5-2017

3-Manifold Perspective on Surface Homeomorphisms for Surfaces with Very Negative Euler Characteristic

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3-Manifold Perspective on Surface Homeomorphisms for Surfaces with Very Negative Euler Characteristic

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Mathematics

by

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May 2017
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This dissertation is approved for recommendation to the Graduate Council.

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Abstract

The goal of this paper is to show for a compact triangulated 3-manifold M with boundary which fibers over the circle that whenever F is a fiber with sufficiently negative Euler characteristic the monodromy maps an essential simple closed curve or an essential simple arc in F to be disjoint from its image (possibly after isotopy). This is shown by applying the theorem of Ichihara, Kobayashi, and Rieck in [10] to the double of M to get a pair of pants. We then find an equivariant pair of pants and use it to find an essential simple closed curve or an essential simple arc which satisfies our theorem. As a corollary, if we add the hypothesis that M is a hyperbolic manifold, we get that the translation distance of the monodromy in the arc and curve complex of F is at most 1 for all but finitely many monodromy maps.

Acknowledgements

I would like to thank the Missouri Southern State University Mathematics Department for offering a friendly environment and helping me take my first steps into the world of mathematics. In particular, I would like to thank Dr. Richard Laird who was always very helpful and helped create a positive environment around the department. I would also like to thank Dr. Charles Curtis who first encouraged me to apply to different graudate programs in mathematics; without this encouragement I am uncertain if I would have ever gone to graduate school.

I would also like to express my appreciation to the Department of Mathematics at the University of Arkansas. I am especially appreciative of the people who work in the mathematics office and the hard work they do to keep everything running well and keeping me informed about important deadlines. I would like to thank both Dr. Matthew Day and Dr. Dan Rutherford who taught differential topology and knot theory classes respectively and helped spark my initial interest in studying topology. I would like to thank Dr. Matt Clay who played an important role in my initial understanding of the curve complex of a surface and also looked at some of the very early work on my dissertation. I would like to thank Dr. Jeremy Van-Horn Morris who taught a class on low dimensional topology which helped expand my knowledge of 3-manifolds, my topic of study. Lastly, I would like to thank my academic advisor, Dr. Yo’av Rieck, who has spent a considerable amount of time looking at various versions of my dissertation and helping me throughout the past few years. He has done an excellent job at pointing out flaws in my logic and has been a great person to work with.

Finally, I would like to thank my family, without whom I would not exist and would be rather lonely.

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1 Introduction

The goal of this paper is to prove a result about fibered 3-manifolds. Our main result is the following (for standard definitions see section 2):

Theorem 1.1. Let F be a connected surface, let $\mu : F \to F$ be a homeomorphism, and let $M := (F \times [0,1]) / \sim$ where $(x,0) \sim (\mu(x),1)$ for $x \in F$. Let $t(M)$ be the minimal number of tetrahedra required to triangulate M. If $\chi(F) \leq -76t(M)+1$ then there is an essential simple closed curve or essential simple arc which is disjoint from its image under μ (possibly after isotopy).

The case when F is closed is due to Ichihara, Kobayashi, and Rieck in [10]. The current paper exclusively deals with the case when F has boundary.

In this section we assume the reader is familar with the curve complex of a surface F denoted $\mathscr{C}(F)$ and the arc and curve complex of a surface F denoted $\mathscr{AC}(F)$. A brief discussion on these topics is included in section 3. Notice that the homeomorphism $\mu : F \to F$ in Theorem 1.1 induces an isometry $\mu_* : \mathscr{AC}(F) \to \mathscr{AC}(F)$ on $\mathscr{AC}(F)$. We define the translation distance of μ in $\mathscr{AC}(F)$ to be $min\{d(v, \mu_*(v)) \mid v \in \mathscr{AC}(F) \text{ is a vertex}\}$ where d is the distance between vertices in the arc and curve complex. Note that μ sends essential simple closed curves to essential simple closed curves and essential simple arcs to essential simple arcs. Since we found in Theorem 1.1 an essential simple closed curve or essential simple arc which is disjoint from its image under μ (possibly after isotopy) we see that the translation distance for μ in $\mathscr{AC}(F)$ is at most 1.

We now describe two types of 3-manifolds. A 3-manifold is simple if it contains no essential surfaces with nonnegative Euler characteristic and, as in [11, p. 42], a compact orientable irreducible 3-manifold is a Haken manifold if it contains a 2-sided incompressible surface. A result due to Haken, as presented in [15], states that for a simple Haken 3-manifold there is a finite number of essential surfaces with a given Euler characteristic. This result also holds when the 3-manifold is simple and not Haken since then the manifold will have no essential surfaces, however, this case is less interesting to us.

As stated in [8, p. 16], an irreducible 3-manifold is called atoroidal if every incompressible torus is boundary parallel. Thurston's hyperbolization theorem, as presented in [12], states that a compact atoroidal Haken 3-manifold M whose boundary consists of a possibly empty collection of tori has interior which admits a complete hyperbolic metric of finite volume. A hyperbolic manifold is a compact 3-manifold whose interior admits a complete finite volume Riemanian metric locally isometric to hyperbolic 3-space. So this theorem says that such an M is a hyperbolic manifold. For some background on hyperbolic manifolds see for example [2].

Going back to the setting in Theorem 1.1, notice that if F is orientable and $\mu : F \to F$ is orientation preserving then M is orientable. We show in Corollary 2.20 that M must be irredicible and moreover we show in Lemma 2.15 that F associated with $F \times \{0\}$ is essential and hence incompressible in M. We conclude in this case that M is a Haken manifold. So under these restraints M satisfies all of the hypotheses in Thurston's hyperbolization theorem except for being atoroidal. Without adding another hypothesis we cannot guarantee that M is atoroidal.

Let M be a 3-manifold. A surface F in M is called a fiber if $M \backslash\backslash F$ is diffeomorphic to $F \times I$. We define for an isotopy class of a fiber F the monodromy of F in M to be the isotopy class of diffeomorphisms $\mu : F \to F$ so that $M := (F \times I)/\sim$ where $(x,0) \sim (\mu(x),1)$ for $x \in F$. If M is a 3-manifold which fibers over S^1 then at least one monodromy exists. In general a 3-manifold M may have more than one monodromy corresponding to distinct fibers. We consider Thurston's hyperbolization theorem for fibered manifolds presented in [16] which states: let M be a 3-manifold that fibers over S^1 whose monodromy is a pseudo Anosov diffeomorphism μ. Then the interior of M admits a complete hyperbolic metric of finite volume. That is, M is a hyperbolic manifold. For a discussion on what it means for a diffeomorphism to be pseudo Anosov see for example [4] and [6].

Let M be a compact 3-manifold which fibers over S^1. A priori it is unknown how many isotopy classes of essential surfaces are fibers for M. Since M is a specific 3-manifold it has a fixed number of tetrahedra, $t(M)$, for a minimal triangulation. Applying Theorem 1.1 to M we get that only fibers, F, with sufficiently high Euler characteristic (that is $\chi(F) > -76t(M)+1$) will have monodromy with translation distance in $\mathscr{AC}(F)$ greater than 1. Now assume that the monodromy of F in M is pseudo Anosov, Thurston's hyperbolization theorem for fibered manifolds tells us that M is hyperbolic. Additionally, since the monodromy of F is pseudo Anosov M is simple. Finally, if we require that M is orientable we get from our argument earlier that M is a Haken manifold. Applying the earlier theorem of Haken's we get that at most finitely many isotopy classes of fibers have sufficiently high Euler characteristic. From our discussion we get the following corollary:

Corollary 1.2. Let M be a compact orientable hyperbolic 3-manifold. Then there are at most finitely many isotopy classes of fibers whose monodromy has translation distance in $\mathscr{AC}(F)$ greater than 1 .

The closed case for Corollary 1.2 is due to Ichihara, Kobayashi, and Rieck in [10]. Now, from just Corollary 1.2, we don't even know if a compact hyperbolic 3-manifold has a single monodromy; in general a compact hyperbolic 3-manifold need not fiber over S^1. Now, under certain conditions, we can use Corollary 1.2 to get a much stronger conclusion. Let $H_2(M, \partial M; \mathbb{R})$ be the second homology of M relative to ∂M. Whenever M is a hyperbolic 3-manifold which fibers over S^1 and $\dim(H_2(M, \partial M; \mathbb{R})) \geq 2$, then there are infinitely many isotopy classes of fibers F in M so that M fibers over S^1 with fiber F. This result is derived from the work of Thurston in [19]. It turns out that hyperbolic 3-manifolds which fiber over S^1 with $\dim(H_2(M, \partial M; \mathbb{R})) \geq 2$ naturally arise as the covering space of a hyperbolic manifold. Indeed, in [1] Agol proves the virtual fibering conjecture which states that for a closed hyperbolic 3-manifold M there exists a finite-sheeted cover $\widetilde{M} \to M$ such that $\widetilde{M}$ fibers over S^1 and satisfies $\dim(H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})) \geq 2$. Moreover, in [20] Wise proves the virtual fibering conjecture when M is a complete finite volume hyperbolic 3-manifold that is not closed; also see [21] and reference within. So there are many compact hyperbolic 3-manifolds M which fiber over S^1 satisfying $\dim(H_2(M,\partial M;\mathbb{R})) \geq 2$. In particular, our next two corollaries are not vacuous. From our discussion along with Corollary 1.2 we conclude the following:

Corollary 1.3. Let M be a compact orientable hyperbolic 3-manifold which fibers over S^1 satisfying $\dim(H_2(M,\partial M;\mathbb{R})) \geq 2$. Then M has infinitely many isotopy classes of fibers of which only finitely many have monodromy with translation distance in $\mathscr{A}\mathscr{C}(F)$ greater than 1.

As in Corollary 1.2 the closed case for Corollary 1.3 also follows from [10], this is due to the fact that the arc and curve complex and the curve complex are equivalent for closed surfaces. We finish this section off by relating our result for the arc and curve complex to a similar result for the curve complex. Applying Lemma 3.4 to Corollary 1.3 we conclude the following:

Corollary 1.4. Let M be a compact orientable hyperbolic 3-manifold which fibers over S^1 satisfying $\dim(H_2(M,\partial M;\mathbb{R})) \geq 2$. Then then M has infinitely many isotopy classes of fibers of which only finitely many have monodromy with translation distance in $\mathscr{C}(F)$ greater than 2.

Again the closed case follows from [10]. In fact, they get the slightly better result that only finitely many isotopy classes of fibers will have monodromy with translation distance in the curve complex greater than 1.

2 Preliminaries

In this paper we use methods from the study of 3-dimensional manifolds. We define briefly some of our basic terms and mention some of the notation we will use. We always assume transversality unless it is clear from context otherwise.

Definition 2.1. An N-manifold M^N is a Hausdorff second countable topological space such that every point in M^N has a neighborhood homeomorphic to an open set of $\mathbb{R}^N_{\geq 0}$, where N is the dimension of M^N. The boundary of M^N, denoted ∂M^N, is the set of all points which do not have a neighborhood homeomorphic to an open set of $\mathbb{R}^N$. Let M_1 and M_2 be manifolds (not necessarily of the same dimension) and let $f : M_1 \to M_2$, we say that f is proper if the preimage of the boundary of M_2 is equal the boundary of M_1.

We include some of our basic notation. We use I to represent the closed interval $[0,1]$. We use S^1 to represent the circle. For a surface F we denote the Euler characteristic of F to be $\chi(F)$. For a topological space X we denote $int(X)$ to be the interior of X and $cl(X)$ to be the closure of X. We use $\cong$ to mean homeomorphic to. For a function f we use $f|_D$ to mean f restricted to D and $im(f)$ to mean the image of f.

Some of the major subjects studied in this paper include that of fiber bundles, essential surfaces, irreducible manifolds, normal surfaces and the double of a manifold. Over the next several pages we will be discussing these terms and related concepts in detail. We first introduce fiber bundles.

Definition 2.2. Let M and N be manifolds. A map $f : M \to N$ is an embedding if it is a homeomorphism onto its image $f(M)$ and $f(M)$ is a submanifold of M. (See [18, p. 4])

Recall for a smooth manifold X and with a point $p \in X$ that the tangent space of X at p is denoted T_pX. Also recall that the tangent bundle of X is denoted by TX where $TX := \{(p, v) \mid p \in X, v \in T_pX\}$.

Definition 2.3. Let F be a properly embedded surface in a 3-manifold M. A coorientation of F

is represented by a continuous function $f : F \to TM$ so that for all $p \in F$, $f(p) \in T_pM \setminus T_pF$. We say that two coorientations g_1 and g_2 are equivalent if there exists a homotopy $H : F \times [0,1] \to TM$ so that $H(s,0) = g_1(s)$ and $H(s,1) = g_2(s)$ for all $s \in F$ and, moreover, for every fixed $t_0 \in [0,1]$ we require that $H(s,t_0)$ is a coorientation. Notice that for each point $p \in F$ the space $T_pM \setminus T_pF$ is disconnected. So F can have at most two coorientations. We say that F is 2-sided if it has two coorientations.

Definition 2.4. Let F, E, and B be topological spaces, and $p : E \to B$ a continuous surjection. We say that $\xi = (F, E, B, p)$ is a fiber bundle (or simply a bundle) if the preimage of every $b \in B$ is homeomorphic to F, and moreover, these preimages satisfy the local triviality condition: every $b \in B$ has an open neighborhood $U \subset B$, so that there is a homeomorphism $F \times U \to p^{-1}(U)$ that maps $F \times \{x\}$ homeomorphically onto $p^{-1}(x)$ for every $x \in U$. We call E the total space, B the base, F the fiber, and p the bundle projection. We say that ξ is an F bundle over B. A bundle (F, E, B, p) is called a trivial bundle if $E \cong F \times B$ and p is the projection onto the second factor.

Definition 2.5. A covering space of a space X is a space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ satisfying the following condition: there exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped by p homeomorphically onto U_α. We do not require p^{-1} to be nonempty, so p need not be surjective. (See [7, p. 56])

Definition 2.6. Let M be an n-manifold. Let S be a submanifold of M of dimension m. A regular neighborhood of S is a submanfold NS of M of dimension n that is the total space of a bundle over S with fiber the $n-m$ ball B^{n-m}. (See [18, p. 60])

In this paper we are primarily interested in fiber bundles with total space a 3-manifold M, base S^1, fiber a surface F, and bundle projection p. It follows from the bundle structure that $p^{-1}(x) \cong F \times \{x\}$ is 2-sided in M for every $x \in S^1$. In this setting since $F \times \{x\}$ is homeomorphic to F for all $x \in S^1$ we often refer to $\{F \times \{x\} \subset M \mid x \in S^1\}$ as the fibers of M without explicitly mentioning the bundle projection. We often associate F with $F \times \{0\}$ using the association of S^1 with $[0,1]/\sim$, where $0 \sim 1$, to make sense of $\{0\}$ in S^1.

We now introduce terminology related to essential surfaces along with related results used in this paper.

Definition 2.7. A properly embedded surface F in a 3-manifold M is called boundary parallel if it is isotopic, fixing ∂F, to a subsurface of ∂M. (See [8, p. 16])

Definition 2.8. A 2-sided connected surface F, which is not a sphere, is called incompressible if for each disk $D \subset M$ with $D \cap F = \partial D$ there is a disk $D' \subset F$ with $\partial D' = \partial D$. If F is a sphere then it is said to be incompressible if it does not bounds a 3-ball. (See [8, p. 13])

Remark. In some references, such as [9, p. 58], the disk may have additional requirements to be incompressible. We will not worry about these details.

Definition 2.9. A properly embedded surface F in a 3-manifold M is boundary incompressible if for each disk $D \subset M$ such that $int(D) \cap (F \cup \partial M) = \emptyset$, $\partial D \cap F$ is an arc α, $\partial D \backslash \alpha \subset \partial M$ is an open arc β, and there exists a disk $D' \subset F$ with $\alpha \subset \partial D'$ and $\partial D' - \alpha \subset \partial F$. (See [8, p. 18])

In addition to the two previous definitions we also say a surface is compressible if it is not incompressible and a surface is boundary compressible if it is not boundary incompressible. Furthermore, in the definition of an incompressible surface, the disk D is a compressing disk for F and in the definition of a boundary incompressible surface, the disk D is called a boundary compressing disk for F. We now define what it means to be an essential surface.

Definition 2.10. A surface S in a 3-manifold M is said to be essential if it is incompressible, boundary incompressible, and not boundary parallel.

Related to the notion of an essential surface we have the notion of an essential simple closed curve and an essential arc. Before we define these, we introduce the notation \\ which means cut open along. In particular, for a manifold X with a properly embedded codimension-1 submanifold Y, we define $X \backslash\backslash Y$ to be the closure of a $X \backslash NY$, where NY is a regular neighborhood of Y.

Definition 2.11. A simple closed curve α in a connected surface F is said to be essential if either $F \backslash \alpha$ is connected or $F \backslash\backslash \alpha$ does not have a disk or an annulus component.

Definition 2.12. A simple arc η is said to be an essential arc in a connected surface F if η is properly embedded and either $F\backslash\eta$ is connected or $F\backslash\backslash\eta$ does not contain a disk component.

We now introduce the concept of an essential subsurface contained in a surface along with a related lemma.

Definition 2.13. Let H be an embedded subsurface of a surface S. We say that H is an essential subsurface of S if no component of ∂H bounds a disk in S. We do not worry about components of ∂H being boundary parallel in S.

Lemma 2.14. Let H be a connected essential subsurface of a connected surface S, where S is not a sphere, then $\chi(S) \leq \chi(H)$.

Proof of Lemma 2.14. Suppose for a contradiction that $\chi(H) < \chi(S)$. If it is not already the case we isotope H to ensure that ∂H is disjoint from ∂S while still ensuring that $H \subset S$. The previous sentence along with the fact that the boundary components of a surface are all simple closed curves allows us to have that $\chi(S) = \chi(H) + \chi(S\backslash H)$ or $\chi(S) - \chi(H) = \chi(S\backslash H)$. Since $\chi(S) - \chi(H) > 0$ we see that $\chi(H\backslash S) > 0$. Since the Euler characteristic is addative over disjoint unions $cl(S\backslash H)$ has a component with positve Euler characteristic and boundary. Hence a component of $cl(S\backslash H)$ is a disk. Since we are considering $H \subset S$ we get that H has a boundary component which bounds this disk. This is a contradiction since we assumed that H was essential. We conclude that the result of the lemma. □

The following lemma is imporant for the main result of this paper.

Lemma 2.15. Let F be a connected surface. Let $\mu : F \to F$ be a homeomorphism. Identify F with $F \times \{0\}$. Then F is an essential surface of $M := (F \times [0,1])/\sim$ where $(x,0) \sim (\mu(x),1)$ for $x \in F$.

Proof of Lemma 2.15. In the case that F is a sphere it is clear from the structure of M that F does not bound a ball and hence is incompressible, In the case that F is not a sphere, suppose for a contradiction that F is compressible. Then there exists a disk $D \subset M$ such that $D \cap F = \partial D$ and there does not exist a disk $D' \subset F$ with $\partial D' = \partial D$. Since $int(D) \cap F = \emptyset$ we conclude that F is also compressible in $F \times I$ with compressing disk D. We assume without loss of generality that $\partial D \subset F = F \times \{0\}$. Notice that $F \times I$ deformation retracts onto F and let $\iota : F \hookrightarrow F \times I$ be the inclusion. It follows from [7, Prop. 1.17, p. 36] that the induced map $\iota_* : \pi_1(F, p) \to \pi_1(F \times I, p)$ is an isomorphism, where $p \in \partial D$. The inclusion of ∂D into F is a nullhomotopic embedding. Since ∂D represents the trivial element in $\pi_1(F \times I, p)$ our isomorphism shows that ∂D also represents the trivial element in $\pi_1(F, p)$. Since ∂D is an embedded curve in the surface F and represents the trivial element in $\pi_1(F, p)$ it must bound an embedded disk in F. We assumed that such a disk does not exist and hence have a contradiction. We conclude that F is incompressible in M.

If $\partial F = \emptyset$ then F is trivially boundary incompressible. Otherwise if $\partial F \neq \emptyset$ we suppose for a contradiction that F is boundary compressible. Then there exists a disk D in M such that $int(D) \cap (F \cup \partial M) = \emptyset$, $\partial D \cap F$ is an arc α, $\partial D \backslash \alpha \subset \partial M$ is an open arc β, and there does not exist a disk $D' \subset F$ with $\alpha \subset \partial D'$ and $\partial D' \backslash \alpha \subset \partial F$. Notice that β is disjoint from F. Since we are considering when F has boundary and these boundary components are simple closed curves we get that ∂M consist of tori and Klein bottles. Let C be the component of ∂M containing β. Notice that $C^* := C \backslash F$ consists of one or more disjoint open annuli. Since β is disjoint from F we get that β is contained in one of the annuli in C^*; we call this component C'. Since $\alpha \subset F$, $\alpha \cup \beta = \partial D$, and $\beta \subset C'$ we get that either β connects distinct ends of C' or β connects an end of C' to the same end. Suppose for a contradiction that β connects distinct ends of C', we consider the following maps:

- $\iota : \partial D = \alpha \cup \beta \cong S^1 \hookrightarrow M$, the inclusion
- $q : M \to S^1$, projection onto the base of the fiber bundle

Since β is disjoint from F and connects distinct ends of C' we get that $q \circ \iota$ not nullhomotopic. We get the following induced homomorphisms:

- $\iota_* : \pi_1(\partial D, p_1) \to \pi_1(M, p_1)$, the 0-map

- $q_* : \pi_1(M, p_1) \to \pi_1(S^1, p_2)$

where $p_1 \in \alpha \subset \partial D$ and $p_2 = q(p_1)$. Since $q \circ \iota$ is not nullhomotopic we get that $(q \circ \iota)_* = q_* \circ \iota_*$ is not the 0-map. Now, since ι_* is the 0-map $q_* \circ \iota_*$ is also the 0-map, a clear contradiction. Thus, β connects an end of C' to the same end of C'. So β and an arc component of ∂F cobound a disk in $cl(C')$. So we may isotope β, in C, into ∂F. This isotopy will also isotope ∂D into F. Doing this isotopy of D we get $\alpha \subset \partial D$ and $\partial D \backslash \alpha \subset \partial F \subset F$. Since F is incompressible there exists a disk $D^* \subset F$ such that $\partial D^* = \partial D$. In particular, $\partial D^* \backslash \alpha \subset \partial F$ and since we have assumed that such a disk D^* did not exist we have a contradiction. We conclude that F is boundary incompressible.

Since F is boundary incompressible it follows that F is not boundary parallel. We conclude that F is essential in M. □

This lemma actually shows that every fiber of 3-manifold which fibers over S^1 is essential. We now introduce the definition of an irreducible manifold along with a useful definition and some results which relate to fibered 3-manifolds.

Definition 2.16. A 3-manifold M is irreducible if every 2-sphere $S^2 \subset M$ bounds a 3-ball $B^3 \subset M$. (See [8, p. 6])

Definition 2.17. For a disk D and a finite collection of pairwise disjoint simple closed curves $\{\gamma_i\}$ which are disjoint from ∂D we say that $\gamma \in \{\gamma_i\}$ is an innermost simple closed curve of D if there exists a disk $D' \subset D$ bounded by γ which is disjoint from any other element in $\{\gamma_i\}$. Similarly, for a 2-sphere S and a finite collection of pairwise disjoint simple closed curves $\{\gamma_i\} \subset S$ we say that $\gamma \in \{\gamma_i\}$ is an innermost simple closed curve of S if there exists a disk $D' \subset S$ bounded by γ which is disjoint from any other element in $\{\gamma_i\}$.

Lemma 2.18. If F is a connected surface with boundary and $\chi(F) \leq 1$, then $F \times I$ is irreducible.

Proof of Lemma 2.18. We induct on $-\chi(F)$. Base case: If $\chi(F) = 1$ then since $\partial F \neq \emptyset$ we get that F is a disk. Thus $F \times I$ is a 3-ball and as shown by Schultens in [18, p. 66-67], which utilizes the

Schonflies Thereom which states that every sphere in $\mathbb{R}^3$ bounds a 3-ball, we get that $F \times I$ is irreducible.

Induction hypothesis: $F \times I$ is irreducible for $-\chi(F) = k$.

Induction step: Suppose $-\chi(F) = k+1$. Recall that $\partial F \neq \emptyset$.

We consider two cases. First consider when ∂F consists of two or more components. Let η be a properly embedded arc in F with $\partial\eta$ in two distict components of ∂F. Notice that $F \backslash\backslash \eta$ is connected. See for example Figure 1.

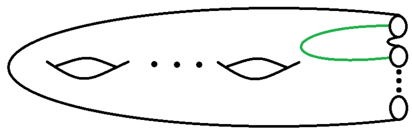

Figure 1: Non-separating essential arc for a surface with more than one boundary component.

Now consider when ∂F consists of exactly one component. Since we have already considered the case when $\chi(F) = 1$ we may assume that $\chi(F) \leq 0$. It follows that F has genus. That is, F consists of a finite connect sum of tori or projective planes along with a single boundary component. It follows that there exists an essential arc η such that $F \backslash\backslash \eta$ is connected. See for example Figure 2 and Figure 3.

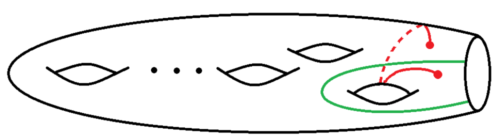

Figure 2: Non-separating essential arc with a curve connecting points on both sides of the arc for a surface with one boundary component.

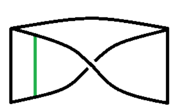

Figure 3: Non-separating essential arc in a mobius strip.

In both cases, for our choice of η, we get that $F\backslash\backslash\eta$ is connected. Since $\chi(\eta) = 1$ we get that $\chi(F\backslash\backslash\eta) = \chi(F) + 1$. Thus,

$$-\chi(F\backslash\backslash\eta) = -\chi(F) - 1 = (k+1) - 1 = k$$

and so $(F\backslash\backslash\eta) \times I$ is irreducible by our induction hypothesis. Let B be a regular neighborhood of $\eta \times I$. Notice that $cl\big((F \times I)\backslash B\big)$ is homeomorphic to $(F\backslash\backslash\eta) \times I$ and hence is irreducible.

To show that $F \times I$ is irreducible, it is sufficient to show that every sphere in $F \times I$ can be isotoped to be disjoint from $\eta \times I$. This is sufficient because then the sphere can be isotoped into $cl\big((F \times I)\backslash B\big)$ which is irreducible and hence the sphere must bound a ball both before and after the isotopy. Suppose for a contradiction that there exist one or more spheres which cannot be isotoped to be disjoint from $\eta \times I$. Of all the spheres which cannot be isotoped to be disjoint from $\eta \times I$, let S be the sphere which, after isotopy, intersects $\eta \times I$ in the fewest components. Let $\mathscr{A}$ be the components of $(\eta \times I) \cap S$. Let $\alpha \in \mathscr{A}$ be an innermost simple closed curve in $\eta \times I$. Since $\eta \times I$ is a disk it is contractible and hence there exists a disk $D \subset \eta \times I$ with $\partial D = \alpha$. Notice that there are open disks D_1 and D_2 such that $S = D_1 \cup \alpha \cup D_2$. Define $S_1 := D_1 \cup D$ and $S_2 := D_2 \cup D$. So S_1 and S_2 are spheres which may be isotoped, in the region around D, to be disjoint from $\eta \times I$ without introducing any new intersections. Thus $S_1 \cap (\eta \times I)$ and $S_2 \cap (\eta \times I)$ have fewer components than $S \cap (\eta \times I)$ and hence we may isotope S_1 and S_2 to be disjoint from $\eta \times I$ due to our assumption that S is the sphere with the fewest components of interesection with $\eta \times I$ after isotopy. It follows that S_1 bounds a 3-ball B_1 and S_2 bounds a 3-ball B_2. Since isotopy will not change the fact that S_1 and S_2 both bound 3-balls we may assume they are in their original position where $S_1 \cap S_2 = D$. It should be noted that B_1 (respectivly B_2) may a priori be on either side of S_1 (respectively S_2), however, a 3-ball only has one boundary component and hence can only be on one side of S_1 (respectively S_2). We conclude that S bounds the 3-ball $B_1 \cup B_2$. Since S bounds a 3-ball we may isotope S to be disjoint from $\eta \times I$. From our assumption about S this is a contradiction. We get that every sphere in $F \times I$ may be isotoped to be disjoint from $\eta \times I$. We conclude that $F \times I$ is irreducible which completes our induction step and the proof. $\square$

Corollary 2.19. If F is a connected surface with boundary and $\chi(F) \le 1$, then $F \times (0,1)$ is irreducible.

Proof of Corollary 2.19. Every sphere in $F \times (0,1)$ is contained in $F \times I$ and hence by Lemma 2.18 must bound a 3-ball (disjoint from the boundary).$\square$

Corollary 2.20. Let F be a connected surface with boundary and $\chi(F) \le 1$. Let $\mu : F \to F$ be a homeomorphism. Define $M := (F \times [0,1])/\sim$ where $(x,0) \ \sim \ (\mu(x),1)$ for $x \in F$. Then M is irreducible.

Proof of Corollary 2.20. Suppose for a contradiction that M is not irreducible. Then there exists a sphere $S \subset M$ which does not bound a ball. Identify F with $F \times \{0\}$. We get, from Lemma 2.15, that F is essential in M. If after isotopy $S \cap F = \emptyset$ then $S \subset M \setminus F = F \times (0,1)$. We know, from Corollary 2.19 that $F \times (0,1)$ is irreducible and hence S must bound a ball. This is a contradiction. Thus, we may assume that $S \cap F \neq \emptyset$ and that S cannot be isotoped to be disjoint from F. Suppose that out of all the spheres in M which cannot be isotoped to be disjoint from F, and hence don't bound a 3-ball, that S is the sphere which, after isotopy, intersects F in the fewest number of simple closed curves. Let $\mathscr{A}$ be this collection of simple closed curves in S. Let $\alpha \in \mathscr{A}$ be an innermost simple closed curve in S. Let D be a disk in S such that $int(D)$ is disjoint from all simple closed curves in $\mathscr{A}$ and $\partial D = \alpha$. At least one such disk D exists since α is innermost. Notice that D is a compressing disk for F. Since F is essential it is also incompressible and hence there exists a disk $D' \subset F$ such that $\partial D' = \alpha$. We first suppose that $S \cap D' = \emptyset$. In this case we see that $D \cup D'$ is a sphere in a manifold homeomorphic to the irreducible manifold $F \times (0,1)$. This follows due to the fact that $D' \subset F$ and $int(D)$ is disjoint from F. So $D \cup D'$ bounds a 3-ball and hence we may isotope D into D' and then off of F in such a way as to reduce the number of intersections of $S \cap F$. This is a contradiction since we have assumed the number of intersections is minimal. Thus, $S \cap D'$

consists of a finite collection of simple closed curves $\mathscr{B}$ in D'. Let $\beta \in \mathscr{B}$ be an innermost simple closed curve in D'. Let D'' be the disk in D' such that $\partial D'' = \beta$ and $int(D'')$ is disjoint from the simple closed curves in $\mathscr{B}$. Notice that $S \backslash \beta$ consists of two open disks, say D_1 and D_2. Notice that $S = D_1 \cup \beta \cup D_2$. Define $S_1 := D_1 \cup D''$ and $S_2 := D_2 \cup D''$. Notice that S_1 and S_2 are spheres and $S_1 \cap S_2 = D''$. Notice that S_1 and S_2 can be isotoped be be disjoint from each other and off of F near D'. This reduces the number of times they intersect F to be fewer than the number of times S intersects F. From our assumption about S we conclude that both S_1 and S_2 can be isotoped to be disjoint from F and hence are isotopic to spheres in in the irreducible submanifold $M \times (0,1) \subset M$. Thus, S_1 must bound a 3-ball B_1 and S_2 must bound a 3-ball B_2 before isotopy. It should be noted that B_1 (respectivly B_2) may a priori be on either side of S_1 (respectively S_2), however, a 3-ball only has one boundary component and hence can only be on one side of S_1 (respectively S_2). We conclude that S bounds the 3-ball $B_1 \cup B_2$. This is a contradiction since we assumed that S did not bound a 3-ball. We conclude that every sphere in M can be isotoped to be disjoint from F and by Corrollary 2.19 must bound a ball. We conclude that M is irreducible. □

Combining Lemma 2.15 with Corollary 2.20 we get that every 3-manifold M which is the total space of a fiber bundle with base S^1 and fiber a surface F with boundary is irreducible with essential fibers. Having established some basic results about irreducible 3-manifolds we move on to normal surface theory.

We now give a brief introduction to normal surface theory. We include some key definitions and finish with a couple of major results which are critical to this paper. To learn more, one may consult for example [18, Ch. 5] which includes a more in-depth discussion. To start with we need the terminology to describe a triangulation of a manifold.

Definition 2.21. The standard k-simplex in $\mathbb{R}^{k+1}$ is given by
$\Delta_k := \{(x_1, x_2, ..., x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1, x_2, ..., x_{k+1} \geq 0, x_1 + x_2 + ... + x_{k+1} = 1\}$. A k-simplex for a 3-manifold M is a continuous map $\psi : \Delta_k \to M$ such that $\psi|_{int(\Delta_k)}$ is a homeomorphism onto

its image.

For the next definition to make sense we note that two spaces are isometric if there exists a map between them which preserves distances. A map with this property is called an isometry.

Definition 2.22. For $0 \le i \le k$ we define a face, Δ_i', of the standard k-simplex, Δ_k, to be a subset of Δ_k isometric to Δ_i which fixes $k-i$ of the coordinates $x_1, x_2, ...x_{k+1}$ in Δ_k to equal 0. A face of a k-simplex $\psi : \Delta_k \to M$ is a map $\psi|_{\Delta_i'} : \Delta_i' \to M$ for any face Δ_i' of the standard k-simplex Δ_k.

Definition 2.23. A simplicial complex for a 3-manifold M is a collection of simplices for M, which we denote K, such that for all simplices $f \in K$, all faces of f are in K and for all pairs of simplices $f_1, f_2 \in K$ the interior of their images are either equal or disjoint. We define the underlying space of the simplicial complex K to be the union of the images the simplices of K. (See [18, p. 15])

Definition 2.24. Let K and L be simplicial complexes whose underlying spaces are given by UK and UL respectively. We say that a continuous map $h : UK \to UL$ is a simplicial map if for every simplex f in K, there is a simplex g in L such that $h \circ f = g$. (See [18, p. 18])

Definition 2.25. Let K and L be simplicial complexes whose underlying spaces are given by UK and UL respectively. We say that $h : UK \to UL$ is a simplicial isomorphism if it is simplicial and a homeomorphism. We say that the two simplicial complexes K and L are isomorphic if there is a simplicial isomorphism $h : UK \to UL$. (See [18, p. 19])

Definition 2.26. A map f defined from a convex set K into a vector space is said to be affine if $f(\sum \alpha_j x_j) = \sum \alpha_j f(x_j)$ when $x_j \in K$, $\alpha_j \ge 0$, and $\sum \alpha_j = 1$. (See [5, p. 151])

Definition 2.27. A triangulation of a 3-manifold M is a pair (M, K), where K is a simplicial complex for M, so that the underlying space of K is equal to M; for every compact subset C of M, the set $\{f \in K \mid C \cap im(f) \neq \emptyset\}$ is finite; and for $f, g \in K$, restricted to the interior of their domain, the map $g^{-1} \circ f$ is affine on its domain. We often denote the pair (M, K) as just $\mathscr{T}$. (See [18, p. 15])

A key component of normal surface theory is the ability to triangulate any compact 3-manifold. This fact follows from the work of Edwin E. Moise, in [14], who showed that a 3-manifold can be triangulated and R.H. Bing, in [3], who extended the result to show that all compact 3-manifolds are triangulable. We now include the basic definitions related to normal surfaces.

Definition 2.28. A properly embedded arc in a 2-dimensional face of the standard 3-simplex is a normal arc if its endpoints lie on distinct edges of of the face. A simple closed curve c in the boundary of the standard 3-simplex is a normal curve if every component of intersection of c with a 2-dimensional face of the standard 3-simplex is a normal arc. (See [18, p. 149])

Definition 2.29. The length of a normal curve in the boundary of the standard 3-simplex Δ_3 is the number of times it intersects a 1-cell of Δ_3. (See [18, p. 149])

Definition 2.30. A normal triangle in the standard 3-simplex is a properly embedded disk whose boundary is a normal curve of length 3. A normal quadrilateral in the standard 3-simplex is a properly embedded disk whose boundary is a normal curve of length 4. A normal disk is a normal triangle or quadrilateral. (See [18, p. 150-151])

Definition 2.31. Let $\mathscr{T}$ be a triangulation of a 3-manifold M. A surface F is said to be normal with respect to $\mathscr{T}$ if for every 3-simplex $\psi : \Delta_3 \to M$ in $\mathscr{T}$ we get that $\psi^{-1}(F \cap \psi(\Delta_3))$ consists of a collection normal disks or is empty.

If M is irredubile and F is an essential, hence incompressible, surface in M we can isotope F to be normal with respect to any triangulation of M. For a proof of this when F is closed see for example [18, p. 152-153]. We will furnish a proof for the similar case when F can have boundary by adapting the proof in [18, p. 152-153]. We add a couple more definitions and a lemma before we begin.

Definition 2.32. For a disk D and a finite collection of properly embedded pairwise disoint arcs $\{\alpha_i\}$ we say $\alpha \in \{\alpha_i\}$ is an outermost arc of D if one (or both) of the disks α cobounds with ∂D is disjoint from any other element in $\{\alpha_i\}$.

Definition 2.33. Let $\mathscr{T}$ be a triangulation of the 3-manifold M. Let F be a properly embedded surface in M. The weight of F with respect to $\mathscr{T}$, denoted $w(F)$, is the number of components in the intersection of F with the 1-skeleton of $\mathscr{T}$. Similary, the measure of F with respect to $\mathscr{T}$, denoted $m(F)$, is the number of components in the intersection of F with the union of the interior of the 2-faces of $\mathscr{T}$ in M.

Lemma 2.34. A closed normal curve on the boundary of a 3-simplex either has length 3 or 4 or it meets some edge more than once.

Proof of Lemma 2.34. For a proof see [18, Lemma 5.2.4, p. 149-150] □

Lemma 2.35. Let M be a compact irreducible 3-manifold with incompressible boundary and triangulation $\mathscr{T}$. If F is an incompressible and boundary incompressible surface in M, that is not a disk, then we can isotope F to be normal with respect to $\mathscr{T}$.

Proof of Lemma 2.35. Let $(w(F), m(F))$ be the weight and measure of F respectively. Let (w_1, m_1) and (w_2, m_2) be different pairs of weight and measure. We say that (w_1, m_1) is greater than (w_2, m_2) if $w_1 > w_2$ or if $w_1 = w_2$ and $m_1 > m_2$. We say that (w_1, m_1) is equal (w_2, m_2) $w_1 = w_2$ and $m_1 = m_2$. Otherwise, (w_1, m_1) is less than (w_2, m_2). We isotope F to be transverse to the triangulation $\mathscr{T}$ and so that $(w(F), m(F))$ is minimal. In particular, $w(F)$ is as small as possible. Let $\psi : \Delta_3 \to M$ be a 3-simplex of $\mathscr{T}$. Define $T := \psi(\Delta_3)$.

Suppose for a contradiction that a 2-cell Δ of T intersects F in a nonempty collection of simple closed curves. Let α be an innermost simple closed curve from this collection. Notice that α is contained in $int(\Delta)$ and hence is contained entirely in ∂M or entirely in $int(M)$. Let D be the disk in Δ that α bounds. Since F is incompressible there exists a disk D' in F such that $\partial D' = \alpha = \partial D$. Notice that $D \cup D'$ is a sphere. Since M is irreducible $D \cup D'$ bounds a 3-ball B. There are two cases to consider. Case 1 occurs if $\alpha \subset \partial M$. In this case $\partial F = \alpha$ and F is a disk which we have assumed is not the case and hence we get a contradiction. Case 2 occurs if $\alpha \subset int(M)$. Then B gives an isotopy of D' into D which we can then isotope off of Δ and this reduces $(w(F), m(F))$ by at least $(0, 1)$ which is a contradiction since we assumed $(w(F), m(F))$

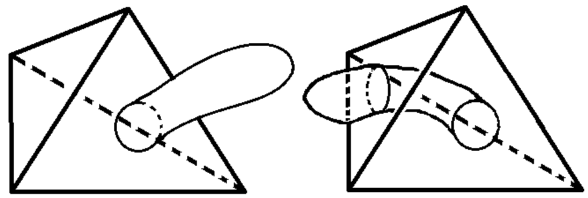

Figure 4: Examples of possible disks in F which can be isotoped off of Δ.

was minimal. See for example Figure 4. Thus, $\Delta \cap F$ consists of no simple closed curves. Since Δ was an arbitrary 2-cell for T and T was arbitrary itself so we may assume that the image of every 2-simplex in $\mathscr{T}$ does not intersect F in a simple closed curve.

Suppose for a contradiction that a 2-cell Δ of T intersects F in a nonempty collection of arcs of which at least one has endpoints in the same 1-cell l of Δ. Let η be an outermost arc in Δ with both endpoints in l. Let D be the disk in Δ cobounded by η and a component l' of l. Notice that $int(D)$ is disjoint from F. Keeping in mind that $\partial D = \eta \cup l'$ and $D \backslash l' \subset int(\Delta)$ we have 3 cases. Case 1 occurs when $D \subset int(M)$. In this case the disk D gives us an isotopy of η into l' which we can then isotope off of l to redcuce $(w(F), m(F))$ by at least $(2,1)$ which is a contradiction since we assumed $(w(F), m(F))$ was minimal. Case 2 occurs when $D \subset \partial M$. In this case D gives us an isotopy of η into l' which we can then isotope off of l to redcuce $(w(F), m(F))$ by at least $(2,1)$ which is a contradiction since we assumed $(w(F), m(F))$ was minimal. Case 3 occurs when $D \cap \partial M = l'$. In this case D is a boundary compressing disk. Since F is boundary incompressible there exists a disk $D' \subset F$ such that $D' \cap D = \eta$ and $\partial D' \backslash \eta \subset \partial F$. Notice that $D'' = D \cup D'$ is a compressing disk for ∂M. Since we have assumed that ∂M is incompressible there exists a disk $D''' \subset \partial M$ such that $D'' \cup D'''$ is a sphere in M. Since M is irreducible $D'' \cup D'''$ bounds a 3-ball. This 3-ball gives us an isotopy of D' into D which we can then isotope off of Δ and this reduces $(w(F), m(F))$ by at least $(2,1)$ which is a contradiction since we assumed $(w(F), m(F))$ was minimal. Thus, $\Delta \cap F$ consists of no arcs whose boundary points shard the same 1-cell. Since Δ was an arbitrary 2-cell for T and T was arbitrary itself we may assume that the image of every 2-face in $\mathscr{T}$ does not intersect F in arcs whose boundary points share the same 1-cell in the 1 skeleton of $\mathscr{T}$.

So far we have shown that the intersection of F with the image of any 2-simplex in $\mathscr{T}$ does not consist of simple closed curves nor does it consist of arcs whose boundary are in the image of the same 1-simplex. Thus, the intersection of F with the image of a 2-simplex is either empty or is the image of a collection of normal arcs. In particular, since F is properly embedded and transverse to $\mathscr{T}$, the intersection of F with the boundary of the image of any 3-simplex of $\mathscr{T}$ is empty or the image of a collection of normal curves. Consider again an arbitrary 3-simplex $\psi : \Delta_3 \to M$ in $\mathscr{T}$ with image given by $T := \psi(\Delta_3)$ so that $T \cap F \neq \emptyset$. Let $\alpha^* \subset \Delta_3$ be a normal curve and define $\alpha := \psi(\alpha^*) \subset \partial T$. Since α is an embedded curve and T is contractible it must bound a disk D. Suppose that F intersects $int(D)$. Let β be an innermost simple closed curve of $D \cap F$ in D. So β bounds a disk D' in D and since F is incompressible there exists another disk $D'' \subset F$ such that $\partial D' = \partial D''$. In particular, $D' \cup D''$ is a sphere in M. Since M is irreducible $D' \cup D''$ bounds a 3-ball B. Due to the existence of B we may isotope D'' into D' and then off of D without increasing $(w(F), m(F))$ and reducing the number of components in $D \cap F$. Continuing in this way we may isotope F to be disjoint from D. Notice that D is a compressing disk for F. Since F is incompressible there exists a disk $D^* \subset F$ such that $\partial D = \beta = \partial D^*$. Since $D \cup D^*$ is a sphere in an irreducible manifold we know that $D \cup D^*$ bounds a 3-ball. Thus, we may isotope D^* into D. If $int(D^*) \cap \partial T \neq \emptyset$ then this isotopy reduces $(w(F), m(F))$ which is a contradiction since we assumed $(w(F), m(F))$ is minimal. Thus, we may assume that $int(D^*) \subset int(T)$. Since T was arbitrary we may assume that the image of every normal curve in a simplex in $\mathscr{T}$ must bound a disk in F. We would like to show that these disks are the images of a normal disk.

Let $\psi : \Delta_3 \to M$ be an arbitrary 3-simplex in $\mathscr{T}$ with $T := \psi(\Delta_3)$. Assume that α^* is a normal curve in Δ_3 and define $\alpha := \psi(\alpha^*)$. By Lemma 2.34, α^* bounds a normal disk or intersects a

1-face of Δ_3 more than once. If α^* bounds a normal disk we are done, otherwise we know that $\alpha \subset T$ intersects a 1-cell of T more than once. Let $D \subset T$ be the disk in F with $\partial D = \alpha$. Since α is an embedded curve in the sphere ∂T the Jordan Curve Theorem tells us that there exist disks D_1 and D_2 in ∂T such that $\partial D_1 = \alpha = \partial D_2$ and $D_1 \cup D_2 = \partial T$. Since α intersects a 1-cell, say l, of T more than once there is an arc η which is a component of either $D_1 \cap l$ with $\partial\eta \subset \partial D_1$ or $D_2 \cap l$ with $\partial\eta \subset \partial D_2$. We may assume without loss of generality the former case. Since $int(D) \subset int(T)$ and $\partial\eta \subset \partial D$ there exists an arc $\eta' \subset D \subset F$ such that $\partial\eta' = \partial\eta$ and $int(\eta') \subset int(D)$. Since $\eta \cup \eta'$ is a simple closed curve in the contractible space T we get that $\eta \cup \eta'$ bounds a disk D^* such that $D^* \backslash \eta \subset int(T)$. Notice that $F \cap int(D^*)$ is empty or consists of simple closed curves. If $F \cap D^* \neq \emptyset$ let ζ be an innermost curve in this collection. Then ζ bounds a disk $D_\zeta \subset D^*$. Notice that D_ζ is a compressing disk for F. Since F is incompressible there exists a disk D'_ζ in F such that $\partial D_\zeta = \partial D'_\zeta$. Since M is irreducible $D_\zeta \cup D'_\zeta$ is a ball bounding sphere. Thus we may isotope D'_ζ into D_ζ and then off of D^*. We remark that since this isotopy may reduce $(w(F), m(F))$, which we assumed was minimal, it must be the case that $D_{\zeta'}$ was already contained in $int(M)$ before isotopy. We continue this process until $F \cap int(D^*) = \emptyset$. Since $\eta \subset l \subset \partial T$ and $D^* \backslash \eta \subset int(T)$ we can break up the remainder of the problem into 2 cases. Case 1 is when $\eta \subset int(M)$. In this case we use D^* to provide an isotopy of η' into η. We do this isotopy followed by an isotopy which moves η' off of l. This will reduce $w(F)$ by 2 and hence reduce $(w(F), m(F))$ which is a contradiction since we assumed that $(w(F), m(F))$ was minimal. Case 2 is when $\eta \subset \partial M$. In this case D^* is a boundary compressing disk. Since F is boundary incompressible there exists a disk D^{**} in F such that $D^* \cap D^{**} = \eta'$ and $\partial D^{**} \backslash \eta' \subset \partial F \subset \partial M$. Notice that $D^* \cup D^{**}$ is a compressing disk for ∂M. Since ∂M is incompressible there exists a disk D^{***} in ∂M such that $\partial D^{***} = \partial(D^* \cup D^{**})$. Since M is irreducible the sphere $D^* \cup D^{**} \cup D^{***}$ bounds a 3-ball. So we may isotope D^{**} into D^*. Following this isotopy we isotope D^{***} off of l. This will reduce $w(F)$ by at least 2 and hence reduce $(w(F), m(F))$ which is a contradiction since we assumed that $(w(F), m(F))$ was minimal. We conclude for both cases that α must intersect a 1-cell of T at most once and following from Lemma 2.34 we conclude that α^* bounds a normal disk in Δ_3. Since ψ, α^*, T, and α were arbitrary choices we conclude that every normal curve bounds a normal disk. In particular we get after our particular isotopy of F that F is normal with respect to the triangulation $\mathscr{T}$. $\square$

The lemma which we have just proved is useful, however, it has the downside of requiring that the boundary of the 3-manifold is incompressible. Thankfully, for the particular class of manifolds that we are interested in this will not be a problem. The following lemma will clarify what is meant by this.

Lemma 2.36. Let F be a connected surface with boundary with $\chi(F) \leq 0$. Let $\mu : F \to F$ be a homeomorphism. Define $M := (F \times [0,1])/\sim$ where $(x,0) \ \sim \ (\mu(x),1)$ for $x \in F$. Then ∂M is incompressible.

Proof of Lemma 2.36. Notice that ∂M consists of a collection of tori and Klein bottles. Suppose for a contradiction that ∂M is compressible, then there exists a component of ∂M, say S, so that S is compressible. Let $D \subset M$ be a compressing disk for S so that $D \cap S = \partial D$, but there does not exists a $D' \subset S$ such that $\partial D = \partial D'$. Consider the following maps:

- $\iota : \partial D \cong S^1 \hookrightarrow M$, the inclusion
- $q : M \to S^1$, projection onto the base of the fiber bundle

We get the following induced homomorphisms:

- $\iota_* : \pi_1(\partial D, p_1) \to \pi_1(M, p_1)$, the 0-map since ∂D bounds a disk $D \subset M$
- $q_* : \pi_1(M, p_1) \to \pi_1(S^1, p_2)$

where $p_1 \in \partial D \subset M$ and $p_2 = q(p_1)$. We consider two cases. Case 1 is when $(q \circ \iota)_* : \pi_1(\partial D, p_1) \to \pi_1(S^1, p_2)$ is a nontrivial homomorphism. Thus $(q \circ \iota)_*$ is not the 0-map. Now, since ι_* is the 0-map $(q \circ \iota)_* = q_* \circ \iota_*$ is also the 0-map, a clear contradiction. Case 2 is when $(q \circ \iota)_* : \pi_1(\partial D, p_1) \to \pi_1(S^1, p_2)$ is the trivial homomorphism and hence $q \circ \iota : \partial D \to S^1$ is nullhomotopic. In this case we may homotope ∂D to be the boundary of a fiber $F \times \{x\}$ where $x \in [0,1)$. In particular, we may homotope ∂D, in S, to be in ∂F where F identified with $F \times \{0\}$. We get from Lemma 2.15 that F is essential and hence incompressible, so there exists a disk $D^* \subset F$ so that $\partial D = \partial D^*$. Since F is connected, $D^* \subset F$, and $\partial D \subset \partial F$ we see that $F = D^*$. So $\chi(F) = 1$, however, we have assumed that $\chi(F) \leq 0$ so we have a contradiction. Since in both case 1 and case 2 we get a contradiction, we see that S is not compressible. Thus, ∂M is incompressible. □

We have established some properties of a 3-manifold M which fibers over S^1 with fiber a surface F. We have also established some of the properties of F when we identify it with $F \times \{0\}$ with S^1 thought of as $[0,1]/\sim$ where $0 \sim 1$. We would now like to understand the properties of the double of M and the double of F thought of as a submanifold of M.

Definition 2.37. Let M' be a copy of M and $\phi : M \to M'$ the homeomorphism which takes a point $x \in M$ to the corresponding point of M'. We define the double of M, DM, as $DM := (M \sqcup M')/\sim$, where $\sim$ is the equivalence relation generated by $x \sim \phi(x)$ for all $x \in \partial M$. Additionally, for F a properly embedded subsurface of M, the double of F, DF, is defined as $DF := (F \sqcup F')/\sim$ where $F' := \phi(F)$ and $\sim$ is the equivalence relation generated by $x \sim \phi(x)$ for all $x \in F \cap \partial M$.

We first remark that by our definition of the double, both DF and DM are closed manifolds. While not neccessarily critical to the our proof it is useful to have the following lemma.

Lemma 2.38. If M is an irreducible 3-manifold and ∂M is incompressible then DM is irreducible.

Proof of Lemma 2.38. Let S be a sphere in DM. Isotope S so that S intersects ∂M minimally and trasversally. From our definition of DM we can see that $M' = cl(DM \setminus M)$. If $S \cap \partial M$ is empty then $S \subset int(M)$ or $S \subset int(M')$. Since M and M' are irreducible S must bound a ball. Now if $S \cap \partial M$ is not empty then S intersects ∂M in a finite number of simple closed curves.

Claim. Together $S \cap M$ and $S \cap M'$ contain at least 2 disk components.

Proof of Claim. Firstly, since S is a sphere, none of the components of $S \cap M$ or $S \cap M'$ has genus. Secondly, since $S \cap \partial M$ is not empty no component of $S \cap M$ or $S \cap M'$ is a sphere. Now $\chi(S) = 2$ and so there exists at least two components of positive Euler characteristic 1 and these components must be disks.

Now take one of the disks from the claim and call it D. If D is isotopic to a disk in ∂M then we can isotope D so that it does not intersect ∂M, but this contradicts the minimality of S intersecting with ∂M and so we may assume that D is not isotopic to a disk in ∂M. Now if D is not isotopic to a disk in ∂M then either there is a disk D' in ∂M with $\partial D = \partial D'$ and hence $S' := D \cup D'$ is a sphere or D is a compressing disk which does not share its boundary with an embedded disk $D' \subset \partial M$. Now in the case that we get the sphere S' we know that since D is not isotopic to a disk in ∂M that S' does not bound a ball. Notice that $S' \subset M$ or $S' \subset M'$ and so by isotopy we can get $S' \subset int(M)$ or $S' \subset int(M')$. We just said that S' does not bound a ball, however, S' is a sphere in M or in M' and hence by irreducibility S' bounds a ball which leads to a contradiction. Thus we only need to concern ourselves with the case in which D is a compressing disk which does not share its boundary with an embedded disk $D' \subset \partial M$. Since $\partial D \subset \partial M$ and D is a compressing disk which does not share its boundary with an embedded disk $D' \subset \partial M$ we see that ∂M is compressible which is a contradiction since we have assumed ∂M is irreducible. This argument tells us that any sphere which intersects ∂M minimally must not intersect ∂M at all and hence by irreducibility of M and M' bound a ball. From this we see than DM is irreducible. □

We now establish that DF is essential in DM whenever F is essential in M.

Lemma 2.39. If F is an essential surface in a 3-manifold M, then DF is essential in DM.

Proof of Lemma 2.39. By the definition of the double of a manifold given in this paper it is trivially true that DF is both boundary incompressible and not boundary parallel since DM has empty boundary. So we only need to show that DF is incompressible in DM. We assume transversality for all of our interesections. Let D be an arbitrary disk in DM such that $\partial D \cap DF \subset DF$. Now $D \cap \partial M$ is empty or consists of a finite collection of simple closed curves and properly embedded arc components. If $D \cap \partial M$ is empty then rename D as D^{**} and go to section 3 of the proof, otherwise $D \cap \partial M$ is nonempty and $|D \cap \partial M| = n + k$ is nonzero with n and k natural numbers where n is the number of simple closed curve components of $D \cap \partial M$ and k is the number of arc components of $D \cap \partial M$.

Section 1. If $n = 0$ we rename D as D^* and go to section 2 of the proof, otherwise if $n \neq 0$ then there exists an innermost simple closed curve component of $D \cap \partial M$ in D which we will call γ_{in}. Now since γ_{in} is the boundary of a disk in D, whose intersection with ∂M is γ_{in}, and ∂M is incompressible it follows that γ_{in} bounds a disk, say $D_{\partial M}$, in ∂M as well. Notice now that $int(D_{\partial M}) \cap D$ is empty or consists of simple closed curve components. Figure 5 gives a partial illustration of the following argument.

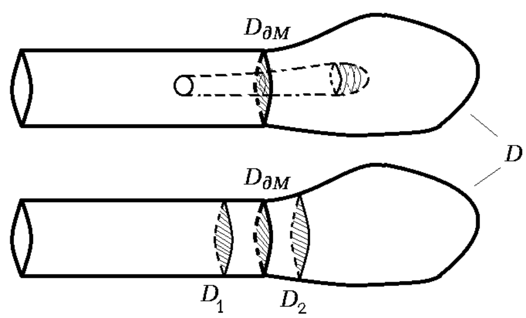

Figure 5: We give an illustration when D intersects $int(D_{\partial M})$ above and another which includes D_1 and D_2 below.

If $int(D_{\partial M}) \cap D$ is empty then we find a regular neighborhood $ND_{\partial M}$ of $D_{\partial M}$ such that $ND_{\partial M} \cap D = A$ and $\partial ND_{\partial M} = D_1 \cup D_2 \cup A$ where A is an annulus in D and D_1 and D_2 are disks with $\partial D_1 \cup \partial D_2 = \partial A$ where without loss of generallity $D_1 \subset int(M)$ while $D_2 \subset int(M')$. Define $D_{temp} := (D \setminus A) \cup D_1 \cup D_2$. Notice that D_{temp} is the disjoint union of a disk and a sphere. Let D' be the disk component of D_{temp}. Notice that $\partial D = \partial D'$, but $|D' \cap \partial M| = n - 1 + k$.

If $int(D_{\partial M}) \cap D$ is not empty then we find an innermost simple closed curve of

$int(D_{\partial M}) \cap D$ in $D_{\partial M}$. Let $D'_{\partial M}$ be the disk in ∂M bounded by this simple closed curve. Following the argument of the previous paragraph with $D_{\partial M}$ replaced with $D'_{\partial M}$ we again end up with a disk D' such that $\partial D = \partial D'$, but $|D' \cap \partial M| \leq n-1+k$.

We repeat the process above to find a disk D'' such that $\partial D' = \partial D''$ and $|D'' \cap \partial M| \leq n-2+k$. We repeat this process until we find a disk $D^{(n)}$ such that $\partial D^{(n-1)} = \partial D^{(n)}$ and $|D^{(n)} \cap \partial M| = k$. We rename $D^{(n)}$ as D^*. Notice that $D^* \cap \partial M$ only contains arc components and $\partial D^* = \partial D$.

Section 2. If $k = 0$ we rename D^* as D^{**} and go to section 3 of the proof, otherwise if $k \neq 0$ then there is an outermost arc component of $D^* \cap \partial M$ in D^* which we call α_{out}. Since α_{out} is outermost there is at least one arc, say α'_{out}, in ∂D^* such that $\partial \alpha_{out} = \partial \alpha'_{out}$, $\alpha'_{out} \subset F$ or $\alpha'_{out} \subset F'$, and α'_{out} does not intersect any components of $D^* \cap \partial M$ except for α_{out}. We assume without loss of generality that $\alpha'_{out} \subset F$. Notice that $\alpha_{out} \cup \alpha'_{out}$ form the boundary of a disk in D^*, we call this disk D^*_{out}. Since $\alpha'_{out} \subset F$ we have $D^*_{out} \subset M$. Notice that D^*_{out} is a boundary compressing disk for F in M with $D^*_{out} \cap F = \alpha'_{out}$. Thus, since F is boundary incompressible in M, there exists a disk $D^*_F \subset F$ with $\partial D^*_F = \alpha'_{out} \cup \alpha_{\partial M}$ where $\alpha_{\partial M}$ is an arc in ∂M such that $\partial \alpha'_{out} = \partial \alpha_{\partial M}$. Since D^*_F exists we isotope ∂D^*, in DF, so that $\alpha'_{out} \subset int(F')$ and $D^* \cap \partial M$ still has k components, but now exactly one of these components is a simple closed curve. Following the result of section 1 of the proof we can find a disk $D^{*(1)}$ such that $|D^{*(1)} \cap \partial M| = k-1$, $D^{*(1)} \cap \partial M$ consists only of arc components and $\partial D^{*(1)}$ is isotopic ∂D^*.

Repeating the process above we may after an appropriate isotopy of $D^{*(1)}$ find a disk $D^{*(2)}$ such that $|D^{*(2)} \cap \partial M| = k-2$, $D^{*(2)} \cap \partial M$ consists only of arc components and $\partial D^{*(2)}$ is isotopic to $\partial D^{*(1)}$. We repeat this process until we get a disk $D^{*(k)}$ such that $D^{*(k)} \cap \partial M$ is empty and $\partial D^{*(k)}$ is isotopic to $\partial D^{*(k-1)}$ in DF. We rename $D^{*(k)}$ as D^{**}.

Section 3. Notice that ∂D^{**} is isotopic to ∂D, in DF, and $D^{**} \cap \partial M$ is empty and so $D^{**} \subset M$ or $D^{**} \subset M'$. Without loss of generality we assume $D^{**} \subset M$. Notice that D^{**} is isotopic to ∂D, in DF; $D^{**} \cap DF \subset F$; and $D^{**} \subset M$. Thus, since F is incompressible as a subsurface of M, we get that D^{**} bounds a disk in F and hence D bounds a disk in DF. Since D was arbitrary we get that DF is incompressible and hence essential. $\square$

We include two more lemmas related to the double of a manifold.

Lemma 2.40. If F is a 2-sided surface in a 3-manifold M, then DF is 2-sided in DM.

Proof of Lemma 2.40. We may assume F and M are smooth. Additionally, in this case, $\phi : M \to M'$ from the definition of the double is a diffeomorphism. By smoothing DM we may assume that ϕ extends to a diffeomorphism $\Phi : DM \to DM$ such that $\Phi|_M = \phi$ and $\Phi|_{M'} = \phi^{-1}$. Let $\Phi_* : T_pDM \to T_{\Phi(p)}DM$ be the pushforward. Since F is 2-sided there is a trivial normal bundle $NF := \{(p, v) | p \in F, v \in T_pM, v \notin T_pF\}$.

To show that DF is 2-sided we take a coorientation on F and extend it a coorientation on DF. Give F a coorientation $f : F \to TM$. For an arbitrary point $p \in F$ we have $f(p) = (p, v_p)$. For this coorientation we require that whenever $p \in \partial F \subset \partial M$ that $v_p \in T_p\partial M \setminus T_pF$. We extend f to $f_{ext} : DF \to T(DM)$ as follows:

$$f_{ext}(p) := \begin{cases} (p, v_p) & p \in F \\ (p, \Phi_*(v_{\Phi(p)})) & p \in int(F') \end{cases}$$

Notice that f_{ext} agrees with the coorientation of f on ∂F. Furthermore, since f_{ext} is continuous on F it must also be continuous on F'. Thus, f_{ext} is a coorientation for DF. Now since f_{ext} agrees with f on F and in particular on ∂F and F, itself, has two distinct coorientations it must be the case that there is another distinct coorientation for DF. We conclude that DF is 2-sided.
$\square$

For the next lemma, assume that M is a triangulated 3-manifold with triangulation $\mathcal{T}$ and that $F \subset M$ is normal with respect to $\mathcal{T}$. Give M' a triangulation $\mathcal{T}'$ such that Φ sends the image of every simplex of $\mathcal{T}$ to the image of a simplex of $\mathcal{T}'$. Notice that F' will be normal with respect to $\mathcal{T}'$ since F is normal with respect to $\mathcal{T}$. Let $D\mathcal{T}$ denote the triangulation of DM using the tetrahedra from $\mathcal{T}$ and $\mathcal{T}'$. It is easy to verify that $D\mathcal{T}$ is in fact a triangulation of

DM.

Lemma 2.41. Given F normal with respect to $\mathscr{T}$ then DF is normal with respect to $D\mathscr{T}$.

Proof of Lemma 2.41. Since F is normal with respect to $\mathscr{T}$ and F' is normal with respect to $\mathscr{T}'$ we get for every 3-simplex $\psi : \Delta_3 \to DM$ in $D\mathscr{T}$ we get that $\psi^{-1}(DF \cap \psi(\Delta_3))$ consists of a collection normal disk or is empty. This holds because every simplex in $D\mathscr{T}$ is a simple in $\mathscr{T}$ or in $\mathscr{T}'$. $\square$

We conclude this section with two basic definitions which are used throughout the remainder of the paper.

Definition 2.42. Let $h : X \to Y$. Let Q be a subset of X. We say that Q is fixed under h if for all $q \in Q$ we have that $h(q) = q$. We say that Q is invariant under h if $h(Q) = Q$. We say that Q is equivariant under h if $h \circ h(Q) = Q$ and either $h(Q) = \emptyset$ or $h(Q) = Q$.

Definition 2.43. A continuous map h is nullhomotopic if it is homotopic to the identity map.

3 Curve Complex and Arc and Curve Complex

In this section we give a brief introduction to the curve complex and the arc and curve complex. For more details one may consult [18] or [17]. We begin with the definition of the curve complex of a surface.

Definition 3.1. The curve complex of a surface F, which we denote $\mathscr{C}(F)$, is a simplicial complex associated with F which relates isotopy classes of essential simple closed curves in F with a 0-simplex (or vertex) in $\mathscr{C}(F)$ and relates any collection of $k+1$ isotopy classes of essential simple closed curves with pairwise disjoint representatives in F to a k-simplex whose $k+1$ vertices in $\mathscr{C}(F)$ each correspond to one of the pairwise disjoint isotopy classes of essential simple closed curves. We define for the surface F the complexity

$$c(F) := 3g(F) + b(F) - 4$$

where $g(F)$ is the genus of F and $b(F)$ is the number of components of ∂F.

We are primarily interested in a surface, F, satisfying $c(F) > 0$. For an example of part of a simplicial complex for a genus 3 surface see Figure 6. It is shown in [18, p. 249]) that the curve complex of a surface is connected whenever the Euler characteristic of the surface is less than or equal to -2. Thus, the types of surfaces we are considering in our paper will all have a connected curve complex. An important concept of the curve complex is distance.

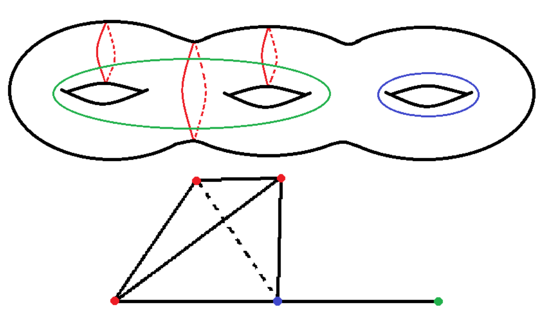

Figure 6: Part of simplicial complex for a genus 3 surface.

Definition 3.2. Let F be a surface. The distance between two vertices v and w in $\mathscr{C}(F)$ is the minimal number of edges in the 1-skeleton of $\mathscr{C}(F)$ required to connect v with w.

It has been shown in [13], for an oriented surface, that for any vertex v in the curve complex there exist another vertex which can have arbitrarily large distance from v. Due to this, we say that the curve complex has infinite diameter.

In addition to the curve complex we are also interested in the arc and curve complex.

Definition 3.3. The arc and curve complex of a surface F, which we denote $\mathscr{AC}(F)$, is a simplicial complex associated with F which relates each isotopy class of essential simple closed curves and each isotopy class of essential simple arcs in F with a 0-simplex (or vertex) in $\mathscr{AC}(F)$. Furthermore a k-simplex in $\mathscr{AC}(F)$ is associated to a collection of $k+1$ isotopy classes of essential simple closed curves and essential simple arcs with pairwise disjoint representatives in F.

It is worth pointing out that for a surface F without boundary $\mathscr{C}(F)$ and $\mathscr{AC}(F)$ are equivalent. Since we are primarily interested in surfaces with boundary this fact is not so useful to us, however, we can relate the two complexes in another way. First we define the translation distance. Let $\mu : F \to F$ be a homeomorphism and $\mu_* : \mathscr{AC}(F) \to \mathscr{AC}(F)$ be the isometry on $\mathscr{AC}(F)$ induced by μ. We define the translation distance of μ in $\mathscr{AC}(F)$ to be $min\{d(v, \mu_*(v)) \mid v \in \mathscr{AC}(F) \text{ is a vertex}\}$ where d is the distance between vertices in the arc and curve complex. Translation distance also makes sense for the curve complex.

Lemma 3.4. Let F be a connected surface with $\chi(F) \leq -16$ and let $\mu : F \to F$ be a homeomorphism. If the translation distance of μ in $\mathscr{AC}(F)$ is at most 1 then the translation distance of μ in $\mathscr{C}(F)$ is at most 2.

Proof of Lemma 3.4. If the translation distance of μ in $\mathscr{AC}(F)$ is at most 1 then there is an essential simple closed curve or essential simple arc $\alpha \subset F$ so that α is disjoint from $\mu(\alpha)$ (possibly after isotopy). It may be the case that α and $\mu(\alpha)$ are in the same isotopy class. We have 2 cases to consider.

Case 1 is when α is an essential simple closed curve. Since α and $\mu(\alpha)$ are essential simple closed curve they are represented by vertices in $\mathscr{C}(F)$ (possible the same one). Thus, as α and $\mu(\alpha)$ are disjoint (possibly after isotopy) the translation distance of μ in $\mathscr{C}(F)$ is at most 1.

Case 2 is when α is an essential simple arc. In this case α and $\mu(\alpha)$ are both essential simple arcs disjoint from each other (possibly after isotopy). Essential simple arcs are not represented in $\mathscr{C}(F)$ and so we need a different strategy than in the previous case. Our goal is to show that we can use α to find an essential simple closed curve β with the property that β and $\mu(\beta)$ intersect in at most 4 points (possibly after isotopy). First we let c be the disjoint union of the boundary components (or component) of F which intersect $\partial\alpha$. So c contains at most 2 components. Now let $N(c \cup \alpha)$ be a regular neighborhood of $c \cup \alpha$ in F. At least one component of $\partial N(c \cup \alpha)$ is not a component of ∂F; we let β be one such component (it does not matter which).

We have 6 subcases to consider. Let c_1, c_2, c_3, and c_4 be an arbitrary collection of components from ∂F. Note that some subcases may not apply to F if it has fewer than 4 boundary components. The following subcases are sufficient since we do not care about the arrangement of the components of ∂F. We first consider subcases in which $\partial\alpha$ is contained in a single component of ∂F. Subcase 1 is when $\partial\alpha \subset c_1$ and $\partial\mu(\alpha) \subset c_1$ and as we traverse c_1 we alternate intersecting $\partial\alpha$ and $\partial\mu(\alpha)$, see Figure 7. Subcase 2 is when $\partial\alpha \subset c_1$ and $\partial\mu(\alpha) \subset c_1$ and as we traverse c_1 we intersect $\partial\alpha$ twice and then intersect $\partial\mu(\alpha)$ twice after picking an appropriate starting point, see Figure 8. Subcase 3 is when $\partial\alpha \subset c_1$ and $\partial\mu(\alpha) \subset c_2$, see Figure 9. We now consider subcases in which $\partial\alpha$ is contained in distinct components of ∂F. Subcase 4 is when one component of $\partial\alpha$ is contained in c_1 and the second is contained in c_2 while one component of $\partial\mu(\alpha)$ is contained in c_1 and the second is contained in c_2, see Figure 10. Subcase 5 is when one component of $\partial\alpha$ is contained in c_1 and the second is contained in c_2 while one component of $\partial\mu(\alpha)$ is contained in c_2 and the second is contained in c_3, see Figure 11. Subcase 6 is when one component of $\partial\alpha$ is contained in c_1 and the second is contained in c_2 while one component of $\partial\mu(\alpha)$ is contained in c_3 and the second is contained in c_4, see Figure 12.

The figures associated to subcase 1 through subcase 6 show, up to isotopy, all the possible ways in which we can obtain β and $\mu(\beta)$ given α. The figures take advantage of the fact that μ is a homeomorphism and hence must map the regular neighborhood $N(c \cup \alpha)$ to a regular neighborhood of $\mu(c \cup \alpha)$, limiting our choice of $\mu(\beta)$ which must be a boundary component of this regular neighborhood. The figures only show the minimum required number of boundary components from F. The possible intersections between β and $\mu(\beta)$ can, after isotopy, always be contained in an annulis in F which is cobounded by a component of ∂F and a boundary parallel simple closed curve and hence we only show the interesting region near ∂F in our figures. From Figure 7 to Figure 12 we can see that depending on our α it is only possibly for β and $\mu(\beta)$ to intersect at most 4 times after an intersection minimizing isotopy. We isotope β and $\mu(\beta)$ to intersect minimally.

Let $K = N(\beta \cup \mu(\beta))$ be a regular neighborhood of $\beta \cup \mu(\beta)$. We get

$$0 \geq \chi(K) = \chi(\beta \cup \mu(\beta)) \geq -4.$$

Now ∂K consists of simple closed curves which have Euler characteristic 0 and so

$$-16 \geq \chi(F) = \chi(K) + \chi(F \backslash K).$$

Therefore,

$$\chi(F \backslash K) \leq -12.$$

Now, since F is connected and $\beta \cap \mu(\beta)$ consists of at most 4 points we see that $F \backslash K$ consists of at most 6 components. Since the Euler characteristic is additive over disjoint unions there is a component K' of $F \setminus K$ so that

$$\chi(K') \leq \frac{-12}{6} \leq -2.$$

Since $K' \leq -2$ there is an essential simple closed curve $\beta' \subset K'$, moreover, since $\beta, \mu(\beta) \subset K$ it must be the case that β and $\mu(\beta)$ are disjoint from β'. We conclude that the translation distance of μ in $\mathscr{C}(F)$ is at most 2. $\square$

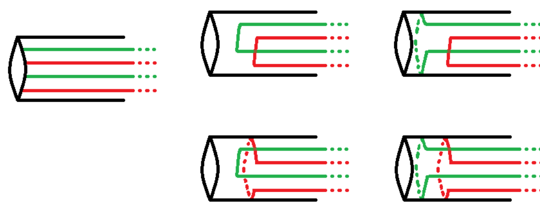

Figure 7: Subcase 1 with α, $\mu(\alpha)$ on the left and possible β, $\mu(\beta)$ on right.

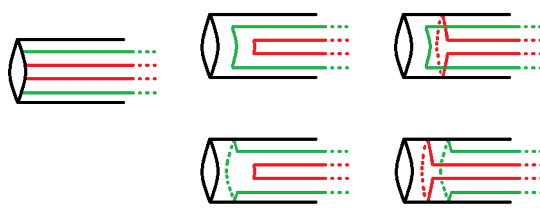

Figure 8: Subcase 2 with α, $\mu(\alpha)$ on the left and possible β, $\mu(\beta)$ on right.

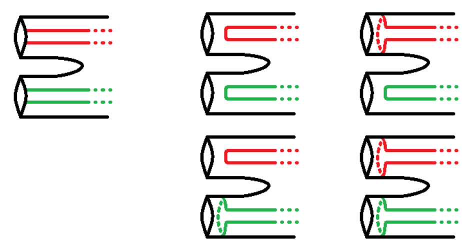

Figure 9: Subcase 3 with α, $\mu(\alpha)$ on the left and possible β, $\mu(\beta)$ on right.

Figure 10: Subcase 4 with α, $\mu(\alpha)$ on the left and possible β, $\mu(\beta)$ on right.

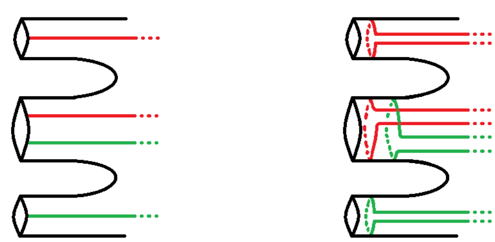

Figure 11: Subcase 5 with α, $\mu(\alpha)$ on the left and possible β, $\mu(\beta)$ on right.

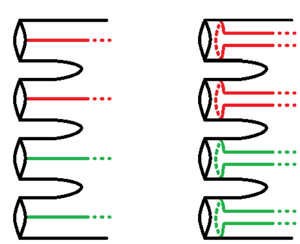

Figure 12: Subcase 6 with α, $\mu(\alpha)$ on the left and possible β, $\mu(\beta)$ on right.

4 Proof

Introduction

Our goal is to generalize the work of Ichihara, Kobayashi, and Rieck in [10]. The statement of their main result given in terms of the Euler characteristic is:

Theorem 4.1. Let M be a connected tame 3-manifold that admits a triangulation using $t(M)$ tetrahedra and $F \subset M$ a 2-sided connected essential closed surface. If $\chi(F) \leq -76t(M) + 2$, then F is strongly cylindrical.

We would like to have a similar result for an essential surface F with boundary. We cannot achieve a result as general as in [10]. Instead we focus on the case when M is the total space of a fibered manifold with base space the circle. We prove Theorem 1.1, as stated below, when F is a surface with boundary; the closed case is due to Ichihara, Kobayashi, and Rieck in [10].

Theorem 1.1. Let F be a connected surface, let $\mu : F \to F$ be a homeomorphism, and let $M := (F \times [0,1])/\sim$ where $(x,0) \sim (\mu(x),1)$ for $x \in F$. Let $t(M)$ be the minimal number of tetrahedra required to triangulate M. If $\chi(F) \leq -76t(M) + 1$ then there is an essential simple closed curve or essential simple arc which is disjoint from its image under μ (possibly after isotopy).

To help achieve this, we will take the double of the manifold M described in the statement of Theorem 1.1 and apply Proposition 11 from the proof of [10]. The statement of Proposition 11 is as follows: there exists a pair of pants $X \subset int(Y)$ or $X \subset int(B)$ so that ∂X is essential in F.

Set up for Proof of Main Theorem

Let F and M be as in Theorem 1.1. Throughout this section we associate F with $F \times \{0\}$. It follows from our construction that F is 2-sided in M and we get from Lemma 2.15 that F is essential in M. Notice that M cut open along F is homeomorphic to $F \times I$; this fact is important later on in the proof.

Let $\mathscr{T}$ be a minimal triangulation of M such that the number of tetrahedra in $\mathscr{T}$ is equal to $t(M)$, where $t(M)$ represents the minimal number of tetrahedra required to triangulate M.

Using Lemma 2.36 we get that ∂M is incompressible. Applying Lemma 2.35 we isotope F to be normal with respect to $\mathcal{T}$. In the following discussion of tetrahedra in a triangulation we will be a bit informal in describing whether we are in the domain of a 3-simplex or in the image of a 3-simplex. Notice that the intersection of F with the tetrahedra of $\mathcal{T}$ gives a decomposition of F into normal disks (normal triangles and normal quadrilaterals). Let N be a regular neighborhood of the 1-skeleton of $\mathcal{T}$. Consider the components of $N \cap F$. These components consist of disks disjoint from ∂F which we call vertex disks and disks having a single arc in ∂F which we call half-vertex disks. For every normal disk D_{normal} we take $cl(D_{normal} \backslash N)$ and call this a truncated normal disk. In this way normal triangles become truncated normal triangles (which are hexagons) and normal quadrilaterals become truncated normal quadrilaterals (which are octagons). Notice that F has a decomposition which consists of truncated normal disks, vertex disks, and half-vertex disks. We shall call the components of this decomposion the faces of F.

As in [10] we use the idea of parallel families. In this case we have 4 distinct types of parallel families. First consider the edges of the 1-skeleton of $\mathcal{T}$. The intersection of an edge of the 1-skeleton of $\mathcal{T}$ with the faces of F will either intersect a collection of vertex disks or half-vertext disks. When this edge is contained in ∂M then this collection is empty or consists of half-vertex disks and hence we call this a parallel family of half-vertex disks. Otherwise, for an edge whose interior is disjoint from ∂M the edge intersects a collection of vertex disks or is empty and this collection is called a parallel family of vertex disks. From normal surface theory there are four types of normal triangles and one type of normal quadrilateral in a given tetrahedron. A type of normal triangle is determined by which vertex is separated from the other three vertices by the normal triangle. We only have one type of normal quadrilateral since if we had more they would intersect each other. It should be noted that we can have many normal disks of the same type in a given tetrahedron. We have in total five types of normal disks in each tetrahedron and since our truncated normal disks are contained in our normal disks we define the type of our truncated normal disk to correspond to the type of the normal disk it is contained in. We call a collection of truncated normal triangles of the same type a parallel family of truncated normal triangles and we call a collection of truncated normal quadrilaterals of the same type a parallel family of truncated normal quadrilaterals. Thus we have four families of truncated normal triangles and one family of truncated normal quadrilaterals in each tetrahedron. It should be noted that a parallel family may be empty.

We define a *RBY* coloring of a normal surface in a triangulated 3-manifold to be a coloring of the faces of the surface such that in each parallel family the outermost faces are colored red and the remaining faces are colored alternately yellow and blue. We color F in this way. Thus at most two faces are red in each parallel family. Let R be the union of red faces, B the union of blue faces, and Y the union of yellow faces of F. Let G be the union of the boundary of the faces of F. By construction G is a trivalent graph and so R, B, and Y are subsurfaces of F.

Let M' be a copy of M and $\phi : M \to M'$ the homeomorphism which takes a point $x \in M$ to the corresponding point of M'. We define the double of M, DM, as $DM := (M \sqcup M')/\sim$, where $\sim$ is the equivalence relation generated by $x \sim \phi(x)$ for all $x \in \partial M$. Additionally, for F in M, the double of F, DF, is defined as $DF := (F \sqcup F')/\sim$ where $F' := \phi(F)$ and $\sim$ is the equivalence relation generated by $x \sim \phi(x)$ for all $x \in F \cap \partial M$. There is a natural map, which we will call the symmetric map, $\Phi : DM \to DM$ such that $\Phi|_M = \phi$ and $\Phi|_{M'} = \phi^{-1}$. Notice that both DM and DF are invariant under Φ and that ∂M and $\partial F = F \cap \partial M$ are fixed under Φ. Since M has a decomposition into tetrahedra and F has a decomposition into faces we give M' a triangulation $\mathcal{T}'$ such that Φ sends a tetrahedron of $\mathcal{T}$ to a tetrahedron of $\mathcal{T}'$ and Φ sends a face $f \subset F$ to a face $\Phi(f) \subset F'$. We color the face $\Phi(f)$ the same color as the face f. For the purpose of convenience we will use the notation $D\mathcal{T}$ to denote the triangulation of DM using the tetrahedra from $\mathcal{T}$ and $\mathcal{T}'$. It is easy to see that $D\mathcal{T}$ is in fact a triangulation of DM since in the double of M we force the faces, edges, and vertices of the tetrahedra of $\mathcal{T}$ which intersect ∂M to match up exactly with a corresponding face, edge, or vertex of a tetrahedra of $\mathcal{T}'$.

From the above argument we have a RBY coloring of the surface DF, with respect to the triangulation $D\mathcal{T}$, which is invariant under Φ. For simplicity we reuse the notation R, B, and Y to represent the union of red, blue, and yellow faces of DF respectively. It is easy to see that $G \cup \Phi(G)$ is the union of the boundary of the faces of DF, however, $G \cup \Phi(G)$ is not trivalent. We replace every half-vertex disk $D_{half} \subset F$ and its corresponding half-vertex disk $\Phi(D_{half}) \subset F'$ with the disk $D_{half} \cup \Phi(D_{half})$. Notice that our new disks are vertex disks. Moreover these new vertex disks are invariant under Φ and have the same color as the two half-vertex disks they contain. Each of these new vertex disks lie in a parallel family of vertex disks where a collection of these vertex disks lie in the same parallel family if its associated half-vertex disks in F lie in the same parallel family. We consider these new vertex disks to be faces of DF. We now have a new decomposition of DF into faces which are truncated normal triangles, truncated normal quadrilaterals, and vertex disks. Our faces no longer include half-vertex disks. Let G' be the union of the boundary components of the faces of DF under our new decomposition of faces; then G' is a trivalent graph and R, B, and Y are subsurfaces of DF. We still have a RBY coloring of DF with the new decomposition of faces.

Since the Euler characteristic is additive over disjoint unions and the intersection of the two copies of F in the double is along simple closed curves which have Euler characteristic zero we get that $\chi(DF) = 2\chi(F)$. Using $t'(DM)$ to represent the number of tetrahedra in $D\mathcal{T}$, with $D\mathcal{T}$ as described above, we see that $t'(DM) = 2t(M)$. Using the hypothesis of Theorem 1.1, that $\chi(F) \leq -76t(M) + 1$, along with the facts above we get the inequality $\chi(DF) \leq -76t'(DM) + 2$.

Lemma 4.2. There exists a RBY coloring of DF invariant under Φ for which there is an essential pair of pants $X \subset DF$ contained in $int(B)$ (or in $int(Y)$).

Proof of Lemma 4.2. We remark that a compact 3-manifold is always tame. Since $\chi(DF) \leq -76t'(DM) + 2$, as seen above; DM is a connected compact 3-manifold with a triangulation of $t'(DM)$ tetrahedra; and $DF \subset DM$ is a 2-sided connected essential closed surface, which follows from Lemma 2.40 and Lemma 2.39; it is clear that $DF \subset DM$ satisfies the hypothesis of the main theorem in [10]. We make four observations. The first observation is that DF is already normal with respect to the triangulation $D\mathcal{T}$ and hence we do not need to isotope it, this follows from Lemma 2.41. The second observation is that any RBY coloring of DF is sufficient for the result of [10]. The third observation is that in the proof of the main theorem in [10] there is a possible need to swap the coloring of every blue and yellow vertex disk. That is, we may need to recolor every blue vertex disk to be yellow and recolor every yellow vertex disk to be blue. Since our RBY coloring of DF is invariant under Φ we will still have an invariant RBY coloring of DF if we swap every blue and yellow vertex disk. So we do not have an issue here and can use our RBY coloring of DF or one with the blue and yellow vertex disks swapped. The fourth observation is that we do not require a minimal triangulation of DM and so $D\mathcal{T}$ is good enough. Now, rather than take the result from the main theorem in [10] we take the result from [10, Prop. 11] to get that there is an essential pair of pants $X \subset DF$ in $int(B)$ (or in $int(Y)$). □

We now want to consider some special pairs of pants that we can have in DM. In particular, we give a definition of the types of pairs of pants that we are interested in and prove some useful properties that they have.

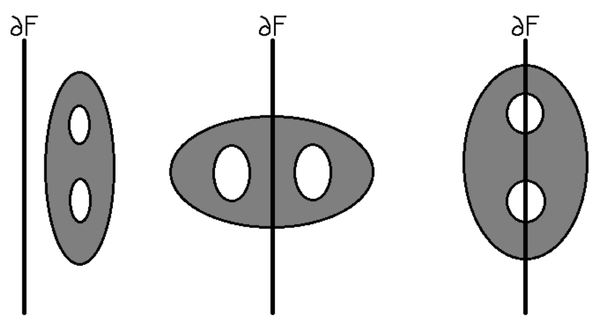

Figure 13: Example of pairs of pants of type T_0, T_1, and T_3 in that order.

Definition 4.3. We define three types of equivariant under Φ pairs of pants in DF. An equivariant pair of pants X is of type T_0 if $X \cap \Phi(X)$ is empty. An equivariant pair of pants X is of type T_1 if there is exactly one component c of ∂X such that $c = \Phi(c)$. Finally, an equivariant pair of pants X is of type T_3 if for each component c of ∂X we get $c = \Phi(c)$. See Figure 13.

Remark. In the case when X is a pair of pants of type T_1 we have that the two components c_1 and c_2 of ∂X which do not map to themselves under Φ have the property $\Phi(c_1) = c_2$ and hence $\Phi(c_2) = c_1$.

Notice that if X is equivariant under Φ then either $\Phi(X) = X$ or $\Phi(X) \cap X$ is empty.

Lemma 4.4. Let X be a pair of pants equivariant under Φ, then X is of type T_0, T_1, or T_3.

Proof of Lemma 4.4. Let c_1, c_2, and c_3 be the boundary components of X. We break this problem into cases based on the number of invariant boundary components of X under Φ.

Case 1: No component of ∂X is invariant under Φ. Suppose for a contradiction that $\Phi(X) = X$. Since $\Phi|_X$ is a homeomorphism it maps boundary components to boundary components and the only way to do this without an invariant boundary component is that without loss of generality $\Phi(c_1) = c_2$, $\Phi(c_2) = c_3$, and $\Phi(c_3) = c_1$ which contradicts the fact that $\Phi \circ \Phi$ is the identity. Thus X and $\Phi(X)$ are disjoint and X is a pair of pants of type T_0.

Case 2: Exactly one component of ∂X is invariant under Φ. Then X is by definition a pair of pants of type T_1.

Case 3: Exactly two components of ∂X are invariant under Φ. We show that no such X exists. Suppose for a contradiction that without loss of generality $\Phi(c_1) = c_1$ and $\Phi(c_2) = c_2$. Since X is equivariant under Φ and $\Phi|_X$ is a homeomorphism, and hence maps boundary components to boundary components, then $\Phi(c_3) = \Phi(c_3)$. So every component of ∂X is invariant under Φ which is a contradiction. Thus there are no equivariant pairs of pants with exactly two invariant boundary components.

Case 4: Every boundary component of X is invariant under Φ. By defintion X is a pair of pants of type T_3. This completes the proof of the lemma. □

Description of Opp map

We construct a special map, similar to the one in [10], $Opp\colon V \to DF$ where V is a connected subsurface of DF contained in $int(B)$ or in $int(Y)$. We show that Opp is injective. Moreover, we show that for $V \subset int(B)$ then $Opp(V) \subset F\backslash B$ or for $V \subset int(Y)$ then $Opp(V) \subset F\backslash Y$. We construct the map below.

This section is very technical and hence we clarify our use of terminology. A vertex disk cannot, in general, be contained in the preimage of a single 3-simplex. So when we refer to a vertex disk we are refering to a subsurface in DM. Thus a parallel family of normal disk is contained in DM. In contrast a normal disk is, by definition, contained in the preimage of a single 3-simplex. So we consider a normal disk as an object in the standard 3-simplex. We are primarily interested in the faces of F, however, and want to think about them all as subsurfaces of DM. Our decomposition of DF into faces consists of vertex disks, truncated normal triangles, and truncated normal quadrilaterals. We already consider a vertex disk as a subsurface of DM. Similarly, in this section we will consider our truncated normal disks as subsurfaces of DM. Thus, a parallel family of faces is contained in DM. A parallel family of truncated normal disks has the additional propery of being properly embedded in the image of a 3-simplex of $D\mathcal{T}$.

Let $\{\psi_j\}_{j=1}^{j=2t(M)}$ be the 3-simplices of $D\mathcal{T}$. Then $\psi_j : \widehat{T} \to M$ where $\widehat{T}$ is the standard 3-simplex in $\mathbb{R}^4$. If we refer to a tetrahedron of $D\mathcal{T}$ we are refering to one of the pairs $(\widehat{T}, \psi_j)$ where $1 \leq j \leq 2t(M)$.

We define a partial vertex disk to be the intersection of a vertex disk with the image of $\widehat{T}$ under a 3-simplex of $D\mathcal{T}$ and a parallel family of partial vertex disks to be the intersection of a parallel family of vertex disks with the image of $\widehat{T}$ under a 3-simplex of $D\mathcal{T}$. In this way we can decompose every vertex disk into partial vertex disks. Let $[F]$ be a nonempty parallel family of truncated normal triangles, truncated normal quadrilaterals, or partial vertex disks with ordered faces $F_1, ..., F_n$ and $n \geq 3$. We define $[\widehat{F}] \subset \widehat{T}$ to be $\psi^{-1}([F])$ where ψ is a simplex in $D\mathcal{T}$ such that $[F] \subset \psi(\widehat{T})$. The components of $[\widehat{F}]$ are given by $\widehat{F}_1 = \psi^{-1}(F_1), ..., \widehat{F}_n = \psi^{-1}(F_n)$ which we call the faces of $[\widehat{F}]$. Let $\widehat{T}^{(1)}$ be the 1-skeleton of $\widehat{T}$ and let $\widehat{T}^{(2)}$ be the 2-skeleton of $\widehat{T}$. For

$1 \le i \le n$ we give the face $\widehat{F}_i \subset [\widehat{F}]$ a cell decomposition as follows:

- 0-cells: components of $\left(\partial\widehat{F}_i \backslash \left(int(\partial\widehat{F}_i \cap \widehat{T}^{(2)}) \cup int(\widehat{T})\right)\right) \cup \left(\partial\widehat{F}_i \cap \partial\widehat{T}^{(1)}\right)$

- 1-cells: components of $\partial\widehat{F}_i \backslash$(0-cells)

- 2-cell: $int(\widehat{F}_i)$

Recall from earlier that N is a regular neighborhood of the 1-skeleton of $\mathscr{T}$. By this we mean that N is a regular neighborhood of the underlying space of the 2-simplices of $\mathscr{T}$. We define $DN := N \cup \Phi(N)$. Notice that DN is a regular neighborhood of the underlying space of the 2-simplices of $D\mathscr{T}$.

When $[F]$ is a parallel family of truncated normal triangles (or quadrilaterals), with at least 3 faces, assume $(\widehat{T}, \psi)$ is the tetrahedron of $D\mathscr{T}$ so that $[F] \subset T$ where $T := \psi(\widehat{T})$ and define $\widehat{C}$ to be the closure of the component of $\left(\widehat{T} \backslash \psi^{-1}(DN \cap T)\right) \backslash \{$outermost faces of $[\widehat{F}]\}$ which intersects $[\widehat{F}]$ nontrivially.

When $[F]$ is a parallel family of partial vertex disks, with at least 3 faces, assume $(\widehat{T}, \psi)$ is the tetrahedron of $D\mathscr{T}$ so that $[F] \subset T$ where $T := \psi(\widehat{T})$ and define $\widehat{C}$ to be the closure of the component of $\psi^{-1}(DN \cap T) \backslash \{$outermost faces of $[\widehat{F}]\}$ which intersects $[\widehat{F}]$ nontrivially.

In both cases $\widehat{C}$ is homeomorphic to $D \times I := C$ where D is a disk simplicially isomorphic to the faces of $[\widehat{F}]$ and $I := [0,1]$. Let $p : D \times I \to D$ be the natural projection. Let $D^{(1)}$ be the 1-skeleton of D and let $D^{(2)}$ be the 2-skeleton of D. Let $h : C \to \widehat{C}$ be a homeomorphism such that for $0 = p_1 < \ldots < p_i < p_{i+1} < \ldots < p_n = 1$ and $\widehat{F}_1, \ldots, \widehat{F}_i, \widehat{F}_{i+1}, \ldots, \widehat{F}_n$, the faces of $[\widehat{F}]$, we have $h|_{D \times \{p_i\}} : D \times \{p_i\} \to \widehat{F}_i$ and:

- For each $q \in D^{(1)} \backslash D^{(0)}$ we have that $h(\{q\} \times I)$ is contained in the interior of a 2-face of $\widehat{T}$ or in the interior of $\widehat{T}$.

- We require the following to hold for all 1-cells l of D and for all $s \in I$. Let $\hat{l}$ be the image under $h \circ (p|_{D \times \{s\}})^{-1}$ of a 1-cell l of D and $\hat{v}$ a boundary point of $\hat{l}$. Define $v := p \circ h^{-1}(\hat{v})$.

Notice that v is a boundary component of l. We require for t a point of distance d from v in l that $h \circ (p|_{D\times\{s\}})^{-1}(t)$ is of distance $\frac{d * length(\hat{l})}{length(l)}$ from $\hat{v}$ in $\hat{l}$. We may assume that C lies in $\mathbb{R}^3$. Thus, since $\widehat{T}$ is the standard 3-simplex in $\mathbb{R}^4$ the notion of length above makes sense.

- Finally for all q in $D^{(1)}$ define $l_q := q \times [0,1]$ and $\hat{l}_q := h(l_q)$. We require that for t a point in l_q of distance d from $q \times \{0\}$ in l_q that $h(t)$ is of distance $\frac{d * length(\hat{l}_q)}{length(l_q)}$ from $h(q \times \{0\})$ in $\hat{l}_q$.

Let U be a, not necessarily connected, subsurface in a blue (or yellow) face $F_i = \psi(\widehat{F}_i)$ of the parallel family $[F]$ above and $U \subset int(B)$ (or $U \subset int(Y)$) where $[F]$ is a parallel family of truncated normal triangles, truncated normal quadrilaterals, or partial vertex disks. We define for all $u \in U$ the points

$$opp_{\pm}(u) := \psi \circ h \circ (p|_{D\times\{p_{i\pm 1}\}})^{-1} \circ p \circ h^{-1} \circ \psi^{-1}(u)$$

and define the map $Opp|_U : U \to DF$ either to be $Opp|_U : u \mapsto opp_+(u)$ for all $u \in U$ or to be $Opp|_U : u \mapsto opp_-(u)$ for all $u \in U$ so that $Opp|_U : U \to DF$ agrees with the coorientation of DF in DM. Notice that $Opp|_U$ sends $U \subset F_i$ either to F_{i-1} or to F_{i+1}. It is easy to see that $Opp|_U$ is injective and hence a homeomorphism.

We would like to define an Opp map for a connected subsurface $V \subset int(B)$ (or $V \subset int(Y)$) such that $Opp : V \to DF$ is a homeomorphism onto its image and behaves similar to $Opp|_U$ when Opp is restricted to a face of DF. We now think of the faces of DF to be truncated normal triangles, truncated normal quadrilaterals and partial vertex disks. Let $\widetilde{F}_1, \dots, \widetilde{F}_w$ be the faces of DF whose interiors have nonempty intersection with V and define for $1 \le i \le w$ that $F_i^* := \widetilde{F}_i \cap V$. Simlar to $Opp|_U$ we have for each F_i^* a map $Opp|_{F_i^*}$ which sends $F_i^* \subset \widetilde{F}_i$ into the face directly above or below $\widetilde{F}_i$ in the parallel family containing $\widetilde{F}_i$ consistent with the coorientation of DF in DM. We define $Opp: V \to DF$ so that

$$Opp(x) := \begin{cases} Opp|_{F_1^*}(x) & x \in F_1^* \\ Opp|_{F_2^*}(x) & x \in F_2^* \\ \vdots & \vdots \\ Opp|_{F_w^*}(x) & x \in F_w^* \end{cases}$$

for all x in V.

By our construction of $Opp: V \to DF$ it is clear that for $V \subset int(B)$ (or $V \subset int(Y)$) that $Opp(V) \subset DF \backslash B$ (or $Opp(V) \subset DF \backslash Y$). We finish this section by proving that Opp is injective.

Remark. Our description of Opp is defined for an arbitrary connected subsurface of DF which is contained in $int(B)$ or in $int(Y)$ and hence we may have multiple Opp maps so long as we specify an appropriate domain. Additionally, for a given domain Opp is well-defined.

Lemma 4.5. For a connected subsurface $S \subset DF$ which is contained in $int(B)$ or in $int(Y)$, Opp is injective and hence a homeomorphism onto its image.

Proof of Lemma 4.5. Suppose without loss of generality that $S \subset int(B)$. By our comment above, S is disjoint from $Opp(S) \subset DF \backslash B$. Suppose for a contradiction that Opp is not injective. Then there exists distinct points $s_1, s_2 \in S$ so that $Opp(s_1) = Opp(s_2)$. Since $Opp(s_1) = Opp(s_2)$ it must be the case that s_1, s_2, and $Opp(s_1)$ are in the same parallel family; let $F_1, ..., F_n$ be the faces of this parallel family. It is not hard to see that if $Opp(s_1) \in F_i$ then without loss of generality $s_1 \in F_{i-1}$ and $s_2 \in F_{i+1}$. Let Ns_1 be a regular neighborhood of s_1 in $S \cap F_{i-1}$ and Ns_2 be a regular neighborhood of s_2 in $S \cap F_{i+1}$. Define $Q_1 := Opp(Ns_1)$ and $Q_2 := Opp(Ns_2)$. Recall that Opp is injective on a single face of DF and hence Q_1 and Q_2 are homeomorphic to Ns_1 and Ns_2 respectively.

There is a natural fiber structure between adjacent parallel faces in a given parallel family; in particular we define a fiber I_p to be the fiber which has initial point $p \in S$ and end point $Opp(p)$. We define an induced coorientation of $Opp(S)$, by S, under Opp to be the

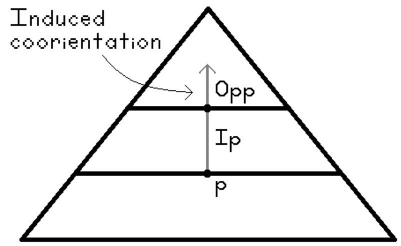

Figure 14: Illustration of induced coorientation of *Opp*(S), by S, under *Opp* relative to I_p.

coorientation so that for every point $p \in S$ the induced coorientation points in the direction opposite of I_p. See Figure 14. Since S is a connected subsurface and *Opp* is continuous, we have that the induced coorientation on *Opp*(S) either agrees with the coorientation of *DF* for all points in *Opp*(S) or disagrees with the coorientation of *DF* for all points in *Opp*(S); we assume without loss of generality that the induced coorientation on *Opp*(S) agrees with the coorientation on *DF*.

Using the notion of an induced coorientation that we just defined there is an induced coorientation of Q_1, by Ns_1, under *Opp* and an induced coorientation of Q_2, by Ns_2, under *Opp*. Notice that the induced coorientations point in opposite directions, particularly at *Opp*(s_1), and hence one of our induced coorientations doesn't agree with the coorientation of *DF*, a contradiction. We conclude that *Opp* is injective and hence a homeomorphism onto its image. □

Main Lemma

Before we prove the Main Lemma we prove a useful lemma and corollary. We use $|\cdot|$ to represent the number of components of.

Lemma 4.6. A simple closed curve in *DF* which is invariant under Φ intersects ∂F exactly twice.

Proof of Lemma 4.6. Let α be a simple closed curve such that $\Phi(\alpha) = \alpha$. Since $DF \backslash \partial F$ consists of the two components $int(F)$ and $int(F')$, it is not connected and hence $|\alpha \cap \partial F| = 2k$, $k \geq 1$.

Now $\alpha\backslash(\alpha\cap\partial F)$ consists of $2k$ components, k components in F and the other k components in F'. Since Φ fixes ∂F it must be the case that for α_1 the closure of a component of $\alpha\backslash(\alpha\cap\partial F)$ in F, there is a corresponding α_2 which is the closure of a component of $\alpha\backslash(\alpha\cap\partial F)$ in F' such that $\Phi(\alpha_1)=\alpha_2$. Thus, $\partial\alpha_1=\partial\alpha_2$ and $\alpha_1\cup\alpha_2$ is a simple closed curve component of α. Since α is connected $\alpha=\alpha_1\cup\alpha_2$ and hence $k=1$. □

Corollary 4.7. A disk in DF which is invariant under Φ intersects ∂F in exactly one arc.

Proof of Lemma 4.7. The defintion of invariance guarantees that any invariant disk must interesect ∂F at least once. Suppose for a contradiction that there exists a disk $D\subset DF$ such that $\Phi(D)=D$ and $|D\cap\partial F|=k$ where $k\geq 2$. Notice that $D\cap\partial F$ consists only of arc components. Indeed if $D\cap\partial F$ contained simple closed curves then D would contain a component of ∂D and no longer be a disk. Thus, $|D\backslash\partial F|=k+1$ and

$$|\partial D\backslash\partial F|\geq k+1\geq 3$$

which is a contradiction by Lemma 4.6 since $\partial D\subset DF$ is an invariant simple closed curve . □

In the next lemma, or Main Lemma, we make use of some terminology. We say a subsurface of *DF* is essentially blue (or yellow) if the subsurface is blue (or yellow) except possibly for a portion which can be contained in the interior of a disk, where the disk is in the interior of the subsurface. We also may say that a subsurface of *DF* is entirely blue (or yellow) if it is contained in $int(B)$ (or in $int(Y)$).

Lemma 4.8. There exists a *RBY* coloring of *DF* invariant under Φ for which there is an entirely blue (or entirely yellow) essential pair of pants $X\subset DF$ of type T_0, T_1, or T_3.

Proof of Lemma 4.8. The *RBY* coloring of *DF* already exists. It remains to show that there exists X satisiying the lemma. We assume without loss of generality that X, from Lemma 4.2, is a pair of pants in $int(B)$. When we talk about intersections we assume transversality, as usual, and in particular transversality of X and $\Phi(X)$. If X does not intersect ∂F it is a pair of pants of type T_0

with essential boundary components and $X \subset int(B)$ and we are done. We assume from now on that X intersects ∂F. We construct a surface, denoted $X^{s,d}$, with the following properties:

- $X^{s,d}$ is a connected subsurface of DF,
- $\Phi(X^{s,d}) = X^{s,d}$ (i.e. $X^{s,d}$ is invariant under Φ),
- $\partial X^{s,d}$ is essential in DF,
- $X \subset X^{s,d}$,
- $X^{s,d}$ is essentially blue, and
- $\chi(X^{s,d}) \leq -1$.

We first define $X^s := X \cup \Phi(X)$. Notice that X^s is invariant under Φ, is a subsurface of DF (this follows from transversality) containing X, and is entirely blue. Let D be the disjoint union of the disks in DF which are bounded by boundary components of X^s. We define $X^{s,d} := X^s \cup D$. Thus $\partial X^{s,d}$ is essential in DF. The only part of $X^{s,d}$ which can be yellow is contained in D and since $X^{s,d}$ is connected we can find a single disk in $X^{s,d}$ which contains all of D. So $X^{s,d}$ is essentially blue. Since X^s is invarint under Φ we get for any boundary component c of X^s which bounds a disk D_c that there is a corresponding boundary component $\Phi(c)$ which bounds a disk $\Phi(D_c)$. It may be the case that $D_c = \Phi(D_c)$. We conclude that $X^{s,d}$ is invariant under Φ. Notice that X is a connected essential subsurface of $X^{s,d}$ and hence Lemma 2.14 tells us that $\chi(X^{s,d}) \leq \chi(X) \leq -1$. Thus, $X^{s,d}$ is a subsurface of DF satisfing the six properties above.

Claim. $X^{s,d}$ contains an entirely blue essential pair of pants of type T_1 or T_3.

Proof of claim. Since X is entirely blue and connected we can define $Opp\colon X \to DF$ and get that $Opp(X) \subset DF \backslash B = int(R \cup Y)$.

Subclaim. For any component α of ∂X we get that α is essential in DF if and only if $Opp(\alpha)$ is essential in DF.

Proof of subclaim. We show one direction using proof by contrapositive; the other direction is similar. If $Opp(\alpha)$ is inessential in DF, then it bounds a disk $D'' \subset DF$. From our construction of the Opp map, α and $Opp(\alpha)$ cobound an annulus A in DM. Thus α is the boundary of the disk $D'' \cup A \subset DM$. Now $int(D'' \cup A) \cap F = D''$ so we use an isotopy to push D'' off of F so that $int(D'' \cup A) \cap F = \emptyset$. So $D'' \cup A$ is a compressing disk for DF whose boundary is α. Since DF is essential and hence incompressible in DM we get that α bounds a disk in DF and hence is inessential in DF. This completes the proof of the subclaim.

We conclude that since X is essential in DF that $Opp(X)$ is essential in DF. We now show that $Opp(X) \cap X^{s,d}$ is empty. Suppose for a contradiction that $Opp(X) \cap X^{s,d}$ is nonempty. Since $Opp(X) \subset int(R \cup Y)$ it is disjoint from $X^s \subset int(B)$. Thus $Opp(X)$ is contained in a disk component of D. Since $Opp(X)$ is essential in this disk Lemma 2.14 tells us that $\chi(Opp(X)) \geq 1$ which is a contradiction. Thus $Opp(X)$ is disjoint from $X^{s,d}$. We conclude that $X^{s,d}$ is not all of DF and hence $X^{s,d}$ has boundary. Let ρ be a boundary component of $X^{s,d}$. Notice that $\partial X^{s,d}$ is contained in ∂X^s and hence ρ is entirely blue. Since $\partial X^{s,d}$ is invariant under Φ we get that ρ is equivariant under Φ and hence is disjoint from ∂F or intersects ∂F exactly twice.

In the case that ρ does not intersect ∂F then due to connectedness of X^s and the invariance under Φ of X^s there is a simple path l in X^s starting at the point p in ρ and ending in $\rho' := \Phi(\rho)$ such that l only intersects ρ at p. Suppose without loss of generality that $\rho \subset F$ and let l' be the component of $l \cap F$ containing p. Notice that $l' \cup \Phi(l')$ is a simple path with one boundary component in ρ, the second boundary component in ρ', and is invariant under Φ. We define $q := \rho \cup l' \cup \Phi(l') \cup \rho'$. See Figure 15. Notice that q is invariant under Φ and is contained in X^s (hence is entirely blue). Let $N(q \cap F)$ be the regular neighborhood of $q \cap F$, in $F \cap X^s$. Notice that there is a regular neighborhood of q, in X^s, which is invariant under Φ given by $Q := N(q \cap F) \cup \Phi(N(q \cap F))$. We see that Q is an entirely blue pair of pants of type T_1.

By our construction of Q we know that since ρ and ρ' are boundary components of X^s they are also boundary components of Q. Moreover, ρ and ρ' are boundary components of $X^{s,d}$

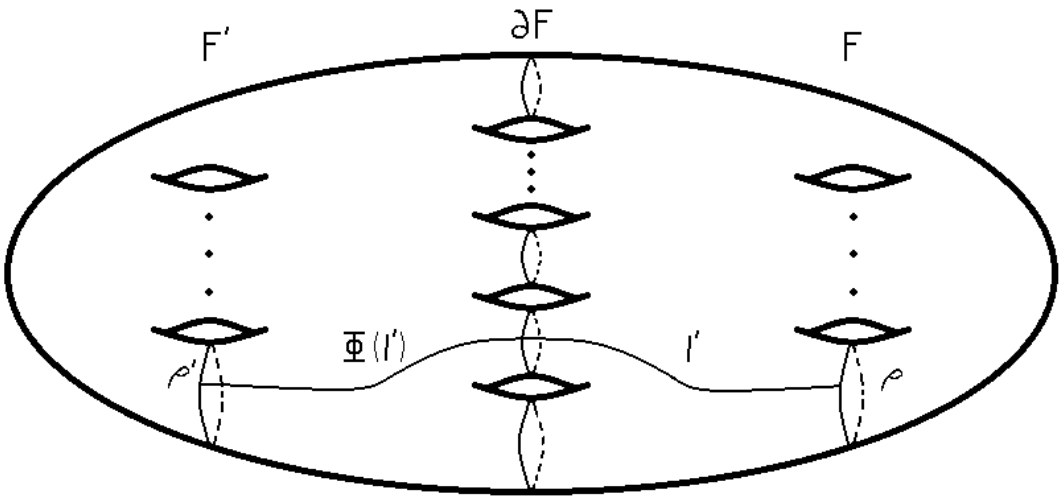

Figure 15: Example of possible decomposition for q.

and hence are essential in DF. Let ρ'' be the third boundary component of Q. Since Q is a regular neighborhood of q and ρ'' was not already a boundary component of $X^{s,d}$ we get from our construction that $\rho'' \subset int(X^s)$.

Claim. ρ'' is essential in DF.

Proof of Claim. Suppose for a contradiction that ρ'' is inessential and hence bounds a disk D'. This disk either contains both ρ and ρ' or neither. If $\rho \sqcup \rho' \subset D'$ then ρ and ρ' both bound a disk and hence are inessential which is a contradiction. Thus D' is disjoint from ρ and ρ'. Now ρ'' is an inessential simple closed curve in $X^s \subset X^{s,d}$. We show that $D' \subset X^{s,d}$. Suppose for a contradiction that $D' \not\subset X^{s,d}$. Since $\partial D' \subset X^{s,d}$, but $D' \not\subset X^{s,d}$ we get that $int(D') \cap X^{s,d}$ is a submanifold of D' with boundary consisting of simple closed curves from $\partial X^{s,d}$. Each boundary component of $X^{s,d}$ in D' bounds a disk in D'; this contradicts the construction of $X^{s,d}$. We conclude that $D' \subset X^{s,d}$.

Since ρ and ρ' are boundary components of $X^{s,d}$, $\rho \sqcup \rho' = \partial(Q \cup D')$, $Q \cup D' \subset X^{s,d}$, and $X^{s,d}$ is connected we get that $X^{s,d} = Q \cup D'$. Thus, since $Q \cup D'$ is an annulus, $\chi(X^{s,d}) = \chi(Q \cup D') = 0$ which is a contradiction since $\chi(X^{s,d}) \leq -1$ and so our claim holds.

Thus ρ'' is essential as desired, moreover, we get that Q is an essential pair of pants of type

T_1 which is entirely blue.

We now consider the case when ρ intersects ∂F. Let p be one of the two points in $\rho \cap \partial F$. There exists a path ρ_2 contained in $\partial F \cap X^{s,d}$ which has endpoints in $\partial X^{s,d}$ one of which is p. The other endpoint of ρ_2 we call p'. We now consider 2 separate cases:

Case 1: p' is in ρ.

Let $\rho_l \subset F'$ and $\rho_r \subset F$ be the oriented paths starting at p and ending in p' such that $\rho = \rho_l \rho_r^{-1}$. Additionally, orient ρ_2 from p to p'.

Claim. The simple closed curves $\rho_l \rho_2^{-1}$ and $\rho_r \rho_2^{-1}$ are essential in DF.

Proof of Claim. Assume for a contradiction that $\rho_l \rho_2^{-1}$ is inessential in DF. Since $\Phi(\rho_l \rho_2^{-1}) = \rho_r \rho_2^{-1}$ it follows that $\rho_r \rho_2^{-1}$ is also inessential in DF. Now with $\sim$ standing for homotopy we get

$$(\rho_l \rho_2^{-1})(\rho_r \rho_2^{-1})^{-1} \sim (\rho_l \rho_2^{-1})(\rho_2 \rho_r^{-1}) \sim \rho_l (\rho_2^{-1} \rho_2) \rho_r^{-1} \sim \rho_l \rho_r^{-1} \sim \rho$$

Now we consider these curves in the fundamental group of DF with base point p and we get for e the identity that

$$e = e * e^{-1} = [\rho_l \rho_2^{-1}] * [\rho_r \rho_2^{-1}]^{-1} = [\rho]$$

and hence ρ is inessential giving us our contradiction. Showing $\rho_r \rho_2^{-1}$ is essential is basically the same proof and so our claim holds.

Recall that $X^{s,d}$ is only essentially blue. We know that $X^{s,d} \setminus int(D) = X^s$ is entirely blue. Let D' be the disjoint union of the components of D which have nontrivial intersection with ρ_2. Since each of these components intersect ∂F and D is invariant under Φ it is clear that D' is invariant under Φ and hence every component of D' intersects ∂F in exactly one arc. Let $N((D' \cup \rho_2) \cap F)$ be a regular neighborhood, in $F \cap X^{s,d}$, of $(D' \cup \rho_2) \cap F$, then $N(D' \cup \rho_2) := N((D' \cup \rho_2) \cap F) \cup \Phi(N((D' \cup \rho_2) \cap F))$ is a regular neighborhood of $D' \cup \rho_2$, in $X^{s,d}$, which is invariant under Φ. We define $\rho_2' := cl((\partial N(D' \cup \rho_2) \cap F) \setminus \rho)$. Notice that

$\rho_2' \subset \partial N(D' \cup \rho_2)$ is entirely blue since the contruction of $N(D' \cup \rho_2)$ was designed so that $\partial N(D' \cup \rho_2)$ was entirely blue. We also remark that $\rho_2' \subset int(F)$ and ρ_2' is homotopic relative the boundary of $X^{s,d}$ to ρ_2. Define $q := \rho \cup \rho_2' \cup \Phi(\rho_2')$. Notice that q is entirely blue and invariant under Φ. Let $q_{p'}$ be the component of $q \setminus (\partial \rho_2' \cup \partial \Phi(\rho_2'))$ containing p'. Define $q' := q \setminus q_{p'}$. Similar to q we get that q' is entirely blue (in fact $q' \subset X^s$) and invariant under Φ. Let $N(q' \cap F)$ be a regular neighborhood, in $F \cap X^s$, of $q' \cap F$; then $Q := N(q' \cap F) \cup \Phi(N(q' \cap F))$ is a regular neighborhood, in X^s, of q', is invariant under Φ, and is entirely blue. Since Q is a pair of pants with boundary components homotopic to ρ, $\rho_l \rho_2^{-1}$, and $\rho_r \rho_2^{-1}$ we conclude that Q is an entirely blue essential pair of pants of type T_1.

Case 2: p' is not in ρ and hence is in a different boundary component of $X^{s,d}$; we call this component ρ' and remark that since it is a boundary component of $X^{s,d}$ it is essential.

The following argument is fairly similar to case 1 and hence we omit some details. Let D' be the disjoint union of the components of D which have nontrivial intersection with ρ_2. Like case 1, D' is invariant under Φ and hence every component of D' intersects ∂F in exactly one arc. We are able to find a regular neighborhood $N(D' \cup \rho_2)$, in $F \cap X^{s,d}$, of $D' \cup \rho_2$ which is invariant under Φ. We define $\rho_2' := cl((\partial N(D' \cup \rho_2) \cap F) \setminus (\rho \cup \rho'))$. By our construction ρ_2' is entirely blue, $\rho_2' \subset int(F)$, and ρ_2' is homotopic relative the boundary of $X^{s,d}$ to ρ_2. We define $q := \rho \cup \rho' \cup \rho_2' \cup \Phi(\rho_2')$. Notice that q is entirely blue and invariant under Φ. Let $q_{p'}$ be the component in $q \setminus (\partial \rho_2' \cup \partial \Phi(\rho_2'))$ containing p'. Define $q' := q \setminus q_{p'}$. Similar to q we get that q' is entirely blue (in fact $q' \subset X^s$) and invariant under Φ. Let $N(q' \cap F)$ be a regular neighborhood, in $F \cap X^s$, of $q' \cap F$, then $Q := N(q' \cap F) \cup \Phi(N(q' \cap F))$ is a regular neighborhood, in X^s, of q', is invariant under Φ, and is entirely blue. Now Q is an entirely blue pair of pants of type T_3.

Claim. Q is essential in DF.

Proof of claim. By construction the essential simple closed curve ρ is a component of ∂Q. Let q_1 and q_2 be the remaining boundary components of Q. By our construction and since ρ_2' is homotopic to ρ_2 it follows that one of the boundary components, say q_1, of Q is homotopic to

$\rho_2\rho'\rho_2^{-1}$ which is the conjugate of an essential curve and hence essential. It remains to show that q_2 is essential. Suppose for a contradiction that q_2 is inessential, then q_2 bounds a disk D'' in DF. Now $\rho \sqcup q_1 \subset D''$ or $(\rho \sqcup q_1) \cap D'' = \emptyset$. If $\rho \sqcup q_1 \subset D''$ then both ρ and q_1 bound a disk in D'' and hence are inessential which is a contradiction. Thus D'' is disjoint from $\rho \sqcup q_1$.

Now $\partial Q \subset X^s \subset X^{s,d}$ and hence $q_2 \subset X^s \subset X^{s,d}$. While not all of q' is disjoint from ∂X^s we know that since Q is a regular neighborhood of q', in X^s, that $Q \backslash q' \subset int(X^s)$. Thus, since q_2 is disjoint from q' we know that $q_2 \subset int(X^s)$. We show that $D'' \subset X^{s,d}$. Suppose for a contradiction that $D'' \not\subset X^{s,d}$. Since $\partial D'' = q_2 \subset int(X^s) \subset int(X^{s,d})$ there exist at least one component of $\partial X^{s,d} \subset int(D'')$ and each component bounds a disk and hence is inessential which is a contradiction. Thus, $Q \cup D''$ is an essential annulus contained in $X^{s,d}$ with $\partial(Q \cup D'') = \rho \sqcup q_1$ where $\rho \subset \partial X^{s,d}$ and q_1 is homotopic to $\rho' \subset \partial X^{s,d}$. Since $X^{s,d}$ is connected we get that $Q \cup D''$ is homotopic, in $X^{s,d}$ to $X^{s,d}$ and hence $\chi(X^{s,d}) = \chi(Q \cup D'') = 0$ which is a contradiction. Thus, q_2 is essential and hence Q is essential which proves the claim.

We conclude that Q is an entirely blue essential pair of pants of type T_3. $\square$

Half of Double

Let X be the pair of pants found in Lemma 4.8, we assume without loss of generality that $X \subset int(B)$. We consider $Opp\colon X \to DF$. Let α_1, α_2, and α_3 be the boundary components of X. Using the bundle structure from the parallelism of faces we have for each $\alpha_i \subset \partial X$ that there is an annulus $A_i \subset DM$ with boundary components α_i and $Opp(\alpha_i)$.

Notation: We write \\ to mean cut open along. We let I stand for the interval $[0,1]$ and let S^1 stand for the circle.

We want to determine what happens to X when we restrict our problem back to the case with just F in M by cutting DM open along ∂M. In other words we want to know what $X \cap F \subset F$ is. Before we look at $DM \backslash\backslash \partial M$, however, we want some results about X and $Opp(X)$ in $DM \backslash\backslash DF$. Using the bundle structure of DM, which is derived from M, we get that $DM \backslash\backslash DF$ is homeomorphic to $DF \times I$ and we use the two expressions interchangably.

Consider $DF \times I$. Since F is 2-sided it follows that DF is 2-sided and hence $\partial(DF \times I)$ has two copies of DF, $DF \times \{0\}$ and $DF \times \{1\}$. We define $DF_0 := DF \times \{0\}$ and $DF_1 := DF \times \{1\}$. Additionally, there are two copies of X; one in DF_0 and the other in DF_1. Suppose without loss of generality that DF_0 is characterized by having a coorientation in $DM\backslash\backslash DF$ which points in while DF_1 is characterized by having a coorientation in $DM\backslash\backslash DF$ which points out. Choose X to be the copy of $X \subset DF_0$ so that the Opp map makes sense. Then, $Opp(X) \subset DF_0$ or $Opp(X) \subset DF_1$. Moreover, A_1, A_2, and A_3 all make sense in $DF \times I$ and the symmetric map $\Phi : DM \to DM$ which we defined earlier will also make sense in $DF \times I$.

Lemma 4.9. For our choice of $X \subset DF_0$ we get that $Opp(X) \subset DF_1$.

Proof of Lemma 4.9. Suppose for a contradiction that $Opp(X) \subset DF_0$. For $i = 1, 2, 3$ let $f_i : S^1 \to DF \times I$ such that $imf_i) = \alpha_i$ and $g_i : S^1 \to DF \times I$ such that $im(g_i) = Opp(\alpha_i)$. Let $H_i : S^1 \times I \to DF \times I$ such that $H_i(s, 0) = f_i(s)$ and $H_i(s, 1) = g_i(s)$. We know H_i exists because of the annulus A_i. Now let $P : DF \times I \to DF \times \{0\}$ be the natural projection, then $P \circ H_i : S^1 \times I \to DF \times \{0\}$ is a homotopy of f_i and g_i in DF_0. The existence of a homotopy in the surface DF_0 guarantees that there is an isotopy in DF_0 of f_i and g_i. Thus, for $i = 1, 2, 3$ we get that α_i is isotopic to $Opp(\alpha_i)$ in DF_0.

Since X is in $int(B)$ we get that $Opp(X)$ is disjoint from B and hence X and $Opp(X)$ are disjoint subsurfaces of DF_0. We get for $i = 1, 2, 3$ that since α_i is isotopic to $Opp(\alpha_i)$ then there exists an annulus, Q_i, in DF_0 between α_i and $Opp(\alpha_i)$ which has interior disjoint from X and $Opp(X)$. Thus we have that $Q := X \cup Opp(x) \cup Q_1 \cup Q_2 \cup Q_3$ is a closed genus 2 subsurface of DF_0 and hence is equal to DF_0. From our hypothesis $\chi(F) \leq -76t(M) + 1$. So,

$$\begin{aligned}
-2 = \chi(Q) = \chi(DF_0) = \chi(DF) &= 2\chi(F) \\
&\leq -152t(M) + 2 \\
&\leq -150
\end{aligned}$$

where the last inequality holds since $t(M) \geq 1$. This is a clear contradiction and hence $Opp(X) \subset DF_1$ as desired. □

We now go back to studying M. Recall that $DM\backslash\backslash\partial M$ is homeomorphic to $M \sqcup M'$. Using the bundle structure of M we conclude that $M\backslash\backslash F$ is homeomorphic to $F \times I$ and use the two expressions interchangably. Thus, since $DM\backslash\backslash DF$ is homeomorphic to $DF \times I$ we talk about $F \times I \subset DF \times I$ instead of $M\backslash\backslash F \subset DM\backslash\backslash DF$. In this setting we consider seperately when X is a pair of pants of type T_0, T_1, or T_3 in $DF_0 \subset DF \times I$.

If X is a pair of pants of type T_0 in DF_0 then $\Phi(X)$ is also a pair of pants of type T_0 in DF_0. Either X or $\Phi(X)$ is contained in $F_0 := F \times \{0\}$. If $X \subset F_0$ then we leave it alone, otherwise if $X \subset DF\backslash F_0$ we redefine X to be $\Phi(X)$. Thus X is an entirely blue essential pair of pants of type T_0 contained in F_0. Recall for $i = 1,2,3$ that A_i is the annulus with α_i and $Opp(\alpha_i)$ as boundary components; the α_i are boundary components of X. Recall that we are associating $F \times I$ with $M\backslash\backslash F$. Since $X \subset F_0 \subset F \times I$ and Opp works with tetrahedra which are contained entirely in M or M', it turns out that $A_i \subset F \times I$ for all i. By Lemma 4.9 we conclude for $i = 1,2,3$ that since $\alpha_i \subset F_0$ that $Opp(\alpha_i) \subset F_1 := F \times \{1\}$.

Lemma 4.10. When $X \subset F_0$ is an essential pair of pants of type T_0, as above, then at least one of A_1, A_2, or A_3 is not boundary parallel to $\partial(F \times I)$.

Proof of Lemma 4.10. Suppose for a contradiction that A_1, A_2, and A_3 are all boundary parallel in $F \times I$. We know for $i = 1,2,3$ that A_i has the boundary components $\alpha_i \subset F_0$ and $Opp(\alpha_i) \subset F_1$. Since A_i is boundary parallel in $F \times I$ there exists for each i an annulus B_i in $\partial(F \times I)$ with boundary components α_i and $Opp(\alpha_i)$.

We need to show that for each i that $int(B_i)$ is disjoint from X and $Opp(X)$. Suppose for now that the annulus B_1 intersects $int(X)$, $int(Opp(X))$, or both. Then it must be the case that $int(X)$, $int(Opp(X))$, or both are contained in the annulus B_1. This is because to be boundary parallel A_1 must, with fixed boundary components, be isotopic to a subsurface of $\partial(F \times I)$ which we are calling B_1 which contains everything in $\partial(F \times I)$ which is on the same side of ∂A_1

as B_1. Thus, B_1 contains one or two essential pairs of pants which is a contradiction by Lemma 2.14. Since B_1 was an arbitrary choice we conclude that $int(B_i)$ is disjoint from X and $Opp(X)$ for $i = 2,3$ as well. We conclude that $Q := B_1 \cup B_2 \cup B_3 \cup X \cup Opp(X)$ is a closed genus 2 subsurface of $\partial(F \times I)$.

Notice that F has boundary and hence $\partial(F \times I)$ is a connected closed surface with $\chi(\partial(F \times I)) = 2\chi(F)$. Since Q is a closed genus 2 subsurface of $\partial(F \times I)$ we get that $Q = \partial(F \times I)$. Thus,

$$\begin{aligned}
-2 &= \chi(Q) \\
&= \chi(\partial(F \times I)) \\
&= 2\chi(F) \\
&\leq 2(-76t(M) + 1), \text{ by the hypothesis of Theorem 1.1} \\
&\leq -150, \text{ since } t \geq 1
\end{aligned}$$

This gives a clear contradiction and we conclude that at least one of A_1, A_2, or A_3 must not be boundary parallel in $F \times I$. □

Using the result of Lemma 4.10 we assume without loss of generality that A_1 is not boundary parallel in $F \times I$. Let $\alpha := \alpha_1 = A_1 \cap F_0$ and so $Opp(\alpha) = Opp(\alpha_1) = A_1 \cap F_1$.

Lemma 4.11. *$Opp(\alpha)$ is isotopic to $\alpha \times \{1\}$ in F_1.*

Proof of Lemma 4.11. Let $f : S^1 \to F \times I$ such that $im(f) = Opp(\alpha)$ and let $g : S^1 \to F \times I$ such that $im(g) = \alpha$. Now let $H : S^1 \times I \to F \times I$ such that $H(s,0) = f(s)$ and $H(s,1) = g(s)$ for all $s \in S^1$. We know H exists because of the annulus A_1. Now let $P : F \times I \to F \times \{1\}$ be the natural projection, then $P \circ H : S^1 \times I \to F \times \{1\}$ is a homotopy of f and $P \circ g$ in F_1. The restrictions of P to F_0 and F_1 are homeomorphisms and since $im(f)$ and $im(g)$ are embeddings we get that $im(P \circ f) = im(f) = Opp(\alpha)$ and $im(P \circ g) = \alpha \times \{1\}$ are embedded. Thus, since f is homotopic to $P \circ g$ and both $im(f)$ and $im(P \circ g)$ are embedded, we get that there exists an isotopy of f

and $P \circ g$ and hence $Opp(\alpha)$ is isotopic to $\alpha \times I$ in F_1. $\square$

The next two paragraphs are used to fix some notation for when X is a pair of pants of type T_1 or T_3. We use similar notation in order to prove Lemma 4.12 and Lemma 4.13 simultaneously for both types of pants.

If X is a T_1 pair of pants in DF_0 then in $F \times I$ we may assume without loss of generality that $A_1 \cap (F \times I)$ is a disk D_1, $A_2 \cap (F \times I) = A_2$, and $A_3 \cap (F \times I)$ is empty. Notice that $\eta := D_1 \cap \alpha_1 = F_0 \cap \alpha_1$ and $Opp(\eta) = D_1 \cap Opp(\alpha_1) = F_1 \cap Opp(\alpha_1)$ are arcs in F_0 and F_1 respectively.

Lastly if X is a T_3 pair of pants in DF_0 then for $i = 1,2,3$ we get that $D_i := A_i \cap (F \times I) \subset F \times I$ is a disk. Notice that $a_i := D_i \cap \alpha_i = F_0 \cap \alpha_i$ and $a_i' := D_i \cap Opp(\alpha_i) = F_1 \cap Opp(\alpha_i)$ are arcs in F_0 and F_1 respectively. We define $\eta := a_1 = D_1 \cap \alpha_1 = F_0 \cap \alpha_1$ and hence $Opp(\eta) = a_1' = D_1 \cap Opp(\alpha_1) = F_1 \cap Opp(\alpha_1)$.

Lemma 4.12. η and $Opp(\eta)$ are essential arcs in F_0 and F_1 respectively.

Proof of Lemma 4.12. Suppose for a contradiction that η is inessential in F_0. Then there is an arc a in ∂F_0 such that $\partial\eta = \partial a$ and $\eta \cup a$ bounds a disk $D_\eta \subset F_0$. We go back to the double DF_0 and notice that this implies that $\Phi(\eta)$ is inessential in $\Phi(F_0)$ and we see that $\Phi(\eta) \cup a$ bounds the disk $\Phi(D_\eta) \subset \Phi(F_0)$. We conclude, since a is a single arc in ∂F_0, that $\alpha_1 = \eta \cup \Phi(\eta)$ bounds the disk $D_\eta \cup \Phi(D_\eta) \subset DF_0$ and hence is inessential which is a contradiction. We can prove, similarly, that $Opp(\eta)$ must be essential. $\square$

Lemma 4.13. $Opp(\eta)$ is isotopic to $\eta \times \{1\}$ in F_1.

Proof of Lemma 4.13. Let $f : [0,1] \to F \times I$ such that $im(f) = Opp(\eta)$ and let $g : [0,1] \to F \times I$ such that $im(g) = \eta$. Now let $H : [0,1] \times I \to F \times I$ such that $H(s,0) = f(s)$ and $H(s,1) = g(s)$ for all s in $[0,1]$ and such that for all t in I we have that $H(0,t)$ and $H(1,t)$ are in $\partial F \times I$. We know H exists because of the disk D_1. Now let $P : F \times I \to F \times \{1\}$ be the natural projection, then $P \circ H : [0,1] \times I \to F \times \{1\}$ is a homotopy relative the boundary of F_1 of f and $P \circ g$ in F_1. Now P restricted to F_0 or F_1 is a homeomorpism and since $im(f)$ and $im(g)$ are embeddings we get that $im(P \circ f) = im(f) = Opp(\eta)$ and $im(P \circ g) = \eta \times \{1\}$ are embedded. Thus, since f is homotopic to $P \circ g$ and $im(f)$ and $im(P \circ g)$ are embedded we get that there exists an isotopy, relative the boundary of F_1, of f and $P \circ g$ and hence $Opp(\eta)$ is isotopic to $\eta \times \{1\}$ in F_1 $\square$

Conclusion

If X is a pair of pants of type T_0 then Lemma 4.10 implies that α and $Opp(\alpha)$ are essential simple closed curves. Recall $\mu : F \to F$ from the statement of Theorem 1.1. In $F \times I$, which is homeomorphic to $M \backslash\backslash F$, we get $\mu(\alpha) = \alpha \times \{1\}$ which, by Lemma 4.11, is isotopic to $Opp(\alpha)$. Since $\alpha \subset int(B)$ (or is in $int(Y)$) we know $Opp(\alpha)$ is disjoint from B (or from Y) and so $\mu(\alpha)$ is (possibly after isotopy) disjoint from α.

If X is a pair of pants of type T_1 or of type T_3 then Lemma 4.12 implies that η and $Opp(\eta)$ are essential arcs in F_0 and F_1 respectively. Recall $\mu : F \to F$ from the statement of Theorem 1.1. In $F \times I$, which is homeomorphic to $M \backslash\backslash F$, we get $\mu(\eta) = \eta \times \{1\}$ which by Lemma 4.13 is isotopic relative to the boundary of F_0 to $Opp(\eta)$. Since $\eta \subset int(B)$ (or is in $int(Y)$) we know $Opp(\eta)$ is disjoint from B (or from Y) and so $\mu(\eta)$ is (possibly after isotopy) disjoint from η.

From the two preceeding paragraphs we conclude the result of Theorem 1.1.

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