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Good Stein Neighborhood Bases for Nonsmooth Pseudoconvex Domains

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Abstract

In 1979, Dufresnoy showed that the existence of a good Stein neighborhood base for $\Omega \subset \mathbb{C}^n$ implies that one can solve the inhomogeneous Cauchy-Riemann equations in $C^\infty(\bar{\Omega})$, even if the boundary of Ω is only Lipschitz. In my thesis, I will show sufficient conditions for the existence of a good Stein neighborhood base on a Lipschitz domain satisfying Property (P).

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Dedication

This dissertation is dedicated to my husband, Michael H. Sullivan who has been a great source of strength and motivation. Also, this thesis is dedicated to my parents, Makoto Iwaki, Toshiko Iwaki, Derald Porter, and Danalene Porter who passed on a respect for education and gave me unconditional love and support.

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1 Introduction

A natural class of domains in several complex variables are pseudoconvex domains. Pseudoconvex domains preserve some nice properties of holomorphic functions from one complex variable. All convex domains are pseudoconvex, and all domains that are locally biholomorphic to convex domains are pseudoconvex. In addition, all domains that can be exhausted from within by pseudoconvex domains are pseudoconvex, and the converse is also true. My research was motivated by finding pseudoconvex domains that can be approximated from outside by pseudoconvex domains. To do so, I need to find a good Stein neighborhood basis, which is defined later. In this section, I introduce some background material including definitions, theorems proved in the past, and motivations. First, we will define defining functions which relate to the differentiability of the boundary of a domain.

Definition 1.1. Let Ω be a C^k domain in $\mathbb{R}^n$ for $k \geq 1$, and r be a C^k function defined in some open neighborhood U of a boundary point p such that $\Omega \cap U = \{x \in U | r(x) < 0\}$, $b\Omega \cap U = \{x \in U | r(x) = 0\}$ and $dr(x) \neq 0$ on $b\Omega \cap U$. The function r is called a local defining function. If U is an open neighborhood of $\overline{\Omega}$, r is called a defining function.

Now let us define the operators ∂ and $\bar{\partial}$.

Definition 1.2. Let $z_j = x_j + iy_j$ for $1 \leq j \leq n$ and f is a function defined on $\mathbb{C}^n$. We use the notations:

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right)$$

$$\frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)$$

Moreover, if we denote $dz_j = dx_j + idy_j$ and $d\overline{z}_j = dx_j - idy_j$,

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j$$

and

$$\bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j,$$

When we have a C^2 defining function r we can define a pseudoconvex domain as follows:

Definition 1.3. A bounded C^2 domain Ω is called pseudoconvex at a point p on the boundary if

$$\sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial \overline{z}_j}(p) t_i \bar{t}_j \geq 0$$

for all $t = (t_1, \cdots, t_n) \in \mathbb{C}^n$ with $\sum_{j=1}^{n} t_j (\frac{\partial r}{\partial z_j})(p) = 0$ where r is a C^2 defining function for Ω.

If the above inequality is strictly greater for $t \neq 0$, then Ω is called a strictly pseudoconvex domain. The definition does not apply for Lipschitz domains, which are the main focus of this thesis, so we need new tools to define pseudoconvexity on Lipschitz domains.

Definition 1.4. A bounded domain is called Lipschitz if locally the boundary of the domain is the graph of a Lipschitz function.

The defining function associated with a Lipschitz domain is called a Lipschitz defining function.

Definition 1.5. Let Ω be a bounded Lipschitz domain. r is a Lipschitz defining function if $r : \mathbb{C}^n \to \mathbb{R}$, $r < 0$ inside of Ω, and $r > 0$ outside of $\overline{\Omega}$, and moreover, for some positive constants C_1 and C_2,

$$C_1 < |dr| < C_2 \text{ a.e. on } b\Omega.$$

Lemma 1.1 from [18] states:

Lemma 1.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists a Lipschitz defining function r.*

Now we can extend the definition of pseudoconvex domains to Lipschitz domains. To do so, we need to define a plurisubharmonic function. First, we define an exhaustion and subharmonic functions.

Definition 1.7. Let Ω be an open domain. A function $\varphi : \Omega \to \mathbb{R}$ is called an exhaustion function for Ω if the closure of $\{x \in \Omega | \varphi(x) < c\}$ for all real c is compact.

Definition 1.8. Let Ω be an open set. A function ρ defined in Ω is subharmonic if for every continuous function h on any compact set $K \subset \Omega$ that is harmonic inside of K, and satisfies $h \geq \rho$ on the boundary of K, we have $\rho \leq h$ in K.

For $z_0 \in \Omega$, choose a small $R > 0$ so that

$$D = \{z_0 + \tau w | \tau \in \mathbb{C}, |\tau| \leq R\} \subset \Omega$$

Now we define a plurisubharmonic function.

Definition 1.9. A function ρ is called plurisubharmonic if for every $z \in \Omega$ and $w \in \mathbb{C}^n$, the function $\tau \to \rho(z + \tau w)$ is subharmonic on D for some $R > 0$.

Another way to characterize a plurisubharmonic function for a C^2 function is shown below:

Theorem 1.10. *A C^2 function ρ on Ω is plurisubharmonic if and only if for all $z \in \Omega$*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) t_j \bar{t}_k \geq 0$$

for all $t = (t_1, \cdots, t_n) \in \mathbb{C}^n$.

One can see Theorem 3.4.3 from [2], for the proof.

Finally, we can define pseudoconvexity on the domain which does not have a smooth boundary.

Definition 1.11. An open domain Ω in $\mathbb{C}^n$ is pseudoconvex if there exists a smooth strictly plurisubharmonic exhaustion function φ on Ω.

One can compare this definition with Definition 1.3, which defines pseudoconvexity on a bounded C^2 domain. The Theorem 3.4.11 [2] shows that these are equivalent definitions on C^2 domains.

Theorem 3.4.10 in [2] states Oka's Lemma:

Lemma 1.12. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Then $-\log \delta$ is plurisubharmonic on Ω for $\delta(z) = dist(z, b\Omega)$.*

Notice that this Lemma does not assume that the boundary of Ω is smooth. Thus, the Lemma still applies on Lipschitz domains. In this thesis, we want to find pseudoconvex domains that can be approximated from outside by pseudoconvex domains, which requires the existence of Stein neighborhood basis.

Definition 1.13. A compact set $K \subset \mathbb{C}^n$ is said to have a Stein neighborhood basis if for any open domain V containing K, there exists a pseudoconvex domain Ω_v such that $K \subset \Omega_v \subset V$.

Note that not all pseudoconvex domains have a Stein neighborhood basis; an example would be a worm domain.

Definition 1.14. For $\beta > \frac{\pi}{2}$, Ω_β is a smooth worm domain if

$$\Omega_\beta = \{(z_1, z_2) \in \mathbb{C}^2 || z_1 + e^{i \log |z_2|^2}|^2 < 1 - \eta(\log |z_2|^2)\}$$

where η is smooth from $\mathbb{R} \to \mathbb{R}$ with the following properties:

1. $\eta(x) \geq 0, \eta$ is even and convex.

2. $\eta^{-1}(0) = I_{\beta - \frac{\pi}{2}}$, where $I_{\beta-\frac{\pi}{2}} = [\beta + \frac{\pi}{2}, \beta - \frac{\pi}{2}]$.

3. There exists an $a > 0$ such that $\eta(x) > 1$ if $x < -a$ or $x > a$.

4. $\eta'(x) \neq 0$ if $\eta(x) = 1$.

The worm domain is a smooth pseudoconvex domain but it does not have a Stein neighborhood basis for $\beta > \frac{3\pi}{2}$ [5]. See also Theorem 6.4.3 in [2].

My motivation for finding the existence of a good Stein neighborhood basis is that one can solve the inhomogeneous Cauchy-Riemann equations smoothly up to the boundary: for every f with $\bar{\partial} f = 0$, there exists u such that $\bar{\partial} u = f$, where $\bar{\partial}$ is the Cauchy-Riemann operator in several complex variables.

In Corollary 4.2.6 in [14], Hörmander used solvability in $L^2(\Omega)$ to show the following theorem.

Theorem 1.15. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. For every $f \in C^\infty_{(p,q)}(\Omega)$ with $\bar{\partial} f = 0$, there exists $u \in C^\infty_{(p,q-1)}(\Omega)$ such that $\bar{\partial} u = f$.*

In [6], Dufresnoy showed that the existence of a good Stein neighborhood basis for $\Omega \subset \mathbb{C}^n$ implies that the Cauchy-Riemann equations are solvable in $C^\infty(\bar{\Omega})$ with $u \in C^\infty_{(p,q-1)}(\bar{\Omega})$, even if the boundary of the domain is only Lipschitz.

Both works showed a similar result, and notice that neither requires any boundary smoothness. In Theorem 4.2.2 in [14], Hörmander used an exhaustion of Ω from inside,which is why his result does not include the boundary. Dufresnoy used a neighborhood basis outside of Ω, which is my motivation to build a neighborhood basis outside of the domain in this thesis. Also, we need more than a just a existence of neighborhood base for Dufresnoy's result to hold. We will also need uniform H-convexity.

Definition 1.16. A compact set $K \subset \mathbb{C}^n$ is said to be uniformly H-convex if there exists a positive sequence $\{\varepsilon_j\}$ that converges to 0, $c > 1$ and a sequence of pseudoconvex domains Ω_j such that $K \subset \Omega_j$ and $\varepsilon_j \leq \text{dist}(K, \mathbb{C}^n \backslash \Omega_j) \leq c\varepsilon_j$ for $j = 1, 2, \cdots$ [3].

Note that uniform H-convexity implies that $\{\Omega_j\}$ is a Stein Neighborhood Basis. Dufresnoy's result can be stated as follows:

Theorem 1.17. *Uniform H-convexity implies that for all $f \in C^{\infty}_{(p,q)}(\bar{\Omega})$ where $0 \leq p \leq n$, $1 \leq q \leq n$ with $\bar{\partial} f = 0$, there exists $u \in C^{\infty}_{(p,q-1)}(\bar{\Omega})$ such that $\bar{\partial} u = f$.*

In [17], Şahutoğlu defined a stronger notion of Stein neighborhood basis:

Definition 1.18. The closure $\bar{\Omega}$ of a nonsmooth pseudoconvex domain Ω has a strong Stein neighborhood basis if Ω has a defining function ρ and there exists $\varepsilon_0 > 0$ such that $\{z \in \mathbb{C}^n : \rho(z) < \varepsilon\}$ is pseudoconvex for $0 \leq \varepsilon \leq \varepsilon_0$.

We will show sufficient conditions for the existence of a strong Stein neighborhood basis. Note that the existence of a strong Stein neighborhood basis implies uniform H-convexity. Since Stein neighborhood bases also have applications to solve the $\bar{\partial}-$Neumann problem, let us give some more definitions.

Definition 1.19. Let $L^2(\Omega)$ denote the space of square integrable functions on Ω and $L^2_{(p,q)}(\Omega)$ denote the space of (p,q)-forms whose coefficients are in $L^2(\Omega)$. The norm $L^2(\Omega)$ is defined by

$$\|f\|^2 = \int_{\Omega} |f|^2 dV$$

for $f \in L^2_{(p,q)}(\Omega)$ and $dV = i^n dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$.

Definition 1.20. We can write a (p,q)-form f as:

$$f = \sum_{|I|=p, |J|=q}{}' f_{I,J} dz^I \wedge d\overline{z}^J,$$

where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$ are multiindicies. Here, $\sum'$ is the summation over strictly increasing multiindices. Also, $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\overline{z}^I = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$. Then, $\bar{\partial} f$ can be defined by

$$\bar{\partial} f = \sum_{I,J}{}' \bar{\partial} f_{I,J} \wedge dz^I \wedge d\overline{z}^J$$

which is $(p, q+1)$-form.

We use $(,)$ to denote the inner product in $L^2_{(p,q)}$.

Definition 1.21. Let the adjoint of $\bar{\partial}$ be the operator

$$\bar{\partial}^* : L^2_{(p,q)}(\Omega) \to L^2_{(p,q-1)}(\Omega).$$

We denote the domain for $\bar{\partial}^*$ as $\mathrm{Dom}(\bar{\partial}^*)$.

We say $f \in \mathrm{Dom}(\bar{\partial}^*)$ if there exists a $g \in L^2_{(p,q-1)}(\Omega)$ such that for every $\psi \in \mathrm{Dom}(\bar{\partial}) \cap L^2_{(p,q-1)}(\Omega)$, we have $(f, \bar{\partial}\psi) = (g, \psi)$. Notice that if we integrate this inner product by parts, the boundary terms, which involves an inner product of normal component of f and components of ψ, will vanish. Thus, any elements in $\mathrm{Dom}(\bar{\partial}^*)$ must satisfy a boundary condition. The condition $f \in \mathrm{Dom}(\bar{\partial}^*)$ is a Dirichlet condition for the normal component of f. Let ρ be a C^1 defining function for Ω. f must satisfy

$$\sum_k f_{I,kK} \frac{\partial \rho}{\partial z_k} = 0 \text{ on } b\Omega \text{ for all } I, K,$$

where $|I| = p$ and $|K| = q - 1$.

Definition 1.22. Define the box operator, $\Box_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ as follows:

$$\Box_{(p,q)} = \bar{\partial}_{(p,q-1)}\bar{\partial}^*_{(p,q)} + \bar{\partial}^*_{(p,q+1)}\bar{\partial}_{(p,q)}.$$

Also, the domain of this box operator is defined as $\mathrm{Dom}(\Box_{(p,q)}) = \{f \in L^2_{(p,q)}(\Omega) | f \in \mathrm{Dom}(\bar{\partial}_{(p,q)}) \cap \mathrm{Dom}(\bar{\partial}^*_{(p,q)}), \bar{\partial}_{(p,q)} \in \mathrm{Dom}(\bar{\partial}^*_{(p,q+1)}) \text{ and } \bar{\partial}^*_{(p,q)} f \in \mathrm{Dom}(\bar{\partial}_{(p,q-1)})\}$.

In [13], Hörmander showed that $\Box$ is invertible on bounded pseudoconvex domains. The operator, $N_{(p,q)} : L^2_{(p,q)}(\Omega) \to \mathrm{Dom}(\Box_{(p,q)})$ such that $N_{(p,q)}\Box = \Box N_{(p,q)} = I$ is called the $\bar{\partial}-$Neumann operator.

Definition 1.23. We define the Fourier transform $\hat{u}$ of u as:

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx \text{ where } x \cdot \xi = \sum_{j=1}^n x_j \xi_j.$$

For any $u \in C_0^\infty(\mathbb{R}^n)$, the Sobolev norm is given by:

$$\|u\|^2_{W^s_{p,q}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$

Thus, $W^s_{p,q}(\mathbb{R}^n)$ can be defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|u\|^2_{W^s_{p,q}(\mathbb{R}^n)}$.

Moreover, let us restrict to the domain $\Omega \subset \mathbb{R}^n$. Then

$$\|u\|_{W^s_{p,q}(\Omega)} = \inf \|\mathcal{U}\|_{W^s_{p,q}(\mathbb{R}^n)}$$

where $\mathcal{U} \in W^s_{p,q}(\mathbb{R}^n)$ and $\mathcal{U}|_\Omega = u$

In [19], Straube showed the following theorem.

Theorem 1.24. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. For $\varepsilon > 0$ let $\Omega_\varepsilon = \{z \in \mathbb{C}^n | dist\,(z, \overline{\Omega}) < \varepsilon\}$. Assume there is a function $r(\varepsilon)$ with $1 - r(\varepsilon) = o(\varepsilon^2)$ as $\varepsilon \to 0^+$ such that, for ε small enough, there exists a pseudoconvex domain $\tilde{\Omega}_\varepsilon$ with $\Omega_{r(\varepsilon)\varepsilon} \subseteq \tilde{\Omega}_\varepsilon \subseteq \Omega_\varepsilon$.*

Then the $\bar{\partial}$-Neumann operators N_q are continuous on $W^s_{(0,q)}(\Omega)$, for all $s \geq 0$ and $1 \leq q \leq n$.

This theorem uses a stronger condition to estimate the $\bar{\partial}$-Neumann operator.

In section 2, I will show the application to an operator called the tangential Cauchy-Riemann complex $\bar{\partial}_b$, which is the Cauchy-Reimann operator restricted to the boundary. The tangential Cauchy-Riemann equations in a pseudoconvex domain can help us in getting information about the boundary values of holomorphic functions.

In [16], Michel and Shaw defined the following: Let Ω be a bounded domain in $\mathbb{C}^n$. Ω is strictly pseudoconvex with Lipschitz boundary $\partial\Omega$ if there exists a Lipschitz defining function ρ for Ω, and there exists a constant $c > 0$ such that $\rho(z) - c|z|^2$ is plurisubharmonic in a neighborhood of $\partial\Omega$.

In section 3, I will show that if the defining function is only strictly plurisubharmonic inside the domain, then under certain conditions, one can still find a strong Stein neighborhood basis outside the domain. Since our functions may not be twice differential, we need to use currents.

Definition 1.25. $w \in C^\infty_{(p,p)}(\Omega)$ is a simple positive form if

$$w = i^p w_1 \wedge \overline{w}_1 \wedge \cdots \wedge w_p \wedge \overline{w}_p$$

where each $w_j \in C^\infty_{(1,0)}(\Omega)$. [15]

Definition 1.26. We say for $\varphi \in C^2$, $i\partial\bar{\partial}\varphi \geq 0$ in the sense of currents if $\int -i\varphi\bar{\partial}\partial w \geq 0$ for all simple positive smooth compactly supported $(n-1, n-1)$-form w.

Notice that this is equivalent to $\int i\bar{\partial}\varphi \wedge \partial w \geq 0$. This implies that $\int i\partial\bar{\partial}\varphi \wedge w \geq 0$. Thus the Definition 1.26 still holds when φ is C^2 function. Moreover, we say $i\partial\bar{\partial}\varphi \geq iK\partial\bar{\partial}|z|^2$ in the sense of currents if $i\partial\bar{\partial}(\varphi - K|z|^2) \geq 0$ for some $K > 0$.

Theorem 1.27. *Let Ω be a bounded Lipschitz domain. Suppose for every $p \in \partial\Omega$, there exists a neighborhood U_p such that*

1. *$\partial\Omega \cap U_p$ is the graph of a Lipschitz function with Lipschitz constant $M > 0$, and*

2. *there is a defining function ρ for Ω such that $i\partial\bar{\partial}\rho(z) \geq iK\partial\bar{\partial}|z|^2$ on Ω in the sense of currents and*

$$\frac{1}{k}|\rho(z)| \leq \delta(z) \leq |\rho(z)|$$

hold on $\Omega \cap U_p$ for some $k > 1$ and $K > 0$.

Under these assumptions, it follows that for every $\beta > 1$, there exists a constant N_β such that if k and M satisfy $N_\beta \geq k\sqrt{1 + M^2} - 1$ then there exists a neighborhood U of $\bar{\Omega}$, and a plurisubharmonic function λ on $U \setminus \Omega$ such that

1. $\lambda \approx \delta^{\beta}$*, and*

2. $i\partial\overline{\partial}\lambda \gtrsim i\delta^{\beta-1}\partial\overline{\partial}|z|^2$ *in the sense of currents.*

Definition 1.28. If $G \leq IH$ for some constant I, independent of z, then we write $H \gtrsim G$. If $G \gtrsim H$ and $H \gtrsim G$, then we write $G \approx H$.

Note that we can take M arbitrarily close to zero if we make U_p small on C^1 domains. I will show in section 4 conditions for the existence of a good Stein neighborhood basis for a Lipschitz domain $\Omega \subset \mathbb{C}^n$ satisfying a special property, called Property (P), which is defined below.

In [1], Catlin defined property (P) as follows:

Definition 1.29. The boundary of a smoothly bounded pseudoconvex domain Ω to satisfies property (P) if for all $M > 0$, there exists a plurisubharmonic function ϕ in $C^{\infty}(\overline{\Omega})$ such that for all $z \in b\Omega$ and for all $t \in \mathbb{C}^n$,

$$\sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_j}(z) t_i \bar{t}_j \geq M|t|^2$$

with $0 \leq \phi \leq 1$.

This implies the existence of a solution operator for $\bar{\partial} u = f$ that is compact.

Definition 1.30. The operator $T : \mathcal{H} \to \mathcal{K}$ is compact if T maps from a closed unit ball, in $\mathcal{H}$ to a set in $\mathcal{K}$, with compact closure.

Here is the main theorem of this thesis.

Theorem 1.31. *Let* Ω *be a bounded Lipschitz pseudoconvex domain, which satisfies Property (P). There exist constants* $V > 1$ *and* $W > 0$ *such that if* $1 < \beta < V$ *and* $\sqrt{1+M^2} < 1 + W(\beta - 1)^2$*, then there exists a neighborhood* U *of* $\overline{\Omega}$ *and a function* λ *on* U/Ω *such that*

1. $\lambda \approx \delta^{\beta}$*, and*

2. $i\partial\overline{\partial}\lambda \gtrsim i\delta^{\beta}\partial\overline{\partial}|z|^2$ *in the sense of currents.*

We will see we can take $W \approx 0.03$ (See Remark 4.3).

My result is analogous to Fornæss and Herbig's result from [9] and [10]. They showed that for a smooth bounded domain, the existence of a defining function that is plurisubharmonic on $\partial\Omega$ implies for all $\eta > 1$ there exists a neighborhood V of $b\Omega$ and smooth defining function r on $V\backslash\bar{\Omega}$ such that r^{η} is strictly plurisubharmonic. This implies the existence of a strong Stein neighborhood basis.

For my thesis, I show the existence of such a defining function by combining techniques developed by Harrington. In 2008, he showed a similar result for C^1 domains satisfying Property (P) [12], and in [11], he built a good defining function inside Ω for all Lipschitz pseudoconvex domains. First, I will build a function on a neighborhood inside and translate the function outside. To do so, I will patch functions together inside of a little neighborhood by taking the supremum over a finite collection of plurisubharmonic functions. Then I will patch whole neighborhoods together around the boundary.

2 Application of $\bar{\partial}$

The $\bar{\partial}_b$ complex, also called the tangential Cauchy-Riemann complex, is the restriction of the $\bar{\partial}$ complex to the boundary. The tangential Cauchy-Riemann equations in pseudoconvex domains can give information about boundary values of holomorphic functions. $\bar{\partial}_b$ provides information about boundary value of holomorphic function in the same way that $\bar{\partial}$ provides information about holomorphic function. For more information, see Chapter 7 in [2]. In [18], Shaw showed an application of $\bar{\partial}$ to solve $\bar{\partial}_b$. First, let us define the space $L^2_{(p,q-1)}(b\Omega)$.

Definition 2.1. Let $I^{p,q} = rH_1 + \bar{\partial}r \wedge H_2$, where r is a defining function, H_1 is a smooth (p,q)-form and H_2 is a smooth $(p,q-1)$-form. Then,

$$\Lambda^{p,q}(b\Omega) = \{\text{the orthogonal complement of } I^{p,q}|_{\partial\Omega} \text{ in } \Lambda^{p,q}(\mathbb{C}^n)\},$$

where $\Lambda^{p,q}(\mathbb{C}^n)$ is the set of (p,q)-forms.

Definition 2.2. Let $L^2(b\Omega)$ denote the space of square integrable functions on $b\Omega$, and $L^2_{(p,q)}(b\Omega)$ denote the space in $\Lambda^{p,q}(b\Omega)$ whose coefficients are in $L^2(b\Omega)$.

On Lipschitz domains, we define $\bar{\partial}_b$ as follows:

Definition 2.3. For any $u \in L^2_{(p,q-1)}(b\Omega)$, if $f \in L^2_{(p,q)}(b\Omega)$ and

$$\int_{b\Omega} u \wedge \bar{\partial}\phi = (-1)^{p+q} \int_{b\Omega} f \wedge \phi \tag{2.1}$$

for every $\phi \in C^\infty_{(n-p,n-1-q)}(\mathbb{C}^n)$, then $\bar{\partial}_b u = f$.

For $1 \leq q \leq n-2$, if we substitute ϕ with $\bar{\partial}\phi$ for some smooth $(n-p, n-q-2)$-form in (2.1), we see that necessary condition to solve $\bar{\partial}_b u = f$ is $\bar{\partial}_b f = 0$ When $q = n-1$, more is needed.

Lemma 3.2 from [18] can be modified to fit the assumptions of this thesis as follows:

Lemma 2.4. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex Lipschitz domain satisfying the assumptions of Theorem 1.31. For any $\alpha \in C^\infty_{(p,q)}(\bar{\Omega})$, such that $\bar{\partial}\alpha = 0$ in Ω, there exists $u \in C^\infty_{(p,q)}(\bar{\Omega})$ with $\bar{\partial}u = \alpha$ in Ω.*

Moreover, for any $-\frac{1}{2} \leq s \leq \frac{1}{2}$, the space $C^\infty_{(p,q)}(\bar{\Omega}) \cap Ker(\bar{\partial})$ is dense in $W^s_{(p,q)}(\Omega) \cap Ker(\bar{\partial})$ in the $W^s_{(p,q)}(\Omega)$ norm.

Proof. Let $\Omega_j = \{z \in \mathbb{C}^n | \lambda(z) < \varepsilon_j\}$ where λ is given by Theorem 1.31. Since λ is strictly plurisubharmonic, each Ω_j is strictly pseudoconvex in the sense of [16]. Thus, $\bar{\Omega}$ has a Stein neighborhood base since $\cap_j \Omega_j = \bar{\Omega}$. By the result from Dufresnoy in [6], we can solve $\bar{\partial}$ smoothly up to the boundary. To prove the second part of the lemma, assume Ω is star-shaped. Recall that a domain Ω is star-shaped if there exists a vantage point $x_0 \in \Omega$ such that if $z \in \Omega$ then the line segment $[x_0, z] \subseteq \Omega$. We let $0 \in \Omega$ be a vantage point. For

$f \in W^s_{(p,q)}(\Omega) \cap Ker(\bar{\partial})$, we can define:

$$f_j(z) = (f * \phi_{\delta_j}) \left(\frac{z}{1 + \frac{1}{j}} \right)$$

for $\phi_{\delta_j} = \frac{1}{\delta_j^{2n}} \phi(\frac{z}{\delta_j})$ where $\phi(z) = \phi(|z|)$ and sufficiently small δ_j.

Then $f_j \in C^\infty(\bar{\Omega})$, $\bar{\partial} f_j \to 0$ in Ω and $f_j \to f$ in $L^2(\Omega)$ by Theorem 7 in Appendix (c)[7].

Moreover, the space $L^2(\mathbb{R}^n)$ is a dense subspace of $W^s(\mathbb{R}^n)$ for all s (See Property ii. from page 302 of [8]). Now recall Definition 1.23. Since for the restricted domain $\Omega \subset \mathbb{R}^n$, every element of $W^s(\Omega)$ can be extend to an element of $W^s(\mathbb{R}^n)$, $f_j \to f$ in $W^s(\Omega)$. Now if Ω is not star-shaped, we let ζ_i be a function supported in an open set U_i so that $U_i \cap \Omega$ is star-shaped. To patch the star-shaped neighborhoods together , we use a partition of unity $\{\zeta_i\}_{i=1}^N$. Then we can find $f_j \in C^\infty_{(p,q)}(\bar{\Omega}_j)$ such that $\|\bar{\partial} f_j\|_{W^s(\Omega_j)} \to 0$ and $f_j \to f$ in $W^s(\Omega)$. Since each Ω_j is strictly pseudoconvex, if we use Corollary 1.3 from [16] for each Ω_j, we can find $u_j \in W^s_{(p,q)}(\Omega_j) \cap C^\infty_{(p,q)}(\Omega_j)$ such that $\bar{\partial} u_j = \bar{\partial} f_j$ in Ω_j and u_j satisfies the estimate:

$$\|u_j\|_{W^s(\Omega_j)} \leq C \|\bar{\partial} f_j\|_{W^s(\Omega_j)},$$

for some constant C. This estimate follows from Corollary 1.3 from [16]. If we call $h_j = f_j - u_j$, then $h_j \in C^\infty_{(p,q)}(\bar{\Omega}) \cap Ker(\bar{\partial})$ and $h_j \to f$ in $W^s_{(p,q)}(\Omega)$. $\square$

Since Lemma 3.2 is the only place in [18] where a plurisubharmonic defining function is needed, the proof of Proposition 3.5 from [18] can be adapted to prove the following proposition.

Proposition 2.5. *Let Ω be a bounded pseudoconvex Lipschitz domain with a Lipschitz defining function satisfying the assumptions of Theorem 1.31 in a neighborhood of $\overline{\Omega}$. For every $f \in W^s_{(0,n)}(\mathbb{C}^n)$, supported in $\overline{\Omega}$ and $-\frac{1}{2} \leq s \leq \frac{1}{2}$, satisfying*

$$\int_\Omega f \wedge g = 0 \text{ for every } g \in C^\infty_{(n,0)}(\overline{\Omega}) \cap ker(\bar{\partial}),$$

we can find $u \in W^s_{(0,n-1)}(\mathbb{C}^n)$ *satisfying* $\bar{\partial} u = f$ *in* $\mathbb{C}^n$. *[18]*

If we can solve $\bar{\partial} u = f$, we can also solve $\bar{\partial}_b$. The main theorem from [18] can be similarly adapted:

Theorem 2.6. *Let* Ω *be a bounded pseudoconvex Lipschitz domain with a Lipschitz defining function satisfying the assumptions of Theorem 1.31 in a neighborhood of* $\overline{\Omega}$. *For every* $\alpha \in L^2_{(p,n-1)}(b\Omega)$, *satisfying*

$$\int_{b\Omega} \alpha \wedge \phi = 0 \textit{ for every } \phi \in C^\infty_{(n-p,0)}(\overline{\Omega}) \cap ker(\bar{\partial})$$

where $0 \leq p \leq n, q = n-1$, *we can find* $u \in L^2_{(p,q)}(b\Omega)$ *satisfying* $\bar{\partial}_b u = \alpha$ *in* $b\Omega$.

3 Lipschitz domains with strictly plurisubharmonic defining functions

In this section, we will prove Theorem 1.27. Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n, n \geq 2$ and for any $x \in \Omega$ denote the distance from x to $\partial\Omega$ by $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Assume for some $K > 0$ there exists a defining function $\rho(z)$ satisfying

$$i\partial\overline{\partial}\rho(z) \geq iK\partial\overline{\partial}|z|^2 \text{ on } \Omega \tag{3.1}$$

in the sense of currents. If $\Omega \subset \mathbb{R}^n$, let $M > 0$ and $R > 0$ be constants such that for all $p \in \partial\Omega$, there exists $B(p,R) \cap \Omega = \{z \in B(p,R) : \mathrm{Im} z_n < \varphi(z', \mathrm{Re} z_n)\}$ for some Lipschitz function φ with Lipschitz constant $M > 0$. We will use z' to denote $\{z_1, \ldots z_{n-1}\}$. Define $r(z) = \mathrm{Im} z_n - \varphi(z', \mathrm{Re} z_n)$. Since $\delta(z)$ is the closest distance to the boundary, $\delta(z) \leq |\varphi(z', \mathrm{Re} z_n) - \mathrm{Im} z_n|$. Let $w \in \partial\Omega$ satisfying $\delta(z) = |z - w|$. Let $\alpha = \varphi(z', \mathrm{Re} z_n) - \mathrm{Im} z_n, \upsilon = \varphi(z', \mathrm{Re} z_n) - \mathrm{Im} w_n$ and $\omega = \alpha - \upsilon = \mathrm{Im} w_n - \mathrm{Im} z_n$. Since $\mathrm{Im} w_n = \varphi(w', \mathrm{Re} w_n)$,

$$|\upsilon| = |\varphi(z', \mathrm{Re} z_n) - \varphi(w', \mathrm{Re} w_n)| \leq M\sqrt{(\mathrm{Re} z_n - \mathrm{Re} w_n)^2 + |z' - w'|^2} = M\sqrt{(\delta(z))^2 - \omega^2}.$$

So $|\upsilon| \leq M\sqrt{(\delta(z))^2 - \omega^2}$. If we square both sides we get $\upsilon^2 \leq M^2(\delta(z))^2 - M^2\omega^2$. Since $\omega = \alpha - \upsilon$, we have $(\alpha-\omega)^2 + M^2\omega^2 \leq M^2(\delta(z))^2$. Notice that the left hand side will reach its minimum when $\omega = \frac{\alpha}{1+M^2}$. Thus, we get:

$$(\alpha - \frac{\alpha}{1+M^2})^2 + M^2(\frac{\alpha}{1+M^2})^2 \leq M^2(\delta(z))^2.$$

This is equivalent to $|\alpha|\frac{1}{\sqrt{1+M^2}} \leq \delta(z)$. Thus, $\frac{1}{\sqrt{1+M^2}}|\varphi(z', \mathrm{Re}z_n) - \mathrm{Im}z_n| \leq \delta(z)$. We have proven:

$$\frac{1}{\sqrt{1+M^2}}|\varphi(z', \mathrm{Re}z_n) - \mathrm{Im}z_n| \leq \delta(z) \leq |\varphi(z', \mathrm{Re}z_n) - \mathrm{Im}z_n|. \tag{3.2}$$

Since ρ is a defining function we assume

$$\frac{1}{k}|\rho(z)| \leq \delta(z) \leq |\rho(z)| \tag{3.3}$$

for $k \geq 1$.

Lemma 3.1. *If $\beta > 1$ and $m > 1$, then*

$$m^\beta \geq 1 - \beta + \beta m \tag{3.4}$$

and

$$m^\beta \left[1 - \beta + \frac{\beta}{m}\right] < 1 \tag{3.5}$$

Proof. Let $f(x) = x^\beta$. Then, $f'(x) = \beta x^{\beta-1}$ and $f''(x) = \beta(\beta-1)x^{\beta-2}$. Since $\beta > 1$ and $x > 0$, $f''(x) > 0$. By linear approximation,

$$f(x) \geq f(1) + f'(1)(x-1) = 1 + \beta x - \beta.$$

If we replace x with m, we have (3.4).

Likewise, if we replace x with $\frac{1}{m}$, we have

$$f\left(\frac{1}{m}\right) \geq 1 + \beta\left(\frac{1}{m}\right) - \beta.$$

This implies that

$$m^{\beta} - \beta m^{\beta} + \beta m^{\beta-1} \leq 1.$$

If we factor out m^{β}, (3.5) holds. □

We will construct a plurisubharmonic function λ and compare it with δ^{β}. For given $b > 0, s \leq b\sqrt{1+M^2}$, and $\varepsilon > 0$, define $z_{\varepsilon} = z - (0, \ldots, 0, is\varepsilon)$, and $\rho_{\varepsilon}(z) = \rho(z_{\varepsilon})$.

Lemma 3.2. *If $z \notin \Omega$ and $\delta(z) \leq b\varepsilon$, then $z_{\varepsilon} \in \overline{\Omega}$. Moreover,*

$$-\rho_{\varepsilon}(z) < -\delta(z) + ks\varepsilon \tag{3.6}$$

and

$$-\rho_{\varepsilon}(z) \geq -\delta(z) + \frac{s\varepsilon}{\sqrt{1+M^2}}. \tag{3.7}$$

Proof. We first show if $z \notin \Omega$, $z_{\varepsilon} \in \Omega$. By definition of $r(z)$, $r(z_{\varepsilon}) = r(z) - s\varepsilon$. By applying an inequality from (3.2), we get

$$r(z) - s\varepsilon \leq \sqrt{1+M^2}\delta(z) - s\varepsilon.$$

This is equivalent to

$$r(z_{\varepsilon}) \leq \sqrt{1+M^2}\delta(z) - s\varepsilon. \tag{3.8}$$

This shows that $r(z_{\varepsilon}) \leq 0$ when $\delta(z) \leq b\varepsilon$. This imply that $z_{\varepsilon} \in \bar{\Omega}$.

Now we will show (3.6). By the definition above, $-\rho_{\varepsilon}(z) = -\rho(z_{\varepsilon})$. Since $z_{\varepsilon} \in \bar{\Omega}$, by (3.3), $-\rho(z_{\varepsilon}) \leq k\delta(z_{\varepsilon})$.

By (3.2),

$$k\delta(z_\varepsilon) \leq -kr(z_\varepsilon) = -k(r(z) - s\varepsilon).$$

Again, by (3.2),

$$-k(r(z) - s\varepsilon) \leq -k\delta(z) + ks\varepsilon < -\delta(z) + ks\varepsilon.$$

So (3.6) follows.

Similarly, by (3.3), $-\rho(z_\varepsilon) \geq \delta(z_\varepsilon)$. By (3.2),

$$\delta(z_\varepsilon) \geq \frac{-r(z_\varepsilon)}{\sqrt{1+M^2}}.$$

We have already shown in (3.8) that

$$\frac{-r(z_\varepsilon)}{\sqrt{1+M^2}} \geq -\delta(z) + \frac{s}{\sqrt{1+M^2}}\varepsilon.$$

Then (3.7) follows.

$\square$

In this section, we assume $s = b\sqrt{1+M^2}$, so if $z \notin \Omega$ and $\delta(z) \leq b\varepsilon$,

$$\delta(z) - kb\varepsilon\sqrt{1+M^2} \leq \rho_\varepsilon(z) \leq \delta(z) - b\varepsilon.$$

Notice that the upper bound and lower bound do not depend on $p \in \partial\Omega$. For $p \in \partial\Omega$, let U_p be a neighborhood where

$$i\partial\overline{\partial}\rho(z) \geq iK\partial\overline{\partial}|z|^2$$

in the sense of currents. Let $B(p, r_p) \subset U_p$ be a ball with radius r_p centered at p. Let $\chi_p \in C_0^\infty(\overline{B(p, \frac{r_p}{2})})$. On $\overline{B(p, \frac{r_p}{3})}$, $\chi_p \equiv 1$, when $0 \leq \chi_p \leq 1$.

Choose $m > 1$ and $\beta > 1$ so that $\frac{\beta}{\beta-1} > m$ and set $a = 1, b = m^2, c = \frac{b}{m}$,

$$A = (c\varepsilon)^\beta - \beta(c\varepsilon)^{\beta-1}(c\varepsilon - b\varepsilon),$$
$$B = \beta(c\varepsilon)^{\beta-1}, \text{ and}$$
$$C = \beta(c\varepsilon)^{\beta-1}b\varepsilon\sqrt{1+M^2}\left(k - \frac{1}{\sqrt{1+M^2}}\right).$$

Notice that $\frac{1}{2m\beta} - \frac{1-\beta+\beta m}{2m\beta m^\beta} > 0$ by (3.4) and $\frac{1}{2m\beta} - \frac{m^\beta[1-\beta+\frac{\beta}{m}]}{2m\beta} > 0$ by (3.5). Since $\frac{1}{\beta} + \frac{1}{m} > 1$, $\frac{1}{2m\beta} - \frac{1}{2m} + \frac{1}{2m^2} > 0$.

Assume k and M are sufficiently small so that

$$k\sqrt{1+M^2} - 1 \leq \frac{1}{2m\beta} - \frac{1-\beta+\beta m}{2m\beta m^\beta} \tag{3.9}$$

and

$$k\sqrt{1+M^2} - 1 \leq \frac{1}{2m\beta} - \frac{1}{2m} + \frac{1}{2m^2}. \tag{3.10}$$

Finally we also assume

$$k\sqrt{1+M^2} - 1 \leq \frac{1}{2m\beta} - \frac{m^\beta[1-\beta+\frac{\beta}{m}]}{2m\beta}. \tag{3.11}$$

By applying the upper bound and lower bound of ρ_ε, we will build a plurisubharmonic function $\lambda_{\varepsilon,p}$ such that

$$\lambda_{\varepsilon,p} = A + B\rho_{\varepsilon,p} + C(\chi_p - 1).$$

We need to show that $\lambda_{\varepsilon,p}$ is plurisubharmonic for small $\varepsilon > 0$, and

$$i\partial\overline{\partial}\lambda_{\varepsilon,p} \geq iH\varepsilon^{\beta-1}\partial\overline{\partial}|z|^2, \tag{3.12}$$

in sense of currents, for some constant $H > 0$. We can compute $\lambda_{\varepsilon,p}$ as

$$i\partial\overline{\partial}\lambda_{\varepsilon,p} = i\partial\overline{\partial}(A + B\rho_{\varepsilon,p} + C(\chi_\rho - 1)) = iB\partial\overline{\partial}\rho_{\varepsilon,p} + iC\partial\overline{\partial}\chi_p.$$

By (3.1) and since $\partial\overline{\partial}\chi_p$ is C^2 function,

$$iB\partial\overline{\partial}\rho_{\varepsilon,p} + iC\partial\overline{\partial}\chi_p \geq iBK\partial\overline{\partial}|z|^2 - iCJ\partial\overline{\partial}|z|^2 = i\partial\overline{\partial}|z|^2(BK - CJ)$$

for some constant J. This implies that by plugging in the values of B and C, we get

$$i\partial\overline{\partial}|z|^2(BK - CJ) = i\partial\overline{\partial}|z|^2\beta(c\varepsilon)^{\beta-1}[K - b\varepsilon\sqrt{1+M^2}\left(k - \frac{1}{\sqrt{1+M^2}}\right)J].$$

Thus,

$$i\partial\overline{\partial}\lambda_{\varepsilon,p} \geq i\partial\overline{\partial}|z|^2\beta(c\varepsilon)^{\beta-1}[K - b\varepsilon\sqrt{1+M^2}\left(k - \frac{1}{\sqrt{1+M^2}}\right)J].$$

One can check that for $0 < H < K\beta c^{\beta-1}$, if $\varepsilon \leq \frac{H - K\beta c^{\beta-1}}{(-b\sqrt{1+M^2}\beta c^{\beta-1}k + b\beta c^{\beta-1})J}$, then (3.12) holds.

Define $L_\varepsilon = \{z \notin \Omega : a\varepsilon \leq \delta(z) \leq b\varepsilon\}$ and $L_{\varepsilon,p} = \{z \in B(p, \frac{r_p}{2})\backslash\Omega : a\varepsilon \leq \delta(z) \leq b\varepsilon\}$. By (3.7) when $\chi = 0$,

$$\lambda_\varepsilon = A + B\rho_\varepsilon - C \leq A + B(\delta(z) - b\varepsilon)) - C = A + B(\delta(z) - kb\varepsilon\sqrt{1+M^2}).$$

By (3.6),when $\chi = 1$,

$$\lambda_\varepsilon = A + B\rho_\varepsilon \geq A + B(\delta(z) - kb\varepsilon\sqrt{1+M^2}).$$

Notice that $\lambda_{\varepsilon,p_1} \geq \lambda_{\varepsilon,p_2}$ on $\overline{B(p_1, \frac{r}{3})} \cap \partial B(p_2, \frac{r}{2})$. So the function

$$\lambda_\varepsilon(z) = \sup_{\{p \in P : z \in L_{\varepsilon,p}\}} \lambda_{\varepsilon,p}(z)$$

is continuous on L_ε, where $\mathcal{P}$ is finite index set so that $\{B(p, \frac{r}{3})\}_{p\in\mathcal{P}}$ covers $\partial\Omega$. By Theorem 1.6.2 in [14] (see also Lemma 2.10 in [4]), λ_ε is plurisubharmonic and satisfies (3.12).

Then we have an inequality

$$A + B\rho_\varepsilon - C \leq \lambda_\varepsilon \leq A + B\rho_\varepsilon.$$

This implies, by (3.6) and (3.7),

$$A + B(\delta(z) - kb\varepsilon\sqrt{1+M^2}) - C \leq \lambda_\varepsilon \leq A + B(\delta(z) - b\varepsilon).$$

By plugging in A, B, and C, we get

$$\begin{aligned}\lambda_\varepsilon \geq (c\varepsilon)^\beta - \beta(c\varepsilon)^{\beta-1}(c\varepsilon - b\varepsilon) + \beta(c\varepsilon)^{\beta-1}(\delta(z) - kb\varepsilon\sqrt{1+M^2}) \\ - \beta(c\varepsilon)^{\beta-1}b\varepsilon\sqrt{1+M^2}\left(k - \frac{1}{\sqrt{1+M^2}}\right) \\ = (c\varepsilon)^\beta\left[1 - \beta + \frac{2\beta b}{c} + \frac{\beta\delta}{c\varepsilon} - \frac{2kb\beta}{c}\sqrt{1+M^2}\right]. \quad (3.13)\end{aligned}$$

Moreover,

$$\lambda_\varepsilon \leq (c\varepsilon)^\beta - \beta(c\varepsilon)^{\beta-1}(c\varepsilon - b\varepsilon) + \beta(c\varepsilon)^{\beta-1}(\delta(z) - b\varepsilon) = (c\epsilon)^\beta\left[1 - \beta + \frac{\beta\delta}{c\varepsilon}\right]. \quad (3.14)$$

To use a patching argument, choose ε_0 sufficiently small so that (3.12) holds and $\{B(p, \frac{r}{3})_{p\in\mathcal{P}}\}$ covers L_{ε_0}. Set $\varepsilon_j = \varepsilon_0 m^{-j}$ and we can check

$$a\varepsilon_j = c\varepsilon_{j+1}, c\varepsilon_j = a\varepsilon_{j-1} = b\varepsilon_{j+1} \text{ and } b\varepsilon_j = c\varepsilon_{j-1}.$$

Then, $\frac{b}{c} = \frac{\varepsilon_{j-1}}{\varepsilon_j}$. At $\delta = b\varepsilon_j = c\varepsilon_{j-1}$, we need to show $\lambda_{\varepsilon_{j-1}} \geq \lambda_{\varepsilon_j}$. By applying inequality

(3.13),

$$\lambda_{\varepsilon_{j-1}} \geq (c\varepsilon_{j-1})^{\beta}\left[1-\beta+\frac{2b\beta}{c}+\frac{c\beta\varepsilon_{j-1}}{c\varepsilon_{j-1}}-\frac{2bk\beta}{c}\sqrt{1+M^2}\right] = (c\varepsilon_{j-1})^{\beta}\left[1+\frac{2b\beta}{c}-\frac{2bk\beta}{c}\sqrt{1+M^2}\right]$$

and by (3.14),

$$\lambda_{\varepsilon_j} \leq (c\varepsilon_j)^{\beta}\left[1-\beta+\frac{b\beta\varepsilon_j}{c\varepsilon_j}\right] = (c\varepsilon_j)^{\beta}\left[1-\beta+\frac{b\beta}{c}\right].$$

Note $m = \frac{b}{c} = \frac{\varepsilon_{j-1}}{\varepsilon_j}$. We need to show

$$m^{\beta}[1+2m\beta-2km\beta\sqrt{1+M^2}] \geq 1-\beta+\beta m,$$

but this follows from (3.9).

Similarly, at $\delta = a\varepsilon_j = c\varepsilon_{j+1}$, we need to show $\lambda_{\varepsilon_{j+1}} \geq \lambda_{\varepsilon_j}$. Again, by (3.13),

$$\begin{aligned}\lambda_{\varepsilon_{j+1}} \geq (c\varepsilon_{j+1})^{\beta}[1-\beta+2\frac{b\beta}{c}+\frac{\beta c\varepsilon_{j+1}}{c\varepsilon_{j+1}}-2\frac{kb\beta}{c}\sqrt{1+M^2}]\\ = (c\varepsilon_{j+1})^{\beta}\left[1+\frac{2b\beta}{c}-\frac{2bk\beta}{c}\sqrt{1+M^2}\right]\end{aligned}$$

and by (3.14)

$$\lambda_{\varepsilon_j} \leq (c\varepsilon_j)^{\beta}\left[1-\beta+\frac{a\beta\varepsilon_j}{c\varepsilon_j}\right] = (c\varepsilon_j)^{\beta}\left[1-\beta+\frac{a\beta}{c}\right].$$

Again, we check $m = \frac{c}{a} = \frac{\varepsilon_j}{\varepsilon_{j+1}} = \frac{b}{c}$. Then we need $m^{\beta}[1-\beta+\beta(\frac{1}{m})] \leq [1+2m\beta - 2km\beta\sqrt{1+M^2}]$.

Again, this follows from (3.11).

Now we have

$$\lambda_{\varepsilon_{j-1}} \geq \lambda_{\varepsilon_j} \text{ when } \delta = b\varepsilon_j = c\varepsilon_{j-1}$$

and

$$\lambda_{\varepsilon_{j+1}} \geq \lambda_{\varepsilon_j} \text{ when } \delta = a\varepsilon_j = c\varepsilon_{j+1}$$

so that the function

$$\lambda(z) = \sup_{\{j: z \in L_{\varepsilon_j}\}} \lambda_{\varepsilon_j}(z)$$

is continuous on $L = \{z \notin \Omega : \delta(z) \leq b\varepsilon_0\}$. We use Theorem 1.6.2 in [14].

Since $a\varepsilon \leq \delta \leq b\varepsilon$, $\frac{\delta}{b} \leq \varepsilon \leq \frac{\delta}{a}$ holds. This implies by (3.12) that

$$i\partial\overline{\partial}\lambda_{\varepsilon,p} \geq iH\left(\frac{1}{b}\right)^{\beta-1}\delta^{\beta-1}\partial\overline{\partial}|z|^2.$$

So by Lemma 2.10 in [4],

$$i\partial\overline{\partial}\lambda \geq iH\left(\frac{1}{b}\right)^{\beta-1}\delta^{\beta-1}\partial\overline{\partial}|z|^2.$$

Furthermore, by (3.13),

$$\lambda \geq \left(\frac{c\delta}{b}\right)^{\beta}[1-\beta+\frac{2\beta b}{c}+\frac{\beta a}{c}-\frac{2kb\beta}{c}\sqrt{1+M^2}].$$

Since $\frac{b}{c} = m$ and $\frac{a}{c} = \frac{1}{m}$,

$$\lambda \geq \left(\frac{\delta}{m}\right)^{\beta}[1-\beta+2\beta m+\frac{\beta}{m}-2k\beta m\sqrt{1+M^2}].$$

Note that right hand side is positive by (3.10). Moreover, by (3.14),

$$\lambda \leq \left(\frac{c\delta}{a}\right)^{\beta}\left[1-\beta+\frac{\beta b}{c}\right],$$

and

$$\lambda \leq (m\delta)^{\beta}[1-\beta+\beta m].$$

We have now proven Theorem 1.27.

4 Lipschitz domains satisfying property (P)

In this section, I will show conditions for the existence of a good Stein neighborhood basis for a Lipschitz domain satisfying property (P), (See Definition 1.29) and prove Theorem 1.31.

As we defined in section 3, let $p \in \partial\Omega$ so that $B(p, R) \cap \Omega = \{z \in B(p, R) : \text{Im} z_n < \varphi(z', \text{Re} z_n)\}$ for some Lipschitz function φ with Lipschitz constant $M > 0$. Also, we will use z' to denote $\{z_1, \dots z_{n-1}\}$ and $r(z) = \text{Im} z_n - \varphi(z', \text{Re} z_n)$. We let $\delta_\varepsilon(z) = \delta(z_\varepsilon)$ where $z_\varepsilon = z - (0, \dots, 0, is\varepsilon)$. Moreover, define $L_\varepsilon = \{z \notin \Omega : a\varepsilon \leq \delta(z) \leq b\varepsilon\}$ and $L_{\varepsilon,p} = \{z \in B(p, \frac{r_p}{2}) \backslash \Omega : a\varepsilon \leq \delta(z) \leq b\varepsilon\}$ for a and b to be chosen later.

First, we need to replace δ_ε with smooth function $\tilde{\delta}_\varepsilon$, such that $-\log \tilde{\delta}_\varepsilon$ is plurisubharmonic and satisfies a related upper and lower bound. Then I will build a function $\lambda_{\varepsilon,p}$ that is plurisubharmonic,

$$i\partial\overline{\partial}\lambda_{\varepsilon,p} \gtrsim i\delta^\beta \partial\overline{\partial}|z|^2$$

and provide the upper and lower bound of $\lambda_{\varepsilon,p}$. As we did in section 3, I will use the patching argument. First I will check the supremum of $\lambda_{\varepsilon,p}$ over $mathcal P$ is continuous to build λ_ε and make sure that $\lambda_{\varepsilon_{j+1}} \geq \lambda_{\varepsilon_j}$ at $\delta = a\varepsilon_j = c\varepsilon_{j+1}$ and $\lambda_{\varepsilon_{j-1}} \geq \lambda_{\varepsilon_j}$ at $\delta = b\varepsilon_j = c\varepsilon_{j+1}$ so that we can build λ.

Lemma 4.1. *On $L_{\varepsilon,p}$, for any $\tilde{s} > s$, there exists b sufficiently small and $\tilde{\delta}_\varepsilon \in C^\infty(L_{\varepsilon,p})$ such that $-\log \tilde{\delta}_\varepsilon$ is plurisubharmonic and $-\delta(z) + \frac{\tilde{s}\varepsilon}{\sqrt{1+M^2}} < \tilde{\delta}_\varepsilon(z) < -\delta(z) + \tilde{s}\varepsilon$*

Proof. We know $-\log \delta_\varepsilon$ is plurisubharmonic and

$$-\delta(z) + \frac{s\varepsilon}{\sqrt{1+M^2}} < \delta_\varepsilon(z) < -\delta(z) + s\varepsilon \tag{4.1}$$

where $a\varepsilon \leq \delta \leq b\varepsilon$ by Lemma 3.2. Let $X = \frac{1}{\sqrt{1+M^2}}$. By assumption, $0 < b < sX$, for b sufficiently small. For P and Q to be chosen later, we let $-\log \tilde{\delta\varepsilon} = -P \log(\delta_\varepsilon) * \eta_\gamma + Q$

where

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & \text{if}|x| < 1 \\ 0 & \text{if}|x| \geq 1 \end{cases}$$

for some constant $C > 0$ and $\eta_\gamma(x) = \frac{1}{\gamma^{2n}}\eta(\frac{x}{\gamma})$ for each $\gamma > 0$. Theorem 7 (iii) in Appendix C of [7] shows that for any $R > 0$, there exists $\gamma > 0$ such that

$$-\log \delta_\varepsilon \leq -\log \delta_\varepsilon * \eta_\gamma \leq -\log \delta_\varepsilon + R$$

where the lower bound follows from the sub-mean value property for plurisubharmonic functions. This implies

$$-P \log \delta_\varepsilon + Q \leq -\log \tilde{\delta_\varepsilon} \leq -P \log \delta_\varepsilon + PR + Q. \tag{4.2}$$

Let $P = \frac{s}{\tilde{s}}$. If $\tilde{s} > s$, then for b sufficiently small,

$$\frac{s}{\tilde{s}} \log{(X - \frac{b}{s})} > \log{(X - \frac{b}{\tilde{s}})}.$$

This is equivalent to

$$(X - \frac{b}{s})^P > X - \frac{b}{\tilde{s}}.$$

If we multiply both sides by $s^P \tilde{s}$, we get

$$\tilde{s}(Xs - b)^P > s^P(X\tilde{s} - b).$$

This implies that

$$\frac{\tilde{s}}{s^P} > \frac{X\tilde{s} - b}{(Xs - b)^P}.$$

Then, there exists $R > 0$ so that

$$\frac{\tilde{s}}{s^P} > \frac{X\tilde{s} - b}{(Xs - b)^P} e^{PR}. \tag{4.3}$$

If we multiply (4.3) by $\frac{\varepsilon}{\varepsilon^P}$, we get

$$\frac{\tilde{s}\varepsilon}{s^P \varepsilon^P} > \frac{X\varepsilon\tilde{s} - b\varepsilon}{(Xs\varepsilon - b\varepsilon)^P} e^{PR}. \tag{4.4}$$

Notice that the left hand side of (4.4) is smaller than $\frac{\varepsilon\tilde{s} - \delta}{(s\varepsilon - \delta)^P}$ since it is increasing in δ when $\delta > \frac{s\varepsilon - P\tilde{s}\varepsilon}{1-P} = 0$. Thus,

$$\frac{\varepsilon\tilde{s} - \delta}{(s\varepsilon - \delta)^P} > \frac{\tilde{s}\varepsilon}{s^P \varepsilon^P} \tag{4.5}$$

holds. Likewise, the right hand side of (4.4) is larger than $\frac{X\varepsilon\tilde{s} - \delta}{(Xs\varepsilon - \delta)^P} e^{PR}$ since it is increasing in δ when $\delta > \frac{\varepsilon X(s - P\tilde{s})}{1-P} = 0$. Thus,

$$\frac{X\varepsilon\tilde{s} - \delta}{(Xs\varepsilon - \delta)^P} e^{PR} < \frac{X\varepsilon\tilde{s} - b\varepsilon}{(Xs\varepsilon - b\varepsilon)^P} e^{PR} \tag{4.6}$$

holds. (4.4) (4.5), and (4.6) imply that there exists Q such that

$$\frac{X\varepsilon\tilde{s} - \delta}{(Xs\varepsilon - \delta)^P} e^{PR} < e^{-Q} < \frac{\varepsilon\tilde{s} - \delta}{(s\varepsilon - \delta)^P}. \tag{4.7}$$

The lower bound on (4.7) implies that

$$-\delta - (-\delta + s\varepsilon X)^P e^{-PR-Q} < -sP^{-1}\varepsilon X \tag{4.8}$$

holds. So we have

$$-\delta - \left(-\delta + \frac{s\varepsilon}{\sqrt{1+M^2}}\right)^P e^{-PR-Q} \leq -\frac{\tilde{s}\varepsilon}{\sqrt{1+M^2}}.$$

Thus, by (4.1),

$$-\delta + \frac{\tilde{s}\varepsilon}{\sqrt{1+M^2}} \leq e^{P\log\delta_\varepsilon - PR - Q}.$$

So

$$-\delta + \frac{\tilde{s}\varepsilon}{\sqrt{1+M^2}} \leq e^{P\log\delta_\varepsilon - PR - Q} \leq \tilde{\delta}_\varepsilon.$$

This implies that $\tilde{\delta}_\varepsilon \geq -\delta + \frac{\tilde{s}\varepsilon}{\sqrt{1+M^2}}$. Likewise, the upper bound on (4.7) implies that, $(-\delta + s\varepsilon)^P e^{-Q} < -\delta + \tilde{s}\varepsilon$ holds. So, by (4.1), we can say

$$e^{P\log\delta_\varepsilon - Q} < -\delta + \tilde{s}\varepsilon.$$

This implies that $\tilde{\delta}_\varepsilon \leq -\delta + \tilde{s}$.

$\square$

Let us replace δ_ε with $\tilde{\delta}_\varepsilon$, and s with $\tilde{s}$ as given by Lemma 4.1 so that we can assume δ_ε is smooth. For χ defined in Section 3 and ϕ defined in 1.29, by applying the upper bound and lower bound of δ_ε, we need to build a plurisubharmonic function $\lambda_{\varepsilon,p}$ on $L_{\varepsilon,p}$ such that

$$\lambda_{\varepsilon,p} = A + B\delta_\varepsilon^{-1} + g(\delta_\varepsilon)\chi + D\phi \tag{4.9}$$

where

$$g(x) = B[(x - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}})^{-1} - x^{-1}] + D$$

for some constants A, B, and D, which are defined later. We need to show that $\lambda_{\varepsilon,p}$ is plurisubharmonic for small $\varepsilon > 0$, and

$$i\partial\bar{\partial}\lambda_{\varepsilon,p} \gtrsim i\delta^\beta \partial\bar{\partial}|z|^2. \tag{4.10}$$

When $D = D_0\varepsilon^\beta$, we let

$$A = (c\varepsilon)^\beta[1 - \beta(-1 + \frac{2s}{c\sqrt{1+M^2}} - \frac{s}{c}) - \frac{2D_0}{c^\beta}]$$

and

$$B = \beta(c\varepsilon)^{\beta-1}(-c\varepsilon + \frac{2s}{\sqrt{1+M^2}}\varepsilon - s\varepsilon)^2.$$

Since $i\partial\bar{\partial}\lambda_{\varepsilon,p} = i\partial\bar{\partial}(A + B\delta_\varepsilon^{-1} + g(\delta\varepsilon)\chi + D\phi)$, we rewrite $i\partial\bar{\partial}\lambda_{\varepsilon,p}$ as

$$i\partial\bar{\partial}\lambda_{\varepsilon,p} = iB\partial\bar{\partial}\delta_\varepsilon^{-1} + i\partial\bar{\partial}(g(\delta_\varepsilon)\chi) + i\partial\bar{\partial}D\phi. \tag{4.11}$$

We want to show that this satisfies (4.10).

Since $i\partial\bar{\partial}(-\log\delta_\varepsilon) \geq 0$, we can say

$$i\partial\bar{\partial}(-\log\delta_\varepsilon) = i\partial(-\delta_\varepsilon^{-1}\bar{\partial}\delta_\varepsilon) = i(-\delta_\varepsilon^{-1}\partial\bar{\partial}\delta_\varepsilon + \delta_\varepsilon^{-2}\partial\delta_\varepsilon \wedge \bar{\partial}\delta_\varepsilon) \geq 0$$

This implies that

$$-i\partial\bar{\partial}\delta_\varepsilon \geq -i\delta_\varepsilon^{-1}\partial\delta_\varepsilon \wedge \bar{\partial}\delta_\varepsilon. \tag{4.12}$$

Now compute

$$i\partial\bar{\partial}(\delta_\varepsilon^{-1}) = i\partial(-\delta_\varepsilon^{-2}\bar{\partial}\delta_\varepsilon) = \delta_\varepsilon^{-2}(-i\partial\bar{\partial}\delta_\varepsilon) + 2i\delta_\varepsilon^{-3}\partial\delta_\varepsilon \wedge \bar{\partial}\delta_\varepsilon.$$

If we apply (4.12), we get:

$$i\partial\bar{\partial}(\delta_\varepsilon^{-1}) \geq \delta_\varepsilon^{-2}(-i\delta_\varepsilon^{-1}\partial\delta_\varepsilon \wedge \bar{\partial}\delta_\varepsilon) + 2i\delta_\varepsilon^{-3}\partial\delta_\varepsilon \wedge \bar{\partial}\delta_\varepsilon = i\delta_\varepsilon^{-3}\partial\delta_\varepsilon \wedge \bar{\partial}\delta_\varepsilon \geq 0$$

Moreover, we can compute

$$i\partial\bar{\partial}(g(\delta_\varepsilon)\chi) = ig''(\delta_\varepsilon)\chi\partial\delta_\varepsilon \wedge \bar{\partial}\delta_\varepsilon + ig'(\delta_\varepsilon)\chi\partial\bar{\partial}\delta_\varepsilon + ig(\delta_\varepsilon)\partial\bar{\partial}\chi + ig'(\delta_\varepsilon)(\partial\delta_\varepsilon \wedge \bar{\partial}\chi + \bar{\partial}\delta_\varepsilon \wedge \partial\chi).$$

Thus, (4.11) can be bounded below by:

$$iB\partial\bar{\partial}\delta_\varepsilon^{-1} + i\partial\bar{\partial}(g(\delta_\varepsilon)\chi) + i\partial\bar{\partial}D\phi \geq iB\delta_\varepsilon^{-3}\partial\delta_\varepsilon \wedge \bar{\partial}\delta_\varepsilon + ig''(\delta_\varepsilon)\chi\partial\delta_\varepsilon \wedge \bar{\partial}\delta_\varepsilon$$
$$+ ig'(\delta_\varepsilon)\chi\partial\bar{\partial}\delta_\varepsilon + ig(\delta_\varepsilon)\partial\bar{\partial}\chi + ig'(\delta_\varepsilon)(\partial\delta_\varepsilon \wedge \bar{\partial}\chi + \bar{\partial}\delta_\varepsilon \wedge \partial\chi) + iD\partial\bar{\partial}\phi.$$

Note that $g'(x) = B(x^{-2} - (x - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}})^{-2}) < 0$ because $s\varepsilon > \frac{s\varepsilon}{\sqrt{1+M^2}}$ implies $x^{-2} < (x - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}})^{-2}$. Using (4.12) again, then the lower bound become

$$i[B\delta_\varepsilon^{-3} + g''(\delta_\varepsilon)\chi + g'(\delta_\varepsilon)\chi\delta_\varepsilon^{-1}](\partial\delta_\varepsilon \wedge \bar{\partial}\delta_\varepsilon) + ig(\delta_\varepsilon)\partial\bar{\partial}\chi + ig'(\delta_\varepsilon)(\partial\delta_\varepsilon \wedge \bar{\partial}\chi + \bar{\partial}\delta_\varepsilon \wedge \partial\chi) + iD\partial\bar{\partial}\phi. \tag{4.13}$$

Let $E = B\delta_\varepsilon^{-3} + g''(\delta_\varepsilon)\chi + g'(\delta_\varepsilon)\chi\delta_\varepsilon^{-1}$ and $F = g'(\delta_\varepsilon)$. E can be written as

$$B(\delta_\varepsilon^{-3}) - 2B\delta_\varepsilon^{-3}\chi + 2B(\delta_\varepsilon - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}})^{-3}\chi + B\chi\delta_\varepsilon^{-3} - B\chi\delta_\varepsilon^{-1}(\delta_\varepsilon - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}})^{-2},$$

which can be simplified as follows:

$$E = B\delta_\varepsilon^{-3}(1-\chi) + (\delta_\varepsilon - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}})^{-3}B\chi(1 + \delta_\varepsilon^{-1}s\varepsilon - \delta_\varepsilon^{-1}\frac{s\varepsilon}{\sqrt{1+M^2}}). \tag{4.14}$$

Since $\delta_\varepsilon \leq -\delta + s\varepsilon \leq -a\varepsilon + s\varepsilon$, we can find the lower bound of (4.14). Then we get:

$$E \geq B((s-a)\varepsilon)^{-3}(1-\chi) + ((\frac{s}{\sqrt{1+M^2}} - a)\varepsilon)^{-3}B\chi(1 + (\varepsilon(s-a))^{-1}(s\varepsilon - \frac{s\varepsilon}{\sqrt{1+M^2}})).$$

By factoring out $B\varepsilon^{-3}$, we can say

$$E \geq B\varepsilon^{-3}[(s-a)^{-3}(1-\chi) + (\frac{s}{\sqrt{1+M^2}} - a)^{-3}\chi(1 + (s-a)^{-1}(s - \frac{s}{\sqrt{1+M^2}}))].$$

Now let $P = (s-a)^{-3}$ and $Q = (\frac{s}{\sqrt{1+M^2}} - a)^{-3}(1 + (s-a)^{-1}(s - \frac{s}{\sqrt{1+M^2}}))$. Since $s > a$, both

P and Q are positive. If $P > Q$,

$$E \geq B\varepsilon^{-3}(Q(1-\chi) + Q\chi) = B\varepsilon^{-3}Q.$$

If $Q > P$,

$$E \geq B\varepsilon^{-3}(P(1-\chi) + P\chi) = B\varepsilon^{-3}P.$$

Thus, for fixed M, s, a,

$$E \geq B\varepsilon^{-3}\min\{P, Q\}$$

on $L_{\varepsilon,p}$.

Since $E > 0$, (4.13) equals

$$i(\sqrt{E}\partial\delta_\varepsilon + \frac{F}{\sqrt{E}}\bar{\partial}\chi) \wedge (\sqrt{E}\bar{\partial}\delta_\varepsilon + \frac{F}{\sqrt{E}}\bar{\partial}\chi) - i\left|\frac{F}{\sqrt{E}}\right|^2 \partial\chi \wedge \bar{\partial}\chi + iD\partial\bar{\partial}\phi + ig(\delta_\varepsilon)\partial\bar{\partial}\chi.$$

Since

$$i(\sqrt{E}\partial\delta_\varepsilon + \frac{F}{\sqrt{E}}\bar{\partial}\chi) \wedge (\sqrt{E}\bar{\partial}\delta_\varepsilon + \frac{F}{\sqrt{E}}\bar{\partial}\chi) \geq 0,$$

$\delta_\varepsilon \approx O(\varepsilon)$, $F \approx O(B\varepsilon^{-2})$, and $E \gtrsim O(B\varepsilon^{-3})$ with a strictly positive constant independent of ε and δ_ε, (4.13) is greater than or equal to

$$-i\frac{|F|^2}{E}\partial\chi \wedge \bar{\partial}\chi + ig(\delta_\varepsilon)\partial\bar{\partial}\chi + iD\partial\bar{\partial}\phi + ig(\delta_\varepsilon)\partial\bar{\partial}\chi \gtrsim O(B\varepsilon^{-1}) + iD\partial\bar{\partial}\phi. \tag{4.15}$$

Thus, we need $D \approx \varepsilon^{-1}B \approx \varepsilon^{-1}\varepsilon^{\beta+1} = \varepsilon^\beta$.

Property (P) implies that we can always pick ϕ, independent of ε, so that for some positive constant G, independent of ε,

$$
\begin{aligned}
&O(B\varepsilon^{-1}) + iD\partial\bar{\partial}\phi \\
&\qquad = O[\beta(c\varepsilon)^{\beta-1}(-c\varepsilon + \frac{2s}{\sqrt{1+M^2}}\varepsilon - s\varepsilon)^2\varepsilon^{-1}] + iD_0\varepsilon^\beta\partial\bar{\partial}\phi \\
&\qquad = O[\beta c^{\beta-1}(-c + \frac{2s}{\sqrt{1+M^2}} - s)^2\varepsilon^\beta] + iD_0\varepsilon^\beta\partial\bar{\partial}\phi \geq G\varepsilon^\beta\partial\bar{\partial}|z|^2 \quad (4.16)
\end{aligned}
$$

in a neighborhood of $\partial\Omega$. Since $\varepsilon^\beta\partial\bar{\partial}|z|^2 \gtrsim \delta^\beta\partial\bar{\partial}|z|^2$ on L_ε,

$$
i\partial\bar{\partial}\lambda_{\varepsilon,p} \gtrsim \delta^\beta\partial\bar{\partial}|z|^2
$$

on L_ε.

By (3.7) when $\chi = 0$,

$$
\lambda_{\varepsilon,p} = A + B\delta_\varepsilon^{-1} + D\phi \geq A + B\left(-\delta(z) + \frac{s\varepsilon}{\sqrt{1+M^2}}\right)^{-1} + D. \qquad (4.17)
$$

By (3.6),when $\chi = 1$, we wish to show:

$$
\lambda_{\varepsilon,p} = A + B\delta_\varepsilon^{-1} + g(\delta_\varepsilon) + D\phi \leq A + B(-\delta(z) + s\varepsilon)^{-1} + g(-\delta(z) + s\varepsilon). \qquad (4.18)
$$

Notice that if we let $h(x) = x^{-1}$, $Bx^{-1} + g(x)$ can be written as

$$
Bh(x) + B(h(x - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}}) - h(x)) + D.
$$

This equals $Bh(x - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}}) + D$. Also, $B, D > 0$ and h is a decreasing function for $x > 0$. Hence $B\delta_\varepsilon^{-1} + g(\delta_\varepsilon)$ is decreasing if $\delta_\varepsilon - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}} > 0$.

By replacing δ_ε with $-\delta(z) + s\varepsilon$, by (3.6) we can say $B(-\delta(z) + s\varepsilon)^{-1} + g(-\delta + s\varepsilon)$ is decreasing as well. Thus, if we can show $\delta_\varepsilon - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}} > 0$, (4.18) is proven.

Let $b = w(\frac{\beta w - w - \beta c}{\beta w - c - c\beta})$, where $w = \frac{2s}{\sqrt{1+M^2}} - s$. Moreover, let $c = \tilde{c}w$ where $0 < \tilde{c} < \frac{\beta-1}{\beta+1} < 1$, and $b = \tilde{b}w$. Then, $\tilde{b} = \frac{\beta-1-\beta\tilde{c}}{\beta-\tilde{c}-\tilde{c}\beta}$. Choose β close enough to 1, so b is small enough as required in Lemma 4.1. Since $\tilde{c} < 1$, $\tilde{b} = \frac{b}{w} \leq 1$ and this implies $b \leq w$.

Since $\frac{\delta}{\varepsilon} \leq b$ on $L_{\varepsilon,p}$, $w = \frac{2s}{\sqrt{1+M^2}} - s > \frac{\delta}{\varepsilon}$. This implies that $-\delta + \frac{2s\varepsilon}{\sqrt{1+M^2}} - s\varepsilon > 0$. Now if we apply (3.7) with $\rho_\varepsilon = -\delta_\varepsilon$ and $k = 1$, we get: $\delta_\varepsilon - s\varepsilon > 0$. Since $\frac{s\varepsilon}{\sqrt{1+M^2}} > 0$, $\delta_\varepsilon - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}} > 0$. Thus (4.18) holds.

So by (4.17) and (4.18),

$$A + B(-\delta(z) + s\varepsilon)^{-1} + g(-\delta + s\varepsilon) \leq A + B\left(-\delta(z) + \frac{s\varepsilon}{\sqrt{1+M^2}}\right)^{-1} + D.$$

Notice that $\lambda_{\varepsilon,p_1} \geq \lambda_{\varepsilon,p_2}$ on $\overline{B(p_1, \frac{r}{3})} \cap \partial B(p_2, \frac{r}{2})$. So the function

$$\lambda_\varepsilon(z) = \sup_{\{p \in P : z \in L_{\varepsilon,p}\}} \lambda_{\varepsilon,p}(z)$$

is continuous on L_ε, where $\mathcal{P}$ is finite index set so that $\{B(p, \frac{r}{3})\}_{p \in \mathcal{P}}$ covers $\partial\Omega$. By Theorem 1.6.2 in [14] (see also Lemma 2.10 in [4]), λ_ε is plurisubharmonic and satisfies (4.10).

By applying (3.6) and (3.7) for $\rho = -\delta$ we get inequalities:

$$\frac{B}{\delta_\varepsilon} + g(\delta_\varepsilon) = \frac{B}{\delta_\varepsilon - s\varepsilon + \frac{s\varepsilon}{\sqrt{1+M^2}}} + D \leq \frac{B}{-\delta + \frac{2s\varepsilon}{\sqrt{1+M^2}} - s\varepsilon} + D = \frac{B}{w\varepsilon - \delta} + D.$$

Thus we have

$$\lambda_\varepsilon \leq A + \frac{B}{w\varepsilon - \delta} + 2D$$

and

$$\lambda_\varepsilon \geq A + \frac{B}{s\varepsilon - \delta}.$$

These imply that

$$\lambda_\varepsilon \leq (c\varepsilon)^\beta[1-\beta(-1+\frac{2s}{c\sqrt{1+M^2}}-\frac{s}{c})+\beta\frac{(-c\varepsilon+\frac{2s}{\sqrt{1+M^2}}\varepsilon-s\varepsilon)^2}{c\varepsilon(w\varepsilon-\delta)}] = (c\varepsilon)^\beta[1-\beta(\frac{w-c}{c})+\beta\frac{\varepsilon(w-c)^2}{c(w\varepsilon-\delta)}] \tag{4.19}$$

and

$$\lambda_\varepsilon \geq (c\varepsilon)^\beta\left[1-\beta\left(-1+\frac{2s}{c\sqrt{1+M^2}}-\frac{s}{c}\right)-\frac{2D_0}{c^\beta}+\beta\frac{(-c\varepsilon+\frac{2s\varepsilon}{\sqrt{1+M^2}}-s\varepsilon)^2}{(c\varepsilon)(-\delta+s\varepsilon)}\right]$$
$$= (c\varepsilon)^\beta[1-\beta(\frac{w-c}{c})-\frac{2D_0}{c^\beta}+\beta\frac{(w-c)^2}{c(s-\frac{\delta}{\varepsilon})})]. \tag{4.20}$$

Let $a=\frac{c^2}{b}$. For ε_0 sufficiently small, set $\varepsilon_j=\varepsilon_0(\frac{a}{c})^j=\varepsilon_0(\frac{c}{b})^j$. We want $\lambda_{\varepsilon_{j+1}}\geq\lambda_{\varepsilon_j}$ at $\delta=a\varepsilon_j=c\varepsilon_{j+1}$ and $\lambda_{\varepsilon_{j-1}}\geq\lambda_{\varepsilon_j}$ at $\delta=b\varepsilon_j=c\varepsilon_{j-1}$.

Let $f(x)=x^{-\beta}[1-\beta(\frac{w}{c}-1)+\beta\frac{(w-c)^2}{c(w-x)}]$. We know $s\geq b\sqrt{1+M^2}$, and $f(c)=c^{-\beta}$. Observe that $f'(c)=0$ and

$$f''(c)=1-\beta+2c(-c+\frac{2s}{\sqrt{1+M^2}}-s)^{-1}<0$$

because this holds if $\beta-1>\frac{2c}{-c+w}$. This is equivalent to $c<\frac{(\beta-1)w}{\beta+1}$. Thus, f has a local maximum at c.

One can observe that

$$\lambda_\varepsilon\leq f\left(\frac{\delta}{\varepsilon}\right)c^\beta\delta^\beta$$

so that when $\delta=a\varepsilon_j$, $\lambda_{\varepsilon_j}\leq f(a)(ca\varepsilon_j)^\beta$ and when $\delta=b\varepsilon_j$, $\lambda_{\varepsilon_j}\leq f(b)(cb\varepsilon_j)^\beta$.

We need to show

$$f(b)<f(c)+\beta(w-c)c^{-\beta-1}\left(\frac{w-s}{s-c}\right) \tag{4.21}$$

and

$$f(a)<f(c)+\beta(w-c)c^{-\beta-1}\left(\frac{w-s}{s-c}\right). \tag{4.22}$$

Note that we can choose D_0 sufficiently small so that (4.22) implies $\lambda_{\varepsilon_{j+1}} \geq \lambda_{\varepsilon_j}$ at $\delta = a\varepsilon_j = c\varepsilon_{j+1}$ because (4.20) implies

$$\begin{aligned}\lambda_{\varepsilon_{j+1}} &\geq (c\varepsilon_{j+1})^{\beta}[1-\beta(\frac{w-c}{c})-\frac{2D_0}{c^{\beta}}+\beta\frac{(w-c)^2}{c(s-c)}]\\ &= (ca\varepsilon_j)^{\beta}[f(c)[1-\beta(\frac{w-c}{c})+\beta\frac{(w-c)^2}{c(s-c)}]]-f(c)2D_0(a\varepsilon_j)^{\beta}\\ &= (ca\varepsilon_j)^{\beta}[f(c)-c^{-\beta-1}\left(\frac{\beta(w-c)(s-w)}{s-c}\right)]-f(c)2D_0(a\varepsilon_j)^{\beta} \quad (4.23)\end{aligned}$$

We can also choose D_0 sufficiently small so that (4.21) implies $\lambda_{\varepsilon_{j-1}} \geq \lambda_{\varepsilon_j}$ at $\delta = b\varepsilon_j = c\varepsilon_{j-1}$ because (4.20) implies

$$\begin{aligned}\lambda_{\varepsilon_{j-1}} &\geq (c\varepsilon_{j-1})^{\beta}[1-\beta(\frac{w-c}{c})-\frac{2D_0}{c^{\beta}}+\frac{(w-c)^2}{c(s-c)}]\\ &= (cb\varepsilon_j)^{\beta}[f(c)[1-\beta(\frac{w-c}{c})+\beta\frac{(w-c)^2}{c(s-c)}]]-f(c)2D_0(b\varepsilon_j)^{\beta}\\ &= (cb\varepsilon_j)^{\beta}[f(c)-c^{-\beta-1}\left(\frac{\beta(w-c)(s-w)}{s-c}\right)]-f(c)2D_0(b\varepsilon_j)^{\beta} \quad (4.24)\end{aligned}$$

Note that if we could show $f(b) > f(a)$, then (4.21) implies (4.22).

First, we will show $f(b) > f(a)$. Choose $0 < t < 1$. so that $\tilde{c} = \frac{c}{w} = t(\frac{\beta-1}{\beta+1})$ and $a = \frac{c^2}{b} = \tilde{a}w$. Then, $b = w(\frac{\beta-1-\beta\tilde{c}}{\beta-\tilde{c}-\tilde{c}\beta})$ and $a = \tilde{c}^2 w(\frac{\beta-\tilde{c}-\tilde{c}\beta}{\beta-1-\beta\tilde{c}})$. Notice that we can rewrite $f(x)$ as:

$$f(x) = x^{-\beta}\left[1+\beta\left(\frac{w}{c}-1\right)\left(\frac{x-c}{w-x}\right)\right].$$

Then if we substitute the values of a, b, and c, $f(b)$ can be written as

$$
\begin{aligned}
f(b) &= b^{-\beta}[1+\beta\left(\frac{w}{c}-1\right)\left(\frac{b-c}{w-b}\right)] \\
&= \left(\frac{\beta-1-\beta t(\frac{\beta-1}{\beta+1})}{\beta-t(\frac{\beta-1}{\beta+1})-t\beta(\frac{\beta-1}{\beta+1})}\right)^{-\beta}\left(1+\frac{\beta^2-\beta-2\beta^2\frac{t(\beta-1)}{\beta+1}+\beta\frac{t^2(\beta-1)^2}{(\beta+1)^2}+\beta^2\frac{t^2(\beta-1)^2}{(\beta+1)^2}}{\frac{t(\beta-1)}{\beta+1}}\right) \\
&= \left(\frac{(\beta-1)(\beta(t-1)-1)}{(t(\beta-1)-\beta)(1+\beta)}\right)^{-\beta}\left(\frac{1}{t}(t+\beta-t^2\beta+(t-1)^2\beta^2)\right).
\end{aligned}
$$

Similarly, $f(a)$ can be written as

$$
\begin{aligned}
f(a) &= a^{-\beta}\left[1+\beta\left(\frac{w}{c}-1\right)\left(\frac{a-c}{w-a}\right)\right] \\
&= \left[\frac{t^2(\beta-1)^2}{(\beta+1)^2}\left(\frac{(\beta-t(\beta-1))(\beta+1)}{(\beta-1)(\beta+1-\beta t)}\right)\right]^{-\beta} \\
&\times\left[1-\beta\left(\frac{t(\beta-1)}{\beta+1}-1\right)\left(\frac{\frac{-\beta t(\beta-1)}{\beta+1}+\beta-1-\frac{t(\beta-1)}{\beta+1}}{\frac{\beta t^2(\beta-1)^2}{(\beta+1)^2}-\beta+\frac{t^2(\beta-1)^2}{(\beta+1)^2}+\frac{t(\beta-1)}{\beta+1}+1}\right)\right] \\
&= \left(\frac{t^2(\beta-1)(\beta-t(\beta-1))}{(\beta+1)(\beta+1-\beta t)}\right)^{-\beta}\left(\frac{-t^2+t-1+t^2\beta^2-2t\beta^2+\beta^2}{t^2(\beta-1)+t-(\beta+1)}\right).
\end{aligned}
$$

Thus, $f(b)>f(a)$ can be written as

$$
\frac{\left(\frac{t^2(t(\beta-1)-\beta)(\beta-1)}{(\beta+1)(-1+(t-1)\beta)}\right)^{-\beta}(1-t+t^2-(t-1)^2\beta^2)}{t-1+t^2(\beta-1)-\beta}
+\frac{1}{t}\left(\frac{(\beta-1)(\beta(t-1)-1)}{(t(\beta-1)-\beta)(1+\beta)}\right)^{-\beta}(t+\beta-t^2\beta+(t-1)^2\beta^2)>0.
$$

Note that since $t+\beta(1-t^2)+(t-1)^2\beta^2>0$, $t+\beta-t^2\beta+(t-1)^2\beta^2>0$. Moreover, since $(t-1)\beta-1<0$ and $t(\beta-1)-\beta<0$,

$$
\left(\frac{t^2(t(\beta-1)-\beta)(\beta-1)}{(\beta+1)(-1+(t-1)\beta)}\right)^{\beta}>0.
$$

Multiplying by

$$\left(\frac{t}{t+\beta-t^2\beta+(t-1)^2\beta^2}\right)\left(\frac{t^2(t(\beta-1)-\beta)(\beta-1)}{(\beta+1)(-1+(t-1)\beta)}\right)^{\beta},$$

we get

$$\left(\frac{t(t+\beta-t\beta)}{1+\beta-t\beta}\right)^{2\beta}+\frac{t(1-t+t^2-(t-1)^2\beta^2)}{(t-1+t^2(\beta-1)-\beta)(t+\beta-t^2\beta+(t-1)^2\beta^2)}>0. \tag{4.25}$$

If we write (4.25) in Taylor Series in β about $\beta=1$, we get

$$\left(\frac{-4t}{(t-3)^3}+\frac{4t^2}{(t-2)^3}+\frac{2t^2\log(\frac{-t}{t-2})}{(t-2)^2}\right)(\beta-1)+O(\beta-1)^2>0 \tag{4.26}$$

Now we let $X=\frac{-t}{t-2}$, $t=\frac{2X}{X+1}$. Notice that $0<X<1$.

Lemma 4.2. $\log(x)-\frac{x}{2}+\frac{1}{2x}>0$ *when* $0<x<1$.

Proof. Let $f(x)=\log(x)-\frac{x}{2}+\frac{1}{2x}$. Since $f(1)=0$ and $f'(x)=\frac{(x-1)^2}{-2x^2}\leq 0$, we can say $f(x)>0$. □

By Lemma 4.2, we have $\log(X)>\frac{X}{2}-\frac{1}{2X}$. This implies (4.26), so (4.25) holds for β sufficiently close to 1. Thus, we can conclude that $f(b)>f(a)$ for β close to 1.

Now we will show (4.21) holds. Let $s=\tilde{s}w$ then $\tilde{s}=\frac{\sqrt{1+M^2}}{2-\sqrt{1+M^2}}$. Recall $\tilde{c}<\frac{\beta-1}{\beta+1}$, $\tilde{b}\leq\frac{1}{2-\sqrt{1+M^2}}$, and $\tilde{b}=\frac{\beta-1-\beta\tilde{c}}{\beta-\tilde{c}-\tilde{c}\beta}$.

We can rewrite (4.21) as

$$f(b)<(\tilde{c}w)^{-\beta}+\beta(w-\tilde{c}w)(\tilde{c}w)^{-\beta-1}\left(\frac{\frac{2\tilde{s}w}{\sqrt{1+M^2}}-2\tilde{s}w}{\tilde{s}w-\tilde{c}w}\right), \tag{4.27}$$

and recall that $f(b) = (\tilde{b}w)^{-\beta}[1 + \beta(\frac{1}{\tilde{c}} - 1) + \frac{\beta(1-2\tilde{c}+\tilde{c}^2)}{\tilde{c}-\tilde{c}\tilde{b}}]$. (4.27) is equivalent to

$$\tilde{b}^{-\beta}[1 - \beta\left(\frac{1}{\tilde{c}} - 1\right) + \frac{\beta(1-2\tilde{c}+\tilde{c}^2)}{\tilde{c}-\tilde{c}\tilde{b}}] < \tilde{c}^{-\beta}[1 + \beta\left(\frac{1-\tilde{c}}{\tilde{c}}\right)\left(\frac{2-2\sqrt{1+M^2}}{\sqrt{1+M^2}-2\tilde{c}+\tilde{c}\sqrt{1+M^2}}\right). \tag{4.28}$$

This can be simplified as

$$\tilde{b}^{\beta}[1 - \beta\left(\frac{1}{\tilde{c}} - 1\right) + \frac{\beta(1-s\tilde{c}+\tilde{c}^2)}{\tilde{c}-\tilde{c}\tilde{b}}] < \tilde{c}^{-\beta}[1 + \beta\left(\frac{1-\tilde{c}}{\tilde{c}}\right)\left(\frac{2-2\sqrt{1+M^2}}{\sqrt{1+M^2}-2\tilde{c}+\tilde{c}\sqrt{1+M^2}}\right)]. \tag{4.29}$$

Note that $\sqrt{1+M^2} - 2\tilde{c} + \tilde{c}\sqrt{1+M^2} > 0$ since $1 - \tilde{c} = \tilde{c} + 1 - 2\tilde{c} > 0$ and $\tilde{c} + 1 - 2\tilde{c} < \sqrt{1+M^2}(\tilde{c}+1) - 2\tilde{c}$

We will call $K = 1 - \beta\left(\frac{1}{\tilde{c}} - 1\right) + \frac{\beta(1-2\tilde{c}+\tilde{c}^2)}{\tilde{c}-\tilde{c}\tilde{b}}$. Since $\tilde{c} = t\left(\frac{\beta-1}{\beta+1}\right)$ for $0 < t < 1$, K can be written as $K = \frac{t-\beta-t^2\beta+(t-1)^2\beta^2}{t}$. If we write K in Taylor series in β about $\beta = 1$, we get:

$$K = (-1 + \frac{2}{t}) + (-4 + \frac{3}{t} + t)(\beta - 1) + (-2 + \frac{1}{t} + t)(\beta - 1)^2. \tag{4.30}$$

Similarly, $\left(\frac{\tilde{b}}{\tilde{c}}\right)^{-\beta} = \left(\frac{1+\beta-t\beta}{t^2+t\beta-t^2\beta}\right)^{-\beta}$ so that

$$\frac{\partial}{\partial\beta}\left[\left(\frac{1+\beta-t\beta}{t^2+t\beta-t^2\beta}\right)^{-\beta}\right]$$
$$= \left(\frac{1+\beta-t\beta}{t^2+t\beta-t^2\beta}\right)^{-\beta}$$
$$\times\left[-\log\left(\frac{1+\beta-t\beta}{t^2+t\beta-t^2\beta}\right) - \beta\left(\frac{t^2+t\beta-t^2\beta}{1+\beta-t\beta}\right)\left(\frac{(1-t)(t^2+t\beta-t^2\beta)-(t-t^2)}{(t^2+t\beta-t^2\beta)^2}\right)\right].$$

When $\beta = 1$,

$$\frac{\partial}{\partial\beta}\left[\left(\frac{1+\beta-t\beta}{t^2+t\beta-t^2\beta}\right)^{-\beta}\right] = -\left(\frac{t}{2-t}\right)[\log\left(\frac{2-t}{t}\right) + \left(\frac{t}{2-t}\right)\left(\frac{-1+2t-t^2}{t}\right)].$$

Therefore,

$$\left(\frac{\tilde{b}}{\tilde{c}}\right)^{-\beta} = \frac{t}{2-t} - \left(\frac{t}{2-t}\right)[\log\left(\frac{2-t}{t}\right) + \left(\frac{t}{2-t}\right)\left(\frac{-1+2t-t^2}{t}\right)](\beta-1) + O(\beta-1)^2. \tag{4.31}$$

Now we can calculate $(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K - 1$ by multiplying (4.30) and (4.31) and subtract 1. We will get

$$\begin{aligned}
&\left(\frac{\tilde{b}}{\tilde{c}}\right)^{-\beta} K - 1 = \\
&\quad \left(\frac{t}{2-t} - \left(\frac{t}{2-t}\right)[\log\left(\frac{2-t}{t}\right) + \left(\frac{t}{2-t}\right)\left(\frac{-1+2t-t^2}{t}\right)](\beta-1) + O(\beta-1)^2\right) \\
&\qquad \times \left((-1+\frac{2}{t}) + (-4+\frac{3}{t}+t)(\beta-1) + (-2+\frac{1}{t}+t)(\beta-1)^2\right) - 1.
\end{aligned}$$

Thus,

$$\left(\frac{\tilde{b}}{\tilde{c}}\right)^{-\beta} K - 1 = (-2(t-1) - \log(\frac{2-t}{t}))(\beta-1) + O(\beta-1)^2 \tag{4.32}$$

Since $\tilde{b} - 1 = \frac{\tilde{c}-1}{\beta-\tilde{c}-\tilde{c}\beta}$, and $\frac{1-\tilde{c}}{\tilde{b}-1} = \beta - \tilde{c} - \tilde{c}\beta$, we can rewrite K as

$$\begin{aligned}
K = 1 - \beta\left(\frac{1}{\tilde{c}} - 1\right) + \frac{\beta(1-2\tilde{c}+\tilde{c}^2)}{\tilde{c}-\tilde{c}\tilde{b}} \\
= 1 - \beta\left(\frac{1}{\tilde{c}} - 1\right)(1 - \frac{1-\tilde{c}}{1-\tilde{b}}) \\
= 1 - \beta\left(\frac{1}{\tilde{c}} - 1\right)(1 - \beta + \tilde{c}(1+\beta)).
\end{aligned}$$

Since $\beta\left(\frac{1}{\tilde{c}} - 1\right) > 0$ and $1 - \beta + \tilde{c}(1+\beta) < 0$, we can say $K > 1$.

Moreover, since $(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K > 0$,

$$\tilde{c}[(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K - 1] + \tilde{c}^2[(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K - 1] + 2\beta - 2\beta\tilde{c} > -\tilde{c} - \tilde{c}^2 + 2\beta - 2\beta\tilde{c}.$$

Note that $\tilde{c} < \frac{\beta-1}{\beta+1} < 1$ implies $-\tilde{c} > -(\frac{\beta-1}{\beta+1})$ and $-\tilde{c}^2 > -(\frac{\beta-1}{\beta+1})^2$. So

$$\begin{aligned}
-\tilde{c}-\tilde{c}^2+2\beta-2\beta\tilde{c} &> -\left(\frac{\beta-1}{\beta+1}\right)-\left(\frac{\beta-1}{\beta+1}\right)^2+2\beta-2\beta\left(\frac{\beta-1}{\beta+1}\right)\\
&= -\left(\frac{\beta-1}{\beta+1}\right)(1+\frac{\beta-1}{\beta+1}+2\beta)+2\beta\\
&= -\left(\frac{\beta-1}{\beta+1}\right)\left(\frac{2\beta+2\beta(\beta+1)}{\beta+1}\right)+2\beta\\
&= -\frac{2\beta^2+2\beta^3-4\beta}{(\beta+1)^2}+\frac{2\beta^3+4\beta^2+2\beta}{(\beta+1)^2}=\frac{2\beta^2+6\beta}{(\beta+1)^2}.
\end{aligned}$$

Since $\beta > 1$, $\frac{2\beta^2+6\beta}{(\beta+1)^2} > 0$.

Therefore,

$$\tilde{c}[(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K-1]+\tilde{c}^2[(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K-1]+2\beta-2\beta\tilde{c} > 0.$$

Now if we solve (4.29) for $\sqrt{1+M^2}$, we will have an upper bound

$$\sqrt{1+M^2} < \frac{2\beta-2\beta\tilde{c}+2\tilde{c}^2[(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K-1]}{\tilde{c}[(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K-1]+\tilde{c}^2[(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K-1]+2\beta-2\beta\tilde{c}} \tag{4.33}$$

When $\tilde{c} = t\left(\frac{\beta-1}{\beta+1}\right)$, the right hand side of (4.33) will be denoted as $F(\beta, t)$. Then

$$F(\beta,t) = \frac{2\beta-2\beta\tilde{c}+2\tilde{c}^2[(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K-1]}{\tilde{c}[(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K-1]+\tilde{c}^2[(\frac{\tilde{b}}{\tilde{c}})^{-\beta}K-1]+2\beta-2\beta\tilde{c}} \tag{4.34}$$

We need to find a critical point with respect to c. To do so we will rewrite $F(\beta, t)$ in Taylor series. Replace $\left(\frac{\tilde{b}}{\tilde{c}}\right)^{-\beta}K-1$ with the right hand side of (4.32), and we will get

$$F(\beta,\tilde{c}) = 1+\frac{1}{4}(-2t+2t^2+t\log(-1+\frac{2}{t}))(\beta-1)^2+O(\beta-1)^3 \tag{4.35}$$

Let $g(t) = -2t+2t^2+t\log(-1+\frac{2}{t})$. Since $g'(t) > 0$ near 0 $g(t) > 0$ near 0. This implies that $F(\beta, t) > 1$.

Now we have (4.21). So the function

$$\lambda(z) = \sup_{\{j : z \in L_{\varepsilon_j}\}} \lambda_{\varepsilon_j}(z)$$

is continuous on $L = \{z \notin \Omega : \delta(z) \leq b\varepsilon_0\}$. We use Theorem 1.6.2 in [14].

Remark 4.3. If we choose V small enough, we can take $t \approx .158645$ and $W \approx 0.03049$. Also, one can observe that $g'(t) < 0$ near 1, $g'(t) = 0$ on $(0, 1)$. So there must be a critical point in $(0, 1)$ when $\beta \approx 1$. Since t is a critical point of $g(t)$, this W is the optimal value. Thus, under this condition, $F(\beta, \tilde{c}) > 1$.

5 References

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