[Journal of the Arkansas Academy of Science](https://scholarworks.uark.edu/jaas)

Volume 42 | Article 33

1988

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EQUATIONS OF VARIATION FOR ORDINARY DIFFERENTIAL EQUATIONS ON MANIFOLDS

For systems of ordinary differential equations in R^n (real Euclidean space of dimension n), it is well known that derivatives of solutions with respect to the initial time, initial position and parameters satisfy certain variational equations. However, for systems of ordinary differential equations on manifolds, only the variational equations for the derivatives of solutions with respect to the initial position seem to have appeared in print (Sternberg, Lectures on Differential Geometry, p. 184, 1964). In this paper, we will derive equations of variation for systems on manifolds. Our methods are different from those employed in the above reference and since no extra effort is involved, all three types will be deduced.

For definitions and properties concerning differentiable manifolds, we refer the reader to (Brickell and Clark, Differentiable Manifolds, 1970). If $f:E_1 \to E_2$ and $g:H_1 \to H_2$ are maps, then $f \times g$ will denote the mapping from $E_1 \times H_1$ into $E_2 \times H_2$ defined by

$$f \times g(x,y) = (f(x),g(y)).$$

Let M and N be C^∞ manifolds of dimensions m and n respective and denote the real numbers by R. We suppose that X is a map with domain (denoted by dom (X)) an open subset of $R \times M \times N$ and range a subset of the tangent bundle of M. We assume further that X is C^∞ and that if (t,p,q) is in dom (X), then X(t,p,q) is an element of the tangent space to M at p. Stated in a different manner, X is a time dependent vector field that also depends on parameters q in N.

We consider the intial value problem

(IV) $$\dot{c}(t) = X(t,c(t),q), \quad c(t_o) = p.$$

For each (t_o,p,q) in dom (X), it is known (Brickell and Clark, Differentiable Manifolds, p. 136, 1970) that the unique solution of (IV) corresponding to (t_o,p,q) is defined in an open interval of R. We will denote the value of this solution at t by $c(t;t_o,p,q)$. It follows that a function may be defined with domain an open subset of $R \times R \times M \times N$ by $C(t,t_o,p,q) = c(t;t_o,p,q)$.

Let (t_o,p,q) be in dom(X) and suppose $u:U \to R^m$ and $v:V \to R^n$ are charts at p and q respectively. Denote identity maps by id. In each case, the domain of a particular id will be clear from the context in which it is used. We ntoe that $id \times u \times v: R \times U \times V \to R \times R^m \times R^n$ is a chart for the product manifold $R \times M \times N$.

Assume that $c:I \to M$ is a solution of (IV) defined on an open interval I of R. Let J be the largest open interval such that t_o is in J and J is contained in I $c^{-1}(U)$. Then t in J implies (t,c(t),q) is in dom(X) $(J \times U \times V)$.

For each j = 1,...,m, let u_j be the jth coordinate function of u and let $X(u_j)$ be the value of the vector field, X, at u_j. Also, define f_j by

(1) $$f_j = X(u_j)o(id \times u^{-1} \times v^{-1}) = X(u_j)o(id \times u \times v)^{-1}$$

and set $f = (f_1,\ldots,f_m)$.

Arguments similar to those given in (Brickell and Clark, Differentiable Manifolds, p. 131, 1970) establish that $c:I \to M$ is a solution of (IV) if and only if uoc, restricted to J, is a solution of

IV') $$x'(t) = f(t,z(t),v(q)), z(t_o) = u(p).$$

Denote the value of this solution at t by $z(t;t_o,u(p),v(q))$. Also, for elements $(t_o,u(p),v(q))$ in dom (f), one may define a function Z on an open subset of $R \times R \times u(U) \times v(V) C R \times R \times R^m \times R^n$ by $Z(t,t_o,u(p),v(q)) = z(t;t_o,u(p),v(q))$. It follows from the preceding discussion that

(2)
$$\begin{aligned} &Z(t,t_o,u(p),v(q)) \\ &= z(t;t_o,u(p),v(q)) \\ &= u(c)t;t_o,p,q)) \\ &= u(C)t,t_o,p,q)) \\ &= [uoCo(id \times id \times u^{-1} \times v^{-1})]\ (t,t_o,u(p),v(q)). \end{aligned}$$

Thus, $Z = uoCo(idxidxu^{-1}xv^{-1})$, and it follows that Z is C^∞. Also, f is C^∞ because we assumed that X is C^∞.

For the purpose of writing partial derivatives, we will denote the arguments of Z by (t,t_o,ξ,η) whre $\xi = (\xi_1,\ldots,\xi_m)$ and $\eta = (\eta_1,\ldots,\eta_n)$. It is well known (Reid, Ordinary Differential Equations. p. 70, 1971) that the first partial derivatives of Z with respect to t_o, ξ_i and η_j satisfy certain variational equations. In the equations that follow, it is understood that the arguments of derivatives of the Z_i are $(t,t_o,u(p),v(q))$ and the arguments of derivatives of the f_i are $(t,Z(t,t_o,u(p),v(q)),v(q))$. Also, δ_{ik} denotes the Kronecker delta. The variational equations are

(V1) $$\frac{\partial}{\partial t}\left(\frac{\partial Z_i}{\partial t_o}\right) = \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}\frac{\partial z_j}{\partial t_o}, \qquad \frac{\partial z_i}{\partial t_o}(t_o,t_o,u(p),v(q)) = -f_i(t_o,u(p),v(q));$$

(V2) $$\frac{\partial}{\partial t}\left(\frac{\partial Z_i}{\partial \xi_k}\right) = \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}\frac{\partial Z_j}{\partial \xi_k}, \qquad \frac{\partial Z_i}{\partial \xi_k}(t_o,t_o,u(p),v(q)) = \delta_{ik};$$

(V3) $$\frac{\partial}{\partial t}\left(\frac{\partial Z_i}{\partial \eta_k}\right) = \sum_{j=1}^{m} \frac{\partial f_i}{\partial \xi_j}\frac{\partial Z_j}{\partial \eta_k} + \frac{\partial f_i}{\partial \eta_k}, \qquad \frac{\partial Z_i}{\partial \eta_k}(t_o,t_o,u(p),v(q)) = 0$$

It follows from (1) and (2) that $\frac{\partial Z_i}{\partial t_o}$, $\frac{\partial Z_i}{\partial \xi_j}$ and $\frac{\partial Z_i}{\partial \eta_k}$ are the respective coordinate expressions for $\frac{\partial C_i}{\partial t_o}$, $\frac{\partial C_i}{\partial u_j}$ and $\frac{\partial C_i}{\partial v_k}$. The derivatives of the Z_i's are evaluated at $(t,t_o,u(p),v(q))$ and the derivatives of the C_i's are evaluated at (t,t_o,p,q). Also, $\frac{\partial f_i}{\partial \xi_j}$ and $\frac{\partial f_i}{\partial \eta_k}$ are coordinate expressions for $\frac{\partial X(u_i)}{\partial u_j}$ and $\frac{\partial X(u_i)}{\partial v_k}$ respectively. The derivatives of the f_i's are evaluated at $(t,Z)t,t_o,u(p),v(q)),v(q))$ and the derivatives of the $X(u_i)$'s are evaluated at $(t,C(t,t_o,p,q),q)$. Consequently, the variational equations for C may be found by substituting in (V1), (V2) and (V3). In the equations that follow, it is understood that the arguments of derivatives of the C_i's are (t,t_o,p,q) and the arguments for the derivatives of the $X(u_i)$'s are $(t,C(t,t_o,p,q),q)$. The variational equations for the system on the manifold are

(VM1) $$\frac{\partial}{\partial t}\left(\frac{\partial C_i}{\partial t_o}\right) = \sum_{j=1}^{m} \frac{\partial X(u_i)}{\partial u_j} \frac{\partial C_j}{\partial t_o}, \qquad \frac{\partial C_i}{\partial t_o}(t_o,t_o,p,q) = X(u_i)(t_o,p,q);$$

(VM2) $$\frac{\partial}{\partial t}\left(\frac{\partial C_i}{\partial u_k}\right) = \sum_{j=1}^{m} \frac{\partial X(u_i)}{\partial u_j} \frac{\partial C_j}{\partial u_k}, \qquad \frac{\partial C_i}{\partial u_k}(t_o,t_o,p,q) = \delta_{ik}$$

(VM3) $$\frac{\partial}{\partial t}\left(\frac{\partial C_i}{\partial v_k}\right) = \sum_{j=1}^{m} \frac{\partial X(u_i)}{\partial u_j} \frac{\partial C_j}{\partial v_k} + \frac{\partial X(u_i)}{\partial v_k}, \qquad \frac{\partial C_i}{\partial v_k}(t_o,t_o,p,q) = 0.$$

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NOTES ON THE BIOLOGY OF *THYANTA CALCEATA* (HEMIPTERA: PENTATOMIDAE) ON *TEPHROSIA VIRGINIANA* (LEGUMINOSAE), A NEW HOST PLANT

Thyanta calceata (Say) is distributed over the eastern United States from New England to Florida and west to Michigan, Illinois and Missouri (McPherson, The Pentatomoidea [Hemiptera] of northeastern North America with emphasis on the fauna of Illinois. S. Ill. Univ. Pr., Carbondale, 1982). The nymphal instars of this pentatomid have been previously described (Paskewitz and McPherson, Great Lakes Entomol. 15(4):231-255, 1982). *Tephrosia virginiana* (L.) ranges from Massachusetts south to Georgia, and west to Minnesota, Texas and Oklahoma (Steyermark, Flora of Missouri, Iowa St. Univ. Pr., Ames, 1975). This study presents additional information on the biology of this insect as it relates to *T. virginiana*, a previously unreported host plant.

McPherson (1982) reported *T. calceata* having been collected from soybean, red clover, blue-grass, cheat, wheat, timothy, winter cress, milkweed, horse-weed, buckbrush, *Lespedeza*, bean, pea, tomato, allegheny blackberry, common bromegrass, mullein, wild raspberry, goldenrod, and evening primrose. We have observed *T. calceata* commonly associated with *T. virginiana* in northeastern Arkansas on Crowley's Ridge and in northern Arkansas and southern Missouri on the Ozark Plateau.

Table 1. Occurrence of *Thyanta calceata* on *Tephrosia virginiana*, 28 April - 7 August 1987.

	PREBLOOM	FLOWERING						POSTBLOOM						
	April	May					June		July					August
	28	1	8	15	22	28	19	26	3	10	16	24	31	7
Adult (N=23)			1	3		6		3	2	1	1	4		2
1st Instar (N=0)														
2nd Instar (N=7)			1	1			1	1				1		2
3rd Instar								2	1					
4th Instar (N=7)					1			2	1			2		1
5th Instar (N=1)								1						