related to* φ *by (17), then (16) defines an isometry of* A^p_α *onto* A^p_α.

A holomorphic self map of $\mathbb{D}$ is called a *full map* if the Lebesgue area measure of $\mathbb{D}\backslash\varphi(\mathbb{D}) = 0$. As the authors in [1] point out, Kolaski proved that if T is a linear isometry then φ is a full map.

Below is the characterization of the onto linear isometries of $\mathcal{B}$ given by Cima and Wogen in [5].

Theorem (Cima, Wogen) *If* T *is an onto isometry of* $\mathcal{B}$ *then there exists* $\theta \in \mathbb{R}$ *and* $\varphi \in$

Aut$(\mathbb{D})$ *such that*

$$Tf(z) = e^{i\theta}\,(\,f \circ \varphi(z) - f(\varphi(0))\,)\,, \quad z \in \mathbb{D}\,.$$

Let $n \geq 2$ and $f \in B_{p,p-2}$. A new norm on $B_{p,p-2}$ that is equivalent to $\|f\|_{p,p-2}$ is the following

$$\|f\| = \sum_{k=0}^{n-1} |f^{(k-1)}(0)| + \left(\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} dA(z)\right)^{\frac{1}{p}},$$

see [27].

Let I denote the identity operator on $B_{p,p-2}$. Given a natural number k define

$$z^k B_{p,p-2} = \{z^k f : f \in B_{p,p-2}\}.$$

Define recursively operators $I_k : A^p_{kp-2} \to z^k B_{p,p-2}$ by

$$I_1(f)(z) = \int_0^z f(u)du$$

and

$$I_k(f) = I_1(I_{k-1}(f))$$

for each integer $k > 1$. Below is the characterization of the linear isometries of $B_{p,p-2}$ that Hornor and Jamison gave in [14].

Theorem (Hornor, Jamison) *Let $p > 1$, $p \neq 2$ and $n \geq 2$. Let $T : B_{p,p-2} \to B_{p,p-2}$ be an isometry. Then there exists a permutation π of the set $\{0, 1, 2, ..., n-1\}$, unimodular complex numbers u_i, a function $g \in A^p_{np-2}$ and a holomorphic function φ on $\mathbb{D}$ and onto a dense subset of $\mathbb{D}$ such that*

$$(Tf)(z) = \sum_{i=0}^{n-1} \frac{u_i f^{(i)}(0) z^{\pi(i)}}{n(i)!} + I_n(g \cdot f^{(n)} \circ \varphi)(z)$$

for all $z \in \mathbb{D}$ and for all $f \in B_{p,p-2}$.

Moreover,

$$\int_{\mathbb{D}}(h\circ\varphi)|g|^p(z)(1-|z|^2)^{np-2}dA(z)=\int_{\mathbb{D}}h(z)(1-|z|^2)^{np-2}dA(z)$$

for every bounded Borel function h on $\mathbb{D}$. Conversely, given a collection u_i of unimodular constants, a permutation π of $\{0,1,...,n-1\}$ and an isometry U of A^p_{np-2}, the formula above defines an isometry on $B_{p,p-2}$.

4.2 Isometries among Composition Operators: Known results

The characterization of isometries among composition operators has been studied on many different spaces. Among these are the Hardy space, the Bergman space, the Bloch space, the space of BMOA and certain Besov spaces.

A holomorphic self-map φ of $\mathbb{D}$ is said to be *inner* if $|\varphi(\zeta)|=1$ for almost every ζ in the unit circle.

In 1968 Nordgren classified all isometries among composition operators on the Hardy Space H^2 (see [20]), also Martin and Vukotic in [18].

Theorem A *The composition operator, C_φ, is an isometry in H^2 if and only if $\varphi(0)=0$ and φ is an inner function.*

Proof: First, assume C_φ is an isometry on H^2. Then, $\|C_\varphi z\|_{H^2}=\|z\|_{H^2}$ and by (8)

$$\int_{\mathbb{T}}|\varphi(\zeta)|^2dm(\zeta)=\int_{\mathbb{T}}|\zeta|^2dm(\zeta).$$

Therefore,

$$\int_{\mathbb{T}}1-|\varphi(\zeta)|^2dm(\zeta)=0.$$

Since φ is a self map of $\mathbb{D}$ we conclude that $|\varphi(\zeta)|=1$ for almost every $\zeta\in\mathbb{T}$ and φ is an inner function. As we will see in Theorem 4.1, $\varphi(0)=0$.

Next, assume that φ is an inner function and that $\varphi(0) = 0$. Then by (10) and [20, Lemma 1] $dm_\varphi(\zeta) = dm(\zeta)$ and C_φ is an isometry on H^2. $\square$

Remark 4.1 *Note that by the proof of Theorem A, C_φ is an isometry on H^2 if and only if $\varphi(0) = 0$ and $\|\varphi\|_{H^2} = 1$.*

Martin and Vukotic also characterized the isometries of the Dirichlet among composition operators in [17].

Theorem B *A composition operator C_φ that acts on $\mathcal{D}$ is an isometry if and only if φ is a univalent full map of $\mathbb{D}$ that fixes the origin.*

Colonna determined the isometries among composition operators in $\mathcal{B}$ as follows (see [6, Theorem 5]).

Theorem (Colonna) *A holomorphic self map φ of $\mathbb{D}$ induces an isometric composition operator on $\mathcal{B}$ if and only if $\varphi(0) = 0$ and $||\varphi||_{\mathcal{B}} = 1$.*

Moreover, Martin and Vukotic determined the isometries among composition operators in $\mathcal{B}$ as follows (see [19, Theorem 1.1]). The *hyperbolic derivative* of φ is

$$\varphi^*(z) = \frac{(1 - |z|^2)\varphi'(z)}{(1 - |\varphi(z)|^2)},$$

and let $\tau^*_\varphi(z) = |\varphi^*(z)|$.

Theorem (Martin, Vukotic) *Let φ be a holomorphic self map of $\mathbb{D}$. Then C_φ is an isometry on $\mathcal{B}$ if and only if $\varphi(0) = 0$ and either φ is a rotation, or for each $\lambda \in \mathbb{D}$ there exists a sequence $(z_n) \in \mathbb{D}$ such that $|z_n| \to 1$, $\varphi(z_n) \to \lambda$ and $|\varphi^*(z_n)| \to 1$.*

The following is immediate by Colonna's theorem; moreover it is implicit in the proof of the theorem by Martin and Vukotic although it is not mentioned there. We give below a direct proof; part of the proof is similar to the one by Martin and Vukotic.

Theorem C *Let φ be a holomorphic self map of $\mathbb{D}$. Then, C_φ is an isometry on $\mathcal{B}$ if and only if $\varphi(0) = 0$ and for each $\lambda \in \mathbb{D}$, $||C_\varphi \alpha_\lambda||_{\mathcal{B}} = 1$.*

Proof: First, assume that $\varphi(0) = 0$ and for each $\lambda \in \mathbb{D}$, $||C_\varphi \alpha_\lambda||_{\mathcal{B}} = 1$. For each given $\lambda \in \mathbb{D}$ there exists a sequence (z_n) in $\mathbb{D}$ such that,

$$1 - \frac{1}{n} < |\alpha'_\lambda(\varphi(z_n))|\, |\varphi'(z_n)|\, (1 - |z_n|^2) \leq 1\,,$$

or equivalently by (1)

$$1 - \frac{1}{n} < (1 - |\alpha_\lambda(\varphi(z_n)|^2)\, \tau_\varphi(z_n) \leq 1\,. \tag{18}$$

We conclude, by taking subsequences if necessary, that either there exists a sequence (z_n) in $\mathbb{D}$ such that $|z_n| \to 1$ and (18) is valid or there exists $z_0 \in \mathbb{D}$ with $z_n \to z_0$ and

$$\lim_{n\to\infty} (1 - |\alpha_\lambda(\varphi(z_n)|^2)\, \tau_\varphi(z_n) = 1\,. \tag{19}$$

First, if $z_n \to z_0$, then by (19) we obtain

$$(1 - |\alpha_\lambda(\varphi(z_0)|^2)\, \tau_\varphi(z_0) = 1\,;$$

since $1 - |\alpha_\lambda(\varphi(z_0)|^2 \leq 1$ and $\tau_\varphi(z_0) \leq 1$, we must have $\lambda = \varphi(z_0)$ and $\tau_\varphi(z_0) = 1$. That is we obtain equality in the Schwarz-Pick Lemma, and there exists $\theta \in \mathbb{R}$ such that $\varphi(z) = e^{i\theta} z$. Next, if for a given $\lambda \in \mathbb{D}$, there exists a sequence (z_n) in $\mathbb{D}$ such that $|z_n| \to 1$ and (18) holds, we conclude that

$$\lim_{n\to\infty} \tau_\varphi(z_n) = 1 \tag{20}$$

and

$$\lim_{n\to\infty} |\alpha_\lambda \circ \varphi(z_n)| = 0$$

or

$$\lim_{n\to\infty} \frac{|\lambda - \varphi(z_n)|}{|1-\bar{\lambda}\varphi(z_n)|} = 0\,;$$

it follows that $\varphi(z_n) \to \lambda$.

We have proved that either φ is a disk rotation or for each $\lambda \in \mathbb{D}$ there exists a sequence $(z_n) \in \mathbb{D}$ such that $|z_n| \to 1$, $\varphi(z_n) \to \lambda$ and (20) holds. We will show that C_φ is an isometry on $\mathcal{B}$.

First, suppose that $\varphi(z) = e^{i\theta} z$, for some $\theta \in \mathbb{R}$. If $f \in \mathcal{B}$, then

$$||C_\varphi f|| = |\varphi(0)| + \sup_{z\in\mathbb{D}} |f'(e^{i\theta} z)|\,(1-|z|^2) = ||f||$$

therefore rotations are isometries on $\mathcal{B}$.

Next, suppose that for each $\lambda \in \mathbb{D}$ there exists a sequence $(z_n) \in \mathbb{D}$ such that $|z_n| \to 1$, $\varphi(z_n) \to \lambda$ and (20) holds. It is enough to prove that if $f \in \mathcal{B}$ with $||f||_{\mathcal{B}} = 1$ then $||C_\varphi f||_{\mathcal{B}} = 1$, see Proposition 4.2 below. Let $f \in \mathcal{B}$ with $||f||_{\mathcal{B}} = 1$. Then, either

(a) there exists $w \in \mathbb{D}$ such that

$$|f'(w)|\,(1-|w|^2) = 1 \tag{21}$$

or

(b) there exists a sequence (w_n) in $\mathbb{D}$ such that $|w_n| \to 1$ and

$$|f'(w_n)|\,(1-|w_n|^2) \to 1\,. \tag{22}$$

Let $h(z) := |f'(z)|\,(1-|z|^2)$. Suppose that (a) holds. Then there exists a sequence $(z_n) \in \mathbb{D}$ such that $|z_n| \to 1$, $\varphi(z_n) \to w$ and (20) holds. Therefore, by (21) and since the function h

is continuous at w, $h(\varphi(z_n))\,\tau_\varphi(z_n) \to 1$ or

$$|(f\circ\varphi)'(z_n)|(1-|z_n|^2) \to 1\,,$$

and $||C_\varphi f||_{\mathcal{B}} \geq 1$. By Theorem 3.3 we obtain $||C_\varphi f||_{\mathcal{B}} = 1$ for each $f \in \mathcal{B}$ with $||f||_{\mathcal{B}} = 1$, and C_φ is an isometry on $\mathcal{B}$. Finally, suppose that (b) holds. Fix $0 < r < 1$. By (22), there exists a natural number n_0 such that $h(w_{n_0}) > (r+1)/2$. By our assumption, there exists a sequence $(z_n) \in \mathbb{D}$ such that $|z_n| \to 1$, $\varphi(z_n) \to w_{n_0}$ and (20) holds. Therefore since h is continuous at w_{n_0}, $h(\varphi(z_n))\,\tau_\varphi(z_n) \to h(w_{n_0})$ or

$$|(f\circ\varphi)'(z_n)|(1-|z_n|^2) \to h(w_{n_0})\,.$$

We conclude that there exists a natural number m such that

$$\begin{aligned}|(f\circ\varphi)'(z_m)|(1-|z_m|^2) &> h(w_{n_0}) - \frac{1-r}{2}\\ &> \frac{r+1}{2} - \frac{1-r}{2}\\ &= r\,.\end{aligned}$$

We have shown that for each $0 < r < 1$ there exists a natural number m such that $|(f\circ\varphi)'(z_m)|(1-|z_m|^2) > r$. Therefore $||C_\varphi f||_{\mathcal{B}} \geq 1$ and by Theorem 3.3 we obtain $||C_\varphi f||_{\mathcal{B}} = 1$ for each $f \in \mathcal{B}$ with $||f||_{\mathcal{B}} = 1$. We have shown that C_φ is an isometry on $\mathcal{B}$.

Conversely, if C_φ is an isometry on $\mathcal{B}$ then as we will see in Proposition 4.2 below, $\varphi(0) = 0$. If $\lambda \in \mathbb{D}$, then by (1)

$$||\alpha_\lambda||_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |\alpha'_\lambda(z)|\,(1-|z|^2) = 1 - |\alpha_\lambda(z)|^2 = 1\,.$$

Therefore $||C_\varphi \alpha_\lambda||_{\mathcal{B}} = 1$. □

Zorboska in [29] characterized the isometries on all Bloch type spaces $\mathcal{B}_\alpha$, $\alpha \neq 1$.

Theorem D *Let φ be a holomorphic self map of $\mathbb{D}$ and let $\alpha > 0$, $\alpha \neq 1$. Then C_φ is an isometry on $\mathcal{B}_\alpha$ if and only if φ is a rotation.*

Remark 4.2 *Examining the proof of Theorem D we see that C_φ is an isometry on $\mathcal{B}_\alpha$ if and only if $\|\varphi\|_{\mathcal{B}_\alpha} = 1$ and $\varphi(0) = 0$. Notice that this was the case in H^2, see Remark 4.1 and Colonna's Theorem in $\mathcal{B}$.*

Isometries among composition operators on the space of BMOA is given by Laitila in the following theorem in [16].

Theorem E *The following are equivalent.*

1. $\|C_\varphi f\|_* = \|f\|_*$ for all $f \in BMOA$.

2. $\|\alpha_w \circ \varphi\|_* = 1$ for all $w \in \mathbb{D}$.

3. *The map φ satisfies the following property:*

   for every $w \in \mathbb{D}$, there is a sequence a_n in $\mathbb{D}$ such that $\varphi(z_n) \to w$ and $\|\varphi_{a_n}\|_2 \to 1$, as $n \to \infty$, where $\varphi_{(a_n)} = \alpha_{\varphi(a_n)} \circ \varphi \circ \alpha_{a_n}$ for $n \in \mathbb{N}$.

In [18] Martin and Vukotic proved the following theorem about the weighted Bergman spaces.

Theorem F *Let $p \geq 1$ and $\alpha > -1$. A composition operator C_φ is an isometry of A^p_α if and only if φ is a rotation.*

Recently, in 2014 Allen, Heller and Pons studied composition operators that are isometries on the Besov space in [1]. Let $\eta_\varphi(\omega)$ denote the cardinality of the set $\varphi^{-1}\{\omega\}$.

Theorem G

1. *Let $1 < p < 2$. Then C_φ is an isometry on B_p if and only if φ is a rotation of the disk.*

2. *Let $p > 2$ and suppose that $n_\varphi = 1$ almost everywhere in some neighborhood of the origin. Then C_φ is an isometry on B_p if and only if φ is a rotation of the disk.*

4.3 Isometries of Banach spaces among composition operators

If C_φ is an isometry on H^2, B_α, $B_{p,p-2}$, A^p_α, BMOA, then $\varphi(0) = 0$, see section 4.2 and [1, Lemma 3.4]. Below we show that under some very general conditions on a Banach space X, the isometries among composition operators fix the origin. In particular, this theorem unifies the proofs of all such known results.

Theorem 4.1 *Let X be a Banach space of holomorphic functions containing the constant functions and $Aut(\mathbb{D})$. Moreover assume that for all $f \in X$, $||f||_X = |f(0)| + ||f||_{sX}$ and for any constant c, $||f + c||_{sX} = ||f||_{sX}$. Then C_φ is an isometry on X if and only if $\varphi(0) = 0$ and for all $f \in X$*

$$||f \circ \varphi||_{sX} = ||f||_{sX}. \tag{23}$$

Proof: First, assume that C_φ is an isometry on X and let $\lambda = \varphi(0)$. We conclude that

$$\begin{aligned} ||\alpha_\lambda \circ \varphi||_{sX} + |\lambda| &= ||\alpha_\lambda \circ \varphi - \lambda||_{sX} + |\lambda| \\ &= ||\alpha_\lambda \circ \varphi - \lambda||_X \\ &= ||C_\varphi(\alpha_\lambda - \lambda)||_X \\ &= ||\alpha_\lambda - \lambda||_X \\ &= ||\alpha_\lambda - \lambda||_{sX} \\ &= ||\alpha_\lambda||_{sX}\,. \end{aligned} \tag{24}$$

By (24) and using one more time that C_φ is an isometry, we conclude that

$$\begin{aligned} |\lambda| + ||\alpha_\lambda||_{sX} = ||\alpha_\lambda||_X &= ||C_\varphi \alpha_\lambda||_X \\ &= ||\alpha_\lambda \circ \varphi||_X \end{aligned}$$

$$
\begin{aligned}
&= ||\alpha_\lambda \circ \varphi||_{sX} \\
&= ||\alpha_\lambda||_{sX} - |\lambda| \qquad (25)
\end{aligned}
$$

and $\lambda = \varphi(0) = 0$. It is now clear that (23) holds.

Conversely, if we assume that $\varphi(0) = 0$ and $\|f \circ \varphi\|_{sX} = \|f\|_{sX}$ for all $f \in X$ then

$$
\begin{aligned}
\|C_\varphi f\|_X &= |f(\varphi(0))| + \|f\|_{sX} \\
&= |f(0)| + \|f\|_{sX} \\
&= \|f\|_X,
\end{aligned}
$$

and C_φ is an isometry on X. $\square$

The *weighted composition operator*, $W_{\psi,\varphi}$, is defined by $W_{\psi,\varphi} f = \psi(f \circ \varphi)$ for $\psi \in H(\mathbb{D})$, and φ an holomorphic self map of $\mathbb{D}$.

Next we extend the result of the theorem above to isometries among weighted composition operators; we derive a necessary and sufficient condition that guarantees equality of seminorms of f and $W_{\psi,\varphi} f$.

Theorem 4.2 *Let X be a Banach space of holomorphic functions containing the constant functions and Aut($\mathbb{D}$). Assume that for all $f \in X$, $\|f\|_X = |f(0)| + \|f\|_{sX}$ and $||f + c||_{sX} = ||f||_{sX}$, for each constant c. Moreover, assume that $W_{\psi,\varphi}$ is an isometry on X. Then $||W_{\psi,\varphi} f||_{sX} = ||f||_{sX}$ for all $f \in X$ if and only if $\psi(z) = \psi(0)$ for all $z \in \mathbb{D}$, $|\psi(0)| = 1$ and $\varphi(0) = 0$.*

Proof: The sufficiency of the conditions for the equality of the semi norms follows easily by Theorem 4.1. For the necessity, if $W_{\psi,\varphi}$ is an isometry and $||W_{\psi,\varphi} f||_{sX} = ||f||_{sX}$ for all $f \in X$ then $|\psi(0)|\,|f(\varphi(0))| = |f(0)|$ for all $f \in X$. Applying this for the functions 1 and $f(z) = z$ we see that $|\psi(0)| = 1$ and $\varphi(0) = 0$ respectively. Next, note that by assumption

$$\begin{aligned} ||1||_X &= ||1||_{sX} + 1 \\ &= ||2||_{sX} + 1 \\ &= ||2||_X - 2 + 1 \\ &= 2||1||_X - 1 \end{aligned} \tag{26}$$

and $||1||_X = 1$. Since $W_{\psi,\varphi}$ is an isometry, $||W_{\psi,\varphi}1||_X = ||1||_X = 1$ and

$$\begin{aligned} 1 = ||\psi||_X &= ||\psi||_{sX} + |\psi(0)| \\ &= ||\psi - \psi(0)||_{sX} + |\psi(0)| \\ &= ||\psi - \psi(0)||_X + |\psi(0)| \,. \end{aligned} \tag{27}$$

Therefore $|\psi(0)| = 1$ if and only if $\psi(z) = \psi(0)$ for all $z \in \mathbb{D}$ and the conclusion follows. $\square$

The only Mobius transformations on $\mathbb{D}$ fixing the origin are disk rotations. Therefore the following is an immediate corollary of Theorem 4.1.

Proposition 4.1 *Let X be a Banach space of holomorphic functions containing the constant functions and $Aut(\mathbb{D})$. Let $\varphi \in Aut(\mathbb{D})$ and assume that $\|f\|_X = |f(0)| + \|f\|_{sX}$ and for any constant c, $||f + c||_{sX} = ||f||_{sX}$. Then C_φ is an isometry on X if and only if φ is a disk rotation.*

Corollary 4.1 *Let $\alpha > -1$, $p > 1$ and $\varphi \in Aut(\mathbb{D})$. Then C_φ is an isometry on $B_{p,\alpha}$ if and only if φ is a rotation.*

Proposition 4.2 *Let X be a Banach space of holomorphic functions containing the identity for $i(z) = z$, $z \in \mathbb{D}$. Then the range of C_φ contains a univalent map if and only if φ is univalent.*

Proof: First, assume there exists $g \in X$ univalent and $f \in X$ such that $C_\varphi(f) = g$. Let $z_1, z_2 \in \mathbb{D}$ be such that $\varphi(z_1) = \varphi(z_2)$. Then

$$g(z_1) = C_\varphi f(z_1) = f(\varphi(z_1)) = f(\varphi(z_2)) = C_\varphi f(z_2) = g(z_2)\,.$$

Since g is univalent, we conclude that $z_1 = z_2$ and thusly φ is univalent.

Next, assume that φ is univalent. Then, since X contains the constants and $\varphi = C_\varphi z$, φ is a univalent map on the range of C_φ. □

The following is an immediate corollary of the above proposition.

Corollary 4.2 *Let $\alpha > -1$, $p > 1$. If C_φ is onto on $B_{p,\alpha}$ then φ is univalent.*

In [1, Lemma 3.4] the authors show that if C_φ is an isometry on $B_{p,p-2}$ then $W_{\varphi,\varphi'}$ is an isometry on A^p_{p-2}. Next we show that the isometries among the composition operators C_φ on all Besov type spaces are precisely the weighted composition operators with symbols φ and φ' on weighted Bergman spaces.

Theorem 4.3 *Let $1 < p < \infty$ and $\alpha > -1$. Then C_φ is an isometry on $B_{p,\alpha}$ if and only if $\varphi(0) = 0$ and $W_{\varphi',\varphi}$ is an isometry on A^p_α.*

Proof: First, assume that C_φ is an isometry on $B_{p,\alpha}$. Then by Theorem 4.1 $\varphi(0) = 0$ and for each $g \in B_{p,\alpha}$, $||C_\varphi g||_{p,\alpha} = ||g||_{p,\alpha}$ or

$$\int_{\mathbb{D}} |g'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^\alpha dA(z) = \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^\alpha dA(z)\,. \tag{28}$$

For each $f \in A^p_\alpha$, pick $g \in B_{p,\alpha}$ such that $f = g'$. Then by (28) we have $||W_{\varphi',\varphi} f||_{A^p_\alpha} = ||f||_{A^p_\alpha}$ and we conclude that $W_{\varphi',\varphi}$ is an isometry on A^p_α.

Next if $W_{\varphi',\varphi}$ is an isometry on A^p_α then for all $f \in A^p_\alpha$

$$\int_{\mathbb{D}} |f(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^\alpha dA(z) = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z)\,.$$

Since $g \in B_{p,\alpha}$ if and only if $g' \in A^p_\alpha$, we conclude that (28) is valid for all $g \in B_{p,\alpha}$ and C_φ is an isometry on $B_{p,\alpha}$. $\square$

The following is an immediate corollary of the proposition above and Kolaski's theorem.

Corollary 4.3 *Let $1 < p < \infty$, $p \neq 2$, and $\alpha > -1$. If C_φ is an isometry on $B_{p,\alpha}$ then $\varphi(0) = 0$ and φ is a full map of $\mathbb{D}$.*

Proposition 4.3 *Let $p > 1$ and $\alpha > -1$. Then all rotations induce isometries on $B_{p,\alpha}$.*

Proof: Let $\theta \in \mathbb{R}$ and $\varphi(z) = e^{i\theta}z$ be a rotation in the unit disk. Then if $f \in B_{p,\alpha}$, by (4) and Corollary 4.3 and applying a change of variables, we have

$$\begin{aligned}
\|C_\varphi f\|^p_{p,\alpha} &= |f(\varphi(0)| + \int_{\mathbb{D}} |f'(\varphi(z)|^p |\varphi'(z)|^p (1-|z|^2)^\alpha \, dA(z) \\
&= |f(0)| + \int_{\mathbb{D}} |f'(e^{i\theta}z)|^p (1-|z|^2)^\alpha \, dA(z) \\
&= |f(0)| + \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^\alpha \, dA(w). \\
&= \|f\|^p_{p,\alpha}.
\end{aligned}$$

Thus rotations are isometries on $B_{p,\alpha}$. $\square$

Proposition 4.4 *Let $1 < p < 2$, $-1 < \alpha \leq p-2$. If C_φ is an isometry on $B_{p,\alpha}$ then φ is univalent on $\mathbb{D}$.*

Proof: By Corollary 4.3, $\varphi(0) = 0$ and $\eta_\varphi(z) \geq 1$ almost everywhere on $\mathbb{D}$. Therefore by Schwarz's Lemma and the Schwarz-Pick Lemma,

$$|\varphi'(z)|^p \, (1-|z|^2)^\alpha \geq (1-|\varphi(z)|^2)^\alpha \, |\varphi'(z)|^2. \tag{29}$$

Then since C_φ is an isometry, by (29) and Proposition 3.2

$$\begin{aligned}
||z||_{p,\alpha}^p = ||\varphi||_{p,\alpha}^p &= \int_{\mathbb{D}} |\varphi'(z)|^p \,(1-|z|^2)^\alpha \, dA(z) \\
&\geq \int_{\mathbb{D}} (1-|\varphi(z)|^2)^\alpha \, |\varphi'(z)|^2 dA(z) \\
&= \int_{\mathbb{D}} (1-|w|^2)^\alpha \, \eta_\varphi(w) \, dA(w) \\
&\geq \int_{\mathbb{D}} (1-|w|^2)^\alpha \, dA(w) \\
&= ||z||_{p,\alpha}^p; \qquad (30)
\end{aligned}$$

therefore

$$||\varphi||_{p,\alpha}^p = \int_{\mathbb{D}} (1-|w|^2)^\alpha \, dA(w) = \int_{\mathbb{D}} (1-|w|^2)^\alpha \, \eta_\varphi(w) \, dA(w) \qquad (31)$$

and $\eta_\varphi(w) = 1$ for almost every $w \in \mathbb{D}$. Equivalently φ is a univalent full map of $\mathbb{D}$. By the proof of the main Theorem in [17] we conclude that φ is univalent on $\mathbb{D}$. □

Theorem 4.4 *Let $1 < p < 2$, $-1 < \alpha \leq p-2$. Then, C_φ is an isometry on $B_{p,\alpha}$ if and only if φ is a rotation.*

Proof: Fix $1 < p < 2$ and $-1 < \alpha \leq p-2$. By Corollary 4.3 and Proposition 4.4, if C_φ is an isometry on $B_{p,\alpha}$ then φ is univalent full map on $\mathbb{D}$ that fixes the origin. Now the calculation in (30) is valid and implies that

$$\begin{aligned}
||\varphi||_{p,\alpha}^p &= \int_{\mathbb{D}} |\varphi'(z)|^p \,(1-|z|^2)^\alpha \, dA(z) \\
&= \int_{\mathbb{D}} (1-|w|^2)^\alpha \eta_\varphi(w) \, dA(w) \\
&= \int_{\mathbb{D}} (1-|\varphi(z)|^2)^\alpha \, |\varphi'(z)|^2 \, dA(z)\,. \qquad (32)
\end{aligned}$$

Let $h(z)$ denote the function below:

$$|\varphi'(z)|^{p-2}(1-|z|^2)^{\alpha} - (1-|\varphi(z)|^2)^{\alpha}\,. \tag{33}$$

Then by (29), $h(z) \geq 0$ for all $z \in \mathbb{D}$ and by (32)

$$\int_{\mathbb{D}} h(z)dA(z) = 0\,;$$

we conclude that $h(z) = 0$ for almost all $z \in \mathbb{D}$, that is

$$|\varphi'(z)|^{p-2}(1-|z|^2)^{\alpha} = (1-|\varphi(z)|^2)^{\alpha}\,.$$

By the Schwarz-Pick Lemma we obtain

$$|\varphi'(z)|^{p-2}\,(1-|z|^2)^{\alpha} \quad \geq \quad (1-|\varphi(z)|^2)^{p-2}\,(1-|z|^2)^{\alpha-p+2}$$

and

$$(1-|\varphi(z)|^2)^{\alpha-p+2} \geq (1-|z|^2)^{\alpha-p+2}\,, \tag{34}$$

for almost all $z \in \mathbb{D}$. Since $\alpha \leq p-2$, this implies that $|z| \leq |\varphi(z)|$. And since $\varphi(0) = 0$, by Schwarz's Lemma, $|\varphi(z)| \leq |z|$. Therefore we obtain equality in Schwarz's Lemma and φ is a rotation. By Proposition 4.3 disk rotations are isometries on $B_{p,\alpha}$. □

Theorem 4.5 *If $p > 2$ and $\alpha \geq p-2$ and φ is univalent then, C_φ is an isometry on $B_{p,\alpha}$ if and only if φ is a rotation.*

Proof: Fix $p > 2$ and $\alpha \geq p-2$. Assume that C_φ is an isometry in $B_{p,\alpha}$. By Proposition 4.1, $\varphi(0) = 0$ and by Corollary 4.3 the Lebesgue area measure of $\mathbb{D} \setminus \varphi(\mathbb{D})$ is 0. Since $p > 2$,

$\alpha \geq p-2$ by both Schwarz's Lemma and the Schwarz-Pick Lemma,

$$\begin{aligned}
||\varphi||_{p,\alpha}^{p} &= \int_{\mathbb{D}} |\varphi'(z)|^{p}\,(1-|z|^{2})^{\alpha}\,dA(z) \\
&= \int_{\mathbb{D}} |\varphi'(z)|^{p-2}\,(1-|z|^{2})^{p-2}\,(1-|z|^{2})^{\alpha-p+2}\,|\varphi'(z)|^{2}\,dA(z) \\
&\leq \int_{\mathbb{D}} (1-|\varphi(z)|^{2})^{p-2}\,(1-|z|^{2})^{\alpha-p+2}\,|\varphi'(z)|^{2}\,dA(z) \\
&\leq \int_{\mathbb{D}} (1-|\varphi(z)|^{2})^{\alpha}\,|\varphi'(z)|^{2}\,dA(z) \\
&= \int_{\varphi(\mathbb{D})} (1-|w|^{2})^{\alpha}\,dA(w) \\
&= \int_{\mathbb{D}} (1-|w|^{2})^{\alpha}\,dA(w) = ||z||_{p,\alpha}^{p} \qquad (35)
\end{aligned}$$

therefore we obtain equalities everywhere above and

$$\int_{\mathbb{D}} (1-|\varphi(z)|^{2})^{p-2}\,(1-|z|^{2})^{\alpha-p+2}\,|\varphi'(z)|^{2}\,dA(z) = \int_{\mathbb{D}} (1-|\varphi(z)|^{2})^{\alpha}\,|\varphi'(z)|^{2}\,dA(z)$$

Since $\alpha - p + 2 \geq 0$ we have for the function $h(z)$ below,

$$\begin{aligned}
h(z) &= (1-|\varphi(z)|^{2})^{p-2}\,\left((1-|\varphi(z)|^{2})^{\alpha} - (1-|z|^{2})^{\alpha-p+2}\right) \\
&= (1-|\varphi(z)|^{2})^{p-2}\,\left((1-|\varphi(z)|^{2})^{\alpha-p+2} - (1-|z|^{2})^{\alpha-p+2}\right) \\
&\geq 0 \qquad (36)
\end{aligned}$$

By (35) we have

$$\int_{\mathbb{D}} |\varphi'(z)|^{2} h(z)\,dA(z) = 0.$$

Since φ is univalent φ' never vanishes. Thus $h(z) = 0$ for almost every $\zeta \in \mathbb{D}$. By Proposition 4.3 disk rotations are isometries in $B_{p,\alpha}$. □

Let $A^{2}(N_{2,\alpha}(w,\varphi)dA(w))$ denote the space of holomorphic functions f on $\mathbb{D}$ with

$$\|f\|_{N_{2,\alpha}}^{2} = \int_{\mathbb{D}} |f(w)|^{2} N_{2,\alpha}(w,\varphi)\,dA(w) < \infty.$$

Below we classify isometries on $B_{2,\alpha}$ in terms of Nevanlinna type counting functions. It extends the main result in [17] and part of the proof is similar to it.

Theorem 4.6 *Let $\alpha > -1$. Then, C_φ is an isometry on $B_{2,\alpha}$ if and only if $\varphi(0) = 0$ and for almost every $w \in \mathbb{D}$,*

$$N_{2,\alpha}(w,\varphi) = \sum_{\varphi(z)=w} (1-|z|^2)^\alpha = (1-|w|^2)^\alpha \,. \tag{37}$$

Proof: First, assume that C_φ is an isometry on $B_{2,\alpha}$. By Theorem 4.3 this is equivalent to $\varphi(0) = 0$ and $W_{\varphi',\varphi}$ being an isometry on A^2_α. Then for all $h \in A^2_\alpha$ we have

$$\int_{\mathbb{D}} |h(\varphi(z))|^2 \, |\varphi'(z)|^2 \, (1-|z|^2)^\alpha \, dA(z) = \int_{\mathbb{D}} |h(z)|^2 \, (1-|z|^2)^\alpha \, dA(z)\,;$$

and by Proposition 3.2 for $g = |h|^2$ we obtain

$$\int_{\mathbb{D}} |h(w)|^2 \, N_{2,\alpha}(w,\varphi) \, dA(w) = \int_{\mathbb{D}} |h(z)|^2 \, (1-|z|^2)^\alpha \, dA(z). \tag{38}$$

By (38), $h \in A^2_\alpha$ if and only if $h \in A^2(N_{2,\alpha}(w,\varphi)dA(w))$ and in fact $\|h\|_{A^2_\alpha} = \|h\|_{N_{2,\alpha}}$. Using the polarization identities in ${A^2}_\alpha$ and in $A^2(N_{2,\alpha}(w,\varphi))dA(w)$ and for functions $f, g \in A^2_\alpha$ (see [10, Lemma 3.3]), we obtain

$$\int_{\mathbb{D}} f(w)\overline{g(w)} \, N_{2,\alpha}(w,\varphi) \, dA(w) = \int_{\mathbb{D}} f(w)\overline{g(w)} \; (1-|z|^2)^\alpha \, dA(z)\,;$$

by using $f(z) = z^m$ and $g(z) = z^n$, $m, n = 0, 1, 2, ...$ we see that for all polynomials $p(z, \overline{z})$

$$\int_{\mathbb{D}} p(z,\overline{z}) \, N_{2,\alpha}(w,\varphi) \, dA(w) = \int_{\mathbb{D}} p(z,\overline{z}) \; (1-|z|^2)^\alpha \, dA(z).$$

By the Stone-Weierstrass Theorem (see [10, Theorem 2.40], for all $h \in C(\overline{\mathbb{D}})$

$$\int_{\mathbb{D}} h(z) \, N_{2,\alpha}(w,\varphi) \, dA(w) = \int_{\mathbb{D}} h(z) \; (1-|z|^2)^\alpha \, dA(z)\,.$$

By the Riesz Representation Theorem (see [22, Theorem 2.14]) we conclude that (37) holds for almost every $w \in \mathbb{D}$. By Theorem 4.1 and (38) the sufficiency of the above condition is clear. $\square$

By Theorem 4.1 and the main result in [17] we obtain the following.

Corollary 4.4 *The composition operator C_φ is an isometry on $B_{2,0}$ if and only if φ is a univalent full map that fixes the origin.*

If C_φ is an isometry on $B_{2,\alpha}$ and $\alpha \geq 0$ then by Theorem 4.6 and for almost every $w \in \mathbb{D}$, $0 < (1-|w|^2)^\alpha \leq \eta_\varphi(w)$. So we obtain the following corollary.

Corollary 4.5 *Let $\alpha > 0$. If C_φ is an isometry on $B_{2,\alpha}$ then $\varphi(0) = 0$ and φ is a full map of $\mathbb{D}$.*

Corollary 4.6 *Let $-1 < \alpha < 0$ and φ a non constant self map of $\mathbb{D}$. Then, C_φ is an isometry on $B_{2,\alpha}$ if and only if φ is a rotation.*

Proof: First, assume that C_φ is an isometry on $B_{2,\alpha}$. By Schwarz's lemma if $z \in \mathbb{D}$ then , $(1-|z|^2)^\alpha \geq (1-|\varphi(z)|^2)^\alpha$. Hence by Theorem 4.6 and for almost all $w \in \mathbb{D}$

$$(1-|w|^2)^\alpha = N_{2,\alpha}(w,\varphi) \geq (1-|w|^2)^\alpha \, \eta_\varphi(w)\,.$$

We conclude $\eta_\varphi(w) \leq 1$. Pick $w \in \mathbb{D}$ with $\eta_\varphi(w) = 1$ and such that (37) holds. We obtain equality in Schwarz's lemma and φ has to be a rotation. By Proposition 4.3 disk rotations are isometries. $\square$

Below we extend Theorem 4.6, include all indices $p > 1$ to $\alpha > -1$.

Theorem 4.7 *Let $p > 1$, $p \neq 2$. Then C_φ is an isometry on $B_{p,\alpha}$ if and only if $\varphi(0) = 0$ and for almost every $w \in \mathbb{D}$, $N_{p,\alpha}(w,\varphi) = (1-|w|^2)^\alpha$.*

Proof: First, assume that C_φ is an isometry on $B_{p,\alpha}$; by Theorem 4.3 this is equivalent to $\varphi(0) = 0$ and $W_{\varphi',\varphi}$ being an isometry on A^p_α. By (17) and for all bounded Borel functions h we have

$$\int_{\mathbb{D}} h(\varphi(z))\,|\varphi'(z)|^p\,(1-|z|^2)^\alpha\,dA(z) = \int_{\mathbb{D}} h(z)\,(1-|z|^2)^\alpha\,dA(z)\,;$$

by Proposition 3.2 we obtain

$$\int_{\mathbb{D}} h(w)\,N_{p,\alpha}(w,\varphi)\,dA(w) = \int_{\mathbb{D}} h(z)\,(1-|z|^2)^\alpha\,dA(z)\,. \tag{39}$$

If h is a continuous function on $\mathbb{D}$ with compact support then it is a bounded Borel function. Therefore (39) holds for all continuous function on $\mathbb{D}$ with compact support and by the Riesz Representation Theorem, (see [22, Theorem 2.14]), we conclude that for almost every $w \in \mathbb{D}$, $N_{p,\alpha}(w,\varphi) = (1-|w|^2)^\alpha$. By (11) the sufficiency of the above condition is obvious. □

Proposition 4.5 *Let $p > 1$, $\alpha > -1$ except $p = 2$, $\alpha = 0$ and let φ be a self map of $\mathbb{D}$ with $\varphi'(0) \neq 0$. Then, C_φ is an isometry on $B_{p,\alpha}$ if and only if φ is a rotation.*

Proof: Since $\varphi'(0) \neq 0$, φ is univalent in a small disk Δ in $\mathbb{D}$ centered at 0. Then by Theorem 4.6, Theorem 4.7, and assuming that C_φ is an isometry on $B_{p,\alpha}$,

$$\int_{\varphi(\Delta)} (1-|w|^2)^\alpha\,dA(w) = \int_{\varphi(\Delta)} N_{p,\alpha}(w,\varphi)\,dA(w)\,.$$

By making non-univalent change of variables as in Proposition 3.2 in both integrals of the above equation we obtain

$$\int_{\Delta} |\varphi'(z)|^2\,(1-|\varphi(z)|^2)^\alpha\,dA(z) = \int_{\Delta} |\varphi'(z)|^p\,(1-|z|^2)^\alpha\,dA(z)\,,$$

equivalently

$$\int_{\Delta} |\varphi'(z)|^2 \left((1-|\varphi(z)|^2)^\alpha - |\varphi'(z)|^{p-2}\,(1-|z|^2)^\alpha\right)\, dA(z) = 0\,.$$

It is clear that the integrand above is non-negative, therefore for almost every $z \in \Delta$,

$$|\varphi'(z)|^{p-2}\,(1-|z|^2)^\alpha = (1-|\varphi(z)|^2)^\alpha\,. \tag{40}$$

Pick a sequence $(z_n) \in \Delta$ such that (40) holds for each z_n and such that z_n converges to 0. By Corollary 4.3, $\varphi(0) = 0$ and we conclude that $|\varphi'(0)| = 1$. By Schwarz's lemma φ is a rotation. The converse is clear. □

In the proof below we use an argument in Proposition 2.5 in [24].

Theorem 4.8 *Let $\alpha > -1$, $\alpha \neq 0$ and φ a non constant self map of $\mathbb{D}$. Then, C_φ is an isometry on $B_{2,\alpha}$ if and only if φ is a rotation.*

Proof: First let $\alpha > 0$ and assume that C_φ is an isometry on $B_{2,\alpha}$. Then, by Theorem 4.6, $\varphi(0) = 0$ and for almost all $w \in \mathbb{D}$

$$\sum_{\varphi(z)=w} (1-|z|^2)^\alpha = (1-|w|^2)^\alpha\,. \tag{41}$$

Since φ is non constant, there exists a natural number n such that $\varphi^{(j)}(0) = 0$ for $j = 0, 1, ..., n-1$ but $\varphi^{(n)}(0) \neq 0$. Moreover there exists an holomorphic and univalent function g in a small disk Δ centered at 0 such that $g(0) = 0$ and $\varphi(z) = g(z)^n$ for all $z \in \Delta$. Moreover $g(\Delta)$ contains a disk Δ' centered at 0. Given $w \in \Delta'$, let $w_1, w_2, ..., w_n$ denote the n-th roots of w that is $w_j^n = w$, for $j = 1, 2, ..., n$. Then $g^{-1}(w_j) \in \varphi^{-1}(w)$, $j = 1, 2, ..., n$ and by (41)

$$(1-|w|^2)^\alpha \geq \sum_{j=1}^{n} \left[(1-|g^{-1}(w_j)|^2)\right]^\alpha\,. \tag{42}$$

Note that

$$g^{-1}(w) = [\,(g^{-1})'(0) + o(1)\,]w = [\,\frac{1}{g'(0)} + o(1)\,]\,w$$

for w sufficiently near 0. By (42) we conclude that

$$\begin{aligned}(1-|w|^2)^{\alpha} &\geq \sum_{j=1}^{n}\left[(1-|\frac{1}{g'(0)}+o(1)|^2)|w_j|^2\right]^{\alpha} \\ &= n\left[(1-|\frac{1}{g'(0)}+o(1)|^2)|w|^{\frac{2}{n}}\right]^{\alpha}\end{aligned}$$

or equivalently that

$$n^{\frac{1}{\alpha}}\,|\,\frac{1}{g'(0)} + o(1)\,|^2)\,|w|^{\frac{2}{n}} - |w|^2 \geq n^{\frac{1}{\alpha}} - 1$$

and therefore

$$n^{\frac{1}{\alpha}}\,|\,\frac{1}{g'(0)} + o(1)\,|^2)\,|w|^{\frac{2}{n}} \geq n^{\frac{1}{\alpha}} - 1$$

for w near 0. Therefore by taking limits as w approaches 0 we get $n = 1$ and φ is univalent near 0 and $\varphi'(0) \neq 0$. By Proposition 4.5 φ must be a disk rotation. By Proposition 4.3 disk rotations are isometries. If $-1 < \alpha < 0$, then Corollary 4.6 applies. □

Example 4.1 *C_{z^2} is an isometry in H^2 but is not an isometry in $B_{2,1}$.*

Let $\varphi(z) = z^2$. Then $\varphi(0) = 0$ and if $|z| = 1$ then $|z|^2 = 1$. We have φ is an inner function and fixes the origin. Therefore by Theorem A, C_{z^2} is an isometry on H^2. By Theorem 4.8, C_{z^2} is not an isometry in $B_{2,1}$.

Remark 4.3 *Note that $B_{2,1} = H^2$ but the isometries in H^2 among composition operators are all the inner functions fixing the origin, while the isometries in the $B_{2,1}$ norm are only the disk rotations.*

Proposition 4.6 *Let φ be an holomorphic full map of $\mathbb{D}$ such that $\varphi(0) = 0$ and for all $z \in \mathbb{D}$, $|\varphi'(z)| \leq 1$. Then φ is locally univalent at 0, that is, $\varphi'(0) \neq 0$.*

Proof: Suppose that $\varphi'(0) = 0$. Then by Schwarz's lemma applied to φ' we see that $|\varphi'(z)| \leq |z|$ for all $z \in \mathbb{D}$. Therefore,

$$\begin{aligned}
|\varphi(z)| &= |\int_0^1 z\varphi'(tz)\,dt| \\
&\leq |z| \int_0^1 |\varphi'(tz)|\,dt \\
&\leq |z|^2 \int_0^1 t\,dt \\
&= \frac{|z|^2}{2} \\
&\leq \frac{1}{2}
\end{aligned} \tag{43}$$

and clearly φ cannot be a full map, and $\varphi'(0) \neq 0$. □

Below we extend Theorem 4.4, Theorem 4.5, and Theorem 4.8 to include all indices $p > 1$.

Theorem 4.9 *If $p > 1$, $p \neq 2$, and $\alpha > -1$ then C_φ is an isometry on $B_{p,\alpha}$ if and only if φ is a rotation.*

Proof: Assume that C_φ is an isometry on $B_{p,\alpha}$. By Corollary 4.3, $\varphi(0) = 0$. By Theorem 4.7, C_φ is an isometry on $B_{p,\alpha}$ if and only if for almost every $w \in \mathbb{D}$

$$N_{p,\alpha}(w, \varphi) = \sum_{\varphi(z)=w} |\varphi'(z)|^{p-2}\,(1 - |z|^2)^\alpha = (1 - |w|^2)^\alpha\,. \tag{44}$$

By Schwarz's lemma $|\varphi'(0)| \leq 1$, and if $|\varphi'(0)| = 1$ then φ is a rotation. So we may assume that $|\varphi'(0)| < 1$.

First, let $\alpha = 0$. We will show that $\varphi'(0) \neq 0$. If $p > 2$, then by (44) and the continuity of φ', for every $z \in \mathbb{D}$, $|\varphi'(z)| \leq 1$. Therefore by Proposition 4.6, $\varphi'(0) \neq 0$. Next, if $1 < p < 2$ then by (44) and the continuity of φ' we obtain that for all $z \in \mathbb{D}$, $|\varphi'(z)| \geq 1$, in particular

$\varphi'(0) \neq 0$. Now by Proposition 4.5, φ is a rotation.

Next, let $\alpha \neq 0$. If $|\varphi'(z)| \leq 1$ for all $z \in \mathbb{D}$ then by Proposition 4.6, $\varphi'(0) \neq 0$ and again by Proposition 4.5, φ is a rotation. On the other hand, if there exists $a \in \mathbb{D}$ such that $|\varphi'(a)| > 1$ then, since $|\varphi'(0)| < 1$, there exists $z_0 \in \mathbb{D}$ such that $|\varphi'(z_0)| = 1$. Therefore φ is also univalent in a small disk Δ with center z_0. By (44), the continuity of φ' and an argument similar to the one for the the derivation of (40) we conclude that for every $z \in \Delta$

$$|\varphi'(z)|^{p-2}\,(1-|z|^2)^\alpha = (1-|\varphi(z)|^2)^\alpha\,;$$

if $z = z_0$ then

$$(1-|z_0|^2)^\alpha = (1-|\varphi(z_0)|^2)^\alpha\,,$$

and by Schwarz's lemma φ is a rotation. That rotations are isometries in any Besov type space follows by Proposition 4.3 and the theorem is proved. □

Let φ be a holomorphic self map of $\mathbb{D}$ with $\varphi(0) = 0$. Martin and Vukotic's main result in [17] implies that for almost every $w \in \mathbb{D}$, $N_{2,0}(w,\varphi) = (1-|w|^2)^0 = 1$ if and only if φ is a univalent full map. By Theorem 4.6 and Theorem 4.9 and for all other indices we have the following.

Corollary 4.7 *Let $p > 1$, $\alpha > -1$, except $p = 2$, $\alpha = 0$. Then, $\varphi(0) = 0$ and for almost every $w \in \mathbb{D}$, $N_{p,\alpha}(w,\varphi) = (1-|w|^2)^\alpha$ if and only if φ is a rotation.*

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