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General Sampling Schemes for the Bergman Spaces

Newton Foster
University of Arkansas, Fayetteville

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Newton H. Foster
University of Central Arkansas
Bachelor of Arts in Mathematics, 2009
University of Arkansas
Master of Science in Mathematics, 2013

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University of Arkansas

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Dr. Daniel H. Luecking
Dissertation Director

Dr. John R. Akeroyd
Committee Member

Dr. Marco M. Peloso
Committee Member

ABSTRACT

A characterization of sampling sequences for the Bergman spaces was originally provided by Seip and later expanded upon by Schuster. We consider a generalized notion of sampling using the infimum norm of the quotient space. Adapting some old techniques, we provide a characterization of general sampling sequences in terms of the lower uniform density.

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1 INTRODUCTION

In this thesis, we will consider a generalized notion of sampling sequences for the Bergman spaces on the complex unit disk. In the most general terms, sampling refers to the process of measuring the value of a function at a series of points in order to obtain a significant amount of information about the overall behavior of the function. A related, and in some ways dual, concept to sampling is interpolation which refers to the process of finding a function that attains some prescribed values at a series of points. While this paper is primarily concerned with the former, any decent mathematical excursion into either sampling or interpolation should provide consideration for its counterpart.

The ideas of sampling and interpolation have enjoyed a rich history going back to signal theory of bandlimited functions. In particular, the Nyquist-Shannon Sampling Theorem provided a complete description of necessary and sufficient conditions to recreate a bandlimited function from a series of samples at regular intervals. Thereafter, many sought to solve similar problems for many classical function spaces.

Beurling explored a problem of balayage of bandlimited functions in terms of Radon measures supported on a given compact set [1]. His work was significant in that it provided necessary and sufficient conditions for the distribution of a discrete sampling set in $\mathbb{R}$. Beurling's method of proof included the concepts of uniformly discrete sequences, weak limits of sequences as well as his formulations of the upper and lower uniform densities.

It was Seip who, inspired by Beurling's results, proceeded to provide a characterization for sampling and interpolation for certain spaces of holomorphic functions on the complex unit disk using an analogous formulation of the upper and lower uniform densities [12]. In particular, he proved the sampling and interpolation theorems for the growth space $A^{-\beta}$. In the latter section of his paper, he outlined a method of proof for the analogous sampling and interpolation theorems for the Bergman space A^2.

In his Ph.D. thesis, and later published in [9], Schuster provided a complete proof of the sampling and interpolation theorems for the Bergman spaces for all $1 \leq p < \infty$. His techniques followed along the lines of those outlined by Beurling and Seip. The case when $0 < p < 1$ required some different techniques but was also proved by Schuster and Varolin in [11].

In the usual formulation of the sampling and interpolation problems for the Bergman spaces, the assumption is made that the points in the sequence be distinct. This is because there is nothing to be gained from measuring the value of a function multiple times. However, one can formulate another version of the sampling problem which also takes into account the functions derivatives at each point. Similarly, for interpolation the problem would be to find a function whose derivatives attain certain prescribed values at each point. Such problems have been considered in the context of various classical function spaces. Krosky and Schuster proved a characterization of multiple interpolation sequences for the Bergman spaces using canonical zero divisors as well as the upper uniform density [4]. However, similar explorations have not been found for so called multiple sampling sequences. Perhaps this is because such a generalization would follow with little effort from the previously considered techniques.

The traditional notions of sampling and interpolation for the Bergman spaces involve a weighted sum. In the case of interpolation, given a sequence of values for which the sum converges the problem is to find a function that attains those values along the sequence of points. For the sampling problem, we want that the sum is bounded above and below by a factor of the function norm. This can be stated in terms of a frame of the function space with respect to point evaluation functionals.

One limitation of such formulations is that they require the consideration of uniformly discrete sequences. Uniformly discrete sequences go all the way back to Beurling. Another limitation of the previously considered formulations of the sampling and interpolation problems is that they deal primarily with discrete values. Even in the case of multiple sampling and interpolation, we have a fixed number of derivatives we are dealing with. One can conceive a problem in which different orders of the derivative are considered at each point in the sequence.

Luecking developed a different formulation of the sampling and interpolation problems in terms of a quotient norm. Instead of just evaluating a function and its derivatives at a point, we can consider the space of all holomorphic functions on some neighborhood of that point and quotient out by the space of all functions that have a zero of the appropriate order at that point. Then a natural choice of norm on this quotient space is the infimum; that is, the infimum of the local Bergman space norm of all functions that agree at that point. We then sum over the sequence and formulate the appropriate sampling and interpolation problems.

One can easily show, and indeed we will later prove, that these more general notions of sampling and interpolation are equivalent in the case of distinct points. This new notion has the added benefit that we can allow points to get arbitrarily close together (in the same neighborhood) as well as having an arbitrary finite order of repetition at any given point in the sequence. In [8], Luecking provided a complete characterization of these so called generalized interpolating sequences. He split up the sequence into these finite clusters contained in neighborhoods over which the quotient space norm is applied. Some consideration must be made for how we choose the clusters and neighborhoods. For example, it is required that any repetitions of a point must be in the same cluster so that we get the appropriate quotient. Also, we cannot allow points to get too close the boundary of the neighborhoods or else we could have arbitrarily large values of point but an arbitrarily small quotient norm.

The choice of clusters and neighborhoods as outlined by Luecking is called a sampling or interpolation scheme. Luecking proved that the necessary and sufficient conditions a generalized interpolation sequence was precisely the density condition $D^+(\Gamma) < 1/p$. Introducing the quotient norm effectively removed the uniformly discrete condition. In the following pages, we turn to the counterpart problem of generalized sampling sequences using the same quotient norm.

In chapter 2, we will introduce notation as well as rigorously define many of the concepts stated here as well as those used throughout. We will also provide a proof that the regular and general notions of sampling are indeed equivalent in the case of uniformly discrete sequences. In chapter 3, we discuss some preliminary results that will play important parts in the proofs that follow. In particular, we discuss equivalent ideas of bounded density in terms of the sampling scheme chosen for that given sequence.

The proofs of the necessity and sufficiency of the density condition follow in chapters 4and 5 respectively. The necessity follows the ideas used by Schuster and Seip but require adaptation based on the sampling schemes. Using the concept of weak limits of Mobius transformations and following the argument used by Luecking in the case of sampling measures, we prove the sufficiency. Finally, in chapter 6 we will provide a consideration of the case $p < 1$, explore examples of general sampling schemes as well as discuss future work.

2 BACKGROUND

Let $\mathbb{D}$ be the complex unit disk and dA be the Lebesgue area measure. We will abuse notation and take $dA = \dfrac{1}{\pi}\, dA$ to be the normalized area measure so that $\int_{\mathbb{D}} dA = 1$.

For $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, let $A^p(\Omega)$ be the space of holomorphic functions such that

$$\|f\|_{p,\Omega} = \left(\int_{\Omega} |f|^p \, dA \right)^{1/p} < \infty.$$

In the case where $\Omega = \mathbb{D}$, we simply write $A^p = A^p(\mathbb{D})$. For the purposes of the proceedings, unless otherwise specified, we will assume that $1 \leq p < \infty$. Note that under this assumption A^p is a Banach space. If $p = 2$, then it is a Hilbert space with reproducing kernel.

An important part of our study of the Bergman Spaces on the unit disk will involve

Möbius transformations of the disk. For any $\xi \in \mathbb{D}$, define $\phi_\xi(z) = \dfrac{\xi - z}{1 - \bar{\xi} z}$. One can easily verify that ϕ_ξ defines a conformal map from $\mathbb{D}$ onto itself and exchanges ξ and 0. We note various properties of ϕ_ξ whose computations shall be omitted:

i. $\phi_\xi(\phi_\xi(z)) = z$, so $\phi_\xi^{-1}(z) = \phi_\xi(z)$

ii. $\phi_\xi'(z) = \dfrac{-(1 - |\xi|^2)}{(1 - \bar{\xi} z)^2}$ and $|\phi_\xi'(z)| = \dfrac{1 - |\phi_\xi(z)|^2}{1 - |z|^2}$

iii. $|\phi_\xi'(z)|^2 = \dfrac{(1 - |\xi|^2)^2}{|1 - \bar{\xi} z|^4}$ is the Jacobian when using ϕ_ξ as a change of variables.

The pseudohyperbolic distance on $\mathbb{D}$ is $\rho(z, \xi) = |\phi_\xi(z)|$ for all $z, \xi \in \mathbb{D}$. One can verify that this is an honest to goodness metric on $\mathbb{D}$. The pseudohyperbolic metric is somewhat natural to study in our case mainly because it is invariant under Möbius transformations. Indeed, we have that

$$\rho(\phi_\xi(z), \phi_\xi(w)) = \rho(z, w)$$

for all $z, w, \xi \in \mathbb{D}$.

Another important property of the pseudohyperbolic metric is that it satisfies the following pseudohyperbolic version of the triangle inequality.

$$\frac{|\rho(z, \xi) - \rho(\xi, w)|}{1 - \rho(z, \xi)\rho(\xi, w)} \leq \rho(z, w) \leq \frac{\rho(z, \xi) + \rho(\xi, w)}{1 + \rho(z, \xi)\rho(\xi, w)}$$

for all $z, w, \xi \in \mathbb{D}$.

We write $\Delta(z, \epsilon) = \{w \in \mathbb{D} : \rho(z, w) < \epsilon\}$ for the psuedohyperbolic disk centered at z of radius ϵ. We can compute the Euclidean area of the pseudohyperbolic disk for reference as well as an estimate in terms of the distance from the boundary.

$$\begin{aligned} |\Delta(z, \epsilon)| &= \int_{\Delta(z,\epsilon)} dA = \int_{\Delta(0,\epsilon)} |\phi_z'(w)|^2 \, dA(w) = \int_{\Delta(0,\epsilon)} \frac{(1 - |z|^2)^2}{|1 - \bar{z} w|^4} \\ &= \epsilon^2 (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{1}{|1 - \bar{z}(\epsilon w)|^4} \, dA(w) = \frac{\epsilon^2 (1 - |z|^2)^2}{(1 - |\epsilon z|^2)^2} \leq \frac{\epsilon^2}{(1 - \epsilon^2)^2} (1 - |z|^2)^2 \end{aligned}$$

In fact, we have

$$\epsilon^2(1-|z|^2)^2 \leq |\Delta(z,\epsilon)| \leq \frac{\epsilon^2}{(1-\epsilon^2)^2}(1-|z|^2)^2$$

for all $z \in \mathbb{D}$ and $\epsilon < 1$.

It will also be useful to compute the hyperbolic area of a pseudohyperbolic disk. We let $\dfrac{1}{(1-|z|^2)^2}\,dA(z)$ be the hyperbolic area measure. Then the hyperbolic area of any set $\Omega \subset \mathbb{D}$ is given by

$$a(\Omega) = \int_\Omega \frac{1}{(1-|z|^2)^2}\,dA(z).$$

Computing the hyperbolic area of an arbitrary pseudohyperbolic disk simply requires polar coordinates integral to get

$$\begin{aligned} a(\Delta(\xi,\epsilon)) &= \int_{\Delta(\xi,\epsilon)} \frac{1}{(1-|z|^2)^2}\,dA(z) = \frac{1}{\pi}\int_0^\epsilon \int_0^{2\pi} \frac{1}{(1-r^2)^2}\,d\theta r\,dr \\ &= \int_0^\epsilon \frac{2r}{(1-r^2)^2}\,dr = \left.\frac{1}{1-r^2}\right|_0^\epsilon = \frac{\epsilon^2}{(1-\epsilon^2)} \end{aligned}$$

Since ϕ_ξ is a linear fractional transformation, we know that the resulting open set $\Delta(z,\epsilon)$ is indeed a Euclidean disk in $\mathbb{D}$. Then the Euclidean radius is

$$\gamma = \frac{\epsilon(1-|z|^2)}{1-\epsilon^2|z|^2}$$

since γ^2 is equal to the area. To compute the center, consider $\Delta(a,\epsilon)$ where a is a real number. Then the boundary of the disk crosses the x-axis at $\dfrac{a-\epsilon}{1-\epsilon a}$ and $\dfrac{a+\epsilon}{1+\epsilon a}$. Therefore, the center of $\Delta(a,\epsilon)$ is

$$\alpha = \frac{1}{2}\left(\frac{a-\epsilon}{1-\epsilon a} + \frac{a+\epsilon}{1+\epsilon a}\right) = \frac{(1-\epsilon^2)a}{1-\epsilon^2 a^2}.$$

Rotating back to an arbitrary z in $\mathbb{D}$, we have

$$\alpha = \frac{(1-\epsilon^2)|z|}{1-\epsilon^2|z|^2}.$$

So with α and γ as calculated above we have $\Delta(z,\epsilon) = D(\alpha,\gamma)$.

Let $\Gamma = \{z_k\}$ be a sequence of distinct points in $\mathbb{D}$. We say that Γ is uniformly discrete if $\inf\{\rho(z_k,z_j) : z_k, z_j \in \Gamma, z_k \neq z_j\} < \infty$ in which case we call

$$\delta(\Gamma) = \inf_{k\neq j} \rho(z_k, z_j)$$

the separation constant of Γ. Uniformly discrete sequences play an important role in the density theorems for sampling and interpolation in the Bergman Spaces.

Now we define the uniform densities of a sequence in $\mathbb{D}$. For a sequence $\Gamma = \{z_k\}$, a point $\xi \in \mathbb{D}$ and a radius r, define

$$D(\Gamma,\xi,r) = \left(\log\frac{1}{1-r}\right)^{-1} \sum_{1/2<|\phi_\xi(z_k)|<r} \log\frac{1}{|\phi_\xi(z_k)|}.$$

Then the upper and lower uniform densities of Γ are given respectively by

$$D^+(\Gamma) = \limsup_{r\to 1^-} \sup_{\xi\in\mathbb{D}} D(\Gamma,\xi,r)$$

$$D^-(\Gamma) = \liminf_{r\to 1^-} \inf_{\xi\in\mathbb{D}} D(\Gamma,\xi,r).$$

Throughout the proceedings, we will want to consider sequences of "bounded density". In order to deal with the concept of bounded density, it is often easier to use an alternative notion of density. Let $n(\Gamma,\xi,r) = |\Delta(\xi,r)\cap\Gamma|$ and define

$$E(\Gamma,\xi,r) = \left(\log\frac{1}{1-r}\right)^{-1} \int_0^r n(\Gamma,\xi,s)ds.$$

Then we can write

$$D^+(\Gamma) = \limsup_{r\to 1^-} \sup_{\xi\in\mathbb{D}} E(\Gamma,\xi,r)$$

$$D^-(\Gamma) = \liminf_{r\to 1^-} \inf_{\xi\in\mathbb{D}} E(\Gamma,\xi,r).$$

The following lemma, proved in [2], gives some equivalent conditions for bounded density.

Lemma 2.1. *Let $\Gamma = \{z_k\}$ be a sequence of distinct points in $\mathbb{D}$. The following are equivalent:*

i. Γ *is the finite union of uniformly discrete sequences*

ii. *there exists* $0 < r < 1$ *such that* $\sup_{\xi\in\mathbb{D}} n(\Gamma,\xi,r) < \infty$

iii. *for all* $0 < r < 1$, $\sup_{\xi\in\mathbb{D}} n(\Gamma,\xi,r) < \infty$

iv. $D^+(\Gamma) < \infty$

Later, we will formulate a new idea of bounded density in terms of the sampling scheme chosen for a given sequence. We will again see that it is equivalent to those mentioned above.

2.1 Sampling Sequences

We are now ready to describe the regular notion of sampling. A sequence $\Gamma = \{z_k\}$ of distinct points in $\mathbb{D}$ is said to be a sampling sequence for A^p is there exists positive constants C_1, C_2 so that

$$C_1\|f\|_p^p \le \sum_k (1-|z_k|^2)^2|f(z_k)|^p \le C_2\|f\|_p^p$$

for all $f \in A^p$. This inequality can be considered a special case of a frame on the function space applied to the point evaluation functionals.

It is well known that the upper sampling inequality is satisfied exactly when the sequence Γ is a finite union of uniformly discrete sequences or equivalently that Γ has bounded density. The problem then is to find necessary and sufficient conditions under which the lower sampling inequality is satisfied. The following theorem was stated in its original form by Seip and later explored by Schuster.

Theorem 2.2. *A sequence Γ of distinct points in $\mathbb{D}$ is a sampling sequence for A^p if and only if it is the finite union of uniformly discrete sequences and it contains a uniformly discrete subsequence Γ' for which $D^-(\Gamma') > 1/p$.*

Now, we can state our more general notion of sampling. Let Γ be a sequence in $\mathbb{D}$, possibly with repeated points. Partition Γ into finite, disjoint, non-empty "clusters" Z_k so that $\cup_k Z_k = \Gamma$. We will require that all repetitions of a certain point are in the same Z_k and that there exists lower bound on the pseudohyperbolic distance between the clusters. The idea here being that sampling a repeated point is equivalent to sampling the values of the derivatives.

We also require a lower bound on the psuedohyperbolic distance between clusters, that is, there exists $\delta > 0$ so that $\inf \{\rho(z, \xi) : z \in Z_k, \xi \in Z_j, k \neq j\} \geq delta$. The idea is that any failure of uniform discreteness is reflected within clusters rather than between them.

With each cluster Z_k, we associate an open set G_k so that $\Delta(z_j, \epsilon) \subset G_k$ for each $z_j \in Z_k$ for some $0 < \epsilon < 1$. We also require an upper bound on the psuedohyperbolic diameter of each G_k, that is, that there exists $R < 1$ so that $\text{diam}_\rho(G_k) = \sup \{\rho(z, \xi) : z, \xi \in G_k\} \leq R$ for all k. We call the collection of pairings $\{(Z_k, G_k)\}$ a sampling scheme for Γ, if it exists.

For each (Z_k, G_k), let $N(Z_k) = N(Z_k, G_k)$ be the subset of functions in $A^p(G_k)$ whose zero set contains Z_k (respecting multiplicity). Then we consider the quotient space $A^p(G_k)/N(Z_k)$ the sets of equivalence classes of functions that agree on Z_k. We say that $f \equiv g$ in $A^p(G_k)/N(Z_k)$ if $f - g \in N(z_k)$. The norm of an equivalence class will be the infimum of the pth power of the $A^p(G_k)$ norm of all elements in that equivalence class. This is the standard norm to take when dealing with a quotient space.

A sequence Γ is said to be a general sampling sequence for A^p with respect to the scheme

$\{(Z_k, G_k)\}$ if there exists a constants C_1, C_2 so that

$$C_1\|f\|_p^p \le \sum_k \inf\left\{\int_{G_k} |g|^p\, dA : g \equiv f \text{ in } A^p(G_k)/N(Z_k)\right\} \le C_2\|f\|_p^p$$

for all $f \in A^p$. It should be noted that the upper inequality is satisfied when the G_k's have bounded overlap, that is when $\sup_{z\in\mathbb{D}} \sum_k \chi_{G_k}(z) < \infty$. We will see, however, that this does not imply bounded density. We will need two more conditions, an upper bound on the size of the Z_k's and a lower bound on the pseudohyperbolic distance between the Z_k's.

For our main result, we will assume all sampling schemes satisfy these conditions and seek necessary and sufficient conditions for which the lower general sampling inequality holds true.

Theorem 2.3. *Suppose Γ has bounded density and $\{(Z_k, G_k)\}$ is a sampling scheme. Then Γ is general sampling with respect to $\{(Z_k, G_k)\}$ if and only if $D^-(\Gamma) > 1/p$.*

Next, we want to show that in the case of uniformly discrete sequences, the two notions of sampling are equivalent. First, we need a lemma that is essentially the pseudohyperbolic form of the sub-mean value property.

Lemma 2.4. *Let $f \in A^p(\Delta(\xi, \epsilon)$, then*

$$\epsilon^2(1-|\xi|^2)^2|f(\xi)|^p \le \int_{\Delta(\xi,\epsilon)} |f|^p\, dA.$$

Proof. First of all, note that the function $f \circ \phi_\xi [\phi_\xi']^{2/p}$ is holomorphic on $\mathbb{D}$. Therefore, $|f \circ \phi_\xi|^p |\phi_\xi'|^2$ is subharmonic and we have

$$\epsilon^2 |f \circ \phi_\xi(0)|^p |\phi_\xi'|^2 \le \int_{D(0,\epsilon)} |f \circ \phi_\xi|^p |\phi_\xi'|^2\, dA.$$

Changing variables in the integral, we see that

$$\epsilon^2(1-|\xi|^2)^2|f(\xi)|^p \leq \int_{\Delta(\xi,\epsilon)} |f|^p \, dA.$$

□

Proposition 2.5. *Let Γ be a uniformly discrete sequence in $\mathbb{D}$ with separation constant δ and $\{(Z_k, G_k)\}$ be a sampling scheme. There exists constants C_1, C_2 such that*

$$C_1 \sum_{z\in\Gamma}(1-|z|^2)^2|f(z)|^p \leq \sum_k \inf\left\{\int_{G_k} |g|^p \, dA : g \equiv f \ \textit{in} \ A^p(G_k)/N(Z_k)\right\}$$
$$\leq C_2 \sum_{z\in\Gamma}(1-|z|^2)^2|f(z)|^p$$

for all $f \in A^p$.

Proof. Let $\sigma = \min\{\delta/2, \epsilon\}$ so that the disks $\Delta(z_j, \sigma)$ are disjoint and $\Delta(z_j, \sigma) \in G_k$ for all $z_j \in Z_k$. The lower inequality is an application of Lemma 2.4 where we have

$$\sigma^2 \sum_{z_j\in Z_k}(1-|z_j|^2)^2|f(z_j)|^p \leq \sum_{z_j\in Z_k}\int_{\Delta(z_j,\sigma)} |g|^p \, dA \leq \int_{G_k} |g|^p \, dA$$

for any $g \in A^p(G_k)$ with $g = f$ on Z_k and all k.

For the upper inequality, consider the function

$$h(z) = \sum_{z_j\in Z_k} f(z_j) \prod_{\substack{z_l\in Z_k \\ z_l\neq z_j}} \frac{z_l - z}{1-\bar{z}_l z}\left(\frac{z_l - z_j}{1-\bar{z}_l z_j}\right)^{-1}.$$

Clearly, $h = f$ on Z_k, so

$$\inf\left\{\int_{G_k} |g|^p \, dA : q \equiv f \ \text{in} \ A^p(G_k)/N(Z_k)\right\} \leq \int_{G_k} |h|^p \, dA.$$

Now, there exists a constant depending on p such that

$$|h(z)|^p \leq C_p \sum_{z_j \in Z_k} |f(z_j)|^p \prod_{\substack{z_l \in Z_k \\ z_l \neq z_j}} \left| \frac{z_l - z_j}{1 - \bar{z}_l z_j} \right|^{-p}.$$

Since Γ is uniformly discrete, $\rho(z_j, z_l) \geq \delta$ for all $z_j, z_l \in Z_k$ with $z_j \neq z_l$. So we have

$$|h(z)|^p \leq C_p \sum_{z_j \in Z_k} |f(z_j)|^p \delta^{-Mp}$$

where M is the upper bound on the number of $z_j \in Z_k$. For each $z_j \in Z_k$, $G_k \subset \Delta(z_j, R)$. Splitting up the integral linearly and estimating the Euclidean area of each $\Delta(z_j, R)$ we get

$$\int_{G_k} |h|^p \, dA \leq C_{p,\delta} |G_k| \sum_{z_j \in Z_k} |f(z_j)|^p \leq C_{p,\delta} \sum_{z_j \in Z_k} |\Delta(z_j, R)| |f(z_j)|^p$$
$$\leq C_{p,\delta,R} \sum_{z_j \in Z_k} (1 - |z_j|^2)^2 |f(z_j)|^p$$

□

A sequence Γ is said to be a zero set for A^p if there exists a non-zero function that vanishes to the appropriate orders on Γ. A sequence that is not a zero set is called a set of uniqueness for A^p since a function is uniquely determined by its values on such a set. As with the regular notion, an important property of general sampling sequences is that they are necessarily a set of uniqueness for A^p. Indeed, if f is a non-trivial function that vanishes on Γ, then the lower sampling inequality is violated.

3 PRELIMINARY RESULTS

3.1 Bounded Density

As mentioned before, our only assumptions on the sampling scheme associated with a given sequence is that there is an upper bound on the pseudohyperbolic diameter of the G_k's, that the points in each Z_k don't get too close to the boundary of G_k and that the clusters don't get too close to one another. We now want to investigate how the choice of a general sampling scheme relates to the idea of bounded density explored in chapter 2. First, we need the following lemma which gives us an upper bound on the number of points in any pseudohyperbolic disk.

Lemma 3.1. *Let Γ be a sequence in $\mathbb{D}$ and $\{(Z_k, G_k)\}$ a general sampling scheme with $\sum_k \chi_{G_k}(z) \leq M$ and $\sup_k \{|Z_k|\} \leq N$, then there exists a constant*

$$n(\Gamma, \alpha, r) \leq C_{M,N,R,\epsilon} \frac{1}{1-r^2}$$

for all $\alpha \in \mathbb{D}$.

Proof. Recall that for all $z_j \in Z_k$, $\Delta(z_j, \epsilon) \subset G_k$ for all k. If $z_j \in \Delta(\alpha, r)$, then $G_k \subset \Delta(\alpha, \frac{r+R}{1+rR})$ where $z_j \in Z_k \subset G_k$. Then

$$\sum_{z \in \Gamma \cap \Delta(\alpha, r)} \chi_{\Delta(z,\epsilon)}(\xi) \leq \sum_{G_k \subset \Delta(\alpha, \frac{r+R}{1+rR})} N \chi_{G_k}(\xi) \leq MN \chi_{\Delta(\alpha, \frac{r+R}{1+rR})}(z).$$

Multiply both sides by $\dfrac{1}{(1-|z|^2)^2}$ and integrate in z, we have

$$n(\Gamma,\alpha,r)\frac{\epsilon^2}{1-\epsilon^2} \leq MN\frac{(\frac{r+R}{1+rR})^2}{1-(\frac{r+R}{1+rR})^2} = MN\frac{(r+R)^2}{(1-r^2)(1-R^2)}$$
$$\leq C_{M,N,R,\epsilon}\frac{1}{1-r^2}.$$

□

Proposition 3.2. *Suppose* Γ *is a sequence in* $\mathbb{D}$ *and* $\{(Z_k, G_k)\}$ *is general sampling scheme, then* $\sum_k \chi_{G_k}(z) \leq M$ *and* $\sup_k \{|Z_k|\} \leq N$ *if and only if* Γ *has bounded density.*

Proof. ($\Longrightarrow$) Applying the previous lemma, we have

$$\int_0^r n(\Gamma,\alpha,s)ds \leq C\int_0^r \frac{1}{1-s^2}ds \leq C\log\frac{1}{1-r}$$

for all $\alpha \in \mathbb{D}$ and all r. Therefore, $D^+(\Gamma) < \infty$.

($\Longleftarrow$) Suppose $|Z_k| \to \infty$ as $k \to \infty$. Let $\xi_k \in Z_k$, then $Z_k \subset \Delta(\xi_k, R)$ for all k. This implies that

$$n(\Gamma,\xi_k,R) = |\Gamma \cap \Delta(\xi_k,R)| \to \infty \text{ as } k \to \infty.$$

Therefore,

$$D^+(\Gamma) \geq \sup_k D(\Gamma,\xi_k,R) = \infty$$

which contradicts our assumption that Γ has bounded density. Now, suppose $\sup_k \{|Z_k|\} \leq N$ and Γ has bounded density. However, suppose $\sum_k \chi_{G_k}(z)$ is unbounded in $\mathbb{D}$. Then there exists a sequence of points $\{\xi_M\}$ such that

$$\sum_k \chi_{G_k}(\xi_M) \geq M.$$

Again, consider $\Delta(\xi_M, R)$. By assumption, $\xi_M \in G_k$ for at least M of the G_k's. Since, the Z_k's are non-empty, we have that

$$n(\Gamma, \xi_M, R) = |\Gamma \cap \Delta(\xi_M, R)| \geq M.$$

Therefore,

$$D^+(\Gamma) \geq \sup_M D(\Gamma, \xi_M, R) = \infty$$

which contradictions our assumption that Γ has bounded density. $\square$

Combining the two properties considered above with those laid out earlier, we can make the following definition. A general sampling scheme $\{(Z_k, G_k)\}$ is said to be an *admissible sampling scheme* if there exists $M, N, R, \epsilon, \delta$ such that

i. $\sup_k \operatorname{diam}_\rho(G_k) \leq R$

ii. for all k, $\Delta(z_j, \epsilon) \subset G_k$ for all $z_j \in Z_k$

iii. for all $j \neq k$, $\rho(z, \xi) > \delta$ for all $z \in Z_j$ and $w \in Z_k$

iv. $\sum_k \chi_{G_k}(z) \leq M$

v. $\sup_k |Z_k| \leq N$.

It is clear that all of the constants mentioned above are invariant under Möbius transformations.

In the regular notion of sampling, we saw that bounded density was necessary and sufficient to get the upper sampling inequality. However, we will see that the upper inequality for general sampling schemes only requires that the G_k's have bounded overlap, which is only half of the bounded density condition. We start off with the observation that the condition of bounded density is independent of the size of the open sets G_k. This is actually a corollary of the bounded density result proved earlier.

Corollary 3.3. *Suppose* $\Gamma = \{z_k\}$ *and* $z_k \in G_k$ *with* $\sup_k \operatorname{diam}_\rho(G_k) < \infty$*, then*

$$D^+(\Gamma) < \infty \text{ if and only if } \sup_{\xi \in \mathbb{D}} \sum_k \chi_{G_k}(\xi) < \infty.$$

Lemma 3.4. *Let* $\{(Z_k, G_k)\}$ *be a sampling scheme with properties* (i.) *and* (ii.)*, then*

$$\sup_{z \in \mathbb{D}} \sum_k \chi_{G_k}(z) < \infty$$

if and only if

$$\sum_k \inf \left\{ \int_{G_k} |g|^p \, dA : g \equiv f \text{ in } A^p(G_k)/N(Z_k) \right\} \le C \|f\|_p^p$$

for all $f \in A^p$.

Proof. ($\Longrightarrow$) Suppose $\sum_k \chi_{G_k}(z) \le M$ for all $z \in \mathbb{D}$, then

$$\sum_k \inf \left\{ \int_{G_k} |g|^p \, dA : g \equiv f \text{ in } A^p(G_k)/N(Z_k) \right\} \le \sum_k \int_{G_k} |f|^p \, dA = \int_{\mathbb{D}} \sum_k \chi_{G_k}(z) |f(z)|^p \, dA(z) \le M \|f\|_p^p.$$

($\Longleftarrow$) Suppose there exists a constant C such that

$$\sum_k \inf \left\{ \int_{G_k} |g|^p \, dA : g \equiv f \text{ in } A^p(G_k)/N(Z_k) \right\} \le C \|f\|_p^p$$

for all $f \in A^p$. Let G_k' be open sets so that $(G_k')_\epsilon \subset G_k$. Suppose that there exists ξ_M so that $\sum_k \chi_{G_k'}(\xi_M) > M/\epsilon^2$. Then applying the upper sampling inequality to the function

$(\phi'_{\xi_M})^{2/p}$, we see that

$$M \leq \sum_{\Delta(\xi_M,\epsilon)\subset G_k} \epsilon^2(1-|\xi_M|^2)^2|\phi'_{\xi_M}(\xi_M)|^2$$
$$\leq \sum_k \inf\left\{\int_{G_k}|g|^p\,dA : g\equiv f \text{ in } A^p(G_k)/N(Z_k)\right\} \leq C\|\phi'_{(\xi_M)})^{2/p}\|_p^p = C$$

for all M. Therefore, we have a contradiction of our assumption. That is, we have that $\sup_{z\in\mathbb{D}}\sum_k \chi_{G'_k}(z) < \infty$. From Lemma 3.4, we see that the G_k's themselves have bounded overlap. □

While it seems natural to consider only sequences of bounded density, one may wonder if it is sufficient. The following conjecture poses this question and is analogous to the theorem of regular sampling which stated that every sampling sequence has a uniformly discrete sequence that is also a sampling sequence.

Conjecture. *Every general sampling sequence for A^p has a subsequence of bounded density that is also a general sampling sequence for A^p.*

3.2 Multiple Sampling

Multiple sampling refers to the idea of not only sampling the value of a function at each point, but also its derivatives. We will see that in our case, this corresponds to having repeated points in the clusters and taking the quotient of functions that have a zero at those points of the respective order.

Before we talk about complete generality, with any finite repetitions in our sequence, let us consider regular notion of multiple sampling sequences. A sequence Γ is said to be multiple sampling for a fixed n, if there exists constants such that

$$C_1\|f\|_p^p \leq \sum_{l=0}^{n-1}\sum_k |f^{(l)}(z_k)|^p(1-|z_k|^2)^{2+lp} \leq C_2\|f\|_p^p$$

for all $f \in A^p$. We can then come up with the following characterization analogous to Theorem 2.2.

Theorem 3.5. *A sequence of distinct points Γ is multiple sampling of order n if and only if Γ is the finite union of uniformly discrete sequences and has a uniformly discrete subsequence Γ' for which $D^-(\Gamma') > 1/np$.*

Another way of thinking about this density condition would be the sequence that repeats every point of Γ n-times has density strictly greater than $1/p$. What we would like to do is allow for each element of the sequence to be repeated an arbitrary (but finite) number of times. Indeed, general sampling allows such sampling problems to be considered.

Before we consider the general case, we want to prove that the regular notion of multiple sampling is equivalent to that of using our new norm in the case of uniformly discrete sequences. We begin with two lemmas.

Proposition 3.6. *Let Γ be a uniformly discrete sequence with separation constant δ. There exists a constant depending on n, p, δ such that*

$$\sum_k \inf\left\{\int_{\Delta(z_k,\delta/2)} |g|^p\, dA : g \in E_k\right\} \leq C \sum_k \sum_{l=0}^{n-1} (1-|z_k|^2)^{2+lp} |f^{(l)}(z_k)|^p$$

for all $f \in A^p$ where $E_k = \{g \in A^p(\Delta(z_k, \delta/2) : g - f$ has zero of order at least n at $z_k\}$.

Proof. Consider the function

$$h(z) = \sum_{l=0}^{n-1} \frac{f^{(l)}(z_k)}{l!}(z-z_k)^l.$$

Then $h \in E_k$, so

$$\inf \left\{ \int_{\Delta(z_k,\delta/2)} |g|^p \, dA : g \in E_k \right\} \leq \int_{\Delta(z_k,\delta/2)} |h|^p \, dA$$
$$\leq |\Delta(z_k,\delta/2)| \sup_{z\in\Delta(z_k,\delta/2)} \{|h(z)|^p\}.$$

There exists a constant depending on n, p, δ such that

$$\leq C(1-|z_k|^2)^2 \sum_{l=0}^{n-1} |f^{(l)}(z_k)|^p \sup_{z\in\Delta(z_k,\delta/2)} \{|z-z_k|\}^{lp}.$$

Let α and γ be the Euclidean center and radius of $\Delta(z_k, \delta/2) = D(\alpha, \gamma)$ respectively, then using the triangle inequality we have

$$|z - z_k| \leq |z - \alpha| + |\alpha - z_k| \leq 2\gamma \leq C_\delta(1 - |z_k|^2)$$

for all $z \in \Delta(z_k, \delta/2)$. Finally, we have that

$$(1-|z_k|^2)^2 \sum_{l=0}^{n-1} |f^{(l)}(z_k)|^p \sup_{z\in\Delta(z_k,\delta/2)} \{|z-z_k|\}^{lp} \leq C_{\delta,n}(1-|z_k|^2)^2 \sum_{l=0}^{n-1} |f^{(l)}(z_k)|^p (1-|z_k|^2)^{lp}$$

□

The following lemma is based on Lemma 2.1 in [5] by Luecking and is stated here for easy reference.

Lemma 3.7. *There exists a constant depending on n, p, ϵ such that for all $f \in \Delta(\xi, \epsilon)$*

$$|f^{(n)}(\xi)|^p (1 - |\xi|^2)^{np+2} \leq C \int_{\Delta(\xi,\epsilon)} |f|^p \, dA.$$

Proof. With $\Delta(\xi, \epsilon) = D(\alpha, \gamma)$, then we see that $|\xi - \alpha| < \epsilon\gamma$. First, we consider $\gamma = 1$ and will later rescale things.

By the Cauchy integral formula, we have

$$f^{(n)}(\xi) = \frac{n!}{2\pi i}\int_{|w-\alpha|=t} \frac{f(w)}{(w-\xi)^{n+1}}dw$$

where $r = (1+\epsilon)/2$ and $t > r$. With $w = \alpha + te^{i\theta}$, $dw = te^{i\theta}\,d\theta$ and taking absolute values, we get

$$|f^{(n)}(\xi)| \leq \frac{n!}{2\pi}\int_0^{2\pi} \frac{|f(\alpha+te^{i\theta})|}{|\alpha+te^{i\theta}-\xi|^{n+1}}t\,d\theta.$$

Now, we can estimate

$$\frac{1}{|\alpha+te^{i\theta}-\xi|} \leq \frac{1}{r-\epsilon} = \frac{2}{1-\epsilon}$$

for all $t > n$ and $0 < \theta < 2\pi$. Applying this estimate then integrating in t

$$\begin{aligned}|f^{(n)}(\xi)|(1-r) = |f^{(n)}(\xi)|\int_r^1 dt &\leq \frac{n!2^{n+1}}{(1-\epsilon)^{n+1}2\pi}\int_0^{2\pi}\int_r^1 |f(\alpha+te^{i\theta})|t\,dt\,d\theta \\ &\leq \frac{n!2^{n+1}}{(1-\epsilon)^{n+1}}\int_{D(\alpha,1)} |f(w)|\,dA(w).\end{aligned}$$

Using $1-r = (1-\epsilon)/2$ and then raising everything to the pth power, we get

$$\begin{aligned}|f^{(n)}(\xi)|^p &\leq \left(\frac{n!2^{n+2}}{(1-\epsilon)^{n+2}}\right)^p \left(\int_{D(\alpha,1)} |f(w)|\,dA(w)\right)^p \\ &\leq \left(\frac{n!2^{n+2}}{(1-\epsilon)^{n+2}}\right)^p \int_{D(\alpha,1)} |f(w)|^p\,dA(w)\end{aligned}$$

the last inequality coming from Hölder's.

As mentioned above, we must now rescale everything by γ so replace f by $f(\alpha+\gamma z)$ on both sides of the inequality to get

$$\gamma^{np+2}|f^{(n)}(\xi)|^p \leq \left(\frac{n!2^{n+2}}{(1-\epsilon)^{n+2}}\right)^p \int_{D(\alpha,\gamma)} |f(w)|^p\,dA(w).$$

Finally, since $\gamma = \dfrac{\epsilon(1-|\xi|^2)}{(1-\epsilon^2|\xi|^2)} > \epsilon(1-|\xi|^2)$, we have the desired result. □

Proposition 3.8. *Let Γ be a uniformly discrete sequence with separation constant δ. There exists a constant depending on n, p, δ such that*

$$\sum_k \sum_{l=0}^{n-1} (1-|z_k|^2)^{2+lp} |f^{(l)}(z_k)|^p \le C \sum_k \inf \left\{ \int_{\Delta(z_k,\delta/2)} |g|^p \, dA : g \in E_k \right\}$$

for all $f \in A^p$ where $E_k = \{g \in A^p(\Delta(z_k, \delta/2) : g - f$ has zero of order at least n at $z_k\}$.

Combining the previous two propositions, we see that in the case of uniformly discrete sequences with each point being repeated n times, the regular notion of multiple sampling is equivalent to our new one.

Finally, we observe that generalized sampling sequences satisfy certain fundamental properties similar to those developed for regular sampling sequences. One important property of sampling sequences, is that they are invariant under Möbius transformations. For a general sampling sequence Γ, let $K(\Gamma)$ be the largest constant that such that the lower sampling inequality holds.

Proposition 3.9. *Let Γ be a general sampling sequence with respect to the scheme $\{(Z_k, G_k)\}$ with sampling constant $K(\Gamma)$. Let ϕ_ξ be a conformal map from $\mathbb{D}$ to $\mathbb{D}$. Then $\phi_\xi(\Gamma)$ is a general sampling sequence with respect to the scheme $\{(Z_k, G_k)\}$ and $K(\phi_\xi(\Gamma)) = K(\Gamma)$.*

Proof. Recall that $|\phi_\xi'|^2$ is the Jacobian so that

$$\int_{\phi_\xi(\Omega)} |f(w)|^p \, dA(w) = \int_\Omega |f(\phi_\xi(z))|^p |\phi_\xi'(z)|^2 \, dA(z).$$

Then the map $f \mapsto f(\phi_\xi)[\phi_\xi']^{\frac{2}{p}}$ is an isometry from $A^p(\phi_\xi(G_k))$ to $A^p(G_k)$. Moreover, it takes functions that vanish on $\phi_\xi(z_k)$ to functions that vanish on z_k. Therefore, we have that

$$\inf\left\{\int_{\phi_\xi(G_k)} |g|^p\, dA : q \equiv f \text{ in } A^p(\phi_\xi(G_k))/N(\phi_\xi(Z_k))\right\}$$
$$= \inf\left\{\int_{G_k} |g|^p\, dA : g \equiv f \text{ in } A^p(G_k)/N(Z_k)\right\}.$$

□

We record the following elementary lemma for we will use this idea several times throughout the proceedings.

Lemma 3.10. *Let $1 < p < \infty$ and $f_1, \ldots, f_n \in L^p$, then*

$$|f_1 + \cdots + f_n|^p \leq n^{p-1}\left(|f_1|^p + \cdots + |f_n|^p\right).$$

Proof. This is a consequence of the fact that x^p is a convex function for $p > 1$:

$$\left|\frac{1}{n}f_1 + \cdots + \frac{1}{n}f_n\right|^p \leq \left(\frac{1}{N}|f_1| + \cdots + \frac{1}{n}|f_n|\right)^p \leq \frac{1}{N}|f_1|^p + \cdots + \frac{1}{n}|f_n|^p.$$

Then

$$|f_1 + \cdots + f_n|^p \leq n^{p-1}\left(|f_1|^p + \cdots + |f_n|^p\right).$$

□

4 NECESSITY

First, we turn to the proof of the necessity in Theorem 2.3. The ideas involved don't deviate far from those used for the regular notion of sampling. However, they require adjustment to the context of the general sampling scheme. Indeed, we will see that choosing the right function to apply to our lower sampling inequality, we can get $D^-(\Gamma) \geq 1/p$. The difficulty lies in show that we can shift Γ outward (toward the boundary) by a pseudohyperbolically small amount and the resulting sequence will also be general sampling. We must also get a lower bound on the pseudohyperbolic size of the shift to see that we have reduced the density enough to get strict inequality.

4.1 Technical Lemmas

We want to show that general sampling sequences are preserved under pseudohyperbolically small outward radial shifts. First, we need the following technical lemmas.

Lemma 4.1. *Let $f \in A^p(\mathbb{D})$ and $\rho(z,\xi) < r/2$, then there exists a constant that depends on r such that*

$$|f(z)|^p \leq \frac{C}{|\Delta(\xi,r)|} \int_{\Delta(\xi,r)} |f|^p \, dA.$$

Proof. Since $|\phi_\xi(z)| < r/2$ and $|f \circ \phi_\xi|^p$ is sub-harmonic,

$$\begin{aligned} |f(z)|^p = |f(\phi_\xi(\phi_\xi))|^p &\leq \frac{1}{D(\phi_\xi(z), r/2)} \int_{D(\phi_\xi(z),r/2)} |f \circ \phi_\xi|^p \, dA \\ &\leq \frac{4}{r^2} \int_{\Delta(0,r)} |f \circ \phi_\xi|^p \, dA. \end{aligned}$$

Now, for $w \in \Delta(0,r)$

$$|\phi_\xi'(w)|^2 = \frac{(1-|\xi|^2)^2}{|1-\bar{\xi}w|^4} \geq \frac{(1-|\xi|^2)^2}{(1+r)^4}$$

and so

$$\begin{aligned} |f(z)|^p &\leq \frac{4(1+r)^4}{r^2(1-|\xi|^2)^2} \frac{(1-|\xi|^2)^2}{(1+r)^4} \int_{\Delta(0,r)} |f \circ \phi_\xi(w)|^p \, dA(w) \\ &\leq \frac{4(1+r)^4}{r^2(1-|\xi|^2)^2} \int_{\Delta(0,r)} |f \circ \phi_\xi(w)|^p |\phi_\xi'(w)|^2 \, dA(w) \\ &= \frac{4(1+r)^4}{r^2(1-|\xi|^2)^2} \int_{\Delta(\xi,r)} |f|^p \, dA. \end{aligned}$$

Finally, we can use

$$\frac{1}{|\Delta(\xi,r)|} = \frac{(1-r^2|\xi|^2)^2}{r^2(1-|\xi|^2)^2} \geq \frac{(1-r^2)^2}{r^2(1-|\xi|^2)^2}$$

to get

$$\begin{aligned} |f(z)|^p &\leq \frac{4(1+r)^4}{(1-r^2)^2}\frac{(1-r^2)^2}{r^2(1-|\xi|^2)^2}\int_{\Delta(\xi,r)}|f|^p\,dA \\ &\leq \frac{4(1+r)^4}{(1-r^2)^2}\frac{1}{|\Delta(\xi,r)|}\int_{\Delta(\xi,r)}|f|^p\,dA. \end{aligned}$$

□

Lemma 4.2. *Let $f \in A^p$ and $\rho(z,\xi) < r/4$, then there exists a constant depending on r such that*

$$|f(z) - f(\xi)|^p \leq C\rho(z,\xi)^p \frac{1}{|\Delta(\xi,r)|}\int_{\Delta(\xi,r)}|f|^p\,dA.$$

Proof. We can write $|f(z) - f(\xi)| = |f_\xi(\phi_\xi(z)) - f_\xi(0)| = |\int_0^{\phi_\xi(z)} f_\xi'(w)dw|$ where $f_\xi(z) = f(\phi_\xi(z))$. Now

$$\begin{aligned} \left|\int_0^{\phi_\xi(z)} f_\xi'(w)dw\right| &\leq |\phi_\xi(z)| \sup\left\{|f_\xi'(w)| : |w| \leq |\phi_\xi(z)|\right\} \\ &= \rho(z,\xi)\sup\left\{\left|\frac{1}{2\pi i}\int_{|t-w|=r/4}\frac{f_\xi(t)}{(t-w)^2}dt\right| : |w| \leq |\phi_\xi(z)|\right\} \\ &\leq \rho(z,\xi)\frac{4}{r}\sup\left\{|f_\xi(t)| : |t| \leq |\phi_\xi(z)| + r/4\right\}. \end{aligned}$$

Finally, we can apply Lemma 2.2 to f_ξ. For $|t| \leq |\phi_\xi(z)| + r/4 < r/2$, since $|t| = \rho(\xi,\phi_\xi(t)) < r/2$ we have

$$|f_\xi(t)| = |f(\phi_\xi(t))| \leq \left(\frac{C}{|\Delta(\xi,r)|}\int_{\Delta(\xi,r)|}|f|^p\,dA\right)^{1/p}$$

□

Lemma 4.3. *Suppose $z, \xi \in \mathbb{D}$ with $\rho(z,\xi) < R < 1$, then there exists a constant depending* *on r and R such that*

$$|\Delta(z,r)| \leq C|\Delta(\xi,r)|.$$

Proof. We will make use of the identity that

$$1 - |\phi_\xi(z)|^2 = \frac{(1-|z|^2)(1-|\xi|^2)}{|1-\bar{\xi}z|^2}$$

which gives us that

$$\frac{1}{(1-|\xi|^2)^2} \leq \frac{(1-|z|^2)^2}{(1-R^2)^2|1-\bar{\xi}z|^4}.$$

Similarly, since $\rho(rz, r\xi) \leq r\rho(z,\xi) < rR$

$$\frac{1}{(1-|rz|^2)^2} \leq \frac{(1-|r\xi|^2)^2}{(1-r^2R^2)^2|1-r^2\bar{\xi}z|^4}.$$

So we can estimate the ratio of the areas as follows.

$$\begin{aligned}\frac{|\Delta(z,r)|}{|\Delta(\xi,r)|} = \frac{r^2(1-|z|^2)^2}{r^2(1-|\xi|^2)^2}\frac{(1-|r\xi|^2)^2}{(1-|rz|^2)^2} &\leq \frac{(1-|z|^2)^4}{(1-R^2)^2|1-\bar{\xi}z|^4}\frac{(1-|r\xi|^2)^2)}{(1-|rz|^2)^2} \\ &\leq \frac{(1-|z|^2)^4}{(1-R^2)^2|1-\bar{\xi}z|^4}\frac{(1-|r\xi|^2)^4}{(1-r^2R^2)^2|1-r^2\bar{\xi}z|^4}.\end{aligned}$$

Since $(1-|z|^2) \leq 2(1-|z|) \leq 2|1-\bar{\xi}z|$, we have

$$\frac{(1-|z|^2)^4}{|1-\bar{\xi}z|^4} \leq 2^4.$$

Finally, we get that

$$\frac{|\Delta(z,r)|}{|\Delta(\xi,r)|} \leq \frac{2^8}{(1-R^2)^2(1-r^2R^2)^2}.$$

□

4.2 Proof

We will consider a linear shift on each open set G_k. Choose $r_k \geq 1$ such that if $\psi_k(z) = r_k z$ for all $z \in G_k$, then $\psi_k(G_k) \subset \mathbb{D}$. We want to show that under such a small shift, the new sequence is also general sampling. First of all, we verify that $\{(\psi_k(Z_k), \psi_k(G_k))\}$ is indeed a sampling scheme. Fix $\xi \in \Gamma$, then $\Delta(\xi, \epsilon) \subset G_k$. Let $z \in \mathbb{D}$ such that $\rho(\xi, z) = \epsilon$, then

$$\begin{aligned}\rho(r_k\xi, r_k z) &= \left|\frac{r_k z - r_k \xi}{1 - r_k^2 \bar{\xi} z}\right| = r_k \left|\frac{z - \xi}{1 - r_k^2 \bar{\xi} z}\right| \\ &\geq \left|\frac{z - \xi}{1 - r_k^2 \bar{\xi} z}\right| \geq \left|\frac{z - \xi}{1 - \bar{\xi} z}\right| = \rho(\xi, z) = \epsilon.\end{aligned}$$

Therefore, $\Delta(r_k\xi, \epsilon) \subset \psi_k(\Delta(\xi, \epsilon)) \subset \psi_k(G_k)$. Also, if we assume $\rho(z, r_k z) \leq \eta$ for all $z \in G_k$, then clearly $\rho(\psi_k(G_k)) = \sup\{\rho(z, w) : z, w \in \psi_k(G_k)\} \leq \dfrac{R + \eta}{1 + \eta R}$.

Proposition 4.4. *Suppose $\Gamma = \cup_k Z_k$ is a general sampling sequence for A^p with respect to the scheme $\{(Z_k, G_k)\}$. Let $\psi_k(z) = r_k z$ for all $z \in G_k$. There exists an $\eta < 1/8$ such that if for all k,*

$$\rho(z, r_k z) \leq \eta \text{ for all } z \in G_k$$

then $\psi(\Gamma) = \cup_k \psi_k(Z_k)$ is a general sampling sequence for A^p with respect to the scheme $\{(\psi(Z_k), \psi(G_k))\}$ and its sampling constant can be approximated by that of Γ.

Proof. First note that is η is sufficiently small, then the pseudohypebolic distance between clusters $\psi(Z_k)$ remains bounded away from 0.

For a $f \in A^p$, consider the map

$$g = f + B \mapsto \tilde{g} = f + r_k^{-2/p} B \circ \psi_k^{-1}.$$

This gives a 1-to-1 correspondence between elements over which we are taking the infimum. We will approximate the difference between any two such pairs which will give us an approximation of the difference between the infimums themselves.

First of all, we change variables and apply Minkowski's inequality:

$$\begin{aligned}\left|\left(\int_{\psi_k(G_k)} |f + r_k^{-2/p} B \circ \psi_k^{-1}|^p \, dA\right)^{1/p} - \left(\int_{G_k} |f + B|^p \, dA\right)^{1/p}\right| \\ = \left|\left(\int_{G_k} |r_k^{2/p} f \circ \psi_k + B|^p \, dA\right)^{1/p} - \left(\int_{G_k} |f + B|^p \, dA\right)^{1/p}\right| \\ \leq \left(\int_{G_k} |r_k^{2/p} f \circ \psi_k - f|^p \, dA\right)^{1/p}.\end{aligned}$$

There exists a constant that depends on p that allows us to split this integral up

$$\begin{aligned}\int_{G_k} |r_k^{2/p} f \circ \psi_k - f|^p \, dA &= \int_{G_k} |r_k^{2/p} f \circ \psi_k - f \circ \psi_k + f \circ \psi_k - f|^p \, dA \\ &\leq C\left(\int_{G_k} |r_k^{2/p} f \circ \psi_k - f \circ \psi_k|^p + |f \circ \psi_k - f|^p \, dA\right).\end{aligned}$$

Now, since $\psi_k(G_k) \subset (G_k)_{1/2}$

$$\begin{aligned}\int_{G_k} |r_k^{2/p} f \circ \psi_k - f \circ \psi_k|^p \, dA &= \frac{(r_k^{2/p} - 1)^p}{r_k^2} \int_{\psi_k(G_k)} |f|^p \, dA \\ &\leq \frac{(r_k^{2/p} - 1)^p}{r_k^2} \int_{(G_k)_{1/2}} |f|^p \, dA.\end{aligned}$$

Here we can assume $|z| > 1/2$ and $1 \leq r_k < 2$, so that

$$\begin{aligned}\eta^p > \rho(z, r_k z)^p &= \left(\frac{(r_k - 1)|z|}{1 - r_k|z|^2)}\right)^p \geq \frac{(r_k - 1)^p |z|^p}{r_k^2} \\ &\geq C\frac{(r_k^2 - 1)^p}{r_k^2} \geq C\frac{(r_k^{2/p} - 1)^p}{r_k^2}.\end{aligned}$$

Finally, we see that

$$\int_{G_k} |r_k^{2/p} f \circ \psi_k - f \circ \psi_k|^p \, dA \leq C\eta^p \int_{(G_k)_{1/2}} |f|^p \, dA.$$

For the other integral, we apply Lemma 2.3 with $r = 1/2$ to get

$$\int_{G_k} |f \circ \psi_k - f|^p \, dA \leq \int_{G_k} \left(C\rho(z, \psi_k(z))^p \frac{1}{|\Delta(z, 1/2)|} \int_{\Delta(z,1/2)} |f(w)|^p \, dA(w) \right) dA(z)$$
$$< C\eta^p \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|\Delta(z, 1/2)|} \chi_{\Delta(z,1/2)}(w) \chi_{G_k}(z) |f(w)|^p \, dA(w) \, dA(z).$$

Next, we apply Fubini's Theorem and use Lemma 2.4. If $z \in G_k$ and $w \in \Delta(z, 1/2)$, then $w \in (G_k)_{1/2}$ and $z \in \Delta(w, 1/2)$. So then

$$\int_{G_k} |f \circ \psi_k - f|^p \, dA$$
$$\leq C\eta^p \int_{(G_k)_{1/2}} |f(w)|^p \int_{\Delta(w,1/2)} \frac{1}{|\Delta(z, 1/2)|} \, dA(z) \, dA(w)$$
$$\leq C\eta^p \int_{(G_k)_{1/2}} |f(w)|^p |\Delta(w, 1/2)| \sup \left\{ \frac{1}{|\Delta(z, 1/2)|} : z \in \Delta(w, 1/2) \right\} dA(w)$$
$$\leq C\eta^p \int_{(G_k)_{1/2}} |f|^p \, dA.$$

Putting this together now, we have shown

$$\left| \left(\int_{\psi_k(G_k)} |f + r_k^{-2/p} B \circ \psi_k^{-1}|^p \, dA \right)^{1/p} - \left(\int_{G_k} |f + B|^p \, dA \right)^{1/p} \right|$$
$$\leq C\eta \left(\int_{(G_k)_{1/2}} |f|^p \, dA \right)^{1/p}$$

Now, let $E_k = A^p(G_k)/N(Z_k)$ and $F_k = A^p(\psi(G_k))/N(\psi(Z_k))$. Using Minkowski's inequality for the sum and the fact that the $(G_k)_{1/2}$'s have bounded overlap, we get

$$\begin{aligned}
&\left| \left(\sum_k \inf \left\{ \int_{\psi(G_k)} |g|^p \, dA : g \equiv f \text{ in } F_k \right\} \right)^{1/p} - \left(\sum_k \inf \left\{ \int_{G_k} |g|^p \, dA : g \equiv f \text{ in } E_k \right\} \right)^{1/p} \right| \\
&\leq \left(\sum_k \left| \left(\inf \int_{\psi(G_k)} |g|^p \, dA : g \equiv f \text{ in } F_k \right)^{1/p} - \left(\inf \int_{G_k} |g|^p \, dA : g \equiv f \text{ in } E_k \right)^{1/p} \right|^p \right)^{1/p} \\
&\qquad \leq \left(\sum_k \left| C\eta \left(\int_{(G_k)_{1/2}} |f|^p \, dA \right)^{1/p} \right|^p \right)^{1/p} \leq C\eta \|f\|_p.
\end{aligned}$$

Recall that $K(\Gamma)$ is the largest constant such that the lower sampling inequality holds. Let $\epsilon > 0$, there exists a function $f \in A^p$ such that $\|f\|_p = 1$ and

$$\left(\sum_k \inf \left\{ \int_{G_k} |g|^p \, dA : g \equiv f \text{ in } E_k \right\} \right)^{1/p} \leq K(\Gamma)^{1/p} + \epsilon.$$

We also have that

$$K(\psi(\Gamma))^{1/p} \leq \left(\sum_k \inf \left\{ \int_{G_k} |g|^p \, dA : g \equiv f \text{ in } F_k \right\} \right)^{1/p}.$$

Therefore,

$$K(\psi(\Gamma))^{1/p} - K(\Gamma)^{1/p} \leq C\eta + \epsilon.$$

We can get the same inequality with Γ and $\psi(\Gamma)$ reversed. And since this is true for all ϵ, we get

$$|K(\psi(\Gamma))^{1/p} - K(\Gamma)^{1/p}| \leq C\eta$$

$\square$

The upper bound on the pseudohyperbolic distance of the shift of the sequence allows for a shifted sequence to remain a general sampling sequence. Now, we want to consider a lower bound on the shift so that we can have

$$D^-(\psi(\Gamma)) < D^-(\Gamma).$$

Lemma 4.5. *For each* $G_k \in \mathbb{D} \setminus D(0, 1/2)$*, choose* $r_k > 1$ *such that* $\sup_{z\in G_k} \rho(r_k z, z) = \eta$ *for all* $z \in G_k$*, then*

$$1/8(1-R)\eta \leq \inf_{z\in G_k} \rho(r_k z, z)$$

for all $z \in G_k$*.*

Proof. Let $a = \inf_{z\in G_k} |z|$, then $\dfrac{a+R}{1+aR} = \sup_{z\in G_k} |z|$. Using $r_k < \dfrac{1+aR}{a+R}$, we estimate

$$\rho(r_k a, a) = \frac{(r_k-1)a}{1-r_k a^2} \geq \frac{(r_k-1)a}{1-a^2}$$

and

$$\eta = \rho\left(r_k \frac{a+R}{1+aR}, \frac{a+R}{1+aR}\right) = \frac{(r_k-1)\dfrac{a+R}{1+aR}}{1-r_k\left(\dfrac{a+R}{1+aR}\right)^2} \leq \frac{(r_k-1)\dfrac{a+R}{1+aR}}{1-\dfrac{a+R}{1+aR}}.$$

Combining this, we see that

$$\begin{aligned}
\frac{\rho(r_k a, a)}{\rho\left(r_k \dfrac{a+R}{1+aR}, \dfrac{a+R}{1+aR}\right)} &\geq \frac{a\left(1-\dfrac{a+R}{1+aR}\right)}{\dfrac{a+R}{1+aR}(1-a^2)} = \frac{\dfrac{a(1+aR)-a^2-aR}{1+aR}}{\dfrac{(a+R)(1-a^2)}{1+aR}} \\
&= \frac{a(1+aR-a-R)}{(a+R)(1-a^2)} = \frac{a(1-R)(1-a)}{(a+R)(1-a^2)} \\
&= \frac{a(1-R)}{(a+R)(1+a)} \geq \frac{1/2(1-R)}{(2)(2)} = 1/8(1-R).
\end{aligned}$$

Then we have

$$1/8(1-R)\eta \le \rho(r_k a, a) \le \rho(r_k z, z)$$

for all $z \in G_k$. □

Lemma 4.6. *Suppose* Γ *has bounded density and* $\rho(\psi(z), z) \ge \alpha$ *for all* $z \in \Gamma$*, then*

$$D(\psi(\Gamma), 0, r) \le (1-\alpha) D(\Gamma, 0, r) + C \left(\log \frac{1}{1-r}. \right)^{-1}$$

Proof. Since $\rho(\psi(z), z) \ge \alpha$ for z, $|\phi(z)| \ge \dfrac{|z|+\alpha}{1+\alpha|z|}$. We want to show that

$$\frac{|z|+\alpha}{1+\alpha|z|} \ge |z|^{(1-\alpha)}.$$

This turns into a real analysis problem to prove that $f(x) = \dfrac{x+\alpha}{1+\alpha x} - x^{(1-\alpha)} \ge 0$ for all $x \in [0,1]$. Observe that $f(1) = \dfrac{1+\alpha}{1+\alpha} - 1^{(1-\alpha)} = 0$. Computing

$$\begin{aligned} f'(x) &= \frac{(1+\alpha x)(1) - (x+\alpha)(\alpha)}{(1+\alpha x)^2} - (1-\alpha)x^{-\alpha} \\ &= \frac{1-\alpha^2}{(1+\alpha x)^2} - (1-\alpha)x^{-\alpha} = (1-\alpha)\left[\frac{1+\alpha}{(1+\alpha x)^2} - x^{-\alpha}\right] \\ &\le \frac{1+\alpha}{(1+\alpha x)^2} - x^{-\alpha} < 0. \end{aligned}$$

Therefore, we have $|\psi(z)| \ge \dfrac{|z|+\alpha}{1+\alpha|z|} \ge |z|^{1-\alpha}$ for all $|z| > 1/2$. And so

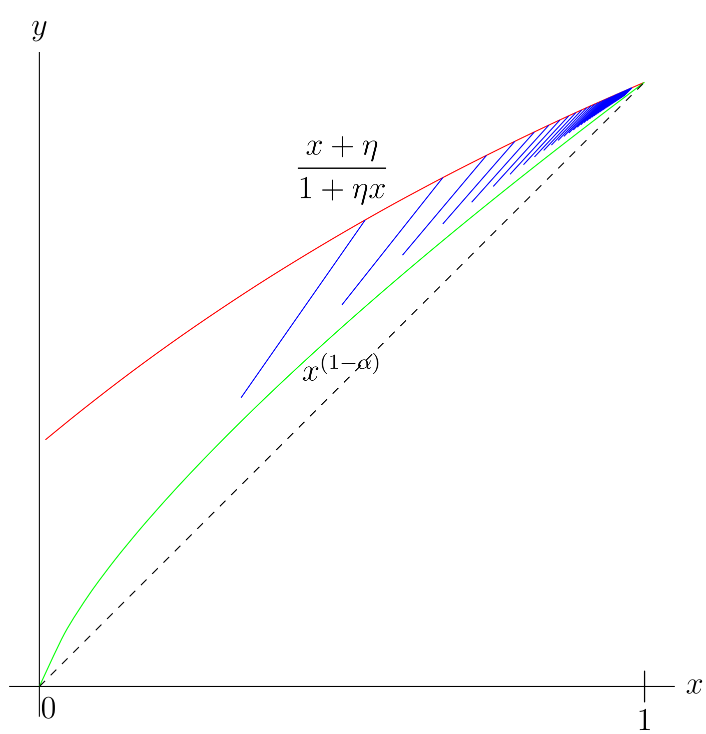
$$\log \frac{1}{|\psi(z)|} \le (1-\alpha) \log \frac{1}{|z|}.$$

To compute the inequality with the densities in question, we observe that

$$
\begin{aligned}
\left(\log\frac{1}{1-r}\right)^{-1} D(\psi(\Gamma),0,r) &= \sum_{1/2<|\psi(z_k)|<r} \log\frac{1}{|\psi(z_k)|} \\
&\leq \sum_{1/2<|z_k|<r} \log\frac{1}{|\psi(z_k)|} + \sum_{1/4<|z_k|<1/2} \log\frac{1}{|\psi(z_k)|} \\
&\leq (1-\alpha) \sum_{1/2<|z_k|<r} \log\frac{1}{|z_k|} + C \\
&= (1-\alpha)\left(\log\frac{1}{1-r}\right)^{-1} D(\Gamma,0,r) + C
\end{aligned}
$$

□

The following graph demonstrates how the r_k's can be chosen and how we get our lower bound. Let $x = |z|$ and y be the modulus of the resulting shift. Here we are using a pseudohyperbolic diameter of $R = 0.25$ and an upper bound on the pseudohyperbolic shift of $\eta = 0.4$.



Theorem 4.7. *Suppose Γ is a general sampling sequence with respect to the admissible sampling scheme $\{(Z_k, G_k)\}$, then*

$$1/p \leq D^-(\Gamma).$$

Proof. Let $\xi_j \in \mathbb{D}$, $r_j < 1$ and $\epsilon_j \to 0$ so that

$$D^-(\Gamma) \leq D(\Gamma, \xi_j, r_j) < D^-(\Gamma) + \epsilon_j.$$

Let $\Gamma_j = \phi_{\xi_j}(\Gamma)$, so then

$$D(\Gamma_j, 0, r_j) = D(\Gamma, \xi_j, r_j).$$

For $\Gamma_j = \{w_{jl}\}_l$, consider the function given by

$$f_j(z) = \prod_{|w_{jl}|<r_j} \frac{1}{|w_{jl}|} \frac{w_{jl} - z}{1 - \bar{w}_{jl} z}$$

where the w_{jl}'s are repeated according to multiplicity. Since $f_j(w_{jl}) = 0$ of the corresponding order, if $\phi_{\xi_j}(Z_k) \subset D(0, r_j)$ then

$$\inf \left\{ \int_{\phi_{\xi_j}(G_k)} |g|^p \, dA : g \equiv f_j \text{ in } A^p(\phi_{\xi_j}(G_k))/N(\phi_{\xi_j}(Z_k)) \right\} = 0$$

Then

$$\sum_k \inf \left\{ \int_{\phi_{\xi_j}(G_k)} |g|^p \, dA : g \equiv f_j \text{ in } A^p(\phi_{\xi_j}(G_k))/N(\phi_{\xi_j}(Z_k)) \right\}$$
$$= \sum_{\phi_{\xi_j}(Z_k) \not\subset D(0,r_j)} \inf \left\{ \int_{\phi_{\xi_j}(G_k)} |g|^p \, dA : g \equiv f_j \text{ in } A^p(\phi_{\xi_j}(G_k))/N(\phi_{\xi_j}(Z_k)) \right\}$$
$$\leq \sum_{\phi_{\xi_j}(Z_k) \not\subset D(0,r_j)} |\phi_{\xi_j}(G_k)| \prod_{|w_{jl}|<r_j} \frac{1}{|w_{jl}|^p}.$$

Notice that each $\phi_{\xi_j}(G_k) \subset \mathbb{D} \setminus D(0, \frac{r_j - R}{1 - r_j R})$ and that the $\phi_{\xi_j}(G_k)$'s have bounded overlap. Therefore, there exists a constant such that

$$\sum_{\phi_{\xi_j}(Z_k) \not\subset D(0,r_j)} |\phi_{\xi_j}(G_k)| \leq C \left| \mathbb{D} \setminus D(0, \frac{r_j - R}{1 - r_j R}) \right|$$

$$= C\left(1 - (\frac{r_j - R}{1 - r_j R})^2\right) = C\frac{1 - 2r_j R + r_j^2 R^2 - (r_j^2 - 2r_j R + R^2)}{(1 - r_j R)^2}$$

$$= C\frac{(1 - r_j^2)(1 - R^2)}{(1 - r_j R)^2} \leq C\frac{(1 - r_j^2)(1 - R^2)}{(1 - R^2)} \leq 2C(1 - r_j)$$

We have

$$\sum_k \inf \left\{ \int_{\phi_{\xi_j}(G_k)} |g|^p \, dA : g \equiv f_j \text{ in } A^p(\phi_{\xi_j}(G_k))/N(\phi_{\xi_j}(Z_k)) \right\}$$
$$\leq C(1 - r_j) \prod_{|w_{jl}| < r_j} \frac{1}{|w_{jl}|^p}.$$

Now, we can rewrite the product as

$$\prod_{|w_{jl}| < r_j} \frac{1}{|w_{jl}|^p} = \exp\left\{ p \sum_{1/2 < |w_{jl}| < r_j} \log \frac{1}{|w_{jl}|} + p \sum_{|w_{jl}| < 1/2} \log \frac{1}{|w_{jl}|} \right\}$$
$$\leq \exp\left\{ pD(\Gamma_j, 0, r_j)\left(\log \frac{1}{1 - r_j}\right) + pC \right\} \leq C(1 - r_j)^{-p(D^-(\Gamma) + \epsilon_j)}.$$

Putting this together, we have

$$\sum_k \inf \left\{ \int_{\phi_{\xi_j}(G_k)} |g|^p \, dA : g \equiv f_j \text{ in } A^p(\phi_{\xi_j}(G_k))/N(\phi_{\xi_j}(Z_k)) \right\} \leq C(1 - r_j)^{1 - p(D^-(\Gamma) + \epsilon_j)}.$$

And since $\|f_j\|_p \geq |f_j(0)| = 1$, in order to not violate the lower sampling inequality, we need

$$1 - pD^-(\Gamma) \leq 0$$

which implies that

$$1/p \leq D^-(\Gamma).$$

□

Finally, to get the strict inequality of the density condition, we apply the above Theorem 4.7 to the sequence shifted outward toward the boundary. To get

$$1/p \leq D^-(\psi(\Gamma)) \leq (1-\alpha)D^-(\Gamma) < D^-(\Gamma).$$

5 SUFFICIENCY

Our proof of sufficiency of the density condition in Theorem 2.3 will follow the argument from [6] in which Luecking gives a characterization of sampling measures. One key component of this argument will be the behavior of weak limits of Möbius transformations of the sequence. But first, we need a few simple lemma's about our sampling norm.

Lemma 5.1. *Let Γ be a sequence with sampling scheme $\{(Z_k, G_k)\}$, and let $C > 0$, then*

$$C^p \inf\left\{\int_{G_k} |g|^p \, dA : g \equiv f/C\right\} = \inf\left\{\int_{G_k} |g|^p \, dA : g \equiv f\right\}.$$

Proof. Let $h \equiv f/C$, then $Ch \equiv f$ so

$$\inf\left\{\int_{G_k} |g|^p \, dA : g \equiv f\right\} \leq \int_{G_k} |Ch|^p \, dA = C^p \int_{G_k} |h|^p \, dA.$$

Therefore,

$$\inf\left\{\int_{G_k} |g|^p \, dA : g \equiv f\right\} \leq C^p \inf\left\{\int_{G_k} |g|^p \, dA : g \equiv f/C\right\}.$$

Similarly, Let $h \equiv f$, then $h/C \equiv f/C$ and so

$$\inf\left\{\int_{G_k} |g|^p\, dA : g \equiv f/C\right\} \leq \int_{G_k} |h/C|^p\, dA = C^{-p}\int_{G_k} |h|^p\, dA.$$

Therefore,

$$\inf\left\{\int_{G_k} |g|^p\, dA : g \equiv f/C\right\} \leq C^{-p}\inf\left\{\int_{G_k} |g|^p\, dA : g \equiv f\right\}$$

□

Lemma 5.2. *Let ϕ be a Möbius transformation and $f \in A^p$, then*

$$\inf\left\{\int_{G_k} |g|^p\, dA : g \equiv f\circ\phi \ \textit{in}\ A^p(G_k)/N(Z_k)\right\}$$
$$= \inf\left\{\int_{\phi(G_k)} |g|^p|\phi'|^2\, dA : g \equiv f \ \textit{in}\ A^p(\phi(G_k))/N(\phi(Z_k))\right\}$$

Proof. Let $h \equiv f$ in $A^p(\phi(G_k))/N(\phi(Z_k))$, then $h\circ\phi \equiv f\circ\phi$ in $A^p(G_k)/N(Z_k)$ and so

$$\inf\left\{\int_{G_k} |g|^p\, dA : g \equiv f\circ\phi\right\} \leq \int_{G_k} |h\circ\phi|^p\, dA = \int_{\phi(G_k)} |h|^p|\phi'|^2\, dA.$$

Similarly, let $h \equiv f\circ\phi$ in $A^p(G_k)/N(Z_k)$, then $h\circ\phi \equiv f$ in $A^p(\phi(G_k))/N(\phi(Z_k))$ and so

$$\inf\left\{\int_{\phi(G_k)} |g|^p|\phi'|^2\, dA : g \equiv f\right\} \leq \int_{\phi(G_k)} |h\circ\phi|^p|\phi'|^2\, dA = \int_{G_k} |h|^p\, dA$$

□

5.1 Weak Limits

As we previously observed, a sequence that is general sampling for A^p is necessarily not a zero set for A^p. In fact, we will use the idea that, in some sense, a sampling sequence is not even close to being a zero set. This idea motivates our method of proof and is central to our intuition about sampling sequences. With this in mind, we proceed to study sequences under weak limits of Möbius transformations.

A sequence $\Gamma = \{z_k\}_k$ is said to be naturally ordered if $|z_k| \leq |z_{k+1}|$ for all k. Suppose $\{\Gamma_j\}$ is a sequence of sequences in $\mathbb{D}$. We say that Γ_j converges weakly to Λ, denoted $\Gamma_j \rightharpoonup \Lambda$, if there exists a natural ordering of each $\Gamma_j = \left\{z_k^j\right\}$ such that

$$\lim_j z_k^j = \lambda_k$$

for all $\lambda_k \in \Lambda$. Such weak limits have been used previously in studying sampling sequences in the case of uniformly discrete sequences. In our generalization, we will need to provide consideration for arbitrary repetitions of point as well as points that may coalesce into a repeated point within a cluster.

We denote $W(\Gamma)$ to be the set of sequences Λ such that $\phi_{\xi_j}(\Gamma) \rightharpoonup \Lambda$ for some $\xi_j \in \mathbb{D}$. Recall that if a sequence of distinct points has bounded density, then it is the finite union of uniformly discrete sequences. When we write the union of sequences that are not disjoint, we mean the union as multiset where the multiplicity of each point in the union is the sum of the multiplicities. Since Γ has bounded density, we can write it as the finite union of distinct sets, each of which can then be split into a finite union of uniformly discrete sequences. Therefore, we have

$$\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_n$$

where each Γ_i is uniformly discrete. Then for each $\xi \in \mathbb{D}$, we have $\phi_\xi(\Gamma) = \phi_\xi(\Gamma_1) \cup \cdots \cup \phi_\xi(\Gamma_n)$. Choose Λ_i so that $\phi_{\xi_j}(\Gamma_i) \rightharpoonup \Lambda_i$, then $\phi_{\xi_j}(\Gamma) \rightharpoonup \Lambda$ where $\Lambda = \cup_i \Lambda_i$.

We want to determine how sequences of bounded density behave under weak limits. First of all, it is clear from the definition of the upper uniform density and the fact that the counting function is additive that $D^+(\Gamma_1 \cup \Gamma_2) \leq D^+(\Gamma_1) + D^+(\Gamma_2)$. We want to generalize the following lemma from [2] for uniformly discrete sequences to sequences of bounded density.

Lemma 5.3. *Suppose* Γ *is uniformly discrete and* $\Lambda \in W(\Gamma)$*, then*

$$D^-(\Gamma) \leq D^-(\Lambda) \text{ and } D^+(\Lambda) \leq D^+(\Gamma).$$

Proposition 5.4. *Suppose* Γ *has bounded density and let* $\Lambda \in W(\Gamma)$*, then*

i. Λ *has bounded density*

ii. $D^-(\Gamma) \leq D^-(\Lambda)$

iii. $D^+(\Lambda) \leq D^+(\Gamma)$.

Proof. (i.) We have that

$$\begin{aligned} D^+(\Lambda) = D^+(\Lambda_1 \cup \ldots \cup \Lambda_n) &\leq D^+(\Lambda_1) + \ldots + D^+(\Lambda_n) \\ &\leq D^+(\Gamma_1) + \ldots + D^+(\Gamma_n) < \infty \end{aligned}$$

where each Γ_i is uniformly discrete and thus has finite density.

(ii.) Since $\lim_{j\to\infty} n(\phi_{\xi_j}(\Gamma), z, s) = n(\Lambda, z, s)$ for all s and z, and by (i.) we have $n(\phi_{\xi_j}(\Gamma), z, s) \leq C\dfrac{1}{1-s}$, by the Lebesgue dominated convergence theorem

$$\lim_{j\to\infty} \int_0^r n(\phi_{\xi_j}(\Gamma), z, s)ds = \int_0^r n(\Lambda, z, s)ds.$$

Therefore,

$$\begin{aligned} \inf_{z\in\mathbb{D}} \int_0^r n(\Gamma, z, s)ds &\leq \lim_{j\to\infty} \int_0^r n(\Gamma, \phi_{\xi_j}(z), s)ds \\ &= \lim_{j\to\infty} \int_0^r n(\phi_{\xi_j}(\Gamma), z, s)ds = \int_0^r n(\Lambda, z, s)ds. \end{aligned}$$

Which implies that $D^-(\Gamma) \leq D^-(\Lambda)$.

(iii.) Similarly to the proof of (ii.), we see that

$$\int_0^r n(\Lambda, z, s)ds = \lim_{j\to\infty} \int_0^r n(\phi_{\xi_j}(\Gamma), z, s)ds$$
$$= \lim_{j\to\infty} \int_0^r n(\Gamma, \phi_{\xi_j}(z), s)ds \le \sup_{z\in\mathbb{D}} \int_0^r n(\Gamma, z, s)ds.$$

Which implies that $D^+(\Lambda) \le D^+(\Gamma)$. □

The following lemma provides consideration for discrete points that coalesce into repeated points under weak limits.

Lemma 5.5. *Let $\{\Gamma_j\}$ be a sequence of sequences in $\mathbb{D}$ such that $\Gamma_j \rightharpoonup \Lambda$. Suppose λ is a point of multiplicity M in Λ. If f_j is a holomorphic function that vanishes on each $z_k^j \to \lambda$ and $f_j \to f$ uniformly on compact sets, then f has a zero of order greater than or equal to M.*

Proof. For any $\delta > 0$, there exists a J such that $D(\lambda, \delta)$ contains each z_k^j for all k and all $j \ge J$. Then

$$M \le \frac{1}{2\pi i} \int_{|z-\lambda|=\delta} \frac{f_j'(z)}{f_j(z)}\, dz.$$

Since $f_j \to f$ uniformly on compact sets,

$$\frac{1}{2\pi i} \int_{|z-\lambda|=\delta} \frac{f_j'(z)}{f_j(z)}\, dz \to \frac{1}{2\pi i} \int_{|z-\lambda|=\delta} \frac{f'(z)}{f(z)}\, dz$$

for any δ. Then

$$\lim_{\delta\to 0} \frac{1}{2\pi i} \int_{|z-\lambda|=\delta} \frac{f'(z)}{f(z)}\, dz = \lim_{\delta\to 0} \left[\lim_{j\to\infty} \frac{1}{2\pi i} \int_{|z-\lambda|=\delta} \frac{f_j'(z)}{f_j(z)}\, dz \right] \ge M.$$

□

It is quite easy to see that every sequence of sequences $\{\Gamma_j\}$ has a subsequence that converges to a (possibly empty) sequence Γ (see for example [2]). Now we can restate our main theorem along with one intermediate step.

Theorem 5.6. *Let Γ be a sequence of points with the admissible scheme $\{(Z_k, G_k)\}$, the following are equivalent:*

i. Γ *is a general sampling sequence with respect to the scheme* $\{(Z_k, G_k)\}$

ii. *there exists a $q < p$ such that every element of $W(\Gamma)$ is a set of uniqueness for A^q.*

iii. $D^-(\Gamma) > 1/p$.

Proof of (iii.) $\implies$ (ii.): Suppose not, then for all $q < p$, there exists an element of $\Lambda_q \in W(\Gamma)$ that is a zero set for A^q. Let f be the function that vanishes on Λ_q, without loss of generality we can assume $f(0) = 1$. By Jenson's formula and since $M_q(r, f) = o((1-r)^{-1/q})$

$$\begin{aligned}\sum_{|z_k|<r} \log \frac{r}{|z_k|} &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})|\, d\theta \\ &\leq \log \left(e^{\epsilon}(1-r)^{-1/q}\right) = 1/q \log \frac{1}{1-r} + \epsilon.\end{aligned}$$

Now, we can get a lower bound in terms of the lower uniform density of Λ_q.

$$\begin{aligned}\sum_{|z_k|<r} \log \frac{r}{|z_k|} &\geq \sum_{|z_k|<r} \log \frac{1}{|z_k|} + n(\Lambda_q, 0, r) \log r \\ &\geq \sum_{1/2<|z_k|<r} \log \frac{1}{|z_k|} + n(\Lambda_q, 0, r) \log r \\ &= \left(\log \frac{1}{1-r}\right) D(\Lambda_q, 0, r) + n(\Lambda_q, 0, r) \log r \\ &\geq \left(\log \frac{1}{1-r}\right) D(\Lambda_q, 0, r) + C\frac{\log r}{1-r} \\ &\geq \left(\log \frac{1}{1-r}\right) D^-(\Lambda_q) + C\frac{\log r}{1-r}.\end{aligned}$$

Putting this together, we have that

$$C\frac{\log r}{1-r} + D^{-}(\Lambda_q)\log\frac{1}{1-r} \le 1/q\log\frac{1}{1-r} + \epsilon$$

which implies that

$$D^{-}(\Gamma) \le D^{-}(\Lambda_q) \le 1/q.$$

Since this is true for all $q < p$, we get the contradiction $D^{-}(\Gamma) \le 1/p$. □

5.2 Proof

Lemma 5.7. *Let $U_\epsilon = \{f \in A^q : \|f\|_q \le 1, |f(0)| \ge \epsilon\}$ and suppose every $\Lambda \in W(\Gamma)$ is a set of uniqueness for A^q. Then there exists a constant $\delta > 0$ such that*

$$\delta < \sum_k \left(\inf\left\{\int_{\phi_\xi(G_k)} |g|^q \, dA : g \equiv f \ \textit{in} \ A^q(\phi_\xi(G_k)/N(\phi_\xi(Z_k)\right\}\right)^{p/q}$$

for all $f \in U_\epsilon$ and $\xi \in \mathbb{D}$.

Proof. Suppose not. Suppose there exists a sequence of functions $\{f_n\}$ in U_ϵ and a sequence $\{\xi_n\}$ in $\mathbb{D}$ such that

$$\lim_{n\to\infty} \sum_k \left(\inf\left\{\int_{\phi_{\xi_n}(G_k)} |g|^q \, dA : g \equiv f_n \text{ in } A^q(\phi_{\xi_n}(G_k))/N(\phi_{\xi_n}(Z_k))\right\}\right)^{p/q} = 0. \qquad (1)$$

Pass to a subsequence such that $\phi_{\xi_n}(\Gamma) \rightharpoonup \Lambda$. Since U_ϵ is a normal family, pass to a further subsequence so that $f_n \to f$ uniformly on compact subsets. Clearly, $|f(0)| \ge \epsilon$ so f is non-trivial function in A^q. If we can demonstrate that $f = 0$ on Λ we would have a contradiction since Λ is a set of uniqueness for A^q.

If we let g_{nk} be the g that realizes the infimum in each term of (1), we can pass to a further subsequence so that $\lim_n g_{nk} = g_k$ in a neighborhood of λ_k. Since $f_n - g_{nk} \to f - g_k$

in a neighborhood of λ, it follows from Lemma 5.5 that $f - g_k$ has a zero at λ_k whose order is at least the number repetitions of λ_k in Λ. But the integral of $|g_{nk}|^q$ over a neighborhood of λ_k tends to 0 by (1). Thus g_k is identically 0 and so f itself has a zero of the appropriate order at λ_k. $\square$

A somewhat subtle point is that if Z_k^n is the cluster that contains z_k^n then, because of the separation between clusters, only z_j^n in the same cluster can converge to λ_k. This prevents λ_k from being repeated more often that accounted for by the g_k in the proof.

The following result is due to Luecking and can be found in [7]. The idea here is we want to use the inequality $|f(\xi)|^q \geq \epsilon \|f \circ \phi_\xi\|_q^q$ which isn't always true. However, we can choose ϵ small enough so that the set over which it is true is significant enough.

Lemma 5.8. *Let $\epsilon > 0$, $q < p$ and $f \in A^p$, if*

$$\Omega = \left\{ \xi \in \mathbb{D} : |f(\xi)|^q < \epsilon \int_{\mathbb{D}} |f(z)|^q |\phi_\xi(z)'|^2 \, dA(z) \right\}$$

then

$$\int_{\Omega} |f|^p \, dA \leq C\epsilon^{p/q} \int_{\mathbb{D}} |f|^p \, dA.$$

Since

$$\int_{\mathbb{D}} |f|^p \, dA = \int_{\Omega} |f|^p \, dA + \int_{\mathbb{D}\setminus\Omega} |f|^p \, dA \leq C\epsilon^{p/q} \int_{\mathbb{D}} |f|^p \, dA + \int_{\mathbb{D}\setminus\Omega} |f|^p \, dA$$

we have

$$(1 - C\epsilon^{p/q}) \int_{\mathbb{D}} |f|^p \, dA \leq \int_{\mathbb{D}\setminus\Omega} |f|^p \, dA.$$

Therefore, we can choose ϵ small enough, independent of f so that

$$1/2 \int_{\mathbb{D}} |f|^p \, dA \leq \int_{\mathbb{D}\setminus\Omega} |f|^p \, dA.$$

We call the set where this is true, $\mathbb{D} \setminus \Omega$, the "*good*" set for f.

Proof of (ii.) $\implies$ (i.): For any $f \in A^p$, with $\|f\|_q \neq 0$, let $\xi \in \mathbb{D}$ where

$$|f \circ \phi_\xi(0)|^q = |f(\xi)|^q \geq \epsilon \int_{\mathbb{D}} |f(z)|^q |\phi_\xi(z)'|^2 \, dA(z) = \epsilon \|f \circ \phi_\xi\|_q^q$$

for ϵ sufficiently small as in the previous lemma. Since $\dfrac{f \circ \phi_\xi}{\|f \circ \phi_\xi\|_q}$ satisfies the hypothesis of the previous lemma, we apply it to get

$$\sum_k \left(\inf \left\{ \int_{\phi_\xi(G_k)} |g|^q \, dA : g \equiv f \circ \phi_\xi \text{ in } A^q(\phi_\xi(G_k))/N(\phi_\xi(Z_k)) \right\} \right)^{p/q} > \delta \|f \circ \phi_\xi\|_q^p.$$

By subharmonicity, we get the lower inequality $\delta \|f \circ \phi_\xi\|_q^p \geq \delta |f(\xi)|^p$. For each k, let g_k be the function such that

$$\int_{G_k} |g_k|^p \, dA = \inf \left\{ \int_{G_k} |g|^p \, dA : g \equiv f \text{ in } A^p(G_k)/N(Z_k) \right\}.$$

Since $g_k = f$ on Z_k, $g_k \circ \phi_\xi = f \circ \phi_\xi$ on $\phi_\xi(Z_k)$. So we have

$$\sum_k \left(\int_{G_k} |g_k|^q |\phi_\xi'|^2 \, dA \right)^{p/q} = \sum_k \left(\int_{\phi_\xi(G_k)} |g_k \circ \phi_\xi|^q \, dA \right)^{p/q} > \delta |f(\xi)|^p.$$

Integrate both sides with respect to ξ over the "*good*" set to get

$$\int_{\mathbb{D}} \sum_k \left(\int_{\mathbb{D}} \chi_{G_k}(z) |g_k(z)|^q |\phi_\xi'(z)|^2 \, dA(z) \right)^{p/q} dA(\xi) > \delta \int_{good} |f(\xi)|^p \, dA \geq \delta/2 \|f\|_p^p.$$

Exchanging the sum and the outer integral and observing that $|\phi_\xi'|^2$ is the kernel of a bounded linear operator on $L^{p/q}(\mathbb{D})$, we finally have

$$\sum_k \int_{G_k} |g_k|^p \, dA \geq C \|f\|_p^p$$

□

To prove that the kernel $|\phi_\xi(z)'|^2 = \dfrac{(1-|\xi|^2)^2}{|1-\bar{\xi}z|^4}$ gives us a bounded linear operator on $L^{p/q}(\mathbb{D})$, we will use the following theorem by Zhu [14].

Lemma 5.9. *Suppose $a, b, c \in \mathbb{R}$, $0 < c$ and $1 < p$ and define*

$$K(\xi, z) = \frac{(1-|\xi|^2)^a(1-|z|^2)^b}{|1-\bar{\xi}z|^c}.$$

Then the integral operator $Tf(\xi) = \int_{\mathbb{D}} K(\xi, z)f(z)\, dA(z)$ defines a bounded linear operator on $L^p(\mathbb{D})$ if and only if

$$c \leq 2 + a + b, \ -pa < 1 < p(b+1).$$

If we take $a = 2, b = 0, c = 4$ and replace p with p/q, we have that $|\phi_\xi(z)'|^2$ defines a bounded linear operator on $L^{p/q}(\mathbb{D})$.

6 FURTHER EXPLORATION

We would like to make note of some assumptions made throughout the dissertation. The first of these assumptions was that $1 \leq p < \infty$. We did make use of Hölders and Minkowski's inequalities several times. However, many of these uses could be circumvented and different, slightly more difficult arguments could be devised.

The second assumption used throughout our characterization was that the sequence have bounded density. The main reason we needed bounded density was to get an upper bound on the growth of $n(\Gamma, \xi, r)$. As conjectured in Chapter 2, we wonder whether it really is sufficient to only consider such sequences. On one hand, in practice it would make sense to only choose a finite number of points in any given disk. And one would think that at a certain point, you have attained enough information about the function from those points. It certainly would seem like a challenge to prove such a thing, given the amount of freedom in any given sampling scheme. For example, you could just choose one point from each cluster and repeat it several times or have points spread out all over the disk.

In addition to exploring the necessity of the assumptions made, one might also want to generalize the spaces of functions used. For example, it is common to study a weighted Bergman Space A^p_α where $\alpha > -1$, the space of functions holomorphic in $\mathbb{D}$ such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha \right)^{1/p} < \infty.$$

Indeed, the Seip inspired characterization of sampling and interpolation sequences for A^p_α has been carried out in [3]. The analogous density condition is $D^-(\Gamma) > (\alpha+1)/p$. An analogous general sampling problem could be formulated and we would expect an analogous result for the weighted Bergman Spaces. Also, going back to the spaces of functions originally considered by Seip, one might devise a sampling problem for the growth spaces A^{-n}, the spaces of analytic functions such that

$$\|f\|_{-n} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^n |f(z)| < \infty$$

where the values of the function and its derivatives are incorporated. Is there then motivation to consider a quotient space similar to the one we considered.

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