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## Closed-Range Composition Operators on Weighted Bergman Spaces and Applications

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Closed-Range Composition Operators on  
Weighted Bergman Spaces and Applications

Closed-Range Composition Operators on  
Weighted Bergman Spaces and Applications

A dissertation submitted in partial fulfillment  
of the requirements for the degree of  
Doctorate of Philosophy in Mathematics

by

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## ABSTRACT

We will discuss necessary and sufficient conditions for the composition operator  $C_\varphi$  to be closed range on the weighted Bergman space  $\mathbb{A}_\alpha^p$  for  $1 \leq p < \infty$  with weights of the form  $(1 - |z|^2)^\alpha$  for  $\alpha > -1$ . The function  $\varphi$  is an analytic self-map of the unit disk  $\mathbb{D}$  and our results extend those previously intended for the classical Bergman space  $\mathbb{A}^2$ . We will also give applications.

## ACKNOWLEDGEMENTS

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## 1 Introduction

Let  $\mathbb{D}$  denote the unit disk  $\{z \in \mathbb{C} : |z| < 1\}$  and let  $\mathbb{T}$  denote the unit circle  $\{z \in \mathbb{C} : |z| = 1\}$ . Let  $m$  denote normalized Lebesgue measure on  $\mathbb{T}$ , and let  $A$  denote normalized two-dimensional Lebesgue measure on the unit disk  $\mathbb{D}$ . For a point  $a$  in  $\mathbb{D}$  and  $r$ ,  $0 < r < 1$ , let  $D(a, r) := \{z \in \mathbb{D} : |z - a| < r\}$  and let  $\Delta(a, r) := \{z \in \mathbb{D} : \rho(z, a) < r\}$  where  $\rho(z, a)$  denotes the *pseudohyperbolic metric* defined for  $z$  and  $a$  in  $\mathbb{D}$  by

$$\rho(z, a) = \frac{|z - a|}{|1 - \bar{z}a|}.$$

For  $a$  in  $\mathbb{D}$  and  $0 \leq b \leq 1$ , let  $D_b(a)$  denote a disk centered at  $a$  with radius  $b(1 - |a|)$ . We let  $\mathcal{H}(\mathbb{D})$  be the set of all functions  $f$  which are analytic in  $\mathbb{D}$ . A function  $\varphi$  is said to be an *analytic self-map of the unit disk*  $\mathbb{D}$  if  $\varphi \in \mathcal{H}(\mathbb{D})$  and  $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ . If  $\varphi$  is an analytic self-map of  $\mathbb{D}$ , then the composition operator  $C_\varphi$  is defined on  $\mathcal{H}(\mathbb{D})$  by  $C_\varphi(f) = f \circ \varphi$ . If  $X$  is a Banach space of analytic functions in  $\mathbb{D}$ , then we say that a composition operator  $C_\varphi$  on a space  $X$  is *compact* if every bounded set in  $X$  is mapped to a set whose closure is compact. The composition operator  $C_\varphi$  is said to be *closed-range* on  $X$  if  $C_\varphi(X)$  is a closed subspace of  $X$ . By the Open Mapping Theorem, for nontrivial  $\varphi$ , this occurs when there exists a constant  $c > 0$  such that  $\|f \circ \varphi\|_X \geq c\|f\|_X$  for all  $f$  in  $X$ . For  $1 \leq p < \infty$ , the Hardy space  $H^p$  is the set of all functions  $f$  in  $\mathcal{H}(\mathbb{D})$  such that

$$\|f\|_{H^p}^p := \sup_{0 < r < 1} \int_{\mathbb{T}} |f(r\xi)|^p dm(\xi) < \infty$$

and  $H^\infty$  is the set of all functions  $f$  in  $\mathcal{H}(\mathbb{D})$  such that

$$\|f\|_{H^\infty} := \sup_{z \in \mathbb{D}} |f(z)| < \infty.$$

For  $\alpha > -1$  we define  $dA_\alpha$  by  $dA_\alpha := c_\alpha \cdot (1 - |z|^2)^\alpha dA(z)$ , where  $c_\alpha = \alpha + 1$ . The weighted Bergman space  $\mathbb{A}_\alpha^p$  is given by:

$$\mathbb{A}_\alpha^p := \{f : f \in \mathcal{H}(\mathbb{D}) \text{ and } \|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f|^p dA_\alpha < \infty\}.$$

In this chapter, we will discuss several classical Banach spaces of analytic functions on the unit disk. We will review standard results describing when a composition operator is bounded, compact, and closed-range on these spaces. In chapter 2, we will give a necessary and sufficient condition for when  $C_\varphi$  is closed-range on the weighted Bergman space  $A_\alpha^p$ . Note that in [28], Nina Zorboska gives conditions for when  $C_\varphi$  is closed-range on the Hardy space  $H^2$  and the weighted Bergman space  $A_\alpha^2$ . These conditions involve the Nevanlinna counting function, which can be difficult to work with.

A good reference for the following discussion is [19]. For  $0 < r < 1$  and a point  $\xi$  in  $\mathbb{T}$ , let  $S(\xi, r)$  denote the interior of the convex hull of the union of  $\{\xi\}$  and  $\{z \in \mathbb{D} : |z| \leq r\}$ . We call  $S(\xi, r)$  the *Stolz region* based at  $\xi \in \mathbb{T}$  with contact angle  $2 \arctan(r)$ .



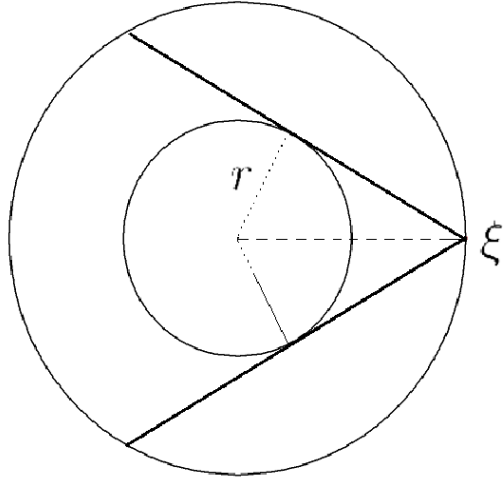


Figure 1: A stolz region

Note that if a curve approaches  $\xi$  from inside the region  $S(\xi, r)$ , then this curve cannot be tangent to the unit circle. For  $f$  in  $H^p$ , we say  $f$  has nontangential limit  $L$  at the point  $\xi$  if for all  $r$  in  $(0, 1)$  and for every sequence  $\{z_n\}$  in  $S(\xi, r)$  that converges to the point  $\xi$ , we have  $\lim_{n \rightarrow \infty} f(z_n) = L$ . Also, for  $r$  in  $(0, 1)$  and for any complex function  $f$  defined on  $\mathbb{D}$ , we define the *nontangential maximal function*  $N_r f$  on  $\mathbb{T}$  by

$$N_r f(\xi) = \sup\{|f(z)| : z \in S(\xi, r)\}.$$

For any  $f$  in  $H^p$ ,  $0 < p < \infty$  and any  $r$  in  $(0, 1)$ , we have  $N_r f \in L^p(\mathbb{T})$ . It is well known (see [19]) that the nontangential limits of  $f$  in  $H^p$ , denoted  $f^*(e^{i\theta})$ , exist almost everywhere  $[m]$  on  $\mathbb{T}$  and  $f^* \in L^p(\mathbb{T})$ . Furthermore,  $\|f^*\|_p = \|f\|_p$  for all  $f$  in  $H^p$ . An *inner function* is a function  $M$  in  $H^\infty$  such that  $|M^*| = 1$  a.e.  $[m]$ . A function of the form

$$S_\mu(z) = \exp\left\{-\int_{-\pi}^{\pi} \frac{\zeta + z}{\zeta - z} d\mu(t)\right\},$$

where  $\mu$  is a positive Borel measure on  $\mathbb{T}$  that is singular with respect to  $m$ , is known as a

*singular inner function*. Note that such a function does not have any zeros in  $\mathbb{D}$ . Let  $\{a_n\}$  be a sequence of points in  $\mathbb{D}$  such that  $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$ . For such sequences, there is an associated *Blaschke product*  $B$  defined on  $\mathbb{D}$  by

$$B(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \bar{a}_n z} \frac{|a_n|}{a_n},$$

where  $\frac{|a_n|}{a_n}$  is taken to be 1 if  $a_n = 0$ . The function  $B$  is in  $H^\infty$  and  $|B^*(e^{i\theta})| = 1$  almost everywhere on  $\mathbb{T}$ . Hence, each Blaschke product is an inner function, as is each singular inner function. Now, every inner function  $M$  can be factored uniquely as the product of a Blaschke product and a singular inner function. That is, every inner function  $M$  may be written in the form

$$M(z) = c \cdot B(z) \cdot S_\mu(z)$$

where  $c$  is a constant such that  $|c| = 1$ . An *outer function* is a function of the form

$$G(z) = c \cdot \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log \varphi(e^{it}) dt \right\}$$

where  $c$  is a constant such that  $|c| = 1$ , and  $\varphi$  is a positive measurable function on  $\mathbb{T}$  such that  $\log \varphi \in L^1(\mathbb{T})$ . For  $0 < p \leq \infty$  and  $f$  in  $H^p$  such that  $f$  is not identically zero, the function  $\log |f^*|$  is in  $L^1(\mathbb{T})$  and

$$G_f(z) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |f^*(e^{it})| dt \right\}$$

is an outer function in  $H^p$ . For such  $f$ , there exists an inner function  $M_f$  such that  $f = M_f G_f$ .

Thus, for all  $p > 0$ , every  $f$  in  $H^p$  may be factored uniquely into the product of an inner function and an outer function. Thus, by our previous statement regarding the factorization of inner functions, we have that every  $f$  in  $H^p$  may be written uniquely in the form

$$f(z) = c \cdot B(z) \cdot S_\mu(z) \cdot G_f(z)$$

for  $z$  in  $\mathbb{D}$ , where  $G_f$  is an outer function in  $H^p$ , see [19].) Note that any analytic self-map  $\varphi$  of the unit disk  $\mathbb{D}$  is in  $H^\infty$ . Hence, by our above discussion,  $\varphi$  may be factored as above and so has only a few possible forms. The function  $\varphi$  may be written as a Blaschke product, a singular inner function, an outer function, or it may be written as a product of these types of functions.

It is natural to ask for which  $\varphi$  is the composition operator  $C_\varphi$  bounded, compact, or closed-range on a Banach space of analytic functions on  $\mathbb{D}$ . We will now catalog such results for various classical Banach spaces. These results are standard in the literature and more information can be found in [22] and [27]. We begin by examining composition operators on the Hardy space  $H^2$ .

### 1.1 Composition Operators on the Hardy Space $H^2$

Littlewood's Subordination Principle (see [22]) states that if  $\varphi$  is an analytic self-map of  $\mathbb{D}$  with  $\varphi(0) = 0$ , then for each  $f$  in  $H^2$ ,  $C_\varphi(f) \in H^2$  and  $\|C_\varphi(f)\| \leq \|f\|$ .

Thus, if  $\varphi$  fixes the origin, then  $C_\varphi$  is bounded on  $H^2$ . To see that this is the case for any holomorphic self-map  $\varphi$  of  $\mathbb{D}$ , we will use  $\alpha_\lambda(z) = \frac{\lambda-z}{1-\bar{\lambda}z}$ , the special automorphism of  $\mathbb{D}$  where  $\alpha_\lambda(\lambda) = 0$ ,  $\alpha_\lambda(0) = \lambda$ , and  $\alpha_\lambda^{-1} = \alpha_\lambda$ . Letting  $\lambda = \varphi(0)$ , we consider the function  $\psi = \alpha_\lambda \circ \varphi$  which is a holomorphic self-map of  $\mathbb{D}$  that fixes the origin. Then,  $\varphi = \alpha_\lambda^{-1} \circ \psi = \alpha_\lambda \circ \psi$  and by Littlewood's Subordination Principle, for all  $f$  in  $H^2$  we have

$$\begin{aligned} \|f \circ \varphi\|^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(\varphi(e^{i\theta}))|^2 d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f \circ \alpha_\lambda \circ \psi(e^{i\theta})|^2 d\theta \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2\pi} \int_0^{2\pi} |f \circ \alpha_\lambda(e^{i\theta})|^2 d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 |\alpha'_\lambda(e^{it})| dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 \frac{1 - |\lambda|^2}{|1 - \bar{\lambda}e^{it}|^2} dt \\
&\leq \frac{1 - |\lambda|^2}{(1 - |\lambda|)^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 dt \\
&= \frac{1 + |\lambda|}{1 - |\lambda|} \cdot \|f\|^2
\end{aligned}$$

Thus we have that  $C_\varphi$  is bounded on  $H^2$  for every analytic self-map  $\varphi$  of  $\mathbb{D}$ .  $\square$

We may also address the question of compactness of composition operators. The *First Compactness Theorem* (page 23 in [22]) states that the composition operator  $C_\varphi$  is a compact operator on  $H^2$  if  $\|\varphi\|_\infty < 1$ . In other words,  $C_\varphi$  is a compact composition operator if  $\varphi(\mathbb{D})$  is relatively compact. The Univalent Compactness theorem (see page 39 in [22]) says that if  $\varphi$  is a univalent self-map of  $\mathbb{D}$ , then,  $C_\varphi$  is compact on  $H^2$  if and only if

$$\lim_{|z| \rightarrow 1^-} \frac{1 - |\varphi(z)|}{1 - |z|} = \infty.$$

It should be noted that necessity in this theorem does not require univalence. As this requirement for compactness deals with a difference quotient, it is reasonable to think that there may be some relationship between this condition and the derivative of  $\varphi$  at the boundary of the disk.

Let  $\varphi$  be a holomorphic self-map of  $\mathbb{D}$ , and let  $\omega$  be a point on  $\partial\mathbb{D}$ . We say that  $\varphi$  has angular limit  $\mathcal{L} = \angle \lim_{z \rightarrow \omega} \varphi(z)$  if  $\varphi(z) \rightarrow \mathcal{L}$  as  $z \rightarrow \omega$  through any stolz region based at  $\omega$ . The map  $\varphi$  has an angular derivative at  $\omega$ , denoted  $\varphi'(\omega)$ , if for some point  $\eta$  in  $\partial\mathbb{D}$ ,

$$\angle \lim_{z \rightarrow \omega} \frac{\eta - \varphi(z)}{\omega - z}$$

exists. This suggests that the angular limit of  $\varphi$  at  $\omega$  exists and is equal to  $\eta$ . Hence, if  $\varphi$  has an angular derivative at any point on  $\partial\mathbb{D}$ , then it must have an angular limit of modulus one at that point.

The *Julia-Caratheodory Theorem* clarifies the relationship between compactness and the existence of angular derivatives. This theorem states that the angular derivative  $\angle \lim_{z \rightarrow \omega} \frac{\eta - \varphi(z)}{\omega - z}$  exists for some  $\eta$  in  $\partial\mathbb{D}$  if and only if  $\liminf_{z \rightarrow \omega} \frac{1 - |\varphi(z)|}{1 - |z|} = \delta$  for some  $\delta$ ,  $0 < \delta < \infty$ . But, by the *Univalent Compactness theorem*,  $\liminf_{z \rightarrow \omega} \frac{1 - |\varphi(z)|}{1 - |z|} < \infty$  implies that  $C_\varphi$  is not compact on  $H^2$ .

Next, we discuss when  $C_\varphi$  is compact on  $H^2$  for arbitrary self-maps  $\varphi$  of  $\mathbb{D}$ . In other words, we want a condition for compactness on  $H^2$  when  $\varphi$  is not necessarily univalent. For a function  $\varphi$  holomorphic on  $\mathbb{D}$ , the Nevanlinna Counting Function of  $\varphi$ , denoted  $N_\varphi$ , is defined as follows:

$$N_\varphi(w) = \begin{cases} \sum_{z \in \varphi^{-1}\{w\}} \log \frac{1}{|z|} & w \in \varphi(\mathbb{D}) \\ 0 & w \notin \varphi(\mathbb{D}) \end{cases}$$

For a function  $f$  analytic on  $\mathbb{D}$ , the *Littlewood-Paley Identity* (see [22]) gives that

$$\|f\|_2^2 = |f(0)|^2 + 2 \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z).$$

The *change-of-variable formula* (see [22]) states that for any analytic map  $\varphi$  on  $\mathbb{D}$ ,

$$\|C_\varphi(f)\|_2^2 = |f(\varphi(0))|_2^2 + 2 \int_{\mathbb{D}} |f'(w)|^2 N_\varphi(w) dA(w).$$

Notice that if  $\varphi$  is univalent, then the change-of-variable formula is just the Littlewood-Paley Identity with the substitution  $w = \varphi(z)$ .

Theorem 2.3 in [20] gives the following result. Suppose  $\varphi$  is a holomorphic self-map of  $\mathbb{D}$ . Then,  $C_\varphi$  is compact on  $H^2$  if and only if

$$\lim_{|w| \rightarrow 1^-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} = 0.$$

If  $\varphi$  is univalent, we have

$$N_\varphi(w) = \log \frac{1}{|z|} \approx 1 - |z|$$

for  $|z|$  large, where  $\varphi(z) = w$ . Thus, in the case that  $\varphi$  is univalent, this theorem is the same as the Univalent Compactness Theorem stated above.

In [3], it is shown that this condition on  $\varphi$  involving the Nevanlinna counting function is equivalent to the condition

$$\lim_{|a| \rightarrow 1^-} \int_{\mathbb{T}} \frac{1 - |a|^2}{|1 - \bar{a}\varphi(z)|^2} dm(z) = 0.$$

In [28], Nina Zorboska gives conditions regarding when a composition operator  $C_\varphi$  will be closed-range on  $H^2$  and on  $\mathbb{A}_\alpha^2$  for  $\alpha > -1$ . The function  $\pi_\varphi(w)$ , having domain  $\mathbb{D} \setminus \varphi(0)$ , is defined by

$$\pi_\varphi(w) = \frac{N_\varphi(w)}{\log \frac{1}{|w|}},$$

where  $N_\varphi(w)$  is the Nevanlinna Counting Function as defined above. For a positive constant  $c$ , the set  $G_c^\varphi$  is defined by

$$G_c^\varphi = \{z : \pi_\varphi(z) > c\}.$$

Theorem 3.4 in [28] states that a composition operator  $C_\varphi$  will be closed-range on  $H^2$  if and only if there exist positive constants  $c$  and  $\delta$  such that

$$A(G_c^\varphi \cap D(\xi, r)) > \delta \cdot A(\mathbb{D} \cap D(\xi, r)) \tag{1}$$

for all  $\xi$  in  $\partial\mathbb{D}$ , where  $D(\xi, r) = \{z \in \mathbb{D} : |z - \xi| < r\}$ . In [14], it is shown that condition (1) may be restated as follows. There exist constants  $\delta_1 > 0$  and  $b$ ,  $0 < b < 1$ , so that

$$A(G_c^\varphi \cap D_b(a)) > \delta_1 \cdot A(D_b(a)) \quad (2)$$

for every  $a$  in  $\mathbb{D}$  where  $D_b(a) = \{z \in \mathbb{D} : |z - a| < b(1 - |a|)\}$ .

A good reference for the following discussion is [28]. Suppose  $\varphi$  is a univalent function such that  $C_\varphi$  does not have closed-range on  $H^2$ . Let  $\psi$  be a holomorphic self-map of  $\mathbb{D}$  such that  $\psi(\mathbb{D})$  is contained in  $\varphi(\mathbb{D})$ . Let  $\omega$  be the self-map of  $\mathbb{D}$  defined by  $\omega := \varphi^{-1} \circ \psi$ . Then,  $\psi = \varphi \circ \omega$ . Let  $\{f_n\}$  be a sequence of functions in  $H^2$  such that  $\|f_n\|_{H^2} = 1$  and  $\|C_\varphi f_n\|_{H^2} \rightarrow 0$ . Then,

$$\|C_\psi f_n\|_{H^2} = \|f_n \circ \varphi \circ \omega\|_{H^2} \leq \|C_\omega\|_{H^2} \|f_n \circ \varphi\|_{H^2} \rightarrow 0.$$

Hence,  $C_\psi$  will not be closed-range on  $H^2$ .

Example 1 in [28] states that if there exists a point  $\xi \in \mathbb{T}$  and a neighborhood  $N_\xi$  about the point  $\xi$  such that  $N_\xi \cap \varphi(\mathbb{D}) = \emptyset$ , then  $C_\varphi$  will not be closed-range on  $H^2$ . To see that this is the case, choose a Euclidean disk  $D(\xi, r)$  to be contained in  $N_\xi$ . Then, for all  $z \in D(\xi, r)$ , we have that  $\gamma_\varphi(z) = 0$  and so the set  $G_c^\varphi$  is empty for all  $c > 0$ . Hence, for any  $c > 0$ ,  $A(G_c^\varphi \cap D(\xi, r)) = 0$  and condition (1) above will not be satisfied.

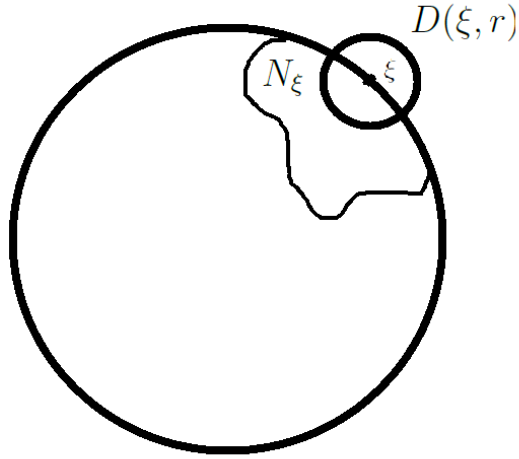


Figure 2: Example

Another example described in [28] is the following. A composition operator  $C_\varphi$  will not be closed-range on  $H^2$  if there is a disk  $D_1$  that is tangent to  $\partial D$  such that  $D_1 \cap \varphi(\mathbb{D}) = \emptyset$ . In this case, for any choice of  $b$ , we can choose a point  $a$  in  $\mathbb{D}$  close enough to the boundary of the disk so  $D_b(a)$  will be contained entirely in the disk  $D_1$ . Then, similar to the previous example, we have that  $\gamma_\varphi(z) = 0$  on  $D_b(a)$ . Hence,  $A(G_c^\varphi \cap D_b(a)) = 0$  for any  $c > 0$  and so condition (2) above is not satisfied.

In [28], Zorboska also remarks that the composition operator  $C_\varphi$  will not be closed-range on  $H^2$  if  $\varphi(\mathbb{D})$  is a proper subset of  $\mathbb{D} \setminus [0, 1)$ .

We will now introduce several other classical spaces of analytic functions in  $\mathbb{D}$ . For more information on the following spaces, see [27].



## 1.2 Composition Operators on the Bloch space $\mathcal{B}$

The *Bloch space*  $\mathcal{B}$  is the space of analytic functions on  $\mathbb{D}$  such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

Under the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)|,$$

$\mathcal{B}$  forms a Banach space. By Proposition 5.1 in [27],  $H^\infty$  is properly contained in  $\mathcal{B}$ , and  $\|f\|_{\mathcal{B}} \leq \|f\|_\infty$  for all  $f \in H^\infty$ . The set of analytic functions in  $\mathbb{D}$  having the property that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) |f'(z)| = 0$$

is called the *little Bloch space* and is denoted by  $\mathcal{B}_0$ . The little Bloch space is a closed subspace of  $\mathcal{B}$ .

For  $z$  in  $\mathbb{D}$ , let  $\tau_\varphi(z) = \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}$ . We can apply the Schwarz-Pick lemma to get  $|\tau_\varphi(z)| \leq 1$  for all  $z$  in  $\mathbb{D}$ . Now, for  $f$  in  $\mathcal{B}$ ,

$$\begin{aligned} (1 - |z|^2) |(f \circ \varphi)'(z)| &= (1 - |z|^2) |f'(\varphi(z))| |\varphi'(z)| \\ &= \frac{(1 - |z|^2) |\varphi'(z)|}{1 - |\varphi(z)|^2} (1 - |\varphi(z)|^2) |f'(\varphi(z))| \\ &= |\tau_\varphi(z)| (1 - |\varphi(z)|^2) |f'(\varphi(z))| \\ &\leq (1 - |\varphi(z)|^2) |f'(\varphi(z))| \end{aligned}$$

Thus  $C_\varphi$  is a bounded composition operator on  $\mathcal{B}$  for every analytic self-map  $\varphi$  of  $\mathbb{D}$ .

It is shown in Theorem 2 in [17] that  $C_\varphi$  will be compact on  $\mathcal{B}$  if and only if for every  $\varepsilon > 0$ , there exists  $r$ ,  $0 < r < 1$ , such that

$$\tau_\varphi(z) = \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} < \varepsilon$$

whenever  $|\varphi(z)| > r$ .

By theorem 2 in [26], for an analytic self-map  $\varphi$  of the unit disk  $\mathbb{D}$ , the composition operator  $C_\varphi$  is compact on  $\mathcal{B}$  if and only if

$$\lim_{n \rightarrow \infty} \|\varphi^n\|_{\mathcal{B}} = 0.$$

In [17], Madigan and Matheson give a similar condition for the compactness of a composition operator on  $\mathcal{B}_0$ . Theorem 1 in [17] states that  $C_\varphi$  is compact on  $\mathcal{B}_0$  if and only if

$$\lim_{|z| \rightarrow 1^-} |\tau_\varphi(z)| = 0.$$

In [11], a necessary and sufficient condition for a composition operator  $C_\varphi$  to be closed-range on  $\mathcal{B}$  is given. Letting  $\mathcal{C}$  be the closed subspace of constant functions, we have  $\|f\|_{\mathcal{B}/\mathcal{C}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)|$ . Theorem 0 in [11] states that  $C_\varphi$  will be closed-range on  $\mathcal{B}$  if and only if

$$\|f \circ \varphi\|_{\mathcal{B}/\mathcal{C}} \geq k \cdot \|f\|_{\mathcal{B}/\mathcal{C}}$$

for a constant  $k > 0$ .

For a subset  $K$  of  $\mathbb{D}$ , if there exists  $k > 0$  with

$$\sup\{(1 - |z|^2)|f'(z)| : z \in \mathbb{D}\} \leq k \cdot \sup\{(1 - |z|^2)|f'(z)| : z \in K\}$$

for every function  $f$  in  $\mathcal{B}$ , then  $K$  is called a *sampling set* for  $\mathcal{B}$ . Define  $F_\varepsilon := \varphi(\Lambda_\varepsilon)$  where  $\Lambda_\varepsilon := \{z \in \mathbb{D} : \tau_\varphi(z) \geq \varepsilon\}$  and  $\varepsilon > 0$ . Theorem 1 in [11] says that a composition operator  $C_\varphi$

will be closed-range on the Bloch space  $\mathcal{B}$  if and only if there exists  $\varepsilon > 0$  such that  $F_\varepsilon$  is a sampling set for  $\mathcal{B}$ . It is also shown in [11] that the set  $F_\varepsilon$  is a sampling set for  $\mathcal{B}$  if it satisfies the *reverse Carleson condition*. That is,  $F_\varepsilon$  is a sampling set for  $\mathcal{B}$  if there exist constants  $c$  and  $s$  with  $0 < c, s < 1$ , such that  $A(F_\varepsilon \cap \Delta(z, s)) \geq c \cdot A(\Delta(z, s))$  for all  $z$  in the unit disk  $\mathbb{D}$ . Hence, by Theorem 1 and Proposition 1 in [11], if  $F_\varepsilon$  satisfies the reverse Carleson condition, then the composition operator  $C_\varphi$  will be closed-range on  $\mathcal{B}$ . If  $\varphi$  is univalent, then, by theorem 2 in [11], the converse of the previous statement also holds.

Let  $\varphi$  be univalent and suppose that the composition operator  $C_\varphi$  is closed-range on  $\mathcal{B}$ . Then, for some  $\varepsilon > 0$ ,  $F_\varepsilon$  satisfies the reverse Carleson condition and by Proposition 3 in [11], there exists  $\delta > 0$  such that for every point  $\omega$  in  $\partial\mathbb{D}$ ,

$$\overline{\lim}_{\varphi(z) \rightarrow \omega} \frac{\text{dist}(\varphi(z), \partial(\varphi(\mathbb{D})))}{|\varphi(z) - \omega|} \geq \delta.$$

Example 1 in [11] shows that this condition is not sufficient for  $C_\varphi$  to be closed-range. In the second example given in [11], we let  $G = \mathbb{D} \setminus [0, 1)$  and  $\varphi$  is chosen to be the Riemann mapping onto  $G$ . By the Koebe One-Quarter Theorem, when  $\varphi$  is univalent, then

$$\tau_\varphi(z) \approx \frac{\text{dist}(\varphi(z), \partial G)}{1 - |\varphi(z)|}.$$

As  $\varphi(z)$  approaches a point  $w$  on the boundary of the disk other than 1, this ratio approaches 1. Then,  $F_\varepsilon$  contains all of  $\mathbb{D}$  except for a pseudohyperbolic neighborhood of the segment  $[0, 1)$ . Thus, we can choose  $r$  large enough so that every point  $z$  in  $\mathbb{D}$  is within pseudohyperbolic distance  $r$  of  $F_\varepsilon$ . So, there exists a constant  $c > 0$  such that  $A(F_\varepsilon \cap \Delta(z, r)) \geq c \cdot A(\Delta(z, r))$  for all  $z$  in the unit disk  $\mathbb{D}$ . Hence,  $F_\varepsilon$  satisfies the reverse Carleson condition and  $C_\varphi$  is closed-range.

Proposition 1 in [10] gives a necessary condition for the composition operator  $C_\varphi$  to be closed-range on  $\mathcal{B}$ . The proposition states that if  $C_\varphi$  is closed-range on  $\mathcal{B}$  then there will exist positive constants  $\varepsilon$  and  $r < 1$  so that, for all  $z$  in  $\mathbb{D}$ ,  $\rho(\varphi(\Lambda_\varepsilon), z) \leq r$ . Recall that  $\rho$  denotes the pseudohyperbolic metric. Theorem 2 in the same source ([10]) also gives a sufficient condition. This theorem gives that  $C_\varphi$  is closed-range on  $\mathcal{B}$  if for some positive constants  $\varepsilon$  and  $r$  with  $r < \frac{1}{4}$ , for all  $w$  in  $\mathbb{D}$  there exists a point  $z_w$  in  $\mathbb{D}$  so that  $\rho(\varphi(z_w), w) < r$  and  $|\tau_\varphi(z_w)| > \varepsilon$ .

### 1.3 Composition Operators on the Besov space $B_p$

For  $0 < p < \infty$ , the Besov space  $B_p$  is the collection of holomorphic functions in  $\mathbb{D}$  such that

$$\begin{aligned} \|f\|_{B_p}^p &= \int_{\mathbb{D}} |f^{(n)}(z)(1 - |z|^2)^n|^p d\lambda(z) \\ &= \int_{\mathbb{D}} |f^{(n)}(z)|^p (1 - |z|^2)^{np} d\lambda(z) < \infty \end{aligned}$$

for any positive integer  $n$  satisfying  $np > 1$  and where

$$d\lambda(z) = \frac{1}{(1 - |z|^2)^2} dA(z).$$

Under the norm  $|f(0)| + \|f\|_{B_p}$ ,  $B_p$  is a Banach space.

Theorem 5.17 in [27] gives the atomic decomposition for  $B_p$ . The theorem states that for  $p > 0$ , there exists a sequence  $\{a_k\}$  in  $\mathbb{D}$  such that for  $b > \max(0, \frac{p-1}{p})$ , the space  $B_p$  is comprised of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \left( \frac{1 - |a_k|^2}{1 - z\bar{a}_k} \right)^b$$

with  $c_k \in \ell^p := \{\{c_k\}_{k=1}^\infty \subset \mathbb{C} : \sum_{k=1}^\infty |c_k|^p < \infty\}$ .

Recall that  $\alpha_\lambda(z) = \frac{\lambda-z}{1-\bar{\lambda}z}$  is the special automorphism of the disk  $\mathbb{D}$  with  $\alpha_\lambda(\lambda) = 0$ ,  $\alpha_\lambda(0) = \lambda$ , and  $\alpha_\lambda^{-1} = \alpha_\lambda$ . By Theorem D in [24], for an analytic self-map  $\varphi$  of  $\mathbb{D}$ ,  $C_\varphi$  is a bounded operator on the Besov space  $B_p$  if and only if

$$\sup_{\lambda \in \mathbb{D}} \|C_\varphi \alpha_\lambda\|_{B_p} < \infty.$$

By theorem 3.5 in [24], for  $1 < p \leq q < \infty$ , when  $\varphi$  is a holomorphic self-map of  $\mathbb{D}$  then the following are equivalent:

1.  $C_\varphi : B_p \rightarrow B_q$  is a compact operator.
2.  $\|C_\varphi \alpha_\lambda\|_{B_q} \rightarrow 0$  as  $|\lambda| \rightarrow 1$ .

Not much is known about conditions for which  $\varphi$  will induce a compact composition operator on  $B_p$  for  $p$  in general.

## 1.4 Composition Operators on the Dirichlet Space

The Dirichlet space  $\mathcal{D}$  is the set of holomorphic functions  $f$  on  $\mathbb{D}$  such that

$$\|f\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

That is, if  $f$  is in  $\mathcal{D}$ , then its derivative is in  $\mathbb{A}^2$ . Note that  $\mathcal{D} = B_2$ , with an equivalent norm.

For  $p > 0$  and  $\mu$  a finite positive Borel measure, if there exists a constant  $0 < c < \infty$  such that

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq c \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)$$

for all  $f$  in  $\mathbb{A}_\alpha^p$ , then  $\mu$  is called a *Carleson measure* for  $\mathbb{A}_\alpha^p$ . As is stated in [15], an equivalent condition for  $\mu$  to be a Carleson measure is

$$\sup_{z \in \mathbb{D}} \frac{\mu(\Delta(z, \eta))}{|\Delta(z, \eta)|} < \infty$$

where, again,  $\Delta(z, \eta)$  denotes the pseudohyperbolic disk. We call  $\mu$  a *compact (or vanishing) Carleson measure* if

$$\sup_{r < |z| < 1} \frac{\mu(\Delta(z, \eta))}{|\Delta(z, \eta)|} \rightarrow 0$$

as  $r \rightarrow 1$ . Let  $n_\varphi$  denote the cardinality of the set  $\varphi^{-1}(w)$ . In [15], Luecking shows that a composition operator  $C_\varphi$  is bounded on the Dirichlet space  $\mathcal{D}$  if  $n_\varphi dA$  is a Carleson measure for  $\mathbb{A}_\alpha^p$  for some  $p > 0$ . By Proposition 5.1 in [16],  $C_\varphi$  is compact on  $\mathcal{D}$  if  $n_\varphi dA$  is a compact Carleson measure. In [15], Luecking also shows that  $C_\varphi$  is closed-range on the Dirichlet space  $\mathcal{D}$  if and only if there exists a constant  $c > 1$  such that

$$\frac{1}{c} \int |f'|^2 dA \leq \int |f'|^2 n_\varphi dA \leq c \int |f'|^2 dA$$

for every  $f$  in  $\mathcal{D}$ . If this condition is satisfied, then there exists  $R$ ,  $0 < R < 1$  and  $\delta > 0$  such that

$$\int_{\Delta(a, r)} n_\varphi dA \leq \delta |\Delta(a, r)|$$

for all  $z \in \mathbb{D}$  where, again,  $|\Delta(a, r)|$  denotes the area of the pseudohyperbolic disk  $\Delta(a, r)$ .

By part 2 of Corollary 2 in [11], when  $\varphi$  is univalent and the composition operator  $C_\varphi$  is bounded below on the Bloch space  $\mathcal{B}$ , then  $C_\varphi$  is also bounded below on the Dirichlet space. That is, if  $\varphi$  is univalent and  $C_\varphi$  is closed-range on  $\mathcal{B}$  then it is also closed-range on  $\mathcal{D}$ .

## 1.5 Composition Operators on BMO

Let  $I$  denote an interval that is contained in  $\mathbb{T}$  and let  $f$  be in  $L^2(\mathbb{T})$ . With  $|I|$  representing the length of the interval  $I$ , the mean of the function  $f$  over  $I$  is given by

$$f_I = \frac{1}{|I|} \int_I f(\theta) d\theta.$$

The space of all functions  $f$  in  $L^2(\mathbb{T})$  that have *bounded mean oscillation* is called  $BMO(\mathbb{T})$ .

A function  $f$  has bounded mean oscillation if

$$\|f\|_{BMO} := \sup_I \frac{1}{|I|} \int_I |f(\theta) - f_I| d\theta < \infty.$$

The space  $BMOA(\mathbb{T})$  is the intersection of  $BMO$  and  $H^2(\mathbb{T})$ . For any function  $f$  in  $L^1(\mathbb{T})$

we can extend  $f$  to a function  $\hat{f}$  on the disk  $\mathbb{D}$  via the *Poisson extension*,

$$\hat{f}(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{1 - |z|^2}{|1 - \bar{z}e^{i\theta}|^2} d\theta$$

for all  $z$  in  $\mathbb{D}$ . Thus,  $BMOA(\mathbb{T})$  can be extended to  $BMOA(\mathbb{D})$ , a space of analytic functions on  $\mathbb{D}$ . Recall the special automorphism of  $\mathbb{D}$ ,

$$\alpha_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$$

where  $\alpha_\lambda(\lambda) = 0$ ,  $\alpha_\lambda(0) = \lambda$ , and  $\alpha_\lambda^{-1} = \alpha_\lambda$ . As is stated in [23], a function  $f$  in  $H^2$  is in  $BMOA$  if

$$\|f\|_* = \sup_{\lambda \in \mathbb{D}} \|f \circ \alpha_\lambda - f(\lambda)\|_2 < \infty.$$

Under the norm  $\|f\|_{BMOA} = \|f\|_* + |f(0)|$ ,  $BMOA$  forms a Banach space.

For  $r$  in  $(0, 1)$ , let  $\Phi_r$  denote the set  $\{z : 1 > |\varphi(z)| > r\}$ , and for the characteristic function of  $\Phi_r$ , we write  $\chi_r(z)$ . By Theorem 3.1 in [6], the composition operator  $C_\varphi$  is

compact on BMOA if and only if for all  $\varepsilon > 0$ , there exists an  $r \in (0, 1)$  such that

$$\int_{R(I)} \chi_{\Phi_r}(z)(1 - |z|^2)|f'(\varphi(z))|^2|\varphi'(z)|^2 dA(z) \leq \varepsilon|I|.$$

By Theorem 4.1 in [23], If  $\varphi$  is a univalent self-map of  $\mathbb{D}$ , then  $C_\varphi$  is compact on BMOA if and only if it is compact on the Bloch space  $\mathcal{B}$ .

By part 1 of Corollary 2 in [11], for univalent  $\varphi$ , if  $C_\varphi$  is bounded below on BMOA, then it is also bounded below on the Bloch space  $\mathcal{B}$ . In other words, if  $\varphi$  is univalent and  $C_\varphi$  is closed-range on BMOA, then  $C_\varphi$  is also closed-range on  $\mathcal{B}$ .

We have defined  $\pi_\varphi(w)$  to be

$$\pi_\varphi(w) := \frac{N_\varphi(w)}{\log(\frac{1}{w})},$$

where  $N_\varphi(w)$  is the Nevalinna counting function. Let  $\pi_{\varphi,\alpha} = (\pi_\varphi(w))^\alpha$  and

$$G_c^{\varphi,\alpha} = \{z : \pi_{\varphi,\alpha+2} > c\}.$$

Theorem 4.1 in [28] states that a composition operator  $C_\varphi$  will be closed-range on the weighted Bergman space  $\mathbb{A}_\alpha^2$ , with  $\alpha > -1$  if and only if there exist constants  $c > 0$  and  $\lambda > 0$  such that

$$A(G^{\varphi,\alpha} \cap D(\xi, r)) > \delta \cdot A(\mathbb{D} \cap D(\xi, r)).$$

Notice that not only does the condition above involve the Nevanlinna counting function, but it also depends on  $\alpha$ . In the next section, we will give a necessary and sufficient condition for when an analytic self-map  $\varphi$  of  $\mathbb{D}$  induces a closed-range composition operator on the weighted Bergman space  $\mathbb{A}_\alpha^p$  for all  $p$  and all  $\alpha > -1$ . This will essentially render all the above conditions equivalent for various values of  $\alpha$ . In [1], J. Akeroyd and P. Ghatage give a necessary and sufficient condition in the case  $p = 2$  and  $\alpha = -1$ .



## 2 Closed-Range Composition Operators on Weighted Bergman Spaces

Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . For any  $\varepsilon$ ,  $0 < \varepsilon < 1$ , define  $\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$ , and let  $G_\varepsilon(\varphi) = G_\varepsilon = \varphi(\Omega_\varepsilon)$ . Note that, in the weighted Bergman space setting,  $G_\varepsilon$  functions in much the same way that  $F_\varepsilon$  did in the Bloch space setting. The set  $G_\varepsilon$  is said to satisfy the *reverse Carleson condition* if there exists a positive constant  $\eta$  so that

$$\int_{G_\varepsilon} |f(z)|^p (1-|z|^2)^p dA \geq \eta \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^p dA$$

for  $f$  analytic in  $\mathbb{D}$  and  $\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^p dA < \infty$ . As shown in [14], this is equivalent to the following condition:

(\*) There exist constants  $c$  and  $s$  with  $0 < c, s < 1$ , such that

$$A(G_\varepsilon \cap \Delta(z, s)) \geq c \cdot A(\Delta(z, s))$$

for all  $z$  in the unit disk  $\mathbb{D}$ . We will show in Theorem 2.3 that a composition operator  $C_\varphi$  is closed-range on  $\mathbb{A}_\alpha^p$  if and only if there exists an  $\varepsilon > 0$  such that  $G_\varepsilon$  satisfies condition (\*).

Recall that we defined the weighted Bergman Spaces  $\mathbb{A}_\alpha^p$  by

$$\mathbb{A}_\alpha^p := \{f : f \text{ is analytic in } \mathbb{D} \text{ and } \|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f|^p dA_\alpha < \infty\}$$

where  $\alpha > -1$  and  $dA_\alpha := c_\alpha \cdot (1-|z|^2)^\alpha dA(z)$  where  $c_\alpha = \alpha + 1$ .

Define  $\mathbb{A}_{\alpha,0}^p$  by

$$\mathbb{A}_{\alpha,0}^p = \{f \in \mathbb{A}_\alpha^p : f(0) = 0\},$$

a closed subspace of  $\mathbb{A}_\alpha^p$ . The following lemma is adapted from Lemmas 2.1, 2.2, and 2.3 in [1], and will be stated without proof.

**Lemma 2.1** *Let  $\varphi$  be an analytic self-map of the unit disk  $\mathbb{D}$ , and let  $\psi_a$  be a conformal automorphism of  $\mathbb{D}$ . Then,*

1.  $C_\varphi$  is closed-range on  $\mathbb{A}_\alpha^p$  if and only if  $C_\varphi$  is closed-range on  $\mathbb{A}_{\alpha,0}^p$
2. If one of  $C_\varphi$ ,  $C_{\varphi \circ \psi_a}$ , or  $C_{\psi_a \circ \varphi}$  is closed-range on  $\mathbb{A}_\alpha^p$ , then so are the other two.
3. If there exists  $\varepsilon > 0$  such that one of  $G_\varepsilon(\varphi)$ ,  $G_\varepsilon(\varphi \circ \psi_a)$ ,  $G_\varepsilon(\psi_a \circ \varphi)$  satisfies condition (\*), then there exists  $\varepsilon > 0$  such that the other two also satisfy condition (\*).

**Lemma 2.2** (Lemma 2.2 in [4]). *Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . If  $C_\varphi$  is closed-range on  $\mathbb{A}_\alpha^p$  then it is closed-range on  $\mathbb{A}_\alpha^{np}$  for any  $n \in \mathbb{N}$ .*

Proof. Assume  $\varphi$  is not constant. Otherwise, the result is trivial. Suppose  $C_\varphi$  is closed-range on  $\mathbb{A}_\alpha^p$ . Then there exists a constant  $c$  such that  $\|f \circ \varphi\|_{\mathbb{A}_{p,\alpha}} \geq c \cdot \|f\|_{\mathbb{A}_{p,\alpha}}$  for all  $f \in \mathbb{A}_\alpha^p$ .

That is,

$$\int_{\mathbb{D}} |f \circ \varphi|^p dA_\alpha \geq c \cdot \int_{\mathbb{D}} |f|^p dA_\alpha$$

for all  $f \in \mathbb{A}_\alpha^p$ . Now, if  $f \in \mathbb{A}_\alpha^{np}$ , then  $f^n \in \mathbb{A}_\alpha^p$ . Thus,

$$\begin{aligned} \int_{\mathbb{D}} |f \circ \varphi|^{np} dA_\alpha &= \int_{\mathbb{D}} |f^n \circ \varphi|^p dA_\alpha \\ &\geq c \cdot \int_{\mathbb{D}} |f^n|^p dA_\alpha \\ &= c \cdot \int_{\mathbb{D}} |f|^{np} dA_\alpha. \end{aligned}$$

and so  $C_\varphi$  is closed-range on  $\mathbb{A}_\alpha^{np}$ .  $\square$

**Theorem 2.3** (1.3 in [5]) *Let  $\varphi$  be an analytic self-map of the unit disk  $\mathbb{D}$ . Suppose  $1 \leq p < \infty$  and  $\alpha > -1$ . Then, the following are equivalent:*

1.  $C_\varphi$  is closed-range on  $\mathbb{A}_\alpha^p$ .

2. There exists  $\varepsilon > 0$  such that  $G_\varepsilon = \varphi(\Omega_\varepsilon)$  satisfies condition (\*).

Proof. By Lemma 2.1 we may assume that  $\varphi(0) = 0$  and we may also restrict our attention to  $C_\varphi$  on  $\mathbb{A}_{\alpha,0}^p$ . By the proof of Theorem 4.28 in [27], there is a constant  $C > 1$  such that

$$\frac{1}{C} \|f\|_{p,\alpha} \leq \left\{ \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^p dA_\alpha(z) \right\}^{\frac{1}{p}} \leq C \|f\|_{p,\alpha}$$

for all  $f$  in  $\mathbb{A}_{\alpha,0}^p$ . We will denote this by

$$\|f\|_{p,\alpha}^p \approx \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^p dA_\alpha(z).$$

We will first show that (2) implies (1). Note that this argument can also be found in the proof of Theorem 2.3 in [4]. Suppose that for some  $\varepsilon > 0$ ,  $G_\varepsilon$  satisfies condition (\*). First consider the case that  $1 \leq p < 2$ . By the Schwarz-Pick lemma (Lemma 1.2 in [8]),  $0 \leq \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \leq 1$  for all  $z$  in  $\mathbb{D}$ . Hence, for all  $z$  in  $\Omega_\varepsilon$ ,  $\varepsilon|\varphi'(z)| < 1$ , and thus, since  $1 \leq p < 2$ , we have  $\varepsilon^2|\varphi'(z)|^2 \leq \varepsilon^p|\varphi'(z)|^p$ . Hence,  $\varepsilon^{2-p} < |\varphi'(z)|^{p-2}$ . Then,

$$\begin{aligned} \|f \circ \varphi\|_{p,\alpha}^p &\approx \int_{\mathbb{D}} |(f \circ \varphi)'(z)|^p (1 - |z|^2)^p dA_\alpha(z) \\ &\geq \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^{p+\alpha} dA(z) \\ &= \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p |\varphi'(z)|^{p-2} |\varphi'(z)|^2 (1 - |z|^2)^{p+\alpha} dA(z) \\ &\geq \varepsilon^{2-p} \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p |\varphi'(z)|^2 (1 - |z|^2)^{p+\alpha} dA(z) \\ &\geq \varepsilon^{\alpha+2} \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p |\varphi'(z)|^2 (1 - |\varphi(z)|^2)^{p+\alpha} dA(z) \\ &= \varepsilon^{\alpha+2} \sum_n \int_{\Omega_\varepsilon \cap R_n} |f'(\varphi(z))|^p |\varphi'(z)|^2 (1 - |\varphi(z)|^2)^{p+\alpha} dA(z) \end{aligned}$$

where  $\mathcal{Z} := \{z \in \mathbb{D} : \varphi'(z) = 0\}$  and  $\{R_n\}$  is a partition of  $\mathbb{D} \setminus \mathcal{Z}$  into at most countably many polar rectangles so that  $\varphi$  is univalent on  $R_n$  for all  $n$ . Let  $S_n = \varphi(\Omega_\varepsilon \cap R_n)$  and let  $\psi_n$  denote the inverse of  $\varphi|_{R_n}$ . Then, letting  $z = \psi_n(w)$ , we have

$$\begin{aligned}
& \varepsilon^{\alpha+2} \sum_n \int_{\Omega_\varepsilon \cap R_n} |f'(\varphi(z))|^p |\varphi'(z)|^2 (1 - |\varphi(z)|^2)^{p+\alpha} dA(z) \\
&= \varepsilon^{\alpha+2} \sum_n \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} \chi_{S_n}(w) dA(w) \\
&= \varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} \left( \sum_n \chi_{S_n}(w) \right) dA(w) \\
&\geq \varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} dA(w)
\end{aligned}$$

Since  $G_\varepsilon$  satisfies condition (\*), we have

$$\begin{aligned}
\varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} dA(w) &\geq \eta \varepsilon^{\alpha+2} \int_{\mathbb{D}} |f'(w)|^p (1 - |w|^2)^{p+\alpha} dA(w) \\
&\approx \int_{\mathbb{D}} |f(w)|^p dA_\alpha(w) \\
&= \|f\|_{p,\alpha}^p
\end{aligned}$$

Hence, for  $1 \leq p < 2$ , we have that  $C_\varphi$  is closed-range on  $\mathbb{A}_\alpha^p$ . We may apply Lemma 2.2 to see that  $C_\varphi$  is closed-range on  $\mathbb{A}_\alpha^p$  for  $1 \leq p < \infty$  and, thus, (1) is satisfied.

We will now show that (1) implies (2) by means of the contrapositive, as is also shown in the proof of Theorem 1.3 in [5]. Suppose that condition (\*) is not satisfied. Then there does not exist  $\varepsilon > 0$  such that  $G_\varepsilon$  satisfies the reverse Carleson condition. In other words,

for any  $\varepsilon > 0$ , there does not exist a positive constant  $\eta$  such that

$$\int_{G_\varepsilon} |f(z)|^p (1 - |z|^2)^p dA \geq \eta \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^p dA$$

for all  $f$  in  $\mathbb{A}_{\alpha,0}^p$ . So, we can find a sequence  $\{f_k\}_{k=1}^\infty \subset \mathbb{A}_{\alpha,0}^p$  such that

$$\int_{\mathbb{D}} |f'_k(w)|^p (1 - |w|^2)^p dA_\alpha(w) = 1$$

for all  $k$ , but

$$\int_{G_k} |f'_k(w)|^p (1 - |w|^2)^p dA_\alpha(w) \rightarrow 0$$

as  $k \rightarrow \infty$ , where  $\Omega_k = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \frac{1}{k}\}$  and  $G_k = \varphi(\Omega_k)$ .

First suppose that  $p \geq 3$ . Note that for all  $z \in \Omega_{j+1} \setminus \Omega_j$ , we have  $\frac{1}{j} \geq \frac{1-|z|^2}{1-|\varphi(z)|^2} > \frac{1}{j+1}$  and,

by the Schwarz-Pick Lemma,  $0 \leq \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \leq 1$  for all  $z$  in  $\mathbb{D}$ . Thus, on  $\Omega_{j+1} \setminus \Omega_j$ ,

$$|\varphi'(z)|^{p-2} \leq (j+1)^{p-2}.$$

Also, on  $\Omega_{j+1} \setminus \Omega_j$ ,

$$(1 - |z|^2)^{p+\alpha-1} \approx \frac{1}{(j+1)^{p+\alpha-1}} \cdot (1 - |\varphi(z)|^2)^{p+\alpha-1}.$$

Now,  $\mathbb{D}$  is equal to the pairwise disjoint union  $\Omega_k \cup (\cup_{j=k}^\infty \Omega_{j+1} \setminus \Omega_j)$ . So,

$$\begin{aligned} \|f_k \circ \varphi\|_{p,\alpha}^p &\approx \int_{\mathbb{D}} |f'_k(\varphi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^p dA_\alpha(z) \\ &= \int_{\Omega_k} |f'_k(\varphi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^p dA_\alpha(z) \\ &\quad + \sum_{j=k}^\infty \int_{\Omega_{j+1} \setminus \Omega_j} |f'_k(\varphi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^p dA_\alpha(z) \end{aligned}$$

Then, by the corollary on page 188 of [22], we have

$$\begin{aligned}
& \int_{\Omega_k} |f'_k(\varphi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^p dA_\alpha(z) \\
& \quad = \int_{\Omega_k} |f'_k(\varphi(z))|^p |\varphi'(z)|^2 |\varphi'(z)|^{p-2} (1 - |z|^2)^{p+\alpha-1} (1 - |z|^2) dA(z) \\
& \leq \int_{\Omega_k} |f'_k(\varphi(z))|^p |\varphi'(z)|^2 (1 - |\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z) \\
& \leq \int_{G_k} |f'_k(w)|^p (1 - |w|^2)^{p+\alpha-1} N_\varphi(w) dA(w) \\
& \leq \int_{G_k} |f'_k(w)|^p (1 - |w|^2)^{p+\alpha} dA(w) \\
& = \int_{G_k} |f'_k(w)|^p (1 - |w|^2)^p dA_\alpha(w) \rightarrow 0
\end{aligned}$$

as  $k \rightarrow \infty$ . Then, again using the corollary on page 188 of [22], we have

$$\begin{aligned}
& \sum_{j=k}^{\infty} \int_{\Omega_{j+1} \setminus \Omega_j} |f'_k(\varphi(z))|^p |\varphi'(z)|^p (1 - |z|^2)^p dA_\alpha(z) \\
& \quad \approx \sum_{j=k}^{\infty} \int_{\Omega_{j+1} \setminus \Omega_j} |f'_k(\varphi(z))|^p |\varphi'(z)|^2 |\varphi'(z)|^{p-2} \left( \frac{1}{(j+1)^{p+\alpha-1}} \right) (1 - |\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z) \\
& \leq \sum_{j=k}^{\infty} \int_{\Omega_{j+1} \setminus \Omega_j} |f'_k(\varphi(z))|^p |\varphi'(z)|^2 ((j+1)^{p-2}) \left( \frac{1}{(j+1)^{p+\alpha-1}} \right) (1 - |\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z) \\
& \leq \frac{1}{k^{\alpha+1}} \sum_{j=k}^{\infty} \int_{\Omega_{j+1} \setminus \Omega_j} |f'_k(\varphi(z))|^p |\varphi'(z)|^2 (1 - |\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z) \\
& = \frac{1}{k^{\alpha+1}} \int_{\mathbb{D} \setminus \Omega_k} |f'_k(\varphi(z))|^p |\varphi'(z)|^2 (1 - |\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z) \\
& \leq \frac{1}{k^{\alpha+1}} \int_{\mathbb{D}} |f'_k(\varphi(z))|^p |\varphi'(z)|^2 (1 - |\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z) \\
& = \frac{1}{k^{\alpha+1}} \int_{\mathbb{D}} |f'_k(w)|^p (1 - |w|^2)^{p+\alpha-1} N_\varphi(w) dA(w) \\
& \leq \frac{c}{k^{\alpha+1}} \int_{\mathbb{D}} |f'_k(w)|^p (1 - |w|^2)^{p+\alpha} dA
\end{aligned}$$

$$= \frac{c}{k^{\alpha+1}} \int_{\mathbb{D}} |f'_k(w)|^p (1 - |w|^2)^p dA_\alpha \rightarrow 0$$

as  $k \rightarrow \infty$ . Thus,  $\|f_k \circ \varphi\|_{p,\alpha} \rightarrow 0$  as  $k \rightarrow \infty$ , even though  $\|f_k\|_{p,\alpha} = 1$  for all  $k$ . Hence, it must be that  $C_\varphi$  is not closed-range on  $\mathbb{A}_{\alpha,0}^p$  for  $p \geq 3$ . By the contrapositive of the previous lemma then, it must be that  $C_\varphi$  is not closed-range on  $\mathbb{A}_{\alpha,0}^p$  for any  $p \geq 1$ . Thus, by means of the contrapositive of what we have just shown, our proof is complete.  $\square$

**Corollary 2.4** *If  $\varphi$  is univalent and  $C_\varphi$  is closed-range on the weighted Bergman space  $\mathbb{A}_\alpha^p$ , then  $C_\varphi$  is closed-range on the Hardy space  $H^2$ .*

Proof. If  $C_\varphi$  is closed-range on any weighted Bergman space  $\mathbb{A}_\alpha^p$ , then, by Theorem 2.3,  $C_\varphi$  is closed-range on  $\mathbb{A}^2$ . Then, by Corollary 4.3 in [28],  $C_\varphi$  is closed-range on  $H^2$ .

### 3 Examples

#### 3.1 An Outer Function

We note that an example of the same type as the following was developed concurrently by P. Ghatage. We also note that  $\varphi$  in the following example is a purely outer function. J. Akeroyd and P. Ghatage discuss the case when  $\varphi$  is a singular inner function in [1].

Let  $\psi$  be a conformal mapping of the unit disk  $\mathbb{D}$  onto the semi-annulus

$$S = \{re^{i\theta} : \frac{1}{2} < r < 1, 0 < \theta < \pi\}.$$

By Theorem 13.2.3 in [12], since  $\mathbb{D}$  and  $S$  are bounded domains in  $\mathbb{C}$ , each bounded by a single Jordan curve, then  $\psi$  extends to a homeomorphism from  $\bar{\mathbb{D}}$  onto  $\bar{S}$ .

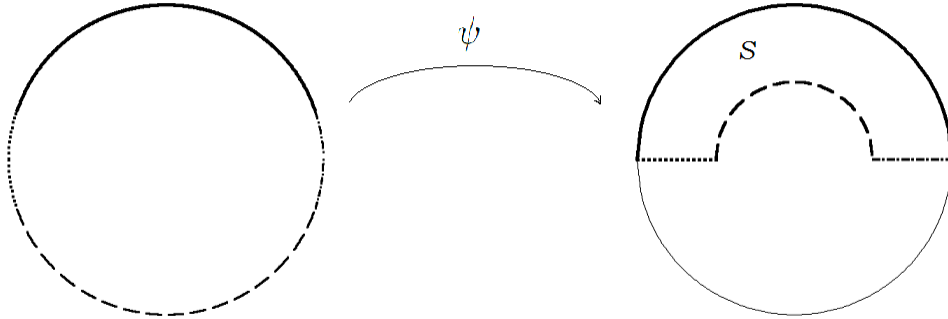


Figure 3:  $\psi : \mathbb{D} \rightarrow S = \{re^{i\theta} : \frac{1}{2} < r < 1, 0 < \theta < \pi\}$

By the Schwarz-Pick Theorem (see [12]), for any  $z$  in  $\mathbb{D}$ ,

$$\frac{1 - |z|^2}{1 - |\psi(z)|^2} \leq \frac{1}{|\psi'(z)|}.$$

We let  $\varphi$  be the analytic self-map of  $\mathbb{D}$  given by  $\varphi(z) := (\psi(z))^{2+\delta}$ ,  $\delta \geq 0$ . First, suppose



that  $\delta = 0$ . Then,  $\varphi$  maps the unit disk  $\mathbb{D}$  to the set

$$S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\} \setminus (\frac{1}{4}, 1).$$

As in the case of the square root function on the upper half plane,  $|\psi'(z)|$  grows without bound for the points that  $\psi$  maps to the corner points of  $\partial S$ .

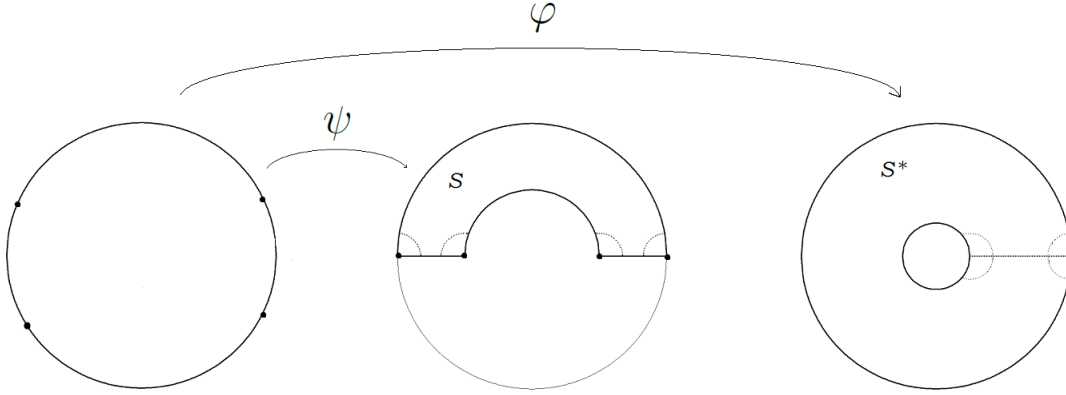


Figure 4:  $\varphi : \mathbb{D} \rightarrow S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\} \setminus (\frac{1}{4}, 1)$

Let  $\xi$  be a point in  $\mathbb{T}$  such that  $\psi(\xi)$  is a corner point of  $\partial S$ . Then, for any  $\varepsilon > 0$ , we can find a region  $R$  about  $\xi$ , such that, for all  $z$  in  $R$ ,

$$\frac{1 - |z|^2}{1 - |\psi(z)|^2} \leq \frac{1}{|\psi'(z)|} < \varepsilon.$$

Since  $|\psi(z)| < 1$ , for all  $z$  in  $R$  we have

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} = \frac{1 - |z|^2}{1 - |\psi(z)|^4} < \frac{1 - |z|^2}{1 - |\psi(z)|^2} \leq \frac{1}{|\psi'(z)|} < \varepsilon.$$

Thus, these points are not contained in

$$\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \geq \varepsilon\}$$

and, hence, the image of this set  $R$  under  $\varphi$  is not contained in  $G_\varepsilon$ . To see that  $\varphi$  does not induce a closed-range composition operator in this case, we need to show that the condition  $(*)$  is not satisfied. To this end, let  $\{p_n\}$  be a sequence of points in  $[0, 1)$  converging to 1 and consider the sequence of pseudohyperbolic disks,  $\Delta(p_n, r)$ , of radius  $r$  with center  $p_n$ . Each pseudohyperbolic disk  $\Delta(p_n, r)$  is a Euclidean disk with radius

$$q_n = \frac{r(1 - p_n^2)}{1 - p_n^2 r^2}$$

and center

$$c_n = \frac{p_n(1 - r^2)}{1 - p_n^2 r^2}.$$

The Euclidean distance from the point 1 to the boundary of  $\Delta(p_n, r)$  is given by  $\frac{(1-p_n)(1-r)}{1+p_n r}$  for each  $n$ . Notice that the ratio of this distance to the Euclidean radius of each  $k$  is  $\frac{r(1+p_n)}{1-r}$ , which approaches the ratio  $\frac{2r}{1-r}$  as  $p_n \rightarrow 1$ . Hence the sequence of pseudohyperbolic disks,  $\Delta(p_n, r)$  is approaching the unit circle  $\mathbb{T}$  nontangentially. In other words, we can find a stolz region which contains each of the disks.

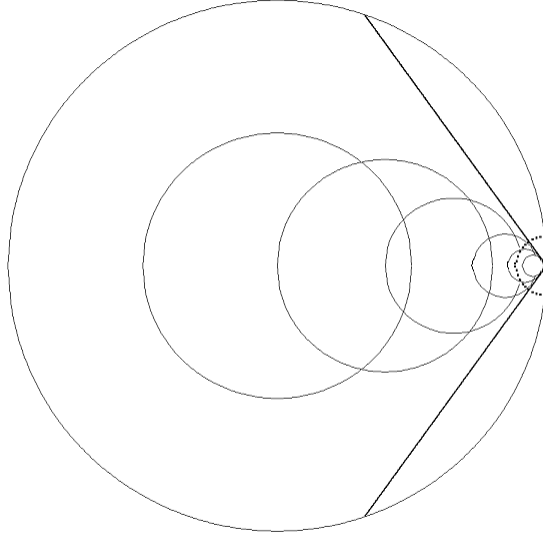


Figure 5: The Sequence  $\Delta(p_n, r)$  approaches  $\partial\mathbb{D}$  nontangentially.

Then, for any  $\varepsilon > 0$  and any  $r$  in  $(0, 1)$ , there exists an  $N$  in  $\mathbb{N}$  with the preimage of  $\Delta(p_n, r)$  under  $\varphi$  contained in  $R$  whenever  $n > N$ . Therefore,  $\Delta(p_n, r) \cap G_\varepsilon = \emptyset$  and condition  $(*)$  fails.

Now, suppose that  $\delta > 0$ . Then,  $\varphi(z) := (\psi(z))^{2+\delta}$  maps the unit disk  $\mathbb{D}$  to the outer annulus  $S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\}$ . We will see that  $\varphi$  now maps a region of points contained in  $\Omega_\varepsilon$  to an outer annulus, and hence, condition  $(*)$  will be satisfied. In fact, provided that  $\varepsilon > 0$  is sufficiently small, we will see that the entire annulus  $S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\}$  is contained in  $G_\varepsilon$ . We define  $\theta_\varepsilon$  to be the smallest angle such that  $\{re^{i\theta} : \pi - \theta_\varepsilon \geq \theta \geq \theta_\varepsilon \text{ and } r \geq \frac{1}{2}\}$  is contained in the image of  $\Omega_\varepsilon$  under  $\psi$ . For the given  $\delta$ , we can choose  $\varepsilon$  small enough such that  $\theta_\varepsilon < \frac{\delta\pi}{4}$ . We let  $\gamma_1$  denote the set of points  $\{re^{i\theta} : 0 < r < 1\}$  and  $\gamma_2$  denote the set of points  $\{re^{i(\pi-\theta)} : 0 < r < 1\}$ .

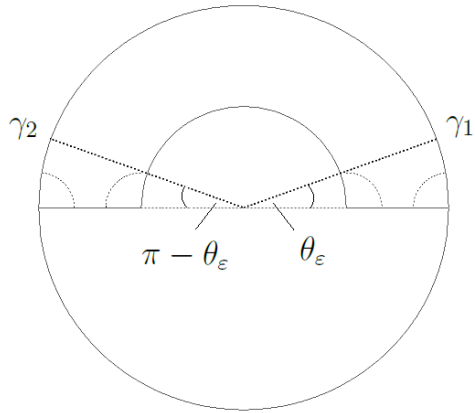


Figure 6: The set of points  $\{re^{i\theta} : \pi - \theta_\epsilon \geq \theta \geq \theta_\epsilon \text{ and } r \geq \frac{1}{2}\}$  is contained in the image of  $\Omega_\epsilon$  under  $\psi$ .

Now, under  $\varphi$ , points in  $\gamma_1$  are mapped to points along the radius from 0 to  $e^{i(2\theta_\epsilon + \delta\theta_\epsilon)}$ . Similarly, points in  $\gamma_2$  are mapped to points along the radius from 0 to  $e^{i(2+\delta)(\pi - \theta_\epsilon)}$ . Since  $\theta_\epsilon < \frac{\delta\pi}{4}$ , we have that the reference angle  $\delta\pi - \theta_\epsilon(2 + \delta)$  is greater than the angle  $(2 + \delta)\theta_\epsilon$ .

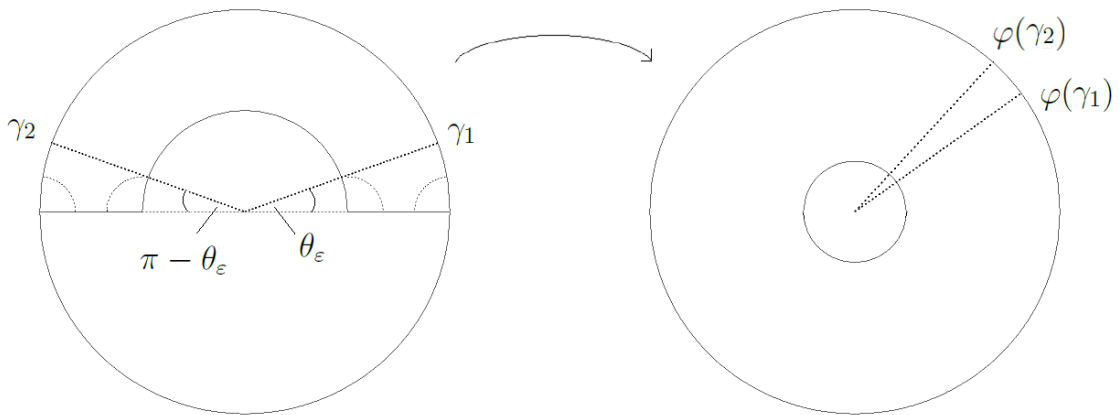


Figure 7:  $G_\epsilon$  contains the entire outer annulus  $S^*$ .

Hence, the image under  $\varphi$  of the set  $\{re^{i\theta} : \pi - \theta_\varepsilon \geq \theta \geq \theta_\varepsilon \text{ and } r \geq \frac{1}{2}\}$  overlaps itself and we have that  $G_\varepsilon$  contains the entire outer annulus  $S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\}$ . Thus, condition  $(*)$  is satisfied and  $\varphi$  induces a closed-range composition operator when  $\delta > 0$ .

### 3.2 Frostman Blaschke Products

Note that the following Frostman Blaschke product example appears in [4]. Remember that we define a Blaschke product  $B$  to be a function of the form

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n| a_n - z}{a_n 1 - \bar{a}_n}$$

where  $z \in \mathbb{D}$ ,  $\{a_n\}$  is a sequence of points in  $\mathbb{D}$  with the property that  $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$ , and  $\frac{|a_n|}{a_n}$  is taken to be 1 if  $a_n = 0$ . For  $\zeta$  in  $\mathbb{T}$ , define  $g_B(\zeta)$  by

$$g_B(\zeta) = \sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|}.$$

By a theorem of Frostman, see [7], the Blaschke product  $B$  has a unimodular nontangential boundary value at  $\zeta$  in  $\mathbb{T}$  exactly when  $g_B(\zeta) = \sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|} < \infty$ . If  $g_B(\zeta)$  converges for every  $\zeta$  in  $\mathbb{T}$ , then we call  $B$  a *Frostman Blaschke Product*. In other words,  $B$  is a Frostman Blaschke product if  $B$  has unimodular nontangential boundary values at every point  $\zeta$  in  $\mathbb{T}$ . Denote the set of accumulation points of the sequence  $\{a_n\}$  in  $\mathbb{T}$  by  $\sigma_B$ . By Theorem 1 in [18], if  $B$  is a Frostman Blaschke product, then  $\sigma_B$  is nowhere dense in  $\mathbb{T}$ . For  $\zeta$  in  $\mathbb{T}$ , we define  $h_B(\zeta)$  by

$$h_B(\zeta) = \sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|^2}.$$

By page 183 in [22],  $B$  has an angular derivative at  $\zeta$  in  $\mathbb{T}$  exactly when  $h_B(\zeta) < \infty$ . Hence, a Frostman Blaschke product  $B$  will have an angular derivative at every point  $\zeta$  in the dense

open set  $\mathbb{T} \setminus \sigma_B$ , but not necessarily at points in  $\sigma_B$ . Let  $\{I_\nu\}$  be the collection of subarcs of  $\mathbb{T} \setminus \sigma_B$ , and for each  $\nu$ , define  $\omega_\nu$  to be the (possibly infinite) number of radians through which  $B$  wraps  $I_\nu$ . If, for some  $\nu_0$ ,  $B(I_{\nu_0}) = \mathbb{T}$ , then the proof of Lemma 3.1 in [1] gives us that

$$\omega_{\nu_0} = \int_{I_{\nu_0}} h_B(\zeta) |d\zeta| > 2\pi.$$

Therefore, by Theorem 3.4 in [1] and Theorem 2.3, we have that  $C_\varphi$  is closed-range on every weighted Bergman space  $\mathbb{A}_\alpha^p$ . Suppose, then, that such a  $\nu_0$  does not exist. In that case, for every  $\nu$ ,  $B(I_\nu)$  is an open subarc of  $\mathbb{T}$ . If  $\bigcup_\nu B(I_\nu) = \mathbb{T}$ , then, since  $\mathbb{T}$  is compact, there exists an integer  $N > 0$  so that  $\bigcup_{\nu=1}^N B(I_\nu) = \mathbb{T}$ . Then, there is a compact subset  $K$  of  $\bigcup_{\nu=1}^N I_\nu$  such that  $B(K) = \mathbb{T}$ . If  $\varepsilon > 0$  is small enough,  $K$  will be contained in the closure of  $\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|B(z)|^2} > \varepsilon\}$ . Hence, by Theorem 2.3, the composition operator  $C_B$  will be closed-range on every weighted Bergman space  $\mathbb{A}_\alpha^p$ . Thus, a sufficient condition for  $C_B$  to be closed-range on each of the weighted Bergman spaces is that  $\bigcup_\nu B(I_\nu) = \mathbb{T}$ .

**Lemma 3.1** (*Lemma 2.5 in [4]*) *Let  $B$  be a Frostman Blaschke product with infinitely many zeros  $\{a_n\}_{n=1}^\infty$ , listed according to multiplicity. Then, for any point  $\zeta^*$  in  $\sigma_B$  and for any  $\delta > 0$ , there exists a subsequence  $\{a_{nk}\}_{k=1}^\infty$  of  $\{a_n\}_{n=1}^\infty$  such that  $|\zeta^* - a_{nk}| < \delta$  for all  $k$  and*

$$\sup_{\zeta \in \sigma_B} \frac{1 - |a_{nk}|^2}{|\zeta - a_{nk}|} \rightarrow 0$$

as  $k \rightarrow \infty$ .

Proof. Suppose there exists  $\delta, c > 0$  and  $\zeta^* \in \sigma_B$  such that

$$\sup_{\zeta \in \sigma_B} \frac{1 - |a_n|^2}{|\zeta - a_n|} \geq c$$

whenever  $|\zeta^* - a_n| < \delta$ . Since  $\zeta^*$  is in  $\sigma_B$ , we can find  $\zeta_1$  in  $\sigma_B$  and  $n_1 > 0$  such that  $|\zeta^* - a_{n_1}| < \delta$  and

$$\frac{1 - |a_{n_1}|^2}{|\zeta_1 - a_{n_1}|} > \frac{c}{2}.$$

So, by choosing  $a_{n_1}$  close enough to  $\zeta^*$ , we can make  $\zeta_1$  as close to  $\zeta^*$  as we want. Since  $\zeta_1$  is in  $\sigma_B$ , we can find  $n_2 > n_1$  so that  $a_{n_2}$  is close enough to  $\zeta_1$  so that  $|\zeta^* - a_{n_2}| < \delta$ . Then, there exists  $\zeta_2$  in  $\sigma_B$  so that

$$\frac{1 - |a_{n_2}|^2}{|\zeta_2 - a_{n_2}|} > \frac{c}{2}.$$

Thus, by choosing  $a_{n_2}$  close enough to  $\zeta_1$ , we can make  $\zeta_2$  to be as close to  $\zeta_1$  as we would like. Hence, for  $j = 1, 2$ , we can force  $|\zeta^* - \zeta_j| < \delta$  and

$$\frac{1 - |a_{n_j}|^2}{|\zeta_j - a_{n_j}|} > \frac{c}{2}.$$

In a similar manner, we can choose  $n_3 > n_2$  so that  $a_{n_3}$  is close enough to  $\zeta_2$  to ensure that  $|\zeta^* - a_{n_3}| < \delta$  and we can find  $\zeta_3$  in  $\sigma_B$  so that  $a_{n_3}$  is close enough to  $\zeta_2$  to ensure that  $|\zeta^* - a_{n_3}| < \delta$ , and we can choose  $\zeta_3$  in  $\sigma_B$  so that  $|\zeta^* - \zeta_3| < \delta$  and

$$\frac{1 - |a_{n_j}|^2}{|\zeta_3 - a_{n_j}|} > \frac{c}{2}$$

for  $j = 1, 2, 3$ . We may then continue in this manner to find a subsequence  $\{a_{n_j}\}_{j=1}^{\infty}$  of  $\{a_n\}_{n=1}^{\infty}$  and a sequence  $\{\zeta_j\}_{j=1}^{\infty}$  in  $\sigma_B$  with the property that

$$\frac{1 - |a_{n_j}|^2}{|\zeta_j - a_{n_j}|} > \frac{c}{2},$$

where  $J \in \mathbb{N}$  and  $1 \leq j \leq J$ . By the compactness of  $\sigma_B$ ,  $\{\zeta_j\}_{j=1}^{\infty}$  has an accumulation point,  $\zeta_0$ , in  $\sigma_B$ , which also fulfills the condition

$$\frac{1 - |a_{n_j}|^2}{|\zeta_0 - a_{n_j}|} \geq \frac{c}{2}.$$

But this means that  $g_B(\zeta_0) = \sum_n \frac{1-|a_n|^2}{|\zeta_0-a_n|}$  diverges, and hence,  $B$  cannot be a Frostman Blaschke product. Thus, the result is proved.  $\square$

**Proposition 3.2** (Proposition 2.6 in [4].) *Let  $B$  be a Frostman Blaschke product with infinitely many zeros  $\{a_n\}_{n=1}^\infty$ , listed according to multiplicity, and let  $\{I_\nu\}_\nu$  be an enumeration of the components of  $\mathbb{T} \setminus \sigma_B$ . Let  $\omega_\nu$  denote the number of radians through which  $B$  wraps  $I_\nu$ , which may be infinite. Then, for any point  $\zeta^*$  in  $\sigma_B$  and any  $\delta > 0$ , at least one of the following hold.*

1. *There is a component  $I_{\nu_0}$  of  $\mathbb{T} \setminus \sigma_B$  such that  $\text{dist}(\zeta^*, I_{\nu_0}) < \delta$  and  $\omega_{\nu_0} = \infty$ .*
2. *There are infinitely many components  $\{I_{\nu_k}\}_{k=1}^\infty$  of  $\mathbb{T} \setminus \sigma_B$  contained in  $\{\zeta \in \mathbb{T} : |\zeta - \zeta^*| < \delta\}$  such that  $\liminf_{k \rightarrow \infty} \omega_{\nu_k} \geq 2\pi$ .*

Proof. By Lemma 3.1, we can find a sequence  $\{a_{n_k}\}_{k=1}^\infty \subset \{a_n\}_{n=1}^\infty$  such that each  $a_{n_k}$  is as close to  $\zeta^*$  as we wish. We can also find a corresponding sequence  $\{I_{\nu_k}\}_{k=1}^\infty$  of not necessarily distinct components of  $\mathbb{T} \setminus \sigma_B$  such that

$$\sup_{\zeta \in \mathbb{T} \setminus I_{\nu_k}} \frac{1 - |a_{n_k}|^2}{|\zeta - a_{n_k}|} \rightarrow 0,$$

as  $k \rightarrow \infty$ . Since, for  $0 < a < b$ ,  $\int_a^b \frac{1}{t^2} = \frac{b-a}{ab} < \frac{1}{a}$ , we have

$$\int_{\mathbb{T} \setminus I_{\nu_k}} \frac{1 - |a_{n_k}^2|}{|\zeta - a_{n_k}|^2} |d\zeta| \rightarrow 0$$

and so

$$\int_{I_{\nu_k}} \frac{1 - |a_{n_k}^2|}{|\zeta - a_{n_k}|^2} |d\zeta| \rightarrow 2\pi$$

as  $k \rightarrow \infty$ . Since, by Lemma 3.1 in [1],

$$\omega_\nu \geq \sum_{\{k: \nu(k)=\nu\}} \int_{I_{\nu_k}} \frac{1 - |a_{n_k}^2|}{|\zeta - a_{n_k}|^2} |d\zeta|,$$



the proposition is proved.  $\square$

If there is a  $\zeta^*$  in  $\sigma_B$  and a  $\delta > 0$  so that condition (1) in proposition 3.2 holds, then the composition operator  $C_B$  will be closed-range on  $\mathbb{A}_\alpha^p$  for every  $p$ ,  $1 \leq p < \infty$ . But then, if  $C_B$  is not closed-range on  $\mathbb{A}_\alpha^p$  for every  $p$ , then it must be that condition (2) in proposition 3.2 holds. It must also be the case that  $\bigcup_\nu B(I_\nu) \neq \mathbb{T}$ . If condition (2) and the previous statement are both true, then, there exists  $\zeta_0$  in  $\mathbb{T}$  so that for an open arc  $\gamma$  in  $\mathbb{T}$  having nonempty intersection with  $\varphi_B$ ,  $B(\gamma \setminus \varphi_B) = \mathbb{T} \setminus \{\zeta_0\}$ . This does not seem very probable and one may suspect that every Frostman Blaschke product will give rise to a closed range composition operator  $C_B$  on every  $\mathbb{A}_\alpha^p$  space. Indeed, J. Akeroyd and P. Ghatage have constructed an example of a Frostman Blaschke product that does not do so. Suppose  $B$  is a Frostman Blaschke product such that  $C_B$  is not closed-range on  $\mathbb{A}_\alpha^p$  for any  $p$ . Since any perturbation of a zero of  $B$  affects the image of each component under  $B$  of  $\mathbb{T} \setminus \varphi_B$  unequally, then for a Blaschke product  $B^*$  obtained by shifting the location of only one of the zeros of  $B$ ,  $C_{B^*}$  will be closed-range on  $\mathbb{A}_\alpha^p$  for any  $p$ . Thus, if the composition operator  $C_B$  is not closed-range on  $\mathbb{A}_\alpha^p$  for every  $p$  then there is a sequence of Frostman Blaschke products  $\{B_k^*\}_{k=1}^\infty$  so that  $C_{B_k^*}$  is closed-range on  $\mathbb{A}_\alpha^p$  for every  $p$ .

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