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Closed-Range Composition Operators on Weighted Bergman Spaces and Applications

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by

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Abstract

We will discuss necessary and sufficient conditons for the composition operator C_φ to be closed range on the weighted Bergman space $\mathbb{A}^p_\alpha$ for $1 \leq p < \infty$ with weights of the form $(1-|z|^2)^\alpha$ for $\alpha > -1$. The function φ is an analytic self-map of the unit disk $\mathbb{D}$ and our results extend those previously intended for the classical Bergman space $\mathbb{A}^2$. We will also give applications.

Acknowledgements

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Contents

1 Introduction

Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Let m denote normalized Lebesgue measure on $\mathbb{T}$, and let A denote normalized two-dimensional Lebesgue measure on the unit disk $\mathbb{D}$. For a point a in $\mathbb{D}$ and r, $0 < r < 1$, let $D(a,r) := \{z \in \mathbb{D} : |z-a| < r\}$ and let $\Delta(a,r) := \{z \in \mathbb{D} : \rho(z,a) < r\}$ where $\rho(z,a)$ denotes the *pseudohyperbolic metric* defined for z and a in $\mathbb{D}$ by

$$\rho(z,a) = \frac{|z-a|}{|1-\bar{z}a|}.$$

For a in $\mathbb{D}$ and $0 \le b \le 1$, let $D_b(a)$ denote a disk centered at a with radius $b(1-|a|)$. We let $\mathcal{H}(\mathbb{D})$ be the set of all functions f which are analytic in $\mathbb{D}$. A function φ is said to be an *analytic self-map of the unit disk* $\mathbb{D}$ if $\varphi \in \mathcal{H}(\mathbb{D})$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. If φ is an analytic self-map of $\mathbb{D}$, then the composition operator C_φ is defined on $\mathcal{H}(\mathbb{D})$ by $C_\varphi(f) = f \circ \varphi$. If X is a Banach space of analytic functions in $\mathbb{D}$, then we say that a composition operator C_φ on a space X is *compact* if every bounded set in X is mapped to a set whose closure is compact. The composition operator C_φ is said to be *closed-range* on X if $C_\varphi(X)$ is a closed subspace of X. By the Open Mapping Theorem, for nontrivial φ, this occurs when there exists a constant $c > 0$ such that $||f \circ \varphi||_X \ge c||f||_X$ for all f in X. For $1 \le p < \infty$, the Hardy space H^p is the set of all functions f in $\mathcal{H}(\mathbb{D})$ such that

$$||f||_{H^p}^p := \sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^p dm(\xi) < \infty$$

and H^∞ is the set of all functions f in $\mathcal{H}(\mathbb{D})$ such that

$$||f||_{H^\infty} := \sup_{z \in \mathbb{D}} |f(z)| < \infty.$$

For $\alpha > -1$ we define dA_α by $dA_\alpha := c_\alpha \cdot (1 - |z|^2)^\alpha dA(z)$, where $c_\alpha = \alpha + 1$. The weighted Bergman space $\mathbb{A}^p_\alpha$ is given by:

$$\mathbb{A}^p_\alpha := \{f : f \in \mathcal{H}(\mathbb{D}) \text{ and } ||f||^p_{p,\alpha} = \int_{\mathbb{D}} |f|^p dA_\alpha < \infty\}.$$

In this chapter, we will discuss several classical Banach spaces of analytic functions on the unit disk. We will review standard results describing when a composition operator is bounded, compact, and closed-range on these spaces. In chapter 2, we will give a necessary and sufficient condition for when C_φ is closed-range on the weighted Bergman space A^p_α. Note that in [28], Nina Zorboska gives conditions for when C_φ is closed-range on the Hardy space H^2 and the weighted Bergman space A^2_α. These conditions involve the Nevanlinna counting function, which can be difficult to work with.

A good reference for the following discussion is [19]. For $0 < r < 1$ and a point ξ in $\mathbb{T}$, let $S(\xi, r)$ denote the interior of the convex hull of the union of $\{\xi\}$ and $\{z \in \mathbb{D} : |z| \leq r\}$. We call $S(\xi, r)$ the *Stolz region* based at $\xi \in \mathbb{T}$ with contact angle $2 \arctan(r)$.

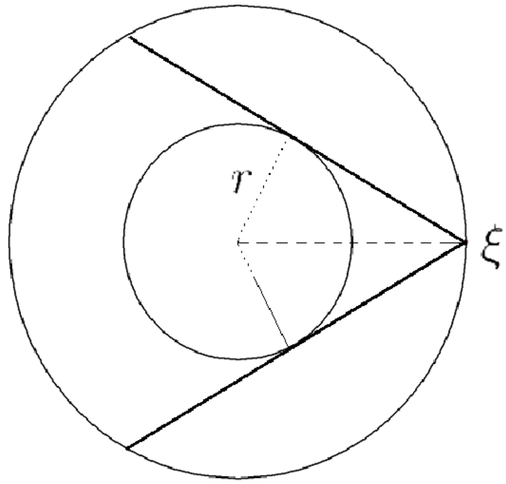

Figure 1: A stolz region

Note that if a curve approaches ξ from inside the region $S(\xi, r)$, then this curve cannot be tangent to the unit circle. For f in H^p, we say f has nontangential limit L at the point ξ if for all r in $(0, 1)$ and for every sequence $\{z_n\}$ in $S(\xi, r)$ that converges to the point ξ, we have $\lim_{n\to\infty} f(z_n) = L$. Also, for r in $(0, 1)$ and for any complex function f defined on $\mathbb{D}$, we define the *nontangential maximal function* $N_r f$ on $\mathbb{T}$ by

$$N_r f(\xi) = \sup\{|f(z)| : z \in S(\xi, r)\}.$$

For any f in H^p, $0 < p < \infty$ and any r in $(0, 1)$, we have $N_r f \in L^p(\mathbb{T})$. It is well known (see [19]) that the nontangential limits of f in H^p, denoted $f^*(e^{i\theta})$, exist almost everywhere $[m]$ on $\mathbb{T}$ and $f^* \in L^p(\mathbb{T})$. Furthermore, $||f^*||_p = ||f||_p$ for all f in H^p. An *inner function* is a function M in H^∞ such that $|M^*| = 1$ a.e. $[m]$. A function of the form

$$S_\mu(z) = \exp\{-\int_{-\pi}^{\pi} \frac{\zeta + z}{\zeta - z} d\mu(t)\},$$

where μ is a positive Borel measure on $\mathbb{T}$ that is singular with respect to m, is known as a *singular inner function.* Note that such a function does not have any zeros in $\mathbb{D}$. Let $\{a_n\}$ be a sequence of points in $\mathbb{D}$ such that $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$. For such sequences, there is an associated *Blaschke product* B defined on $\mathbb{D}$ by

$$B(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \bar{a_n} z} \frac{|a_n|}{a_n},$$

where $\frac{|a_n|}{a_n}$ is taken to be 1 if $a_n = 0$. The function B is in H^∞ and $|B^*(e^{i\theta})| = 1$ almost everywhere on $\mathbb{T}$. Hence, each Blaschke product is an inner function, as is each singular inner function. Now, every inner function M can be factored uniquely as the product of a Blaschke product and a singular inner function. That is, every inner function M may be written in the form

$$M(z) = c \cdot B(z) \cdot S_\mu(z)$$

where c is a constant such that $|c| = 1$. An *outer function* is a function of the form

$$G(z) = c \cdot \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log \varphi(e^{it}) dt \right\}$$

where c is a constant such that $|c| = 1$, and φ is a positive measurable function on $\mathbb{T}$ such that $\log \varphi \in L^1(\mathbb{T})$. For $0 < p \leq \infty$ and f in H^p such that f is not identically zero, the function $\log |f^*|$ is in $L^1(\mathbb{T})$ and

$$G_f(z) = \exp\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |f^*(e^{it})| \} dt$$

is an outer function in H^p. For such f, there exists an inner function M_f such that $f = M_f G_f$. Thus, for all $p > 0$, every f in H^p may be factored uniquely into the product of an inner function and an outer function. Thus, by our previous statement regarding the factorization of inner functions, we have that every f in H^p may be written uniquely in the form

$$f(z) = c \cdot B(z) \cdot S_\mu(z) \cdot G_f(z)$$

for z in $\mathbb{D}$, where G_f is an outer function in H^p, see [19].) Note that any analytic self-map φ of the unit disk $\mathbb{D}$ is in H^∞. Hence, by our above discussion, φ may be factored as above and so has only a few possible forms. The function φ may be written as a Blaschke product, a singular inner function, an outer function, or it may be written as a product of these types of functions.

It is natural to ask for which φ is the composition operator C_φ bounded, compact, or closed-range on a Banach space of analytic functions on $\mathbb{D}$. We will now catalog such results for various classical Banach spaces. These results are standard in the literature and more information can be found in [22] and [27]. We begin by examining composition operators on the Hardy space H^2.

1.1 Composition Operators on the Hardy Space H^2

Littlewood's Subordination Principle (see [22]) states that if φ is an analytic self-map of $\mathbb{D}$ with $\varphi(0) = 0$, then for each f in H^2, $C_\varphi(f) \in H^2$ and $||C_\varphi(f)|| \leq ||f||$.

Thus, if φ fixes the origin, then C_φ is bounded on H^2. To see that this is the case for any holomorphic self-map φ of $\mathbb{D}$, we will use $\alpha_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}$, the special automorphism of $\mathbb{D}$ where $\alpha_\lambda(\lambda) = 0$, $\alpha_\lambda(0) = \lambda$, and $\alpha_\lambda^{-1} = \alpha_\lambda$. Letting $\lambda = \varphi(0)$, we consider the function $\psi = \alpha_\lambda \circ \varphi$ which is a holomorphic self-map of $\mathbb{D}$ that fixes the origin. Then, $\varphi = \alpha_\lambda^{-1} \circ \psi = \alpha_\lambda \circ \psi$ and by Littlewood's Subordination Principle, for all f in H^2 we have

$$\begin{aligned} ||f \circ \varphi||^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(\varphi(e^{i\theta}))|^2 d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f \circ \alpha_\lambda \circ \psi(e^{i\theta})|^2 d\theta \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2\pi}\int_0^{2\pi} |f\circ\alpha_\lambda(e^{i\theta})|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2|\alpha_\lambda'(e^{it})|dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 \frac{1-|\lambda|^2}{|1-\overline{\lambda}e^{it}|^2}dt \\
&\le \frac{1-|\lambda|^2}{(1-|\lambda|)^2}\cdot\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dt \\
&= \frac{1+|\lambda|}{1-|\lambda|}\cdot ||f||^2
\end{aligned}
$$

Thus we have that C_φ is bounded on H^2 for every analytic self-map φ of $\mathbb{D}$. □

We may also address the question of compactness of composition operators. The *First Compactness Theorem* (page 23 in [22]) states that the composition operator C_φ is a compact operator on H^2 if $||\varphi||_\infty < 1$. In other words, C_φ is a compact composition operator if $\varphi(\mathbb{D})$ is relatively compact. The Univalent Compactness theorem (see page 39 in [22]) says that if φ is a univalent self-map of $\mathbb{D}$, then, C_φ is compact on H^2 if and only if

$$\lim_{|z|\to 1^-} \frac{1-|\varphi(z)|}{1-|z|} = \infty.$$

It should be noted that necessity in this theorem does not require univalence. As this requirement for compactness deals with a difference quotient, it is reasonable to think that there may be some relationship between this condition and the derivative of φ at the boundary of the disk.

Let φ be a holomorphic self-map of $\mathbb{D}$, and let ω be a point on $\partial\mathbb{D}$. We say that φ has angular limit $\mathcal{L} = \angle\lim_{z\to\omega}\varphi(z)$ if $\varphi(z)\to\mathcal{L}$ as $z\to\omega$ through any stolz region based at ω. The map φ has an angular derivative at ω, denoted $\varphi'(\omega)$, if for some point η in $\partial\mathbb{D}$,

$$\angle\lim_{z\to\omega}\frac{\eta-\varphi(z)}{\omega-z}$$

exists. This suggests that the angular limit of φ at ω exists and is equal to η. Hence, if φ has an angular derivative at any point on $\partial\mathbb{D}$, then it must have an angular limit of modulus one at that point.

The *Julia-Caratheodory Theorem* clarifies the relationship between compactness and the existence of angular derivatives. This theorem states that the angular derivative $\angle \lim_{z\to\omega} \frac{\eta-\varphi(z)}{\omega-z}$ exists for some η in $\partial\mathbb{D}$ if and only if $\liminf_{z\to\omega} \frac{1-|\varphi(z)|}{1-|z|} = \delta$ for some δ, $0<\delta<\infty$. But, by the *Univalent Compactness theorem*, $\liminf_{z\to\omega} \frac{1-|\varphi(z)|}{1-|z|} < \infty$ implies that C_φ is not compact on H^2.

Next, we discuss when C_φ is compact on H^2 for arbitrary self-maps φ of $\mathbb{D}$. In other words, we want a condition for compactness on H^2 when φ is not necessarily univalent. For a function φ holomorphic on $\mathbb{D}$, the Nevanlinna Counting Funtion of φ, denoted N_φ, is defined as follows:

$$N_\varphi(w) = \begin{cases} \sum_{z\in\varphi^{-1}\{w\}} \log\frac{1}{|z|} & w \in \varphi(\mathbb{D}) \\ 0 & w \notin \varphi(\mathbb{D}) \end{cases}$$

.

For a function f analytic on $\mathbb{D}$, the *Littlewood-Paley Identity* (see[22]) gives that

$$||f||^2 = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|} dA(z).$$

The *change-of-variable formula* (see [22]) states that for any analytic map φ on $\mathbb{D}$,

$$||C_\varphi(f)||_2^2 = |f(\varphi(0))|_2^2 + 2\int_{\mathbb{D}} |f'(w)|^2 N_\varphi(w) dA(w).$$

Notice that if φ is univalent, then the change-of-variable formula is just the Littlewood-Paley Identity with the substitution $w=\varphi(z)$.

Theorem 2.3 in [20] gives the following result. Suppose φ is a holomorphic self-map of $\mathbb{D}$. Then, C_φ is compact on H^2 if and only if

$$\lim_{|w|\to 1^-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} = 0.$$

If φ is univalent, we have

$$N_\varphi(w) = \log \frac{1}{|z|} \approx 1 - |z|$$

for $|z|$ large, where $\varphi(z) = w$. Thus, in the case that φ is univalent, this theorem is the same as the Univalent Compactness Theorem stated above.

In [3], it is shown that this condition on φ involving the Nevalinna counting function is equivalent to the condition

$$\lim_{|a|\to 1^-} \int_{\mathbb{T}} \frac{1-|a|^2}{|1-\bar{a}\varphi(z)|^2} dm(z) = 0.$$

In [28], Nina Zorboska gives conditions regarding when a composition operator C_φ will be closed-range on H^2 and on $\mathbb{A}^2_\alpha$ for $\alpha > -1$. The function $\pi_\varphi(w)$, having domain $\mathbb{D} \setminus \varphi(0)$, is defined by

$$\pi_\varphi(w) = \frac{N_\varphi(w)}{\log \frac{1}{|w|}},$$

where $N_\varphi(w)$ is the Nevanlinna Counting Function as defined above. For a positive constant c, the set G^φ_c is defined by

$$G^\varphi_c = \{z : \pi_\varphi(z) > c\}.$$

Theorem 3.4 in [28] states that a composition operator C_φ will be closed-range on H^2 if and only if there exist positive constants c and δ such that

$$A(G^\varphi_c \cap D(\xi, r)) > \delta \cdot A(\mathbb{D} \cap D(\xi, r)) \tag{1}$$

for all ξ in $\partial\mathbb{D}$, where $D(\xi, r) = \{z \in \mathbb{D} : |z - \xi| < r\}$. In [14], it is shown that condition (1) may be restated as follows. There exist constants $\delta_1 > 0$ and b, $0 < b < 1$, so that

$$A(G_c^{\varphi} \cap D_b(a)) > \delta_1 \cdot A(D_b(a)) \tag{2}$$

for every a in D where $D_b(a) = \{z \in \mathbb{D} : |z - a| < b(1 - |a|)\}$.

A good reference for the following discussion is [28]. Suppose φ is a univalent function such that C_φ does not have closed-range on H^2. Let ψ be a holormorphic self-map of $\mathbb{D}$ such that $\psi(\mathbb{D})$ is contained in $\varphi(\mathbb{D})$. Let ω be the self-map of $\mathbb{D}$ defined by $\omega := \varphi^{-1} \circ \psi$. Then, $\psi = \varphi \circ \omega$. Let $\{f_n\}$ be a sequence of functions in H^2 such that $||f_n||_{H^2} = 1$ and $||C_\varphi f_n||_{H^2} \to 0$. Then,

$$||C_\psi f_n||_{H^2} = ||f_n \circ \varphi \circ \omega||_{H^2} \leq ||C_\omega||_{H^2} ||\dot{f_n} \circ \varphi||_{H^2} \to 0.$$

Hence, C_ψ will not be closed-range on H^2.

Example 1 in [28] states that if there exists a point $\xi \in \mathbb{T}$ and a neighborhood N_ξ about the point ξ such that $N_\xi \cap \varphi(\mathbb{D}) = \emptyset$, then C_φ will not be closed-range on H^2. To see that this is the case, choose a Euclidean disk $D(\xi, r)$ to be contained in N_ξ. Then, for all $z \in D(\xi, r)$, we have that $\gamma_\varphi(z) = 0$ and so the set G_c^φ is empty for all $c > 0$. Hence, for any $c > 0$, $A(G_c^\varphi \cap D(\xi, r)) = 0$ and condition (1) above will not be satisfied.

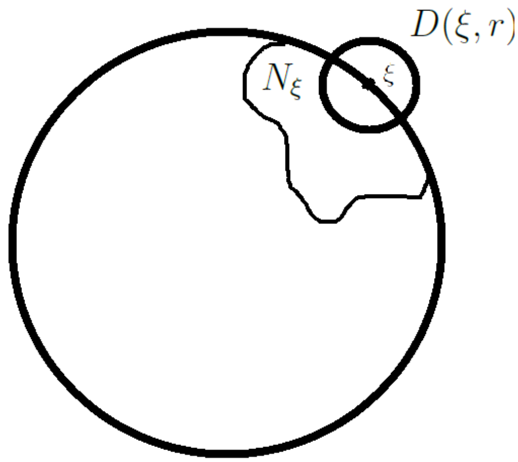

Figure 2: Example

Another example described in [28] is the following. A composition operator C_φ will not be closed-range on H^2 if there is a disk D_1 that is tangent to ∂D such that $D_1 \cap \varphi(\mathbb{D}) = \emptyset$. In this case, for any choice of b, we can choose a point a in $\mathbb{D}$ close enough to the boundary of the disk so $D_b(a)$ will be contained entirely in the disk D_1. Then, similar to the previous example, we have that $\gamma_\varphi(z) = 0$ on $D_b(a)$. Hence, $A(G_c^\varphi \cap D_b(a)) = 0$ for any $c > 0$ and so condition (2) above is not satisfied.

In [28], Zorboska also remarks that the composition operator C_φ will not be closed-range on H^2 if $\varphi(\mathbb{D})$ is a proper subset of $\mathbb{D} \setminus [0, 1)$.

We will now introduce several other classical spaces of analytic functions in $\mathbb{D}$. For more information on the following spaces, see [27].

1.2 Composition Operators on the Bloch space $\mathcal{B}$

The *Bloch space* $\mathcal{B}$ is the space of analytic functions on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty.$$

Under the norm

$$||f||_{\mathcal{B}}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|,$$

$\mathcal{B}$ forms a Banach space. By Proposition 5.1 in [27], H^∞ is properly contained in $\mathcal{B}$, and $||f||_{\mathcal{B}}\leq||f||_\infty$ for all $f\in H^\infty$. The set of analytic functions in $\mathbb{D}$ having the property that

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)|=0$$

is called the *little Bloch space* and is denoted by $\mathcal{B}_0$. The little Bloch space is a closed subspace of $\mathcal{B}$.

For z in $\mathbb{D}$, let $\tau_\varphi(z)=\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}$. We can apply the Schwarz-Pick lemma to get $|\tau_\varphi(z)|\leq 1$ for all z in $\mathbb{D}$. Now, for f in $\mathcal{B}$,

$$\begin{aligned}(1-|z|^2)|(f\circ\varphi)'(z)|&=(1-|z|^2)|f'(\varphi(z))||\varphi'(z)|\\&=\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}(1-|\varphi(z)|^2)|f'(\varphi(z))|\\&=|\tau_\varphi(z)|(1-|\varphi(z)|^2)|f'(\varphi(z))|\\&\leq(1-|\varphi(z)|^2)|f'(\varphi(z))|\end{aligned}$$

Thus C_φ is a bounded composition operator on $\mathcal{B}$ for every analytic self-map φ of $\mathbb{D}$.

It is shown in Theorem 2 in [17] that C_φ wll be compact on $\mathcal{B}$ if and only if for every $\varepsilon > 0$, there exists r, $0 < r < 1$, such that

$$\tau_\varphi(z) = \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} < \varepsilon$$

whenever $|\varphi(z)| > r$.

By theorem 2 in [26], for an analytic self-map φ of the unit disk $\mathbb{D}$, the composition operator C_φ is compact on $\mathcal{B}$ if and only if

$$\lim_{n\to\infty} ||\varphi^n||_{\mathcal{B}} = 0.$$

In [17], Madigan and Matheson give a similar condition for the compactness of a composition operator on $\mathcal{B}_0$. Theorem 1 in [17] states that C_φ is compact on $\mathcal{B}_0$ if and only if

$$\lim_{|z|\to 1^-} |\tau_\varphi(z)| = 0.$$

In [11], a necessary and sufficient condition for a composition operator C_φ to be closed-range on $\mathcal{B}$ is given. Letting $\mathcal{C}$ be the closed subspace of constant functions, we have $||f||_{\mathcal{B}/\mathcal{C}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$. Theorem 0 in [11] states that C_φ will be closed-range on $\mathcal{B}$ if and only if

$$||f \circ \varphi||_{\mathcal{B}/\mathcal{C}} \geq k \cdot ||f||_{\mathcal{B}/\mathcal{C}}$$

for a constant $k > 0$.

For a subset K of $\mathbb{D}$, if there exists $k > 0$ with

$$\sup\{(1-|z|^2)|f'(z)| : z \in \mathbb{D}\} \leq k \cdot \sup\{(1-|z|^2)|f'(z)| : z \in K\}$$

for every function f in $\mathcal{B}$, then K is called a *sampling set* for $\mathcal{B}$. Define $F_\varepsilon := \varphi(\Lambda_\varepsilon)$ where $\Lambda_\varepsilon := \{z \in \mathbb{D} : \tau_\varphi(z) \geq \varepsilon\}$ and $\varepsilon > 0$. Theorem 1 in [11] says that a composition operator C_φ

will be closed-range on the Bloch space $\mathcal{B}$ if and only if there exists $\varepsilon > 0$ such that F_ε is a sampling set for $\mathcal{B}$. It is also shown in [11] that the set F_ε is a sampling set for $\mathcal{B}$ if it satisfies the *reverse Carleson condition*. That is, F_ε is a sampling set for $\mathcal{B}$ if there exist constants c and s with $0 < c, s < 1$, such that $A(F_\varepsilon \cap \Delta(z,s)) \geq c \cdot A(\Delta(z,s))$ for all z in the unit disk $\mathbb{D}$. Hence, by Theorem 1 and Proposition 1 in [11], if F_ε satisfies the reverse Carleson condition, then the composition operator C_φ will be closed-range on $\mathcal{B}$. If φ is univalent, then, by theorem 2 in [11], the converse of the previous statement also holds.

Let φ be univalent and suppose that the composition operator C_φ is closed-range on $\mathcal{B}$. Then, for some $\varepsilon > 0$, F_ε satisfies the reverse Carleson condition and by Proposition 3 in [11], there exists $\delta > 0$ such that for every point ω in $\partial\mathbb{D}$,

$$\overline{\lim}_{\varphi(z)\to\omega} \frac{\mathrm{dist}(\varphi(\mathrm{z}), \partial(\varphi(\mathbb{D})))}{|\varphi(z) - \omega|} \geq \delta.$$

Example 1 in [11] shows that this condition is not sufficient for C_φ to be closed-range. In the second example given in [11], we let $G = \mathbb{D} \setminus [0, 1)$ and φ is chosen to be the Riemann mapping onto G. By the Koebe One-Quarter Theorem, when φ is univalent, then

$$\tau_\varphi(z) \approx \frac{\mathrm{dist}(\varphi(\mathrm{z}), \partial\mathrm{G})}{1 - |\varphi(z)|}.$$

As $\varphi(z)$ approaches a point w on the boundary of the disk other than 1, this ratio approaches 1. Then, F_ε contains all of $\mathbb{D}$ except for a pseudohyperbolic neighborhood of the segment $[0, 1)$. Thus, we can choose r large enough so that every point z in $\mathbb{D}$ is within pseudohyperbolic distance r of F_ε. So, there exists a constant $c > 0$ such that $A(F_\varepsilon \cap \Delta(z, r)) \geq c \cdot A(\Delta(z, r))$ for all z in the unit disk $\mathbb{D}$. Hence, F_ε satisfies the reverse Carleson condition and C_φ is closed-range.

Proposition 1 in [10] gives a necessary condition for the composition operator C_φ to be closed-range on $\mathcal{B}$. The proposition states that if C_φ is closed-range on $\mathcal{B}$ then there will exist positive constants ε and $r < 1$ so that, for all z in $\mathbb{D}$, $\rho(\varphi(\Lambda_\varepsilon), z) \leq r$. Recall that ρ denotes the pseudohyperbolic metric. Theorem 2 in the same source ([10]) also gives a sufficient condition. This theorem gives that C_φ is closed-range on $\mathcal{B}$ if for some positive constants ε and r with $r < \frac{1}{4}$, for all w in $\mathbb{D}$ there exists a point z_w in $\mathbb{D}$ so that $\rho(\varphi(z_w), w) < r$ and $|\tau_\varphi(z_w)| > \varepsilon$.

1.3 Composition Operators on the Besov space B_p

For $0 < p < \infty$, the Besov space B_p is the collection of holomorphic functions in $\mathbb{D}$ such that

$$\begin{aligned} ||f||_{B_p}^p &= \int_{\mathbb{D}} |f^{(n)}(z)(1-|z|^2)^n|^p d\lambda(z) \\ &= \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np} d\lambda(z) < \infty \end{aligned}$$

for any positive integer n satisfying $np > 1$ and where

$$d\lambda(z) = \frac{1}{(1-|z|^2)^2} dA(z).$$

Under the norm $|f(0)| + ||f||_{B_p}$, B_p is a Banach space.

Theorem 5.17 in [27] gives the atomic decomposition for B_p. The theorem states that for $p > 0$, there exists a sequence $\{a_k\}$ in $\mathbb{D}$ such that for $b > \max(0, \frac{p-1}{p})$, the space B_p is comprised of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \left(\frac{1-|a_k|^2}{1-z\bar{a}_k} \right)^b$$

with $c_k \in l^p := \{\{c_k\}_{k=1}^{\infty} \subset \mathbb{C} : \sum_{k=1}^{\infty} |c_k|^p < \infty\}$.

Recall that $\alpha_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}$ is the special automorphism of the disk $\mathbb{D}$ with $\alpha_\lambda(\lambda) = 0$, $\alpha_\lambda(0) = \lambda$, and $\alpha_\lambda^{-1} = \alpha_\lambda$. By Theorem D in [24], for an analytic self-map φ of $\mathbb{D}$, C_φ is a bounded operator on the Besov space B_p if and only if

$$\sup_{\lambda \in \mathbb{D}} ||C_\varphi \alpha_\lambda||_{B_p} < \infty.$$

By theorem 3.5 in [24], for $1 < p \leq q < \infty$, when φ is a holomorphic self-map of $\mathbb{D}$ then the following are equivalent:

1. $C_\varphi : B_p \to B_q$ is a compact operator.

2. $||C_\varphi \alpha_\lambda||_{B_q} \to 0$ as $|\lambda| \to 1$.

Not much is known about conditions for which φ will induce a compact composition operator on B_p for p in general.

1.4 Composition Operators on the Dirichlet Space

The Dirichlet space $\mathcal{D}$ is the set of holomorphic functions f on $\mathbb{D}$ such that

$$||f||_{\mathcal{D}}^2 = \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

That is, if f is in $\mathcal{D}$, then its derivative is in $\mathbb{A}^2$. Note that $\mathcal{D} = B_2$, with an equivalent norm.

For $p > 0$ and μ a finite positive Borel measure, if there exists a constant $0 < c < \infty$ such that

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq c \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)$$

for all f in $\mathbb{A}^p_\alpha$, then μ is called a *Carleson measure* for $\mathbb{A}^p_\alpha$. As is stated in [15], an equivalent condition for μ to be a Carleson measure is

$$\sup_{z\in\mathbb{D}} \frac{\mu(\Delta(z,\eta))}{|\Delta(z,\eta)|} < \infty$$

where, again, $\Delta(z,\eta)$ denotes the pseudohyperbolic disk. We call μ a *compact (or vanishing) Carleson measure* if

$$\sup_{r<|z|<1} \frac{\mu(\Delta(z,\eta)}{|\Delta(z,\eta)|} \to 0$$

as $r \to 1$. Let n_φ denote the cardinality of the set $\varphi^{-1}(w)$. In [15], Luecking shows that a composition operator C_φ is bounded on the Dirichlet space $\mathcal{D}$ if $n_\varphi dA$ is a Carleson measure for $\mathbb{A}^p_\alpha$ for some $p > 0$. By Proposition 5.1 in [16], C_φ is compact on $\mathcal{D}$ if $n_\varphi dA$ is a compact Carleson measure. In [15], Luecking also shows that C_φ is closed-range on the Dirichlet space $\mathcal{D}$ if and only if there exists a constant $c > 1$ such that

$$\frac{1}{c}\int |f'|^2 dA \le \int |f'|^2 n_\varphi dA \le c\int |f'|^2 dA$$

for every f in $\mathcal{D}$. If this condition is satisfied, then there exists R, $0 < R < 1$ and $\delta > 0$ such that

$$\int_{\Delta(a,r)} n_\varphi dA \le \delta|\Delta(a,r)|$$

for all $z \in \mathbb{D}$ where, again, $|\Delta(a,r)|$ denotes the area of the pseudohyperbolic disk $\Delta(a,r)$. By part 2 of Corollary 2 in [11], when φ is univalent and the composition operator C_φ is bounded below on the Bloch space $\mathcal{B}$, then C_φ is also bounded below on the Dirichlet space. That is, if φ is univalent and C_φ is closed-range on $\mathcal{B}$ then it is also closed-range on $\mathcal{D}$.

1.5 Composition Operators on BMO

Let I denote an interval that is contained in $\mathbb{T}$ and let f be in $L^2(\mathbb{T})$. With $|I|$ representing the length of the interval I, the mean of the function f over I is given by

$$f_I = \frac{1}{|I|}\int_I f(\theta)d\theta.$$

The space of all functions f in $L^2(\mathbb{T})$ that have *bounded mean oscillation* is called BMO($\mathbb{T}$). A function f has bounded mean oscillation if

$$||f||_{BMO} := \sup_I \frac{1}{|I|}\int_I |f(\theta) - f_I|d\theta < \infty.$$

The space BMOA($\mathbb{T}$) is the intersection of BMO and $H^2(\mathbb{T})$. For any function f in $L^1(\mathbb{T})$ we can extend f to a function $\hat{f}$ on the disk $\mathbb{D}$ via the *Poisson extension*,

$$\hat{f}(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{1-|z|^2}{|1-\bar{z}e^{i\theta}|^2}d\theta$$

for all z in $\mathbb{D}$. Thus, BMOA($\mathbb{T}$) can be extended to BMOA($\mathbb{D}$), a space of analytic functions on $\mathbb{D}$. Recall the special automorphism of $\mathbb{D}$,

$$\alpha_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}$$

where $\alpha_\lambda(\lambda) = 0$, $\alpha_\lambda(0) = \lambda$, and $\alpha_\lambda^{-1} = \alpha_\lambda$. As is stated in [23], a function f in H^2 is in BMOA if

$$||f||_* = \sup_{\lambda\in\mathbb{D}} ||f\circ\alpha_\lambda - f(\lambda)||_2 < \infty.$$

Under the norm $||f||_{BMOA} = ||f||_* + |f(0)|$, BMOA forms a Banach space.

For r in $(0,1)$, let Φ_r denote the set $\{z : 1 > |\varphi(z)| > r\}$, and for the characteristic function of Φ_r, we write $\chi_r(z)$. By Theorem 3.1 in [6], the composition operator C_φ is compact on BMOA if and only if for all $\varepsilon > 0$, there exists an $r \in (0,1)$ such that

$$\int_{R(I)} \chi_{\Phi_r}(z)(1-|z|^2)|f'(\varphi(z))|^2|\varphi'(z)|^2 dA(z) \leq \varepsilon |I|.$$

By Theorem 4.1 in [23], If φ is a univalent self-map of $\mathbb{D}$, then C_φ is compact on BMOA if and only if it is compact on the Bloch space $\mathcal{B}$.

By part 1 of Corollary 2 in [11], for univalent φ, if C_φ is bounded below on BMOA, then it is also bounded below on the Bloch space $\mathcal{B}$. In other words, if φ is univalent and C_φ is closed-range on BMOA, then C_φ is also closed-range on $\mathcal{B}$.

We have defined $\pi_\varphi(w)$ to be

$$\pi_\varphi(w) := \frac{N_\varphi(w)}{\log(\frac{1}{w})},$$

where $N_\varphi(w)$ is the Nevalinna counting function. Let $\pi_{\varphi,\alpha} = (\pi_\varphi(w))^\alpha$ and

$$G_c^{\varphi,\alpha} = \{z : \pi_{\varphi,\alpha+2} > c\}.$$

Theorem 4.1 in [28] states that a composition operator C_φ will be closed-range on the weighted Bergman space $\mathbb{A}^2_\alpha$, with $\alpha > -1$ if and only if there exist constants $c > 0$ and $\lambda > 0$ such that

$$A(G^{\varphi,\alpha} \cap D(\xi, r)) > \delta \cdot A(\mathbb{D} \cap D(\xi, r)).$$

Notice that not only does the condition above involve the Nevanlinna counting function, but it also depends on α. In the next section, we will give a necessary and sufficient condition for when an analytic self-map φ of $\mathbb{D}$ induces a closed-range composition operator on the weighted Bergman space $\mathbb{A}^p_\alpha$ for all p and all $\alpha > -1$. This will essentially render all the above conditions equivalent for various values of α. In [1], J. Akeroyd and P. Ghatage give a necessary and sufficient condition in the case $p = 2$ and $\alpha = -1$.

2 Closed-Range Composition Operators on Weighted Bergman Spaces

Let φ be an analytic self-map of $\mathbb{D}$. For any ε, $0 < \varepsilon < 1$, define $\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$, and let $G_\varepsilon(\varphi) = G_\varepsilon = \varphi(\Omega_\varepsilon)$. Note that, in the weighted Bergman space setting, G_ε functions in much the same way that F_ε did in the Bloch space setting. The set G_ε is said to satisfy the *reverse Carleson condition* if there exists a positive constant η so that

$$\int_{G_\varepsilon} |f(z)|^p(1-|z|^2)^p dA \geq \eta \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^p dA$$

for f analytic in $\mathbb{D}$ and $\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^p dA < \infty$. As shown in [14], this is equivalent to the following condition:

($*$) There exist constants c and s with $0 < c, s < 1$, such that

$$A(G_\varepsilon \cap \Delta(z,s)) \geq c \cdot A(\Delta(z,s))$$

for all z in the unit disk $\mathbb{D}$. We will show in Theorem 2.3 that a composition operator C_φ is closed-range on $\mathbb{A}^p_\alpha$ if and only if there exists an $\varepsilon > 0$ such that G_ε satisfies condition ($*$). Recall that we defined the weighted Bergman Spaces $\mathbb{A}^p_\alpha$ by

$$\mathbb{A}^p_\alpha := \{f : f \text{ is analytic in } \mathbb{D} \text{ and } ||f||^p_{p,\alpha} = \textstyle\int_{\mathbb{D}} |f|^p dA_\alpha < \infty\}$$

where $\alpha > -1$ and $dA_\alpha := c_\alpha \cdot (1-|z|^2)^\alpha dA(z)$ where $c_\alpha = \alpha + 1$.

Define $\mathbb{A}^p_{\alpha,0}$ by

$$\mathbb{A}^p_{\alpha,0} = \{f \in \mathbb{A}^p_\alpha : f(0) = 0\},$$

a closed subspace of $\mathbb{A}^p_\alpha$. The following lemma is adapted from Lemmas 2.1, 2.2, and 2.3 in [1], and will be stated without proof.

Lemma 2.1 *Let φ be an analytic self-map of the unit disk $\mathbb{D}$, and let ψ_a be a conformal automorphism of $\mathbb{D}$. Then,*

1. *C_φ is closed-range on $\mathbb{A}^p_\alpha$ if and only if C_φ is closed-range on $\mathbb{A}^p_{\alpha,0}$*

2. *If one of C_φ, $C_{\varphi\circ\psi_a}$, or $C_{\psi_a\circ\varphi}$ is closed-range on $\mathbb{A}^p_\alpha$, then so are the other two.*

3. *If there exists $\varepsilon > 0$ such that one of $G_\varepsilon(\varphi)$, $G_\varepsilon(\varphi\circ\psi_a)$, $G_\varepsilon(\psi_a\circ\varphi)$ satisfies condition $(*)$, then there exists $\varepsilon > 0$ such that the other two also satisfy condition $(*)$.*

Lemma 2.2 *(Lemma 2.2 in [4]). Let φ be an analytic self-map of $\mathbb{D}$. If C_φ is closed-range on $\mathbb{A}^p_\alpha$ then it is closed-range on $\mathbb{A}^{np}_\alpha$ for any $n \in \mathbb{N}$.*

Proof. Assume φ is not constant. Otherwise, the result is trivial. Suppose C_φ is closed-range on $\mathbb{A}^p_\alpha$. Then there exists a constant c such that $||f\circ\varphi||_{\mathbb{A}_{p,\alpha}} \geq c\cdot||f||_{\mathbb{A}_{p,\alpha}}$ for all $f \in \mathbb{A}^p_\alpha$. That is,

$$\int_{\mathbb{D}} |f\circ\varphi|^p dA_\alpha \geq c\cdot\int_{\mathbb{D}} |f|^p dA_\alpha$$

for all $f \in \mathbb{A}^p_\alpha$. Now, if $f \in \mathbb{A}^{np}_\alpha$, then $f^n \in \mathbb{A}^p_\alpha$. Thus,

$$\begin{aligned}\int_{\mathbb{D}} |f\circ\varphi|^{np} dA_\alpha &= \int_{\mathbb{D}} |f^n\circ\varphi|^p dA_\alpha \\ &\geq c\cdot\int_{\mathbb{D}} |f^n|^p dA_\alpha \\ &= c\cdot\int_{\mathbb{D}} |f|^{np} dA_\alpha.\end{aligned}$$

and so C_φ is closed-range on $\mathbb{A}^{np}_\alpha$. □

Theorem 2.3 *(1.3 in [5]) Let φ be an analytic self-map of the unit disk $\mathbb{D}$. Suppose $1 \leq p < \infty$ and $\alpha > -1$. Then, the following are equivalent:*

1. C_φ *is closed-range on* $\mathbb{A}^p_\alpha$.

2. *There exists* $\varepsilon > 0$ *such that* $G_\varepsilon = \varphi(\Omega_\varepsilon)$ *satisfies condition* $(*)$.

Proof. By Lemma 2.1 we may assume that $\varphi(0) = 0$ and we may also restrict our attention to C_φ on $\mathbb{A}^p_{\alpha,0}$. By the proof of Theorem 4.28 in [27], there is a constant $C > 1$ such that

$$\frac{1}{C}||f||_{p,\alpha} \leq \left\{ \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p dA_\alpha(z) \right\}^{\frac{1}{p}} \leq C||f||_{p,\alpha}$$

for all f in $\mathbb{A}^p_{\alpha,0}$. We will denote this by

$$||f||^p_{p,\alpha} \approx \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p dA_\alpha(z).$$

We will first show that (2) implies (1). Note that this argument can also be found in the proof of Theorem 2.3 in [4]. Suppose that for some $\varepsilon > 0$, G_ε satisfies condition $(*)$. First consider the case that $1 \leq p < 2$. By the Schwarz-Pick lemma (Lemma 1.2 in [8]), $0 \leq \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \leq 1$ for all z in $\mathbb{D}$. Hence, for all z in Ω_ε, $\varepsilon|\varphi'(z)| < 1$, and thus, since $1 \leq p < 2$, we have $\varepsilon^2|\varphi'(z)|^2 \leq \varepsilon^p|\varphi'(z)|^p$. Hence, $\varepsilon^{2-p} < |\varphi'(z)|^{p-2}$. Then,

$$\begin{aligned}
||f \circ \varphi||^p_{p,\alpha} &\approx \int_{\mathbb{D}} |(f\circ\varphi)'(z)|^p (1-|z|^2)^p dA_\alpha(z) \\
&\geq \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^{p+\alpha} dA(z) \\
&= \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p |\varphi'(z)|^{p-2} |\varphi'(z)|^2 (1-|z|^2)^{p+\alpha} dA(z) \\
&\geq \varepsilon^{2-p} \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p |\varphi'(z)|^2 (1-|z|^2)^{p+\alpha} dA(z) \\
&\geq \varepsilon^{\alpha+2} \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^{p+\alpha} dA(z) \\
&= \varepsilon^{\alpha+2} \sum_n \int_{\Omega_\varepsilon \bigcap R_n} |f'(\varphi(z))|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^{p+\alpha} dA(z)
\end{aligned}$$

where $\mathcal{Z} := \{z \in \mathbb{D} : \varphi'(z) = 0\}$ and $\{R_n\}$ is a partition of $\mathbb{D} \setminus \mathcal{Z}$ into at most countably many polar rectangles so that φ is univalent on R_n for all n. Let $S_n = \varphi(\Omega_\varepsilon \bigcap R_n)$ and let ψ_n denote the inverse of $\varphi|_{R_n}$. Then, letting $z = \psi_n(w)$, we have

$$\begin{aligned}
&\varepsilon^{\alpha+2} \sum_n \int_{\Omega_\varepsilon \bigcap R_n} |f'(\varphi(z))|^p |\varphi'(z)|^2 (1 - |\varphi(z)|^2)^{p+\alpha} dA(z) \\
&\qquad = \varepsilon^{\alpha+2} \sum_n \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} \chi_{S_n}(w) dA(w) \\
&\qquad = \varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} (\sum_n \chi_{S_n}(w)) dA(w) \\
&\qquad \geq \varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} dA(w)
\end{aligned}$$

Since G_ε satisfies condition $(*)$, we have

$$\begin{aligned}
\varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1 - |w|^2)^{p+\alpha} dA(w) &\geq \eta \varepsilon^{\alpha+2} \int_{\mathbb{D}} |f'(w)|^p (1 - |w|^2)^{p+\alpha} dA(w) \\
&\approx \int_{\mathbb{D}} |f(w)|^p dA_\alpha(w) \\
&= ||f||_{p,\alpha}^p
\end{aligned}$$

Hence, for $1 \leq p < 2$, we have that C_φ is closed-range on $\mathbb{A}_\alpha^p$. We may apply Lemma 2.2 to see that C_φ is closed-range on $\mathbb{A}_\alpha^p$ for $1 \leq p < \infty$ and, thus, (1) is satisfied.

We will now show that (1) implies (2) by means of the contrapositive, as is also shown in the proof of Theorem 1.3 in [5]. Suppose that condition $(*)$ is not satisfied. Then there does not exist $\varepsilon > 0$ such that G_ε satisfies the reverse Carleson condition. In other words, for any $\varepsilon > 0$, there does not exist a positive constant η such that

$$\int_{G_\varepsilon} |f(z)|^p (1-|z|^2)^p dA \geq \eta \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^p dA$$

for all f in $\mathbb{A}^p_{\alpha,0}$. So, we can find a sequence $\{f_k\}_{k=1}^\infty \subset \mathbb{A}^p_{\alpha,0}$ such that

$$\int_{\mathbb{D}} |f_k'(w)|^p (1-|w|^2)^p dA_\alpha(w) = 1$$

for all k, but

$$\int_{G_k} |f_k'(w)|^p (1-|w|^2)^p dA_\alpha(w) \to 0$$

as $k \to \infty$, where $\Omega_k = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \frac{1}{k}\}$ and $G_k = \varphi(\Omega_k)$.

First suppose that $p \geq 3$. Note that for all $z \in \Omega_{j+1} \setminus \Omega_j$, we have $\frac{1}{j} \geq \frac{1-|z|^2}{1-|\varphi(z)|^2} > \frac{1}{j+1}$ and, by the Schwarz-Pick Lemma, $0 \leq \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \leq 1$ for all z in $\mathbb{D}$. Thus, on $\Omega_{j+1} \setminus \Omega_j$,

$$|\varphi'(z)|^{p-2} \leq (j+1)^{p-2}.$$

Also, on $\Omega_{j+1} \setminus \Omega_j$,

$$(1-|z|^2)^{p+\alpha-1} \approx \frac{1}{(j+1)^{p+\alpha-1}} \cdot (1-|\varphi(z)|^2)^{p+\alpha-1}.$$

Now, $\mathbb{D}$ is equal to the pairwise disjoint union $\Omega_k \cup (\cup_{j=k}^\infty \Omega_{j+1} \setminus \Omega_j)$. So,

$$\begin{aligned}
||f_k \circ \varphi||_{p,\alpha}^p &\approx \int_{\mathbb{D}} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^p dA_\alpha(z) \\
&= \int_{\Omega_k} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^p dA_\alpha(z) \\
&\qquad + \sum_{j=k}^\infty \int_{\Omega_{j+1} \setminus \Omega_j} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^p dA_\alpha(z)
\end{aligned}$$

Then, by the corollary on page 188 of [22], we have

$$\int_{\Omega_k} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^p dA_\alpha(z)$$
$$= \int_{\Omega_k} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 |\varphi'(z)|^{p-2} (1-|z|^2)^{p+\alpha-1} (1-|z|^2) dA(z)$$

$$\leq \int_{\Omega_k} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^{p+\alpha-1} \log \tfrac{1}{|z|} dA(z)$$

$$\leq \int_{G_k} |f_k'(w)|^p (1-|w|^2)^{p+\alpha-1} N_\varphi(w) dA(w)$$

$$\leq \int_{G_k} |f_k'(w)|^p (1-|w|^2)^{p+\alpha} dA(w)$$

$$= \int_{G_k} |f_k'(w)|^p (1-|w|^2)^p dA_\alpha(w) \to 0$$

as $k \to \infty$. Then, again using the corollary on page 188 of [22], we have

$$\sum_{j=k}^{\infty} \int_{\Omega_{j+1}\setminus\Omega_j} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^p dA_\alpha(z)$$
$$\approx \sum_{j=k}^{\infty} \int_{\Omega_{j+1}\setminus\Omega_j} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 |\varphi'(z)|^{p-2} (\frac{1}{(j+1)^{p+\alpha-1}})(1-|\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z)$$
$$\leq \sum_{j=k}^{\infty} \int_{\Omega_{j+1}\setminus\Omega_j} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 ((j+1)^{p-2})(\frac{1}{(j+1)^{p+\alpha-1}})(1-|\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z)$$
$$\leq \frac{1}{k^{\alpha+1}} \sum_{j=k}^{\infty} \int_{\Omega_{j+1}\setminus\Omega_j} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z)$$
$$= \frac{1}{k^{\alpha+1}} \int_{\mathbb{D}\setminus\Omega_k} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z)$$
$$\leq \frac{1}{k^{\alpha+1}} \int_{\mathbb{D}} |f_k'(\varphi(z))|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^{p+\alpha-1} \log \frac{1}{|z|} dA(z)$$
$$= \frac{1}{k^{\alpha+1}} \int_{\mathbb{D}} |f_k'(w)|^p (1-|w|^2)^{p+\alpha-1} N_\varphi(w) dA(w)$$
$$\leq \frac{c}{k^{\alpha+1}} \int_{\mathbb{D}} |f_k'(w)|^p (1-|w|^2)^{p+\alpha} dA$$

$$= \frac{c}{k^{\alpha+1}} \int_{\mathbb{D}} |f_k'(w)|^p (1-|w|^2)^p dA_\alpha \to 0$$

as $k \to \infty$. Thus, $||f_k \circ \varphi||_{p,\alpha} \to 0$ as $k \to \infty$, even though $||f_k||_{p,\alpha} = 1$ for all k. Hence, it must be that C_φ is not closed-range on $\mathbb{A}^p_{\alpha,0}$ for $p \geq 3$. By the contrapositive of the previous lemma then, it must be that C_φ is not closed-range on $\mathbb{A}^p_{\alpha,0}$ for any $p \geq 1$. Thus, by means of the contrapositive of what we have just shown, our proof is complete. □

Corollary 2.4 *If φ is univalent and C_φ is closed-range on the weighted Bergman space $\mathbb{A}^p_\alpha$, then C_φ is closed-range on the Hardy space H^2.*

Proof. If C_φ is closed-range on any weighted Bergman space $\mathbb{A}^p_\alpha$, then, by Theorem 2.3, C_φ is closed-range on $\mathbb{A}^2$. Then, by Corollary 4.3 in [28], C_φ is closed-range on H^2.

3 Examples

3.1 An Outer Function

We note that an example of the same type as the following was developed concurrently by P. Ghatage. We also note that φ in the following example is a purely outer function. J. Akeroyd and P. Ghatage discuss the case when φ is a singular inner function in [1].

Let ψ be a conformal mapping of the unit disk $\mathbb{D}$ onto the semi-annulus

$$S = \{re^{i\theta} : \frac{1}{2} < r < 1, 0 < \theta < \pi\}.$$

By Theorem 13.2.3 in [12], since $\mathbb{D}$ and S are bounded domains in $\mathbb{C}$, each bounded by a single Jordan curve, then ψ extends to a homeomorphism from $\bar{\mathbb{D}}$ onto $\bar{S}$.

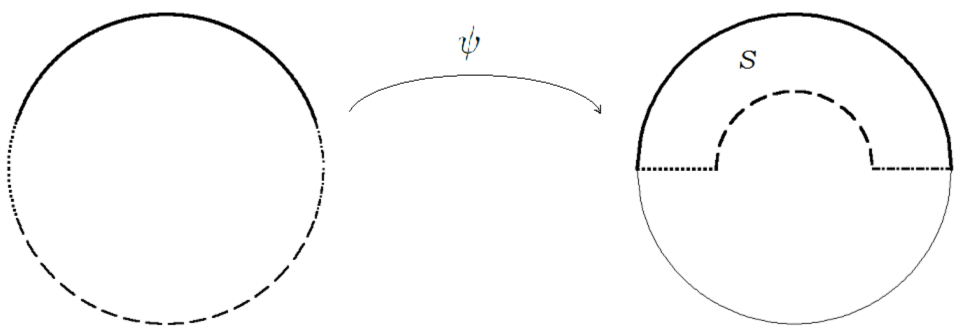

Figure 3: $\psi : \mathbb{D} \to S = \{re^{i\theta} : \frac{1}{2} < r < 1, 0 < \theta < \pi\}$

By the Schwarz-Pick Theorem (see [12]), for any z in $\mathbb{D}$,

$$\frac{1 - |z|^2}{1 - |\psi(z)|^2} \leq \frac{1}{|\psi'(z)|}.$$

We let φ be the analytic self-map of $\mathbb{D}$ given by $\varphi(z) := (\psi(z))^{2+\delta}$, $\delta \geq 0$. First, suppose that $\delta = 0$. Then, φ maps the unit disk $\mathbb{D}$ to the set

$$S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\} \setminus (\frac{1}{4}, 1).$$

As in the case of the square root function on the upper half plane, $|\psi'(z)|$ grows without bound for the points that ψ maps to the corner points of ∂S.

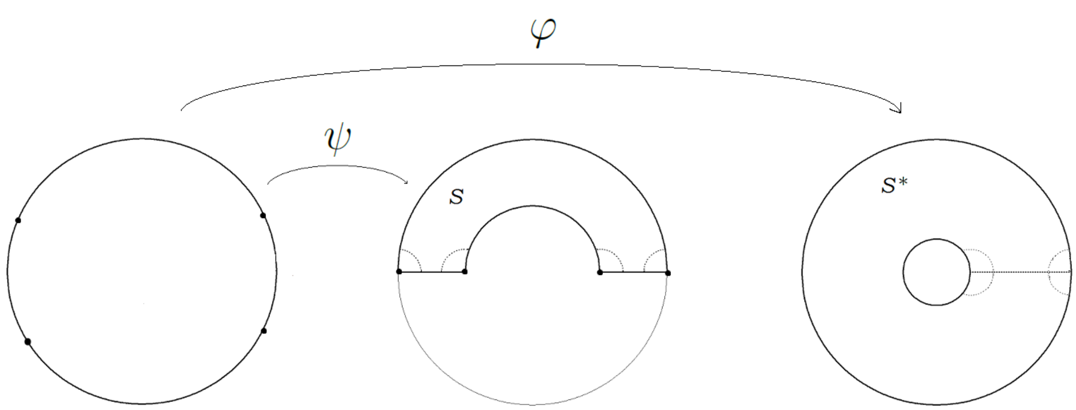

Figure 4: $\varphi : \mathbb{D} \to \mathrm{S}^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\} \setminus (\frac{1}{4}, 1)$

Let ξ be a point in $\mathbb{T}$ such that $\psi(\xi)$ is a corner point of ∂S. Then, for any $\varepsilon > 0$, we can find a region R about ξ, such that, for all z in R,

$$\frac{1 - |z|^2}{1 - |\psi(z)|^2} \leq \frac{1}{|\psi'(z)|} < \varepsilon.$$

Since $|\psi(z)| < 1$, for all z in R we have

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} = \frac{1 - |z|^2}{1 - |\psi(z)|^4} < \frac{1 - |z|^2}{1 - |\psi(z)|^2} \leq \frac{1}{|\psi'(z)|} < \varepsilon.$$

Thus, these points are not contained in

$$\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \geq \varepsilon\}$$

and, hence, the image of this set R under φ is not contained in G_ε. To see that φ does not induce a closed-range composition operator in this case, we need to show that the condition $(*)$ is not satisfied. To this end, let $\{p_n\}$ be a sequence of points in $[0, 1)$ converging to 1 and consider the sequence of pseudohyperbolic disks, $\Delta(p_n, r)$, of radius r with center p_n. Each pseudohyperbolic disk $\Delta(p_n, r)$ is a Euclidean disk with radius

$$q_n = \frac{r(1-p_n^2)}{1-p_n^2 r^2}$$

and center

$$c_n = \frac{p_n(1-r^2)}{1-p_n^2 r^2}.$$

The Euclidean distance from the point 1 to the boundary of $\Delta(p_n, r)$ is given by $\frac{(1-p_n)(1-r)}{1+p_n r}$ for each n. Notice that the ratio of this distance to the Euclidean radius of each k is $\frac{r(1+p_n)}{1-r}$, which approaches the ratio $\frac{2r}{1-r}$ as $p_n \to 1$. Hence the sequence of pseudohyperbolic disks, $\Delta(p_n, r)$ is approaching the unit circle $\mathbb{T}$ nontangentially. In other words, we can find a stolz region which contains each of the disks.

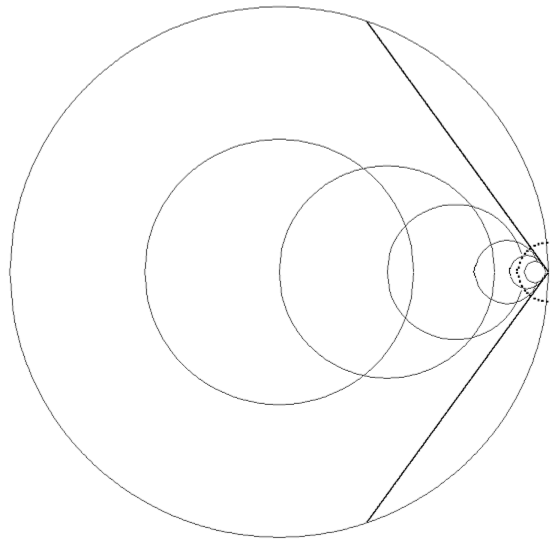

Figure 5: The Sequence $\Delta(p_n, r)$ approaches $\partial\mathbb{D}$ nontangentially.

Then, for any $\varepsilon > 0$ and any r in $(0,1)$, there exists an N in $\mathbb{N}$ with the preimage of $\Delta(p_n, r)$ under φ contained in R whenever $n > N$. Therefore, $\Delta(p_n, r) \bigcap G_\varepsilon = \emptyset$ and condition $(*)$ fails.

Now, suppose that $\delta > 0$. Then, $\varphi(z) := (\psi(z))^{2+\delta}$ maps the unit disk $\mathbb{D}$ to the outer annulus $S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\}$. We will see that φ now maps a region of points contained in Ω_ε to an outer annulus, and hence, condition $(*)$ will be satisfied. In fact, provided that $\varepsilon > 0$ is sufficiently small, we will see that the entire annulus $S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\}$ is contained in G_ε. We define θ_ε to be the smallest angle such that $\{re^{i\theta} : \pi - \theta_\varepsilon \geq \theta \geq \theta_\varepsilon \text{ and } r \geq \frac{1}{2}\}$ is contained in the image of Ω_ε under ψ. For the given δ, we can choose ε small enough such that $\theta_\varepsilon < \frac{\delta\pi}{4}$. We let γ_1 denote the set of points $\{re^{i\theta} : 0 < r < 1\}$ and γ_2 denote the set of points $\{re^{i(\pi-\theta)} : 0 < r < 1\}$.

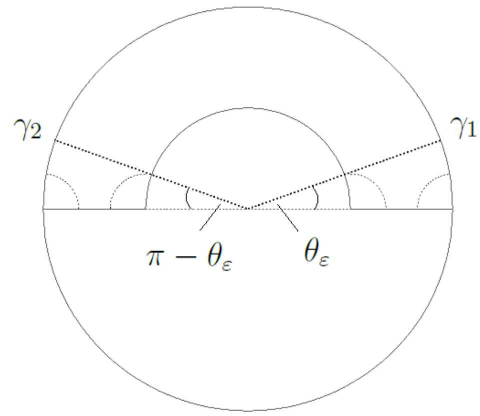

Figure 6: The set of points $\{re^{i\theta} : \pi - \theta_\varepsilon \geq \theta \geq \theta_\varepsilon \text{ and } r \geq \frac{1}{2}\}$ is contained in the image of Ω_ε under ψ.

Now, under φ, points in γ_1 are mapped to points along the radius from 0 to $e^{i(2\theta_\varepsilon + \delta\theta_\varepsilon)}$. Similarly, points in γ_2 are mapped to points along the radius from 0 to $e^{i(2+\delta)(\pi - \theta_\varepsilon)}$. Since $\theta_\varepsilon < \frac{\delta\pi}{4}$, we have that the reference angle $\delta\pi - \theta_\varepsilon(2+\delta)$ is greater than the angle $(2+\delta)\theta_\varepsilon$.

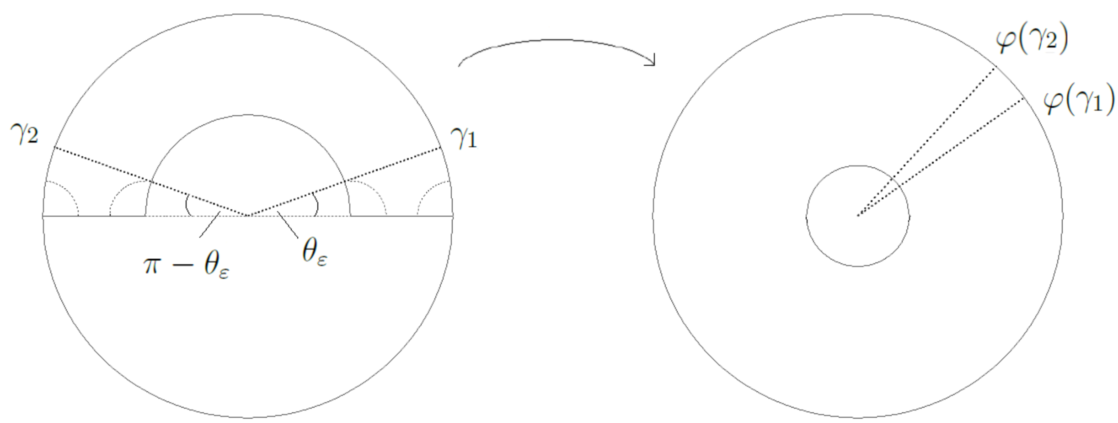

Figure 7: G_ε contains the entire outer annulus S^*.

Hence, the image under φ of the set $\{re^{i\theta} : \pi - \theta_\varepsilon \geq \theta \geq \theta_\varepsilon \text{ and } r \geq \frac{1}{2}\}$ overlaps itself and we have that G_ε contains the entire outer annulus $S^* := \{z \in \mathbb{D} : |z| > \frac{1}{4}\}$. Thus, condition $(*)$ is satisfied and φ induces a closed-range composition operator when $\delta > 0$.

3.2 Frostman Blaschke Products

Note that the following Frostman Blaschke product example appears in [4]. Remember that we define a Blaschke product B to be a function of the form

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a_n}}$$

where $z \in \mathbb{D}$, $\{a_n\}$ is a sequence of points in $\mathbb{D}$ with the property that $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$, and $\frac{|a_n|}{a_n}$ is taken to be 1 if $a_n = 0$. For ζ in $\mathbb{T}$, define $g_B(\zeta)$ by

$$g_B(\zeta) = \sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|}.$$

By a theorem of Frostman, see [7], the Blaschke product B has a unimodular nontangential boundary value at ζ in $\mathbb{T}$ exactly when $g_B(\zeta) = \sum_n \frac{1-|a_n|^2}{|\zeta - a_n|} < \infty$. If $g_B(\zeta)$ converges for every ζ in $\mathbb{T}$, then we call B a *Frostman Blaschke Product*. In other words, B is a Frostman Blaschke product if B has unimodular nontangential boundary values at every point ζ in $\mathbb{T}$. Denote the set of accumulation points of the sequence $\{a_n\}$ in $\mathbb{T}$ by σ_B. By Theorem 1 in [18], if B is a Frostman Blaschke product, then σ_B is nowhere dense in $\mathbb{T}$. For ζ in $\mathbb{T}$, we define $h_B(\zeta)$ by

$$h_B(\zeta) = \sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|^2}.$$

By page 183 in [22], B has an angular derivative at ζ in $\mathbb{T}$ exactly when $h_B(\zeta) < \infty$. Hence, a Frostman Blaschke product B will have an angular derivative at every point ζ in the dense open set $\mathbb{T} \setminus \sigma_B$, but not necessarily at points in σ_B. Let $\{I_\nu\}$ be the collection of subarcs of $\mathbb{T} \setminus \sigma_B$, and for each ν, define ω_ν to be the (possibly infinite) number of radians through which B wraps I_ν. If, for some ν_0, $B(I_{\nu_0}) = \mathbb{T}$, then the proof of Lemma 3.1 in [1] gives us that

$$\omega_{\nu_0} = \int_{I_{\nu_0}} h_B(\zeta)|d\zeta| > 2\pi.$$

Therefore, by Theorem 3.4 in [1] and Theorem 2.3, we have that C_φ is closed-range on every weighted Bergman space $\mathbb{A}^p_\alpha$. Suppose, then, that such a ν_0 does not exist. In that case, for every ν, $B(I_\nu)$ is an open subarc of $\mathbb{T}$. If $\bigcup_\nu B(I_\nu) = \mathbb{T}$, then, since $\mathbb{T}$ is compact, there exists an integer $N > 0$ so that $\bigcup_{\nu=1}^N B(I_\nu) = \mathbb{T}$. Then, there is a compact subset K of $\bigcup_{\nu=1}^N I_\nu$ such that $B(K) = \mathbb{T}$. If $\varepsilon > 0$ is small enough, K will be contained in the closure of $\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|B(z)|^2} > \varepsilon\}$. Hence, by Theorem 2.3, the composition operator C_B will be closed-range on every weighted Bergman space $\mathbb{A}^p_\alpha$. Thus, a sufficient condition for C_B to be closed-range on each of the weighted Bergman spaces is that $\bigcup_\nu B(I_\nu) = \mathbb{T}$.

Lemma 3.1 *(Lemma 2.5 in [4]) Let B be a Frostman Blaschke product with infinitely many zeros $\{a_n\}_{n=1}^\infty$, listed according to multiplicity. Then, for any point ζ^* in σ_B and for any $\delta > 0$, there exists a subsequence $\{a_{nk}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ such that $|\zeta^* - a_{nk}| < \delta$ for all k and*

$$\sup_{\zeta \in \sigma_B} \frac{1 - |a_{nk}|^2}{|\zeta - a_{nk}|} \to 0$$

as $k \to \infty$.

Proof. Suppose there exists $\delta, c > 0$ and $\zeta^* \in \sigma_B$ such that

$$\sup_{\zeta \in \sigma_B} \frac{1 - |a_n|^2}{|\zeta - a_n|} \geq c$$

whenever $|\zeta^* - a_n| < \delta$. Since ζ^* is in σ_B, we can find ζ_1 in σ_B and $n_1 > 0$ such that $|\zeta^* - a_{n1}| < \delta$ and

$$\frac{1 - |a_{n1}|^2}{|\zeta_1 - a_{n1}|} > \frac{c}{2}.$$

So, by choosing a_{n1} close enough to ζ^*, we can make ζ_1 as close to ζ^* as we want. Since ζ_1 is in σ_B, we can find $n_2 > n_1$ so that a_{n2} is close enough to ζ_1 so that $|\zeta^* - a_{n2}| < \delta$. Then, there exists ζ_2 in σ_B so that

$$\frac{1 - |a_{n2}|^2}{|\zeta_2 - a_{n2}|} > \frac{c}{2}.$$

Thus, by choosing a_{n2} close enough to ζ_1, we can make ζ_2 to be as close to ζ_1 as we would like. Hence, for $j = 1, 2$, we can force $|\zeta^* - \zeta_2| < \delta$ and

$$\frac{1 - |a_{nj}|^2}{|\zeta_2 - a_{nj}|} > \frac{c}{2}.$$

In a similar manner, we can choose $n_3 > n_2$ so that a_{n_3} is close enough to ζ_2 to ensure that $|\zeta^* - a_{n_3}| < \delta$ and we can find ζ_3 in σ_B so that a_{n_3} is close enough to ζ^2 to ensure that $|\zeta^* - a_{n_3}| < \delta$, and we can choose ζ_3 in σ_B so that $|\zeta^* - \zeta_3| < \delta$ and

$$\frac{1 - |a_{nj}|^2}{|\zeta_3 - a_{nj}|} > \frac{c}{2}$$

for $j = 1, 2, 3$. We may then continue in this manner to find a subsequence $\{a_{n_j}\}_{j=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ and a sequence $\{\zeta_j\}_{j=1}^{\infty}$ in σ_B with the property that

$$\frac{1 - |a_{n_j}|^2}{|\zeta_J - a_{n_j}|} > \frac{c}{2},$$

where $J \in \mathbb{N}$ and $1 \leq j \leq J$. By the compactness of σ_B, $\{\zeta_j\}_{j=1}^{\infty}$ has an accumulation point, ζ_0, in σ_B, which also fulfills the condition

$$\frac{1 - |a_{n_j}|^2}{|\zeta_0 - a_{n_j}|} \geq \frac{c}{2}.$$

But this means that $g_B(\zeta_0) = \sum_n \frac{1-|a_n|^2}{|\zeta_0 - a_n|}$ diverges, and hence, B cannot be a Frostman Blaschke product. Thus, the result is proved. $\square$

Proposition 3.2 *(Proposition 2.6 in [4].) Let B be a Frostman Blaschke product with infinitely many zeros $\{a_n\}_{n=1}^{\infty}$, listed according to multiplicity, and let $\{I_\nu\}_\nu$ be an enumeration of the components of $\mathbb{T} \setminus \sigma_B$. Let ω_ν denote the number of radians through which B wraps I_ν, which may be infinite. Then, for any point ζ^* in σ_B and any $\delta > 0$, at least one of the following hold.*

1. *There is a component I_{ν_0} of $\mathbb{T} \setminus \sigma_B$ such that $dist(\zeta^*, I_{\nu_0}) < \delta$ and $\omega_{\nu_0} = \infty$.*

2. *There are infinitely many components $\{I_{\nu_k}\}_{k=1}^{\infty}$ of $\mathbb{T}\setminus\sigma_B$ contained in $\{\zeta \in \mathbb{T} : |\zeta - \zeta^*| < \delta\}$ such that $\liminf_{k\to\infty} \omega_{\nu_k} \geq 2\pi$.*

Proof. By Lemma 3.1, we can find a sequence $\{a_{n_k}\}_{k=1}^{\infty} \subset \{a_n\}_{n=1}^{\infty}$ such that each a_{n_k} is as close to ζ^* as we wish. We can also find a corresponding sequence $\{I_{\nu_k}\}_{k=1}^{\infty}$ of not necessarily distinct components of $\mathbb{T} \setminus \sigma_B$ such that

$$\sup_{\zeta \in \mathbb{T}\setminus I_{\nu_k}} \frac{1 - |a_{n_k}|^2}{|\zeta - a_{n_k}|} \to 0,$$

as $k \to \infty$. Since, for $0 < a < b$, $\int_a^b \frac{1}{t^2} = \frac{b-a}{ab} < \frac{1}{a}$, we have

$$\int_{\mathbb{T}\setminus I_{\nu_k}} \frac{1 - |a_{n_k}^2|}{|\zeta - a_{n_k}|^2} |d\zeta| \to 0$$

and so

$$\int_{I_{\nu_k}} \frac{1 - |a_{n_k}^2|}{|\zeta - a_{n_k}|^2} |d\zeta| \to 2\pi$$

as $k \to \infty$. Since, by Lemma 3.1 in [1],

$$\omega_\nu \geq \sum_{\{k:\nu(k)=\nu} \int_{I_{\nu_k}} \frac{1 - |a_{n_k}^2|}{|\zeta - a_{n_k}|^2} |d\zeta|,$$

the proposition is proved. □

If there is a ζ^* in σ_B and a $\delta > 0$ so that condition (1) in proposition 3.2 holds, then the composition operator C_B will be closed-range on $\mathbb{A}^p_\alpha$ for every p, $1 \leq p < \infty$. But then, if C_B is not closed-range on $\mathbb{A}^p_\alpha$ for every p, then it must be that condition (2) in proposition 3.2 holds. It must also be the case that $\bigcup_\nu B(I_\nu) \neq \mathbb{T}$. If condition (2) and the previous statement are both true, then, there exists ζ_0 in $\mathbb{T}$ so that for and open arc γ in $\mathbb{T}$ having nonempty intersection with φ_B, $B\left(\gamma \setminus \varphi_B\right) = \mathbb{T} \setminus \{\zeta_0\}$. This does not seem very probably and one may suspect that every Frostman Blaschke product will give rise to a closed range composition operator C_B on every $\mathbb{A}^p_\alpha$ space. Indeed, J. Akeroyd and P. Ghatage have constructed an example of a Frostman Blaschke product that does not do so. Suppose B is a Frostman Blaschke product such that C_B is not closed-range on $\mathbb{A}^p_\alpha$ for any p. Since any pertubation of a zero of B affects the image of each component under B of $\mathbb{T} \setminus \varphi_B$ unequally, then for a Blaschke product B^* obtained by shifting the location of only one of the zeros of B, C_{B^*} will be closed-range on $\mathbb{A}^p_\alpha$ for any p. Thus, if the composition operator C_B is not closed-range on $\mathbb{A}^p_\alpha$ for every p then there is a sequence of Frostman Blaschke products $\{B^*_k\}_{=1}^\infty$ so that C_{B^*} is closed-range on $\mathbb{A}^p_\alpha$ for every p.

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