8-2017

Π-Operators in Clifford Analysis and its Applications

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Mathematics

by

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Wuhan University
Bachelor of Science in Mathematics, 2009

August 2017
University of Arkansas

This dissertation is approved for recommendation to the Graduate Council.

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Abstract

In this dissertation, we studies Π-operators in different spaces using Clifford algebras. This approach generalizes the Π-operator theory on the complex plane to higher dimensional spaces. It also allows us to investigate the existence of the solutions to Beltrami equations in different spaces.

Motivated by the form of the Π-operator on the complex plane, we first construct a Π-operator on a general Clifford-Hilbert module. It is shown that this operator is an L^2 isometry. Further, this can also be used for solving certain Beltrami equations when the Hilbert space is the L^2 space of a measure space. This idea is applied to examples of some conformally flat manifolds, the real projective space, cylinders, Hopf manifolds and n-dimensional hyperbolic upper half space.

It is worth pointing out that the proof for the L^2 isometry of Π-operator on the unit sphere is different from the idea mentioned above. In that idea, it requires the Dirac operator and its dual operator commute to prove the L^2 isometry of the Π-operator. However, this is no longer true for the spherical Dirac operator. Hence, we use the spectrum of spherical Dirac operator to overcome this problem. Since the real projective space can be defined as a projection from the unit sphere, Π-operator theory in the real projective space can be induced from the one on the unit sphere. Similarly, Π-operator theory on cylinders (Hopf manifolds) is derived from the one on n-dimensional Euclidean space via a projection map.

Classical Clifford analysis is centered at the study of functions on n-dimensional Euclidean space taking values in Clifford numbers. In contrast, Clifford analysis in higher spin spaces is the study of functions on n-dimensional Euclidean space taking values in arbitrary irreducible representations of the Spin group. At the end of this thesis, we construct an L^2 isometric Π-operator in higher spin spaces. Further, we provide an Ahlfors-Beurling type inequality in higher spin spaces to conclude the thesis.

Acknowledgements

I would like to express my deep appreciation and gratitude to my advisor, Dr. John Ryan, for the patient guidance he provided me, all the way from when I was first considering applying to the PhD program in the Mathematics Department, through to completion of this degree. I am truly fortunate to have had the opportunity to work with him. I also want to thank Dr. Uwe Kähler, who gave me precious advice and guidance on my research.

I would also like to express my gratitude my committee members, Dr. Maria Tjani and Dr. Daniel Luecking for the friendly guidance, thought provoking suggestions, and the general collegiality offered to me over the years. I also wish to thank the staff in the department of Mathematical Science at University of Arkansas for the contributions that each of them made to my intellectual growth during my years of study at the University of Arkansas.

I would like to thank to all my research group members: Chao Ding, Junxia Li and Jing Li. I would also like to thank my student colleagues for providing a stimulating and fun environment in which to learn and grow.

Finally, words can not express how grateful I am to my mother, my father and my grandparents, for all of the love, education and sacrifice from my birth. In particularly, I wish to express my deepest gratitude to my caring, loving, and supportive husband, Chao Ding. You are the most reliable partner in my life and your support and encouragement is the greatest contribution to my research.

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1 Introduction

With the help of functional analytic methods, complex analysis has been used as a powerful tool to study linear and nonlinear first order partial differential equations in the complex plane. Some of the most important of these partial differential equations are called the Beltrami equations. This is because of the fact that the theory of Beltrami equations is connected with many problems in geometry and analysis, for instance,

- The general theory of linear and quasilinear elliptic system,
- Problems of conformal mappings of Riemannian manifolds,
- Related problems of conformal and almost complex structure on general Riemannian manifolds,
- The classical theory of uniformization and the theory of Teichmüller spaces,
- Problems in the conformally invariant string theories in theoretical physics.

More details about the applications of Beltrami equations can be found in [11].

In one dimensional complex analysis, the Beltrami equation is given by

$$\mu \frac{\partial \omega}{\partial \overline{z}} = \frac{\partial \omega}{\partial z},$$

where $\mu = \mu(z)$ is a given complex function, and $z \in \mathbb{C}$. The solutions to this Beltrami equation are also called *quasiconformal mappings.* If we let

$$\omega(z) = \overline{z} + T_\Omega h,$$

then we have

$$\mu\frac{\partial\omega}{\partial\overline{z}} = \mu(z)\frac{\partial(\overline{z}+T_\Omega h)}{\partial\overline{z}} = \mu(z)(I+\Pi_\Omega h),$$
$$\frac{\partial\omega}{\partial z} = \frac{\partial(\overline{z}+T_\Omega h)}{\partial z} = \frac{\partial T_\Omega h}{\partial z} = h,$$

where $T_\Omega h(z) = -\frac{1}{\pi}\int_\Omega \frac{h(\zeta)}{\zeta - z}d\zeta_1 d\zeta_2$, $\zeta = \zeta_1 + i\zeta_2$, with$\frac{\partial(T_\Omega h)}{\partial z} = h$ and

$$\Pi_\Omega h(z) = \frac{\partial T_\Omega h}{\partial\overline{z}} = -\frac{1}{\pi i}\int_\Omega \frac{h(\xi)}{(\xi - z)^2}d\xi_1 d\xi_2.$$

Hence, the Beltrami equation is transformed to the fixed-point equation

$$\mu(z)(I+\Pi_\Omega h) = h.$$

Here Π_Ω is called the *complex* Π-*operator* (also known as the Beurling-Ahlfors transform), defined as a complex partial derivative of T_Ω. Recall that

Theorem 1.1. ***(Banach Fixed Point Theorem)****[24]*

Let (X, d) be a non-empty complete metric space. A mapping $T : X \longrightarrow X$ is called a contraction mapping on X if there exists $q \in [0, 1)$, such that $d(T(x), T(y)) \leq qd(x, y)$. Such T admits a unique fixed-point x^* in X, which means $T(x^*) = x^*$.

Hence, by the Banach fixed-point theorem, the unique solution of the Beltrami equation exists if $\|\mu\| \leq \mu_0 < \frac{1}{\|\Pi_\Omega\|}$, where μ_0 is a constant. Indeed, this can be observed easily from the inequalities below.

$$||\mu(z)(I+\Pi_\Omega h_1) - \mu(z)(I+\Pi_\Omega h_2)|| = ||\mu(z)\Pi_\Omega(h_1 - h_2)|| \leq ||\mu\Pi_\Omega|| \cdot ||h_1 - h_2||.$$

Therefore, the existence of the solution of the Beltrami equation turns to the estimate of the complex Π-operator, and many results have already been established, for instance, [9, 42].

From the description above, we notice that quasiconformal mapping is closely related to Beltrami equations. Quasiconformal maps are special kind of complex homeomorphisms, which are generalizations of the well known conformal maps. Conformal maps impose a very strong condition on the differential (approximating linear map), whereas quasiconformal maps relax this condition considerably. Indeed, every conformal map is also a quasiconformal map. Quasiconformal maps retain many aspects of conformal maps. They have many applications in areas of heat conduction, electrostatic potential and fluid flow, and are also a valuable tool in the field of complex dynamics. However, the study of quasiconformal maps can be extended to higher dimensions.

With the help of Clifford algebras, the classical Beltrami equation and Π-operator with some well known results can be generalized to higher dimensions. Abundant results in Euclidean space have been found. For instance, in [26], Gürlebeck, Kähler and Shapiro considered a class of generalizations of the complex one-dimensional Π-operator in spaces of quaternion-valued functions depending on four real variables. In [25], Gürlebeck and Kähler provide a hypercomplex generalization of the complex Π-operator which turns out to have most of the properties of its original in one dimensional complex analysis. Kähler studied Beltrami equations in the case of quaternions in [28]. This gave an overview of possible generalizations of the complex Beltrami equation in the quaternionic case and their properties. In [10], the authors studied the Π-operator in Clifford analysis by the use of two orthogonal bases of a Euclidean space. This allowed one to find expressions of the jump of the generalized Π-operator across the boundary of a domain. The case of the Π-operator and the Beltrami equation on the n-sphere has also been discussed in [15] with most useful properties inherited from the complex Π-operator.

From the work of Π-operator theory mentioned above, we notice that Π-operator is usually defined as D^*T, where D^* is the dual of a Dirac type operator and T is an integral operator constructed via the fundamental solution of the Dirac type operator D. More importantly, if D and D^* commute, then this gives a L^2 isometry property to our Π-operator D^*T. This idea brings us to the definition of a Π-operator on a general Clifford-Hilbert module. Hence, we actually complete the work of constructing an isometric Π-operator on a general Clifford-Hilbert module. Further, a Beltrami equation can also be constructed here when the Hilbert space is $L^2(X)$ for some measure space X, and the norm estimate of our isometric Π-operator can solve this Beltrami equation. See more details in Section 4.5. We apply this general setting to consider several practical examples on conformally flat manifolds and hyperbolic upper half space in higher dimensions.

To generalize results to the unit sphere, we define two Π-operators related to the conformally invariant spherical Dirac operator. The idea to consider the n-sphere is not only motivated by being the classic example of a manifold and being invariant under the conformal group, but also by the fact that in the case of $n = 3$ due to the recently proved Poincaré conjecture there is a wide class of manifolds which are homeomorphic to the 3-sphere. This makes our results much more general and valid for any simply connected closed 3-manifold. In particular, results on local and global homeomorphic solutions of the spherical Beltrami equation carry over to such manifolds. The Π-operator theory in Euclidean space, many authors use the commutativity of Dirac operator and its dual operator to prove the L^2 isometry property for the Π-operator. However, the spherical Dirac operator D_s and its dual operator no longer commute. Hence, we introduce the spectrum technique for the spherical Dirac operator and Cauchy transform to overcome this problem. More details can be found in our paper [15].

Conformally flat manifolds are manifolds with atlases whose transition maps are Möbius transformations. Some can be parametrized by U/Γ where U is a simply connected subdomain of either $\mathbb{S}^n$ or $\mathbb{R}^n$ and Γ is a Kleinian group acting discontinuously on U. Examples of such manifolds treated here include the real projective space $\mathbb{R}P^n$, cylinders and Hopf manifolds $\mathbb{S}^1 \times \mathbb{S}^n$. More details for these conformally flat manifolds can be found in [29, 30]. In our recent paper [16], we give the property of L^2 isometry to the Π-operator and generalize the results in the the complex plane, Euclidean space ([25]) and on the unit sphere ([15]) to the previous conformally flat manifolds through some projection maps.

Hyperbolic function theory in the upper half space is a modification of standard Clifford analysis. It is based on the hyperbolic metric rather than the Euclidean one. Abundant results related to the hyperbolic Dirac operator in the upper half space have already been found. For instance, the expression of the hyperbolic Dirac operator, Cauchy integral formula, Borel-Pompeiu formula, etc. We refer the reader to [19, 20, 31, 38] for more details. In [16] , we introduce a Beltrami equation on the upper half space and a hyperbolic Π-operator that keeps the property of L^2 isometry. This hyperbolic Π-operator inherits many properties from the one dimensional complex analysis case.

Clifford analysis in higher spin spaces is the study of functions on n-dimensional Euclidean space taking values in arbitrary irreducible representations of the Spin group. It is first studied by Bures et al. in 2002, where they construct Rarita-Schwinger operator as the generalization of the Dirac operator in higher spin spaces. With the idea we provide for the Π-operator on a general Clifford-Hilbert module, we can construct an L^2 isometric Π-operator in higher spin spaces as well. As a result found during this work, an Ahlfors-Beurling type inequality is also established.

1.1 Dissertation Outline

This dissertation is organized as follows:

In Section 2, we introduce Clifford algebras with some well known properties; we then introduce some real subgroups in real Clifford algebras, in particular, the special orthogonal group and Spin group which is the double covering group of the special orthogonal group.

In Section 3, we introduce the Dirac operator on the Euclidean space and some classical results, such as fundamental solutions, Cauchy's integral formula, Cauchy's theorem and Borel-Pompeiu theorem. In order to construct the conformally invariant spherical Dirac operator, Möbius transformations and Ahlfors-Vahlen matrices are introduced. Then we use a Cayley transformation to induce the Dirac operator and those integral formulas to the unit sphere.

Since the main topic of this thesis is constructing Π-operators in different spaces, some well known results of Π-operator theory are introduced in section 4. We first give the Π-operator on the 1-dimensional complex plane and its application to solve the complex Beltrami equation. We also point out that the unique solution is a quasiconformal mapping and an introduction of quasiconformal mappings is also provided. Then we extend the results from the 1-dimensional case to higher dimensional spaces. In other words, the Π-operator and the Beltrami equation on $n+1$-dimensional space are introduced. We also give the explanation for quasiconformal mappings related to the solutions of the Beltrami equation. At the end of this section, we demonstrate the construction of a Π-operator on general Clifford-Hilbert modules.

Through sections 5-9, we investigate specific examples of constructing Π-operators on different spaces following the idea given at the end of previous section.

In section 5, we create two spherical Π-operators. The first one is an L^2 isometry up to isomorphism. We give the application to solve the first type of spherical Beltrami equation. Then we create the second type of spherical Π-operator with the idea given in Section 4.5. This Π-operator is an L^2 isometric operator. Since $D_s^*D_s \neq D_sD_s^*$, we use a spectrum technique instead. At the end, we give the Beltrami equation and the condition to the existence of the unique solution.

In section 6, we induce the Π-operator from the unit sphere to the real projective space. In [29], the real projective space is a conformally flat manifold defined by $\mathbb{R}P^n = \mathbb{S}^n/\{\pm 1\}$. Hence we induce the generalized spherical Dirac operator to the real projective space and create the Π-operator on $\mathbb{R}P^n$. When we prove the property of L^2 isometry, we use a projection technique by deriving the spectrum of Dirac operator over the real projective space via the spectrum of the Dirac operator over the sphere. Finally, we also show how to use $\Pi_{\mathbb{R}P^n}$ to solve the Beltrami equation on $\mathbb{R}P^n$.

In [29], the Dirac operators with their fundamental solutions are induced to cylinders and Hopf manifolds. In section 7, we induce the Π-operators to cylinders and Hopf manifolds as well with similar arguments as in the previous section. The conditions to the existence of the unique solutions of the Beltrami equations are also given.

The Dirac operator in the hyperbolic upper half space is defined with respect to the hyperbolic metric given in [19, 20, 38]. Then we give the generalized hyperbolic Dirac operator, some integral formulas and the construction of Π-operator in upper half space in Section 8. The proof of L^2 isometry of our Π-operator is also provided.

In the last section, we generalized the Π-operator to higher spin spaces with respect to the Rarita-Schwinger operator. This is the generalization of the Dirac operator on the higher spin spaces. We also give a uniform estimation of the generalized Cauchy transformation on the higher spin spaces, which is called an Ahlfors-Beurling type Inequality.

2 Clifford Algebras

2.1 Definitions and Properties

Clifford algebras are the algebras that form the basis of this thesis. They are naturally associated with bilinear forms on vector spaces. A bilinear form can be considered as a generalization of an inner product and is defined as follows:

Definition 2.1. Suppose V is a vector space over $\mathbb{R}$. A bilinear form B is a map

$$\mathcal{B}:\ V\times V\longrightarrow\mathbb{R};\ (u,v)\mapsto\mathcal{B}(u,v),$$

which is linear in both arguments:

$$\mathcal{B}(au_1 + bu_2, v) = a\mathcal{B}(u_1, v) + b\mathcal{B}(u_2, v);$$
$$\mathcal{B}(u, av_1 + bv_2) = a\mathcal{B}(u, v_1) + b\mathcal{B}(u, v_2).$$

One can associate a matrix $B = (a_{ij})_{ij} \in \mathbb{R}^{n\times n}$ to every bilinear form on an n-dimensional vector space:

$$\mathcal{B}(u, v) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_i a_{ij} v_j = u^T B v,$$

where $u, v \in V$. If the matrix B is symmetric, the associated bilinear form is called symmetric and if $det(B) \neq 0$, the associated form is called non-degenerate, i.e. for each non-zero vector $u \in V$ there exists a non-zero vector $v \in V$ such that $\mathcal{B}(u, v) \neq 0$.

Definition 2.2. If V is a real vector space equipped with a symmetric, non-degenerate bilinear form $\mathcal{B}$, then $(V, \mathcal{B})$ is called a non-degenerate orthogonal space.

Note that with a proper choice of a basis for V, every non-degenerate orthogonal space can be reduced to a space $\mathbb{R}^{p,q}$ with $p + q = n = dim(V)$. (p, q) are called the signature of the orthogonal space $(V, \mathcal{B})$. The physical interpretation of the numbers p and q are the number of time-like and space-like dimensions respectively. This means that there exist a basis $\{e_1, \cdots, e_p, e_{p+1}, \cdots, e_{p+q}\}$ such that:

$$\mathcal{B}(e_i, e_j) = 0,\ if\ i \neq j;$$
$$\mathcal{B}(e_i, e_i) = 1,\ if\ 1 \leq i \leq p;$$
$$\mathcal{B}(e_i, e_i) = -1,\ if\ p + 1 \leq i \leq p + q.$$

For instance, the Minkowski space has signature $(1, 3)$ or $(3, 1)$, depending on the convention, while the classical Euclidean space has signature $(3, 0)$. We always assume that the basis is orthonormal. In other words, the associated matrix B is diagonal and of the type $B = diag(1, \cdots, 1, -1, \cdots, -1)$. We are now in a position to give a definition for a Clifford algebra. First, the most general definition is given. Afterwards, a more useful definition that we will continue using throughout this thesis will be given.

Definition 2.3. Suppose that $\mathcal{B}$ is a non-degenerate bilinear form on a real vector space V. The Clifford algebra $\mathcal{C}l(V, \mathcal{B})$ associated to the bilinear form $\mathcal{B}$ is a associative algebra with unit $1 \in \mathbb{R}$ defined as

$$\mathcal{C}l(V, \mathcal{B}) := T(V)/I(V, \mathcal{B}).$$

Here, $T(V)$ is the universal tensor-algebra

$$T(V) := \bigoplus_{k\in\mathbb{N}} (\bigotimes^{k} V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots,$$

and $I(V, \mathcal{B})$ is the two-sided ideal generated by all elements of the form $u \otimes u - \mathcal{B}(u, u)1$, with $u \in V$.

From now on, we will drop the tensor symbol $u \otimes v$, i.e. we will simply write uv instead. Moreover, we will also drop the unit because we only work with fields $\mathbb{R}$ and $\mathbb{C}$. Earlier, we showed that real non-degenerate orthogonal spaces can be classified according to their signature and that generates a universal real Clifford algebra. After a proper choice for a basis for V, $\mathcal{C}l(V, \mathcal{B})$ can be reduced to the Clifford algebra $\mathcal{C}l(\mathbb{R}^{p,q}, \mathcal{B}_{p,q})$.

Lemma 2.1. *For every (p, q) with $p + q = n$, a basis for $\mathcal{C}l(\mathbb{R}^{p,q}, \mathcal{B}_{p,q})$ is given by the set*

$$\{1, e_1, \cdots, e_n, e_{1,2}, \cdots, e_{n-1,n}, \cdots, e_{12\cdots n}\},$$

where $e_{i_1\cdots e_k}$ is a shorthand for $e_{i_1} \cdots e_{i_k}$.

If $(V, \mathcal{B}) = \mathbb{R}^{p,q}$, the associated Clifford algebra $\mathcal{C}l(\mathbb{R}^{p,q}, \mathcal{B}_{p,q})$ will be denoted by $\mathcal{C}l_{p,q}$. An alternative and much more useful definition for this Clifford algebra is the following:

Definition 2.4. For all $(p, q) \in \mathbb{N} \times \mathbb{N}$ with $p + q = n$, the algebra $\mathbb{R}_{p,q}$ is an associative algebra (with unit) that is multiplicatively generated by the basis $\{e_1, \cdots, e_n\}$ satisfying the following multiplication rules:

$$e_i^2 = 1, \ if \ 1 \leq i \leq p;$$

$$e_i^2 = -1, \ if \ p+1 \leq i \leq p+q;$$

$$e_i e_j + e_j e_i = 0, \ if \ i \neq j.$$

These are called the universal Clifford algebra for the space $\mathbb{R}^{p,q}$ with $dim_{\mathbb{R}}(\mathcal{C}l_{p,q}) = 2^n$.

It is clear that a basis for the algebra is given by

$$\mathcal{C}l_{p,q} = Span\{e_{i_1 \cdots i_k} : 1 \leq i_1 < \cdots < e_k \leq n\}.$$

Let $k \in \mathbb{N}$ and $A = \{i_1, \cdots, i_k\} \subset \{1, \cdots, n\}$, then every element of $\mathcal{C}l_{p,q}$ is of the form $\sum_A a_A e_A$ with $a_A \in \mathbb{R}$. If $A = \emptyset$, we let $e_\emptyset = 1$. Elements of a Clifford algebra are called Clifford numbers. We usually use $\mathcal{C}l_n$ as a shorthand notation for $\mathcal{C}l_{0,n}$. We also define the following spaces:

Definition 2.5. For all $0 \leq k \leq n$, we define the space $\mathcal{C}l_{p,q}^{(k)}$ of k-vectors as:

$$\mathcal{C}l_{p,q}^{(k)} := Span_{\mathbb{R}}\{e_A : |A| = k\},$$

with $\mathcal{C}l_{p,q}^{(0)} = \mathbb{R}$. In particular, the space of $\mathcal{C}l_{p,q}^{(1)}$ is called the space of vectors and the space of $\mathcal{C}l_{p,q}^{(2)}$ is called the space of bivectors. Hence, we have

$$\mathcal{C}l_{p,q} = \oplus \mathcal{C}l_{p,q}^{(k)}.$$

The above decomposition can also be rewritten as

$$\mathcal{C}l_{p,q} = \mathcal{C}l^e_{p,q} \oplus \mathcal{C}l^o_{p,q}$$

where $\mathcal{C}l^e_{p,q} = \oplus\mathcal{C}l^{(2n)}_{p,q}$, and $\mathcal{C}l^o_{p,q} = \oplus\mathcal{C}l^{(2n-1)}_{p,q}$. This tells us $\mathcal{C}l_{p,q}$ is a $\mathbb{Z}_2$-graded algebra.

To conclude this section, we introduce some (anti-)involutions on $\mathcal{C}l_{p,q}$. We first define them on the basis elements, the action on arbitrary Clifford numbers follows by linear extension.

1. The *inversion* on $\mathcal{C}l_{p,q}$ is defined as $\hat{e}_{i_1\cdots i_k} := (-1)^k e_{i_1\cdots i_k}$.
2. The *reversion* on $\mathcal{C}l_{p,q}$ is defined as $\tilde{e}_{i_1\cdots i_k} := e_{i_k\cdots i_1}$.
3. The *conjugation* on $\mathcal{C}l_{p,q}$ is defined as $\bar{e}_{i_1\cdots i_k} := \tilde{\hat{e}}_{i_1\cdots i_k} = (-1)^{\frac{k(k+1)}{2}} e_{i_1\cdots i_k}$.

In the rest of this thesis, we only deal with $\mathcal{C}l_n$ over $\mathbb{R}$ unless otherwise specified.

2.2 Real Subgroups of Real Clifford Algebras

One of many applications of Clifford algebras $\mathcal{C}l_n$ is the following: they can be used to introduce some important groups which define double coverings of orthogonal group $O(n)$ and special orthogonal group $SO(n)$. These groups are crucial in the study of *the spinor representations.*

Definition 2.6. The orthogonal group $O(n)$ is the group of linear transformations on $\mathbb{R}^n$ which leave the bilinear form invariant, i.e.,

$$\{\varphi \in End(\mathbb{R}^n) : \mathcal{B}(u,v) = \mathcal{B}(\varphi(u), \varphi(v)), \forall u, v \in V\} = \{A \in \mathbb{R}^{n\times n} : A^T A = Id\}.$$

An important subgroup of $O(n)$ is the special orthogonal group

$$SO(n) = \{A \in O(n) : detA = 1\}.$$

Suppose a is a unit vector on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, if we consider axa, we may decompose

$$ax\tilde{a} = ax_{a_\parallel}\tilde{a} + ax_{a_\perp}\tilde{a} = -x_{a_\parallel} + x_{a_\perp}.$$

So, the action $ax\tilde{a}$ describes a reflection R_a of x in the direction of a. These reflections are the building blocks for the entire group $O(n)$:

Theorem 2.2. ***(Cartan-Dieudonné)****[23] Every element of $O(n)$ is a composition of at most n reflections with respect to hyperplanes in $\mathcal{C}l_n$, i.e. For any $\varphi \in O(n)$, there exist $k \leq n$ and $a_1, \cdots, a_k \in \mathbb{S}^{n-1}$, such that*

$$\varphi = R_{a_1} \circ R_{a_2} \circ \cdots \circ R_{a_k}.$$

If k is even, then φ is a rotation and if k is odd then φ is an anti-rotation.

Hence, we are motivated to define

$$Pin(n) := \{a = y_1 \cdots y_p : \; p \in \mathbb{N} \text{ and } y_1, \cdots, y_p \in \mathbb{S}^{n-1}\}$$

where for $a \in Pin(n)$ we have $ax\tilde{a} = O_a x$ for appropriate $O_a \in O(n)$. Under Clifford multiplication $Pin(n)$ is a group. Further, we have a group homomorphism as follows.

$$\theta : \; Pin(n) \longrightarrow O(n); \; a \mapsto O_a.$$

We also define

$$Spin(n) := \{a \in Pin(n) : \; for \; some \; q \in \mathbb{N}, \; a = y_1 \cdots y_{2q}\}.$$

The $Spin(n)$ group is a subgroup of $Pin(n)$ and θ is also a group homomorphism from $Spin(n)$ to $SO(n)$. Indeed, it can be shown that θ is surjective with $Ker\theta = \{-1, 1\}$.

Thus, $Pin(n)$ and $Spin(n)$ are double covers of $O(n)$ and $SO(n)$ respectively. See more details in [36].

3 Clifford Analysis

Now we have established Clifford algebras and some of their properties, we are concerned with defining a differential operator and performing analysis with Clifford algebras.

3.1 Dirac Operators and Clifford Analyticity in $\mathcal{C}l_n$

We identify the Euclidean space $\mathbb{R}^{n+1}$ with the direct sum $\Lambda^0\mathbb{R}^n \oplus \Lambda^1\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^{n+1}$ is a domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$. Then functions f defined in Ω with values in $\mathcal{C}l_n$ are considered. These functions can be written as

$$f(x) = \sum_{A\subseteq\{e_1,e_2,...e_n\}} e_A f_A(x), \quad x \in \Omega,\ f_A(x) \in \mathbb{R}.$$

Properties such as continuity, differentiability, integrability, and so on, which are ascribed to f have to be possessed by all components $f_A(x)$, $(A \subseteq \{e_1, e_2, ...e_n\})$. The spaces $C^k(\Omega, \mathcal{C}l_n), L_p(\Omega, \mathcal{C}l_n)$ are defined as right Banach modules with the corresponding traditional norms. In particular, the space $L_2(\Omega, \mathcal{C}l_n)$ is a right Hilbert module equipped with a $\mathcal{C}l_n$-valued sesquilinear form

$$\langle u, v\rangle = \int_\Omega \overline{u(\eta)}v(\eta)\, d\Omega_\eta.$$

Furthermore, $W_p^k(\Omega, \mathcal{C}l_n), k \in \mathbb{N}, 1 \leq p < \infty$ denotes the Sobolev spaces as the right module of all functionals whose derivatives belong to $L_p(\Omega, \mathcal{C}l_n)$, with norm

$$\|f\|_{W_p^k(\Omega,\mathcal{C}l_n)} := \Big(\sum_A \sum_{\|\alpha\|\leq k} \|D^\alpha f_A\|^p_{L_p(\Omega,\mathcal{C}l_n)}\Big)^{1/p}.$$

The closure of the space of test functions $C_0^\infty(\Omega, \mathcal{C}l_n)$ in the W_p^k-norm will be denoted by $\overset{\circ}{W}{}_p^k(\Omega, \mathcal{C}l_n)$.

Definition 3.1. Consider $\mathbb{R}^n$ as a subset of $\mathcal{C}l_n$ and write $x \in \mathbb{R}^n$ as $x = x_1e_1 + \cdots + x_ne_n$. Then we define

$$D_x := \sum_{j=1}^{n} e_j \partial_{x_j}$$

to be the *Dirac operator* for $\mathbb{R}^n$, where ∂_{x_j} is the partial derivative with respect to x_j.

Notice that $D_x^2 = -\Delta = -\sum_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$, where Δ is the Laplacian in $\mathbb{R}^n$. This definition suggests we should also consider the following two differential operators in $\mathcal{C}l_n$

$$D_0 := \partial_{x_0} + \sum_{j=1}^{n} e_j \partial_{x_j} = e_0 \partial_{x_0} + D_x;$$
$$\overline{D_0} := \partial_{x_0} - \sum_{j=1}^{n} e_j \partial_{x_j} = e_0 \partial_{x_0} - D_x.$$

which have the property $D_0\overline{D_0} = \overline{D_0}D_0 = \Delta_{n+1}$. D_0 and $\overline{D_0}$ are also called *the generalized Dirac operator* and *the conjugate generalized Dirac operator*, respectively. In particular, when $n = 1$, this is the one complex variable case. This tells us that the Dirac operator is the generalization of the Cauchy-Riemann operator in analysis of one complex variable to higher dimensions.

Definition 3.2. A $\mathcal{C}l_n$-valued function $f(x)$ defined on a domain Ω in $\mathbb{R}^{n+1}$ is called *left monogenic* if

$$D_x f(x) = \sum_{i=1}^{n} e_i \partial_{x_i} f(x) = 0.$$

Similarly, f is called a *right monogenic* function if it satisfies

$$f(x) D_x = \sum_{i=1}^{n} \partial_{x_i} f(x) e_i = 0.$$

3.2 Integral Formulas and Fundamental Solutions for The Euclidean Dirac Operator

In complex analysis, some of the most important properties of analytic functions are Cauchy's integral formula, Cauchy's theorem and Borel-Pompeiu formula. Since analytic functions can also be considered as solutions for the Cauchy-Riemann operator, as the generalization of Cauchy-Riemann operator to higher dimensions, the Euclidean Dirac operator also has such integral formulas.

Theorem 3.1. ***(Cauchy's theorem)*** *[18]* Fix a domain $\Omega \subset \mathbb{R}^n$ and $\overline{\Omega} \subseteq \mathbb{R}^n$ with its boundary $\partial\Omega$ a C^1 hypersurface. Suppose $f, g: \ \Omega \longrightarrow \mathcal{C}l_n$ are C^1 and $gD_x = 0 = D_x f$ on all of Ω. Then

$$\int_{\partial\Omega} g(x)n(x)f(x)d\sigma(x) = 0,$$

where $n(x)$ is the outer normal vector and $d\sigma(x)$ is the surface measure on ∂V.

Define $G: \ \mathbb{R}^n\backslash\{0\} \longrightarrow \mathbb{R}^n\backslash\{0\}$ by $G(x) := \frac{x}{||x||^n}$. Note that this function, considered as a function to $\mathcal{C}l_n$, is left and right monogenic. Indeed, $G(x-y) = \frac{x-y}{||x-y||^n}$ is the fundamental solution of D_x.

Theorem 3.2. ***(Cauchy's Integral Formula)*** *[18]* Fix a domain $\Omega \subset \mathbb{R}^n$ and $\overline{\Omega} \subseteq \mathbb{R}^n$ with its boundary $\partial\Omega$ a C^1 hypersurface. Suppose $f: \ \Omega \longrightarrow \mathcal{C}l_n$ is C^1 and $D_x f = 0$ on all of Ω. Then for $y \in \Omega$, we have

$$f(y) = \frac{1}{\omega_{n-1}} \int_{\partial\Omega} G(x-y)n(x)f(x)d\sigma(x),$$

where ω_{n-1} is the area of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$.

In analogy to complex analysis, the Clifford analysis version of Cauchy's integral formula immediately gives one a great deal of results, such as the analyticity (interpreted in the appropriate sense) of monogenic functions.

Now given $f(x)$, a C^1 function defined in a neighborhood of a bounded domain Ω, we define its *Cauchy transform* by the convolution integral

$$T_\Omega f(x) = \frac{1}{\omega_n}\int_\Omega G(x-y)f(y)dy.$$

Theorem 3.3. *[27] Suppose f and Ω are as above. Then for each $x \in \Omega$, it holds that*

$$f(x) = \frac{1}{\omega_{n-1}} D_x \int_\Omega G(x-y)f(y)dy^n.$$

Here T is the generalization to Euclidean space of the Cauchy transform in the complex plane, and it is the right inverse of D_x, that is $D_x T = I$. Also, the non-singular boundary integral operator is given by

$$F_{\partial\Omega} f(x) = \frac{1}{\omega_{n-1}}\int_{\partial U} G(x-y)n(y)f(y)d\sigma(y).$$

We have the *Borel-Pompeiu formula* as follows.

Theorem 3.4. ***(Borel-Pompeiu formula)****[27] For $f \in C^1(\Omega, \mathcal{C}l_n)$, we have*

$$f(x) = \frac{1}{\omega_{n-1}}\int_{\partial\Omega} G(x-y)n(y)f(y)d\sigma(y) + \frac{1}{\omega_{n-1}}\int_\Omega G(x-y)D_y f(y)dy.$$

In particular, if $f \in W_0^k(\Omega, \mathcal{C}l_n)$, then

$$f(x) = \frac{1}{\omega_{n-1}}\int_\Omega G(x-y)D_y f(y)dy.$$

Corollary 3.5. *[27] Suppose $f \in C_0^\infty(\mathbb{R}^n, \mathcal{C}l_n)$. Then the previous theorem says that*

$$D_x T f = f, \ T D_x f = f.$$

In this sense, D and T are inverses of each other over $C_0^\infty(\mathbb{R}^n, \mathcal{C}l_n)$.

3.3 Generalized Dirac Operators and Integral Formulas

If we identify the Euclidean space $\mathbb{R}^{n+1}$ with the direct sum $\mathbb{R} \oplus \mathbb{R}^n$, we can derive basic results for the generalized Dirac operator, which are similar to the previous section. Let Ω be a bounded smooth domain in $\mathbb{R}^{n+1}$ and $f \in C^1(\Omega, \mathcal{C}l_n)$, $G(x-y) = \dfrac{\overline{x-y}}{\|x-y\|^{n+1}}$ is the fundamental solution of D_0. Hence the Cauchy transform is defined as

$$T_\Omega f(x) = \frac{1}{\omega_n} \int_\Omega G(x-y) f(y) dy,$$

where T is the generalization to Euclidean space of the Cauchy transform in the complex plane, and it is the right inverse of D_0, that is $D_0 T = I$. Also, the non-singular boundary integral operator is given by

$$F_{\partial\Omega} f(x) = \frac{1}{\omega_n} \int_{\partial\Omega} G(x-y) n(y) f(y) d\sigma(y).$$

We have the *Borel-Pompeiu formula* as follows.

Theorem 3.6. *For $f \in C^1(\Omega, \mathcal{C}l_n)$, we have*

$$f(x) = \frac{1}{\omega_n} \int_{\partial\Omega} G(x-y) n(y) f(y) d\sigma(y) + \frac{1}{\omega_n} \int_\Omega G(x-y) D_0 f(y) dy,$$

In particular, if $f \in W_0^k(\Omega, \mathcal{C}l_n)$, then

$$f(x) = \frac{1}{\omega_n} \int_\Omega G(x-y) D_0 f(y) dy.$$

Hence, D_0 and T are inverse operators for function $f \in W_0^k(\Omega, \mathcal{C}l_n)$, which says $D_0 T = T D_0 = I$.

Similarly the fundamental solution of $\overline{D_0}$ is $\overline{G(x-y)} = \dfrac{x-y}{\|x-y\|^{n+1}}$. Using $\overline{G(x-y)}$ we could define the conjugate of the Cauchy transform as follows:

$$\overline{T}_\Omega f(x) = \frac{1}{\omega_n} \int_\Omega \overline{G(x-y)} f(y) dy,$$

which is the right inverse of $\overline{D_0}$. Also, the non-singular boundary integral operator is given by

$$\overline{F}_{\partial\Omega} f(x) = \frac{1}{\omega_n} \int_{\partial\Omega} \overline{G(x-y)} n(y) f(y) d\sigma(y).$$

For $f \in W_0^k(\Omega, \mathcal{C}l_n)$, $\overline{D_0 T} = \overline{T D_0} = I$.

3.4 Möbius Transformations and Ahlfors-Vahlen Matrices

In analysis of one complex variable, a function f sending a region in $\mathbb{R}^2 = \mathbb{C}$ into $\mathbb{C}$ is conformal at z if it is complex analytic and has a non-zero derivative, $f'(z) \neq 0$ (we only consider sense-preserving conformal mappings). The only conformal transformations of the whole plane $\mathbb{C}$ are affine linear transformations: compositions of rotations, dilations and translations. The *Möbius mapping*

$$f(z) = \frac{az+b}{cz+d}, \; a, b, c, d \in \mathbb{C}, \; ad - bc \neq 0,$$

is affine linear when $c = 0$; otherwise, it is conformal at each $z \in \mathbb{C}$ except $z = -\dfrac{d}{c}$. The Möbius mapping f sends $\mathbb{C}\backslash\{-\dfrac{d}{c}\}$ onto $\mathbb{C}\backslash\{\dfrac{a}{c}\}$. If we agree that $f(-\dfrac{d}{c}) = \infty$ and $f(\infty) = \dfrac{a}{c}$, then f becomes a (one-to-one) transformation of $\mathbb{C} \cup \infty$, the complex plane compacting by the point at infinity. These transformations are called *Möbius transformations* of $\mathbb{C} \cup \{\infty\}$. Möbius transformations are compositions of rotations, translations, dilations and inversions. Möbius transformations send circles (and affine lines) to circles (or affine lines). The derivative of a Möbius transformation is a composition of a rotation and a dilation.

In the higher dimensional case, a conformal mapping preserves angles between intersecting curves. Formally, a differomorphism $\phi : U \longrightarrow \mathbb{R}^n$ is said to be *conformal* if for each $x \in U \subset \mathbb{R}^n$ and each $\mathbf{u}, \mathbf{v} \in TU_x$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is preserved under $D\phi_x$. When the dimension $n > 2$, Liouville's Theorem states that any smooth conformal mapping on a domain of $\mathbb{R}^n$ can be expressed as compositions of translations, dilations, orthogonal transformations and inversions: they are Möbius transformations. Ahlfors and Vahlen find a connection between Möbius transformations and a particular matrix group, when the dimension $m > 2$. They show that given a Möbius transformation on $\mathbb{R}^n \cup \{\infty\}$ it can be expressed as $y(x) = (ax + b)(cx + d)^{-1}$ where $a,\ b,\ c,\ d \in \mathcal{C}l_n$ and satisfy the following conditions:

1. $a,\ b,\ c,\ d$ are all products of vectors in $\mathbb{R}^n$;
2. $a\tilde{b},\ c\tilde{d},\ \tilde{b}c,\ \tilde{d}a$ in $\mathbb{R}^n$;
3. $a\tilde{d} - b\tilde{c} = \pm 1$.

The associated matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called a *Vahlen matrix* of the Möbius transformation $y(x)$ of $\mathbb{R}^n$, see more details in [36]. All Vahlen matrices form a group under matrix multiplication, the Vahlen group. Notice that

$y(x) = (ax+b)(cx+d)^{-1} = ac^{-1} + (b - ac^{-1}d)(cx+d)^{-1}$, this suggests that a conformal transformation can be decomposed as compositions of translation, dilation, reflection and inversion. This is called the *Iwasawa decomposition* for the Möbius transformation $y(x)$.

3.5 Conformal Invariance of Dirac Operators

One important fact about the conformal mapping is that it preserves monogenic functions, which also means the conformal invariance of the Dirac equation. This has been established for many years, see [35, 39, 40].

Theorem 3.7. ***(Conformal invariance of Dirac equation)****[27]*

Assume $f \in C^1(\mathbb{R}^n, \mathcal{C}l_n)$ and $D_y f(y) = 0$. If $y = M(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation, then

$$D_x \frac{\widetilde{cx+d}}{||cx+d||^m} f(M(x)) = 0.$$

In other words, the kernel of the Dirac operator is invariant under Möbius transformations.

Further, we have intertwining operators for the Euclidean Dirac operator, i.e., it is conformally invariant.

Proposition 3.8. *[35] If $y = (ax+b)(cx+d)^{-1}$, then we have*

$$\frac{\widetilde{cx+d}}{||cx+d||^{m+2}} D_y f(y) = D_x \frac{\widetilde{cx+d}}{||cx+d||^m} f((ax+b)(cx+d)^{-1}).$$

We just reviewed the first order conformally invariant differential operator in classical Clifford analysis with some properties. Recall that, in harmonic analysis, as a second order differential operator, the Laplacian Δ is also conformally invariant, and we already knew that $-\Delta = D_x^2$. Hence, we expect that D_x^j is conformally invariant as well for $j > 2$. This has been confirmed and similar results on fundamental solutions and intertwining operators have also been established. First, let $y = M(x) = (ax+b)(cx+d)^{-1}$ be a Möbius transformation, we denote

$$G_k(x) = \frac{x}{||x||^{n-2l}},\ if\ k = 2l+1;\ G_k(x) = ||x||^{n-2l},\ if\ k = 2l;$$
$$J_k(M,x) = \frac{\widetilde{cx+d}}{||cx+d||^{n-2l}},\ if\ k = 2l+1,\ J_k(M,x) = ||cx+d||^{n-2l},\ if\ k = 2l;$$
$$J_{-k}(M,x) = \frac{\widetilde{cx+d}}{||cx+d||^{n+2l}},\ if\ k = 2l+1,\ J_k(M,x) = ||cx+d||^{n+2l},\ if\ k = 2l.$$

Then we have

Proposition 3.9. *[35]**(Intertwining operators for j-Dirac operator)***

If $y = M(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation, then

$$J_{-k}(M,x)D_y^j f(y) = D_x^j J_k(M,x) f((ax+b)(cx+d)^{-1}).$$

Notice that conformal invariance of the j-Dirac equation $D_x^j f(x) = 0$ can be deduced from this easily.

Proposition 3.10. *[35]**(Fundamental solutions for D_x^j)***

The fundamental solution of D_x^j is $G_j(x)$ (up to a multiplicative constant), where $G_j(x)$ is defined as above. However, when the dimension m is even, we require that $j < m$.

Notice that, for instance, when the dimension m is even and $m = j$, then the candidate for the fundamental solution $G_j(x)$ is a constant, which can not be a fundamental solution.

3.6 Clifford Analysis on the Unit Sphere

On the unit n-sphere [32], the spherical Dirac operator D_s is defined as follows:

$$D_s = x(\Gamma - \frac{n-1}{2}),$$

where $x \in \mathbb{S}^{n-1}$, $\Gamma = -\sum_{i=1,j>i}^{n-1} e_i e_j L_{i,j}$, and here the operators $L_{i,j} = x_i \partial_{x_j} - x_j \partial_{x_i}$ are called the *angular momentum operators.* This D_s is the conformally invariant differential operator corresponding to D_x. It can be derived from the Cayley transformation $C : \mathbb{R}^n \longrightarrow \mathcal{C}l_n$, where

$$C(x) = (e_{n+1}x + 1)(x + e_{n+1})^{-1} = x_s.$$

Here $x \in \mathbb{R}^n$ and e_{n+1} is a unit vector in $\mathbb{R}^{n+1}$ orthogonal to $\mathbb{R}^n$, and $x_s \in \mathbb{S}^n$. The Cayley transformation is a Möbius transformation, then intertwining operators of D_s are

$$J(C, x) = \frac{x + e_{n+1}}{\|x + e_{n+1}\|^n},$$

and

$$J_{-1}(C, x) = \frac{x + e_{n+1}}{\|x + e_{n+1}\|^{n+2}}.$$

We have the following intertwining relations of D_x and D_s as

$$J_{-1}(C, x)D_s = D_x J(C, x).$$

It is well known that the fundamental solution of D_s is $G_s(x, y) = \frac{-1}{\omega_{n-1}} \frac{x - y}{\|x - y\|^{n-1}}$, $x, y \in \mathbb{S}^{n-1}$.

Assume Ω is a bounded smooth domain in $\mathbb{S}^{n-1}$ and $f \in C^1(\Omega, \mathcal{C}l_n)$. Similar as in the Euclidean space, we can define a Cauchy transform with respect to D_s as follows ([32]).

$$T_\Omega f(x) = \int_\Omega G_s(x, y)f(y)dy = -\frac{1}{\omega_{n-1}} \int_\Omega \frac{x - y}{\|x - y\|^{n-1}} f(y)dy.$$

Here, T_Ω is also a right inverse for D_s, which is $D_s T = I$. Also, we have two non-singular boundary integral operators

$$F_{\partial\Omega}f(x)=\int_{\partial\Omega}G_s(x,y)n(y)f(y)d\sigma(y).$$

Then the Borel-Pompeiu formula for D_s is stated as follows.

Theorem 3.11. *([32])**(Borel-Pompeiu formula)***

For $f\in C^1(\Omega)\cap C(\overline{\Omega})$, we have

$$f(x)=\int_{\partial\Omega}G_s(x,y)n(v)f(v)d\sigma(y)+\int_{\Omega}G_s(x,y)D_sf(y)dy,$$

in other words, $f=F_{\partial\Omega}f+T_{\Omega}D_sf$. In particular, if f has compact support, then $TD_s=I$.

4 Π-Operator in Euclidean Space and on General Clifford-Hilbert Modules

In this section, we will first recall some basic results for Beltrami equation and Π-operator on the complex plane. A geometric explanation for quasiconformal mappings is also provided here. Then, we investigate these in higher dimensional Euclidean space. At the end, we introduce our L^2 isometric Π-operator on a general Clifford-Hilbert module with its application to a Beltrami equation defined over a Clifford-valued Hilbert space $L^2(X,\mathcal{C}l_n)$, where X is some measure space. This motivates the constructions of Π-operator in the next several sections.

4.1 Beltrami equation and the Π-Operator on the Complex Plane

The Π-operator is one of the tools used to study smoothness of functions over Sobolev spaces and to solve the Beltrami equation. In one dimensional complex analysis, the Beltrami equation, is the partial differential equation given by

$$\frac{\partial w}{\partial\overline{z}}=\mu\frac{\partial w}{\partial z},$$

where $z \in \mathbb{C}$, $w = w(z)$ is a complex function in some open set $U \subseteq \mathbb{C}$ with derivatives that are locally L^2, and $\mu = \mu(z)$ is a given complex function in $L^\infty(U)$, $\|\mu\| < 1$, called the Beltrami coefficient. The solution of the Beltrami equation relies on a singular integral operator defined on $L^p(\mathbb{C})$ for all $1 < p < \infty$, which is called the Beurling transform–the complex Π-operator,

$$\Pi_\Omega h(z) = \frac{\partial}{\partial \overline{z}} \frac{1}{\pi} \int_\Omega \frac{h(\xi)}{\|\xi - z\|} d\xi = -\frac{1}{\pi i} \int_\Omega \frac{h(\xi)}{(\xi - z)^2} d\xi.$$

This singular integral operator is a conformally invariant operator and acts as an isometry from $L^2(\mathbb{C})$ to $L^2(\mathbb{C})$. The Beltrami equation shares the same solution with the singular integral equation $h = q(z)(I + \Pi_\Omega)h$. By the Banach fixed point theory, $h = \mu(z)(I + \Pi_\Omega)h$ has a unique solution when $\|\mu\| < \mu_0 < \dfrac{1}{\|\Pi\|}$, where μ_0 is a constant. More details could be found in the Introduction.

4.2 Quasiconformal Maps on the Complex Plane

In one dimensional complex analysis, a quasiconformal mapping is a generalized conformal mapping and named by Ahlfors ([4]). It is a homeomorphism between plane domains, which to first order maps small circles to small ellipses of bounded eccentricity. More specifically, if we let $f : \Omega_1 \longrightarrow \Omega_2$ be a function between bounded plane domains in complex space, $f = u(x, y) + iv(x, y)$ is differentiable in the real sense, which means all the partial derivatives u_x, u_y, v_x, v_y are continuous in an open subdomain $U \subseteq \Omega_1$. Let $z_0 = x_0 + iy_0, z_0 \in \Omega_1$, and $z \in \Omega_1$, $z \to z_0$, we have the formula

$$f(z) = f(z_0) + \partial f(z_0)(z - z_0) + \overline{\partial} f(z_0)(\overline{z} - \overline{z_0}) + o(z - z_0).$$

Consider the function $\widetilde{L}(z - z_0) = \partial f(z_0)(z - z_0) + \overline{\partial} f(z_0)(\overline{z} - \overline{z_0})$. The norm of $\widetilde{L}(z - z_0)$ represents the distance between $f(z)$ and $f(z_0)$ when z is sufficiently close to z_0. Hence,

$\widetilde{L}(z) = \partial f(z_0)z + \overline{\partial} f(z_0)\overline{z}$ maps the distance between z and z_0 to the distance between $f(z)$ and $f(z_0)$. Further, when the Jacobian $J(z_0) = |\partial f(z_0)|^2 - |\overline{\partial} f(z_0)|^2 \geq 0$, $\tilde{L}$ is sense-preserving, which means $J(z_0) > 0$ and $|\partial f(z_0)| \geq |\overline{\partial} f(z_0)|$.

Let $\partial f(z_0) = |\partial f(z_0)|e^{i\alpha} = Ae^{i\alpha}$, $\overline{\partial} f(z_0) = |\overline{\partial} f(z_0)|e^{i\beta} = Be^{i\beta}$, and $z = re^{i\theta}$, then

$$\widetilde{L}(z) = A(z + \frac{B}{A}\overline{z}) = Ae^{i(\alpha+\frac{\beta}{2})}r\big(e^{i(\theta-\frac{\beta}{2})} + \frac{|B|}{|A|}e^{-i(\theta-\frac{\beta}{2})}\big).$$

Thus if we let $g(z) = Ae^{i(\theta-\frac{\beta}{2})} \cdot z$, $h(z) = Be^{i\beta} \cdot z$. Both maps g and h are simiply rotations. Let $L_\mu(z) = z + \mu\overline{z} = (1+\mu)x + i(1-\mu)y$. It maps x to $(1+\mu)x$ and y to $(1-\mu)y$, which is an expansion in the x-direction and a compression in the y-direction. Therefore, we have that

$$\widetilde{L}(z) = g \circ L_{\|\mu\|} \circ h.$$

If f satisfies the Beltrami equation $\dfrac{\partial f}{\partial \overline{z}} = \mu \dfrac{\partial f}{\partial z}$, it is easy to see, f maps an infinitesimal circle centered as z_0 to an infinitesimal ellipse centered at $f(z_0)$, and the ratio of the major axis to the minor axis of the infinitesimal ellipse is

$$\widetilde{K}_f(z_0) = \frac{1 + |\mu(z_0)|}{1 - |\mu(z_0)|} = \frac{|\partial f(z_0)| + |\overline{\partial} f(z_0)|}{|\partial f(z_0)| - |\overline{\partial} f(z_0)|}.$$

$\widetilde{K}_f(z_0)$ is the ratio of the maximum stretch to the minimum stretch of an infinitesimal circle around z_0 under f. Let $f : \Omega_1 \longrightarrow \Omega_2$, if there exists a constant $K \geq 1$, $\widetilde{K}_f = \sup_{z\in\Omega_1} \widetilde{K}_f(z) \leq K$, we call f $K-$*quasiconformal.*

4.3 Π-Operator in Euclidean Space

It is well known that in complex analysis, the Π-operator can be realized as the composition of $\partial_{\bar{z}}$ and the Cauchy transform. Hence, the generalization of Π-operator in higher dimensional Euclidean space via Clifford algebra can be defined as follows.

Definition 4.1. The Π-operator in Euclidean space $\mathbb{R}^{n+1}$ is defined as

$$\Pi = \overline{D_0}T.$$

In [25, 26], we have an integral expression of Π as follows.

Theorem 4.1. *Assume $f \in W_0^p(\Omega)(1 < p < \infty)$, then we have*

$$\Pi f(z) = -\frac{1}{\omega_{n+1}} \int_{\Omega} \frac{(n-1) + (n+1)\frac{\overline{\zeta - z}^2}{\|\zeta - z\|^2}}{\|\zeta - z\|^{n+1}} f(\zeta) d\zeta + \frac{1-n}{1+n} f(z).$$

The following are some well known properties for the Π-operator.

Theorem 4.2. *([25]) Suppose $f \in W_0^p(\Omega)(1 < p < \infty)$, then*

1. $D_0 \Pi f = \overline{D_0} f$,
2. $\Pi D_0 f = \overline{D_0} f - \overline{D_0} F_{\partial\Omega} f$,
3. $F_{\partial\Omega} \Pi f = (\Pi - T\overline{D_0}) f$,
4. $D_0 \Pi f - \Pi D_0 f = \overline{D_0} F_{\partial\Omega} f$.

The following is the decomposition of $L^2(\Omega, \mathcal{C}l_n)$, see more details in [25].

Theorem 4.3. ***($L^2(\Omega, \mathcal{C}l_n)$ Decomposition)***

$$L^2(\Omega, \mathcal{C}l_n) = L^2(\Omega, \mathcal{C}l_n) \bigcap Ker\overline{D_0} \bigoplus D_0(W_0^{1,2}(\Omega, \mathcal{C}l_n)),$$

and

$$L^2(\Omega, \mathcal{C}l_n) = L^2(\Omega, \mathcal{C}l_n) \bigcap Ker D_0 \bigoplus \overline{D_0}(W_0^{1,2}(\Omega, \mathcal{C}l_n)).$$

Notice that

$$\Pi(L^2(\Omega,\mathcal{C}l_n)\bigcap Ker\overline{D_0}) = L^2(\Omega,\mathcal{C}l_n)\bigcap Ker D_0,$$
$$\Pi(D_0(W_0^{1,2}(\Omega,\mathcal{C}l_n))) = \overline{D_0}(W_0^{1,2}(\Omega,\mathcal{C}l_n)).$$

Hence, Π-operator maps from $L^2(\Omega,\mathcal{C}l_n)$ to $L^2(\Omega,\mathcal{C}l_n)$.

One key property of the Π-operator is that it is an L^2 isometry. In other words,

Theorem 4.4. *([25]) For functions in $L^2(\Omega,\mathcal{C}l_n)$, we have*

$$\Pi^*\Pi = I.$$

To complete this section, we introduce the application of the Π-operator to solve the Beltrami equation. Let $\Omega \subseteq \mathbb{R}^{n+1}$, $q:\Omega\to\mathcal{C}l_n$ and $\omega:\Omega\to\mathcal{C}l_n$ be a sufficiently smooth function. The generalized Beltrami equation

$$D_0\omega = q\overline{D_0}\omega$$

has a solution $\omega = Th+\phi$, where ϕ is an arbitrary monogenic function. Substitute such w into the Beltrami equation, we have the following

$$\begin{aligned} D_0w &= D_0(Th+\phi) = h \\ &= q\overline{D_0}(\phi+Th) = q(\overline{D_0}\phi+\Pi h). \end{aligned}$$

Therefore, we transform the Beltrami equation into an integral equation $h = q(\overline{D_0}\phi+\Pi h)$. By the Banach fixed point theorem, this equation has a unique solution if $\|q\| \le \dfrac{1}{\|\Pi\|}$, see [25] and the Introduction. This tells us that the existence of a unique solution to the Beltrami equation depends on the norm estimate for Π-operator. By [42], we have

$\|\Pi\|_{L^p(\mathbb{R}^{n+1})} \leq (n+1)(p^*-1)$, where $p^* = max(p, p/(p-1))$.

4.4 Quasiconformal Mappings and Beltrami Equations in Higher Dimensional Spaces

Similarly as the case in $\mathbb{C}$, we extend the case from the complex plane to the $(n+1)-$dimensional space. Let $f : \Omega_1 \longrightarrow \Omega_2$ be a paravector-valued function, that is $f = \sum_{i=0}^{n} f_i e_i$, where $\Omega_1, \Omega_2 \subseteq \mathbb{R}^{n+1}$. We shall discuss the quasiconformal structure induced by $f(x)$ in the neighborhood of $x = 0$. By Taylor's series expansion, for $x \to 0$,

$$f(x) = f(0) + \sum_{i=0}^{n} x_i \partial_{x_i} f(0) + o(\|x\|^2).$$

Let $\widetilde{L}(x) = \sum_{i=0}^{n} x_i \partial_{x_i} f(0)$, then $||\widetilde{L(x)}||$ represents the distance between $f(x)$ and $f(0)$ when x is sufficiently close to the origin. We have

$$\begin{aligned} \widetilde{L}(x) = \sum_{i=0}^{n} x_i \partial_{x_i} f(0) &= \frac{(x+\overline{x})}{2} \cdot \frac{(D+\overline{D})}{2} f(0) + \frac{(x-\overline{x})}{2} \cdot \frac{(D-\overline{D})}{2} f(0) \\ &= \frac{1}{2}(x \cdot Df(0) + \overline{x} \cdot \overline{D} f(0)). \end{aligned}$$

If f is a solution of the Beltrami equation $Df = q\overline{D}f$, where q is a measurable function, we defined

$$\widetilde{K}_f(0) = \frac{||\widetilde{L(x)}||_{max}}{||\widetilde{L(x)}||_{min}} = \frac{\|Df(0)\| + \|\overline{D}f(0)\|}{\|Df(0)\| - \|\overline{D}f(0)\|} = \frac{1 + \|q(0)\|}{1 - \|q(0)\|}.$$

Let x be on a sphere centered at 0 with radius $\|x\|$. When $x \to 0$, f maps the infinitesimal sphere to an infinitesimal elliptical sphere centered at $f(0)$ with radius $\|\widetilde{L}(x)\|$. The ratio of the maximum stretch to the minimum stretch of the ellipse around $f(0)$ is $\widetilde{K}_f(0)$. Similarly we could define the $K-$quasiconformal of the paravector-valued function f, if there exists a $K \geq 1$, such that $\widetilde{K}_f = \sup_{z \in \Omega_1} \widetilde{K}_f(z) \leq K$.

From the argument above, we notice that every solution to the Beltrami equation is a quasiconformal mapping, the existence of solutions to the Beltrami equations becomes an important topic for studying quasiconformal mapping.

4.5 Π-operator on Clifford-Hilbert Modules

In this section, we will provide a Π-operator defined on a general Clifford-Hilbert module. This Π-operator also has an isometry property. It motivates the definitions of Π-operators in different conformally flat manifolds in the following several sections.

Let H be a real Hilbert space with inner product $\langle\ ,\ \rangle$, then $H\otimes\mathcal{C}l_n$ is a Clifford-Hilbert module. More details for Clifford-Hilbert module can be found in [10, 34]. Let E be a dense subspace of H, and $f,g\in E\otimes\mathcal{C}l_n$. Suppose an operator D acts on $E\otimes\mathcal{C}l_n$, which also satisfies $D^*D=DD^*$ where D^* is the dual operator of D in the sense of

$$\langle Df,g\rangle=\langle f,D^*g\rangle.$$

Let T bs an operator acting on $E\otimes\mathcal{C}l_n$. It is called the inverse of D if it satisfies $DT=TD=I$.

Definition 4.2. The Π-operator on $E\otimes\mathcal{C}l_n$ is defined as

$$\Pi=D^*T.$$

On the unit sphere $\mathbb{S}^n$, the spherical Π operator is defined as $\Pi_s=\overline{D_s}T$, [15]. By Theorem 9 in [15], $D_s^*=-\overline{D_s}$.

Recall that E is dense in H, so we can induce a Π-operator on $H\otimes\mathcal{C}l_n$ immediately. The Π-operator defined above also possesses an important property that it has in one dimensional complex analysis. That is

Theorem 4.5. *The operator $\Pi=D^*T$ is an isometric operator on $H\otimes\mathcal{C}l_n$.*

Proof.

$$\begin{aligned}\langle \Pi f, \Pi g\rangle &= \langle D^*Tf, D^*Tg\rangle = \langle Tf, DD^*Tg\rangle \\ &= \langle Tf, D^*DTg\rangle = \langle DTf, DTg\rangle = \langle f, g\rangle.\end{aligned}$$

□

We next show that our generalized Π-operator can be used to solve certain Beltrami equations. More specifically, if we let H be $L^2(X)$, where X is a measure space with a measure η. Hence, we can define a Beltrami equation over $H \otimes \mathcal{C}l_n$ i.e., $L^2(X, \mathcal{C}l_n)$ as follows.

$$Df = qD^*f,$$

where $q \in L^\infty(X, \mathcal{C}l_n)$, which is defined similarly as in Euclidean space with the essential supremum norm with respect to η. By the substitution $f = \phi + Th$ where ϕ is a solution for $D\phi = 0$, we transform the Beltrami equation in the following way.

$$D(\phi + Th) = h = qD^*(\phi + Th) = q(D^*\phi + \Pi h).$$

Hence, if h is the unique solution of the equation

$$h = q(D^*\phi + \Pi h),$$

$f = \phi + Th$ is the unique solution of the Beltrami equation. The Banach fixed point theory tells us this equation has a unique solution if $\|q\| \leq q_0 < \dfrac{1}{\|\Pi\|}$, with q_0 being a constant. Hence, as in the classical case, the problems of the existence of the solution to the Beltrami equation becomes the norm estimate of our Π-operator.

As special cases of this general Hilbert space approach, one has the L^2 isometry of the usual Π-operator in one complex variable and the Π-operator in $\mathbb{R}^n$ described in [10, 25, 26] and elsewhere. The next sections describe the Π-operator acting over L^2 spaces over other manifolds.

5 Spherical Π-Type Operators

5.1 Spherical Π-Type Operator with Generalized Spherical Dirac Operator

Let $\{e_0, e_1, \cdots, e_n\}$ be the standard orthogonal basis of $\mathbb{R}^{n+1}$ with $e_0^2 = 1$ and $e_i^2 = -1,\ i = 1, \cdots, n$. We should use the generalized Dirac operator

$$D_0 = e_0 \frac{\partial}{\partial_{x_0}} + \sum_{j=1}^{n} e_j \frac{\partial}{\partial_{x_j}} = e_0 \frac{\partial}{\partial_{x_0}} + D_x.$$

The spherical Dirac operator D_s on $\mathbb{S}^n$ is defined as follows.

$$\overline{x} D_0 = \sum_{j=1}^{n} e_0 e_j (x_0 \partial_{x_j} - x_j \partial_{x_0}) - \sum_{i=1, j>i}^{n} e_i e_j (x_i \partial_{x_j} - x_j \partial_{x_i}) + \sum_{j=0}^{n} (x_j \partial_{x_j}).$$

Denote $\Gamma_0 = \sum_{j=1}^{n} e_0 e_j ((x_0 \partial_{x_j} - x_j \partial_{x_0})) - \sum_{i=1, j>i}^{n} e_i e_j ((x_i \partial_{x_j} - x_j \partial_{x_i}))$. Hence,

$$D_s = \frac{x}{\|x\|^2} (E_r + \Gamma_0) = \xi (D_r + \frac{\Gamma_0}{r}),$$

where $r D_r = E_r$ and $r = \|x\|$.

In particular, we have the conformally invariant spherical Dirac operator as follows,

$$D_s = w(\Gamma_0 - \frac{n}{2}), \qquad w \in \mathbb{S}^n.$$

Similarly, we have $\overline{D_s} = \overline{\xi}(D_r + \frac{\overline{\Gamma_0}}{r})$, and since $\overline{D_s}$ is also conformally invariant, we have $\overline{D_s} = \overline{w}(\overline{\Gamma_0} - \frac{n}{2})$, where

$$\overline{\Gamma_0} = -\sum_{j=1}^{n} e_0 e_j (x_0 \partial_{x_j} - x_j \partial_{x_0}) - \sum_{i=1, j>i}^{n} e_i e_j (x_i \partial_{x_j} - x_j \partial_{x_i}).$$

Lemma 5.1.

$$\Gamma_0 \overline{w} = n\overline{w} - \overline{w}\overline{\Gamma_0};$$

$$\overline{\Gamma_0} w = nw - w\Gamma_0.$$

Proof. Since we have

$$D_0 \overline{w} = n + 2E_r - w\overline{D_0},$$

$$\overline{D_0} w = n + 2E_r - \overline{w} D_0,$$

then we have

$$\Gamma_0 \overline{w} = (\overline{w} D_0 - E_r)\overline{w} = \overline{w}(D_0 \overline{w}) - E_r \overline{w} = n\overline{w} + \overline{w} E_r - \overline{D_0} = n\overline{w} - \overline{w}\overline{\Gamma_0};$$

$$\overline{\Gamma_0} w = (w\overline{D_0} - E_r) w = w(\overline{D_0} w) - E_r w = nw - wE_r - D_0 = nw - w\Gamma_0.$$

□

Theorem 5.2.

$$D_s \overline{w} = -w\overline{D_s},\ \overline{D_s} w = -\overline{w} D_s.$$

Proof. A straight forward calculation completes the proof. Indeed,

$$D_s \overline{w} = w(\Gamma - \frac{n}{2})\overline{w} = w\Gamma\overline{w} - \frac{n}{2} = w(n\overline{w} - \overline{w}\overline{\Gamma}) - \frac{n}{2} = -w\overline{w}(\overline{\Gamma} - \frac{n}{2}) = -w\overline{D_s}$$

$$\overline{D_s}w = \overline{w}(\overline{\Gamma}w - \frac{n}{2}w) = \overline{w}\overline{\Gamma}w - \frac{n}{2} = \overline{w}(nw - w\Gamma) - \frac{n}{2} = -\overline{w}w(\Gamma - \frac{n}{2}) = -\overline{w}D_s.$$

□

Theorem 5.3. *Since D_s and $\overline{D_s}$ are both conformally invariant, we have their fundamental solutions as follows:*

$$D_sG_s(w-v) = D_s\frac{1}{\omega_n}\frac{\overline{w-v}}{\|w-v\|^n} = \delta(w-v),$$

$$\overline{D_s}\overline{G_s(w-v)} = \overline{D_s}\frac{1}{\omega_n}\frac{w-v}{\|w-v\|^n} = \delta(w-v).$$

Proof. First, we assume $w \neq v$. Since
$||w-v||^2 = \sum_{j=0}^{n}(w_j - v_j)^2 = \sum_{j=0}^{n} w_j^2 + \sum_{j=0}^{n} v_j^2 - 2\sum_{j=0}^{n} w_jv_j = 2 - 2\langle w, v\rangle$, by $D_s = w(\Gamma - \frac{n}{2})$, and $\Gamma\langle w, v\rangle = \overline{w}v - \langle w, v\rangle$, we can have

$$\begin{aligned}
& D_s\frac{\overline{w-v}}{\|w-v\|^n} = w(\Gamma - \frac{n}{2})(\overline{w-v})(\|w-v\|^2)^{-\frac{n}{2}} \\
= \ & 2^{-\frac{n}{2}}\big(w(\Gamma - \frac{n}{2})\overline{w}(1-\langle w,v\rangle)^{-\frac{n}{2}} - w(\Gamma - \frac{n}{2})\overline{v}(1-\langle w,v\rangle)^{-\frac{n}{2}}\big) \\
= \ & 2^{-\frac{n}{2}}\big((nw - \overline{\Gamma}w - \frac{n}{2}w)\overline{w}(1-\langle w,v\rangle)^{-\frac{n}{2}} - w\Gamma(1-\langle w,v\rangle)^{-\frac{n}{2}}\overline{v} + \frac{n}{2}w\overline{v}(1-\langle w,v\rangle)^{-\frac{n}{2}}\big) \\
= \ & 2^{-\frac{n}{2}}\big(-\overline{\Gamma}(1-\langle w,v\rangle)^{-\frac{n}{2}} - w\Gamma(1-\langle w,v\rangle)^{-\frac{n}{2}}\overline{v} + \frac{n}{2}(1+w\overline{v})(1-\langle w,v\rangle)^{-\frac{n}{2}}\big) \\
= \ & 2^{-\frac{n}{2}}\big(-\frac{n}{2}(1-\langle w,v\rangle)^{-\frac{n}{2}-1}(w\overline{v} - \langle w,v\rangle) - \frac{n}{2}(1-\langle w,v\rangle)^{-\frac{n}{2}-1}(1 - w\overline{v}\langle w,v\rangle) \\
& +\frac{n}{2}(1+w\overline{v})(1-\langle w,v\rangle)^{-\frac{n}{2}}\big) \\
= \ & 2^{-\frac{n}{2}-1}[(-w\overline{v} + \langle w,v\rangle - 1 + w\overline{v}\langle w,v\rangle + (1+w\overline{v})(1-\langle w,v\rangle))](1-\langle w,v\rangle)^{-\frac{n}{2}-1} \\
= \ & 0
\end{aligned}$$

Similarly, by $\overline{D_s} = \overline{w}(\overline{\Gamma} - \frac{n}{2})$ and $\overline{\Gamma}\langle w, v\rangle = w\overline{v} - \langle w, v\rangle$, we obtain that

$$\begin{aligned}
& \overline{D_s}\frac{w-v}{\|w-v\|^n} = \overline{w}(\overline{\Gamma} - \frac{n}{2})(w-v)(\|w-v\|^2)^{-\frac{n}{2}} \\
= \;& 2^{-\frac{n}{2}}\big(\overline{w}(\overline{\Gamma} - \frac{n}{2})w(1-\langle w,v\rangle)^{-\frac{n}{2}} - \overline{w}(\overline{\Gamma} - \frac{n}{2})v(1-\langle w,v\rangle)^{-\frac{n}{2}}\big) \\
= \;& 2^{-\frac{n}{2}}\big((n\overline{w} - \Gamma\overline{w} - \frac{n}{2}\overline{w})w(1-\langle w,v\rangle)^{-\frac{n}{2}} - \overline{w}\overline{\Gamma}(1-\langle w,v\rangle)^{-\frac{n}{2}}v + \frac{n}{2}\overline{w}v(1-\langle w,v\rangle)^{-\frac{n}{2}}\big) \\
= \;& 2^{-\frac{n}{2}}\big(-\Gamma(1-\langle w,v\rangle)^{-\frac{n}{2}} - \overline{w}\overline{\Gamma}(1-\langle w,v\rangle)^{-\frac{n}{2}}v + \frac{n}{2}(1+\overline{w}v)(1-\langle w,v\rangle)^{-\frac{n}{2}}\big) \\
= \;& 2^{-\frac{n}{2}}\big(-\frac{n}{2}(1-\langle w,v\rangle)^{-\frac{n}{2}-1}(\overline{w}v - \langle w,v\rangle) - \frac{n}{2}(1-\langle w,v\rangle)^{-\frac{n}{2}-1}(1-\overline{w}v\langle w,v\rangle) \\
& + \frac{n}{2}(1+\overline{w}v)(1-\langle w,v\rangle)^{-\frac{n}{2}}\big) \\
= \;& 2^{-\frac{n}{2}-1}[(-\overline{w}v + \langle w,v\rangle - 1 + \overline{w}v\langle w,v\rangle + (1+\overline{w}v)(1-\langle w,v\rangle))](1-\langle w,v\rangle)^{-\frac{n}{2}-1} \\
= \;& 0
\end{aligned}$$

$$\overline{D_s}\overline{G(w-v)} = \overline{D_s}\frac{w-v}{\|w-v\|^n} = 0, \;\; w \neq v.$$

Since we have the fact that for $x \in \mathbb{R}^{n+1}$, $||x||^\alpha$ is weak differentiable if $\alpha > -n+2$ with weak derivative $\partial_{x_i}||x||^\alpha = \alpha x_i ||x||^{\alpha-2}$, the calculation above is also true in the distribution sense. Therefore,$D_sG(w-v)$ and $\overline{D_s}\overline{G(w-v)}$ both have only support at the origin, since they both have degree $-n$ and the only distribution having degree $-n$ in $\mathbb{S}^{n-1}$ is $\delta(x)$, this completes the proof. $\square$

Let Ω be a bounded smooth domain in $\mathbb{S}^n$ and $f \in C^1(\Omega, \mathcal{C}l_n)$, we have the Cauchy transforms for both D_s and $\overline{D_s}$,

$$\begin{aligned}
T_\Omega f(w) &= \int_\Omega G_s(w-v)f(v)dv = \frac{1}{\omega_n}\int_\Omega \frac{\overline{w-v}}{\|w-v\|^n}f(v)dv, \\
\overline{T}_\Omega f(w) &= \int_\Omega \overline{G_s(w-v)}f(v)dv = \frac{1}{\omega_n}\int_\Omega \frac{w-v}{\|w-v\|^n}f(v)dv.
\end{aligned}$$

Also, the non-singular boundary integral operators are given by

$$F_{\partial\Omega}f(w) = \int_{\partial\Omega} G_s(w-v)n(v)f(v)d\sigma(v),$$
$$\overline{F}_{\partial\Omega}f(w) = \int_{\partial\Omega} \overline{G_s(w-v)}n(v)f(v)d\sigma(v).$$

Then we have Borel-Pompeiu Theorem as follows.

Theorem 5.4. ***(Borel-Pompeiu Theorem)***

For $f \in C^1(\Omega, \mathcal{C}l_{n-1})$, we have

$$f(w) = \int_{\partial\Omega} G_s(w-v)n(v)f(v)d\sigma(v) + \int_{\Omega} G_s(w-v)D_sf(v)dv,$$

in other words, $f = F_{\partial\Omega}f + T_\Omega D_s f$. Similarly, $f = \overline{F}_{\partial\Omega}f + \overline{T}_\Omega\overline{D_s}f$

$$f(w) = \int_{\partial\Omega} \overline{G_s(w-v)}n(v)f(v)d\sigma(v) + \int_{\Omega} \overline{G_s(w-v)}\overline{D_s}f(v)dv,$$

If f is a function with compact support, then $TD_s = \overline{TD_s} = I$.

Since the conformally invariant spherical Laplace operator Δ_s has the fundamental solution $H_s(w-v) = -\frac{1}{(n-2)\omega_n}\frac{1}{\|w-v\|^{n-2}}$, see [32]. We have factorizations of Δ_s as follows.

Theorem 5.5. $\Delta_s = \overline{D_s}(D_s + w) = D_s(\overline{D_s} + \overline{w})$.

Proof.

$$
\begin{aligned}
& (D_s + w)\frac{1}{\|w-v\|^{n-2}} \\
= \; & w(\Gamma - \frac{n}{2})(\|w-v\|^2)^{-\frac{n-2}{2}} + w(\|w-v\|^2)^{-\frac{n-2}{2}} \\
= \; & 2^{-\frac{n-2}{2}} w\Gamma(1-\langle w,v\rangle)^{-\frac{n-2}{2}} + (1-\frac{n}{2})w\frac{2-2\langle w,v\rangle}{\|w-v\|^n} \\
= \; & (n-2)w2^{-\frac{n}{2}}(\overline{w}v - \langle w,v\rangle)(1-\langle w,v\rangle)^{-\frac{n}{2}} - (n-2)w\frac{1-\langle w,v\rangle}{\|w-v\|^n} \\
= \; & (n-2)w\frac{\overline{w}v - \langle w,v\rangle}{\|w-v\|^n} - (n-2)w\frac{1-\langle w,v\rangle}{\|w-v\|^n} \\
= \; & -(n-2)\frac{w-v}{\|w-v\|^n}.
\end{aligned}
$$

Hence, $(D_s + w)H_s(w-v) = \frac{1}{\omega_n}\frac{w-v}{\|w-v\|^n} = \overline{G_s(w-v)}$.

Similarly, $(\overline{D_s} + \overline{w})H_s(w-v) = G_s(w-v)$ by

$$
\begin{aligned}
& (\overline{D_s} + \overline{w})\frac{1}{\|w-v\|^{n-2}} \\
= \; & \overline{w}(\overline{\Gamma} - \frac{n}{2})(\|w-v\|^2)^{-\frac{n-2}{2}} + \overline{w}(\|w-v\|^2)^{-\frac{n-2}{2}} \\
= \; & 2^{-\frac{n-2}{2}} \overline{w}\overline{\Gamma}(1-\langle w,v\rangle)^{-\frac{n-2}{2}} + (1-\frac{n}{2})\overline{w}\frac{2-2\langle w,v\rangle}{\|w-v\|^n} \\
= \; & (n-2)\overline{w}2^{-\frac{n}{2}}(w\overline{v} - \langle w,v\rangle)(1-\langle w,v\rangle)^{-\frac{n}{2}} - (n-2)\overline{w}\frac{1-\langle w,v\rangle}{\|w-v\|^n} \\
= \; & (n-2)\overline{w}\frac{w\overline{v} - \langle w,v\rangle}{\|w-v\|^n} - (n-2)\overline{w}\frac{1-\langle w,v\rangle}{\|w-v\|^n} \\
= \; & -(n-2)\frac{\overline{w-v}}{\|w-v\|^n}.
\end{aligned}
$$

□

We also have the duality of D_s as follows.

Theorem 5.6. $D_s^* = -\overline{D_s}$.

Proof. Let $f, g : \Omega \to \mathcal{C}l_{n-1}$ both have compact supports,

$$
\begin{aligned}
& < D_s f, g > \\
= \ & < w(\Gamma_0 - \frac{n}{2}) f, g > \\
= \ & < (\Gamma_0 - \frac{n}{2}) f, \overline{w} g > \\
= \ & < \Gamma_0 f, \overline{w} g > - \frac{n}{2} < f, \overline{w} g > \\
= \ & < f, \Gamma_0 \overline{w} g > - \frac{n}{2} < f, \overline{w} g > \\
= \ & < f, (n\overline{\omega} - \overline{\omega}\overline{\Gamma_0}) g > - \frac{n}{2} < f, \overline{w} g > \\
= \ & < f, -\overline{w}(\overline{\Gamma_0} - \frac{n}{2}) g > \\
= \ & < f, -\overline{D_s} g > .
\end{aligned}
$$

□

Definition 5.1. Define the generalized spherical Π-type operator as

$$\Pi_{s,0} f = (\overline{D_s + w}) T f.$$

We have some properties of $\Pi_{s,0}$ as follows.

Proposition 5.7.

$$
\begin{aligned}
D_s \Pi_{s,0} &= \overline{D_s - w}, \\
\Pi_{s,0} D_s &= \overline{D_s + w}.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& D_s \Pi_{s,0} = D_s (\overline{D_s + w}) T = (\overline{D_s - w}) D_s T = \overline{D_s - w}, \\
& \Pi_{s,0} D_s = (\overline{D_s + w}) T D_s = \overline{D_s + w}.
\end{aligned}
$$

$\square$

From the proposition above, we can have decompositions of $L^2(\Omega, \mathcal{C}l_{n-1})$ as follows.

Theorem 5.8.

$$L^2(\Omega, \mathcal{C}l_{n-1}) = L^2(\Omega, \mathcal{C}l_n) \bigcap Ker(\overline{D_s - w}) \bigoplus D_s(W_0^{1,2}(\Omega, \mathcal{C}l_n)),$$

$$L^2(\Omega, \mathcal{C}l_{n-1}) = L^2(\Omega, \mathcal{C}l_n) \bigcap Ker D_s \bigoplus (\overline{D_s + w})(W_0^{1,2}(\Omega, \mathcal{C}l_n)).$$

Notice that

$$\Pi_{s,0}(L^2(\Omega, \mathcal{C}l_n) \bigcap Ker(\overline{D_s - w}) = L^2(\Omega, \mathcal{C}l_n) \bigcap Ker D_s,$$
$$\Pi_{s,0} D_s(W_0^{1,2}(\Omega, \mathcal{C}l_n)) = (\overline{D_s + w})(W_0^{1,2}(\Omega, \mathcal{C}l_n)).$$

Hence, $\Pi_{s,0}$ operator is from $L^2(\Omega, \mathcal{C}l_n)$ to $L^2(\Omega, \mathcal{C}l_n)$. The proof is similar to Theorem 4.3.

Definition 5.2. We define the Π_s^+ operator as

$$\Pi_s^+ f = \overline{D_s} T^+ f,$$

where $T^+ f = \displaystyle\int_\Omega G^+(w - v) f(v) dv$,

$$G^+(w - v) = G_s(w - v) + w H_s(w - v) - 2G_s^{(3)}(w - v),$$

and

$$G_s^{(3)}(w - v) = \frac{1}{(n-2)(n-4)\omega_n} \frac{\overline{w - v}}{\|w - v\|^{n-4}}.$$

Notice that $G_s^{(3)}(w - v)$ is actually the reproducing kernel of $D_s^{(3)} = (D_s - w)\overline{D_s}(D_s + w)$ and the proof is similar as in [32].

Proposition 5.9.

$$
\begin{aligned}
\Pi_{s,0}(L^2(\Omega, \mathcal{C}l_n) \bigcap Ker D_s) &= L^2(\Omega, \mathcal{C}l_n) \bigcap Ker(\overline{D_s - w}), \\
\Pi_{s,0}(\overline{D_s + w})(W_0^{1,2}(\Omega, \mathcal{C}l_n)) &= D_s(W_0^{1,2}(\Omega, \mathcal{C}l_n)).
\end{aligned}
$$

Theorem 5.10. Π_s *is an isometry on* $W_0^{1,2}(\Omega, \mathcal{C}l_n)$ *up to isomorphism.*

Proof. Let $f \in L^2(\Omega, \mathcal{C}l_n)$, then

$$
\begin{aligned}
& \langle \Pi_s f, \Pi_s^+ g \rangle \\
= \; & \langle (\overline{D_s} + \overline{w}) T f, \overline{D_s} T^+ g \rangle \\
= \; & \langle T f, (-D_s + w) \overline{D_s} T^+ g \rangle \\
= \; & -\langle T f, (D_s - w) \overline{D_s} T^+ g \rangle \\
= \; & -\langle T f, \overline{D_s}(D_s + w) T^+ g \rangle \\
= \; & \langle D_s T f, (D_s + w) T^+ g \rangle = \langle f, g \rangle.
\end{aligned}
$$

□

5.2 Application of $\Pi_{s,0}$ to the Solution of a Beltrami Equation

We have a Beltrami equation related to $\Pi_{s,0}$ as follows. Let $\Omega \subseteq \mathbb{S}^n$ be a bounded, simply connected domain with sufficiently smooth boundary, $q : \Omega \longrightarrow \mathcal{C}l_n$ a measurable function. Let $f : \Omega \longrightarrow \mathcal{C}l_n$ be a sufficiently smooth function. The spherical Beltrami equation is as follows:

$$D_s f = q(\overline{D_s + w}) f.$$

By substitute $f = \phi + Th$, where ϕ is an arbitrary left-monogenic function such that $D_s\phi = 0$, we have

$$D_s(\phi + Th) = h = q(\overline{D_s + w})(\phi + Th) = \big((\overline{D_s + w})\phi + \Pi_{s,0}h\big).$$

Therefore, the Beltrami equation is transformed into a singular integral equation as

$$h = q\big((\overline{D_s + w})\phi + \Pi_{s,0}h\big).$$

Similar argument could be found in introduction. By the Banach fixed point theorem, this equation has a unique solution in the case where

$$\|q\| \leq q_0 < \frac{1}{\|\Pi_{s,0}\|},$$

with q_0 being a constant. Hence, for the rest of this section, we will estimate the L^p norm of $\Pi_{s,0}$ with $p > 1$.

Since $\overline{D_s} = \overline{w}(\overline{\Gamma} - \frac{n}{2}) = \overline{w}(w\overline{D_0} - E_r - \frac{n}{2}) = \overline{D_0} - wE_r - \frac{n}{2}\overline{w}$, then

$$\Pi_{s,0}f(w) = \overline{(D_s + w)}Tf(w) = (\overline{D}T + \overline{w}(1 - E_w)T - \frac{n}{2}T)f(w).$$

It is easy to see that

$$\frac{\partial}{\partial w_j}\int_{\mathbb{S}^n} \frac{\overline{w - v}}{\|w - v\|^n} f(v)dv = \int_{\mathbb{S}^n} \frac{\overline{e_j} - n(w_j - v_j)\frac{\overline{w-v}}{\|w-v\|^2}}{\|w - v\|^n} f(v)dv + \omega_n \frac{\overline{e_j}}{n} f(v),$$

since

$$\frac{\partial}{\partial w_j}\frac{\overline{w - v}}{\|w - v\|^n} = \frac{\overline{e_j} - n(w_j - v_j)\frac{\overline{w-v}}{\|w-v\|^2}}{\|w - v\|^n},$$

and using Chapter IX § 7 in [41]

$$\int_S \frac{\overline{w-v}}{\|w-v\|} \cos(r, w_j) dS = \omega_n \frac{\overline{e_j}}{n},$$

where S is a sufficiently small neighborhood of w on $\mathbb{S}^n$.

Hence, we have

$$\begin{aligned}
\overline{D}Tf(w) &= \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{\sum \overline{e_j}^2 - n\sum(w_j - v_j)\overline{e_j}\frac{\overline{w-v}}{\|w-v\|^2}}{\|w-v\|^n} f(v)dv + \frac{\sum \overline{e_j}^2}{n} f(v) \\
&= \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{(1-n) - n\frac{\overline{w-v}^2}{\|w-v\|^2}}{\|w-v\|^n} f(v)dv + \frac{1-n}{n} f(v) \\
E_w Tf(w) &= \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{\sum w_j\overline{e_j} - n\sum w_j(w_j - v_j)\frac{\overline{w-v}}{\|w-v\|^2}}{\|w-v\|^n} f(v)dv + \frac{\sum w_j\overline{e_j}}{n} f(v) \\
&= \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{\overline{w} - n < w, w-v > \frac{\overline{w-v}}{\|w-v\|^2}}{\|w-v\|^n} f(v)dv + \frac{\overline{w}}{n} f(v).
\end{aligned}$$

Therefore, we have an integral expression of $\Pi_{s,0}$ as follows.

Theorem 5.11.

$$\begin{aligned}
\Pi_{s,0} f(w) &= (\overline{D}T + \overline{w}(1 - E_w)T - \frac{n}{2}T) f(w) \\
&= \frac{1}{\omega_n}\int_{\mathbb{S}^n} \frac{1 - n - \overline{w}^2}{\|w-v\|^n} f(v)dv + \frac{n}{\omega_n}\int_{\mathbb{S}^n} \frac{\overline{v} - \langle w, v\rangle \overline{w}}{\|w-v\|^{n+1}} \cdot \frac{\overline{w-v}}{\|w-v\|} f(v)dv \\
&+ (1 - \frac{n}{2})\frac{\overline{w}}{\omega_n}\int_{\mathbb{S}^n} \frac{\overline{w-v}}{\|w-v\|^n} f(v)dv + \frac{1-n}{n} f(v).
\end{aligned}$$

Since

$$\Pi_{s,0} = (\overline{D_s + w})T = (\overline{w}(\overline{\Gamma_0} - \frac{n}{2}) + \overline{w})T = \overline{w}\overline{\Gamma_0}T + (1 - \frac{n}{2})\overline{w}T,$$

where $\overline{\Gamma_0} = -\sum_{j=1}^n e_0 e_j (x_0\partial_{x_j} - x_j\partial_{x_0}) - \sum_{i=1, j>i}^n e_i e_j (x_i\partial_{x_j} - x_j\partial_{x_i})$. To estimate the L^p norm of $\Pi_{s,0}$, we need the following result.

Theorem 5.12. *Suppose p is a positive integer and $p > 1$, then $\|T\|_{L^p} \leq \frac{\omega_{n-1}}{4}$.*

Proof. Since

$$\begin{aligned}
\|Tf\|^p_{L^p} &= (\frac{1}{\omega_n})^p \int_\Omega \| \int_\Omega G_s(w-v)f(v)dv^n\|^p dw^n \\
&= (\frac{1}{\omega_n})^p \int_\Omega \| \int_\Omega G_s(w-v)^{\frac{1}{q}} G_s(w-v)^{\frac{1}{p}} f(v)dv^n\|^p dw^n \\
&\leq (\frac{1}{\omega_n})^p \int_\Omega \big((\int_\Omega \|G_s(w-v)\|dv^n)^{\frac{p}{q}} \cdot \int_\Omega \|G_s(w-v)\|\|f(v)\|^p dv^n \big) dw^n \\
&\leq (\frac{1}{\omega_n})^p C_1^{\frac{p}{q}} \int_\Omega \int_\Omega \|G_s(w-v)\|\|f(v)\|^p dv^n dw^n \\
&= (\frac{1}{\omega_n})^p C_1^{\frac{p}{q}} \int_\Omega \|f(v)\|^p (\int_\Omega \|G_s(w-v)\|dw^n) dv^n \\
&\leq (\frac{1}{\omega_n})^p C_1^{\frac{p}{q}+1} \int_\Omega \|f(v)\|^p (\int_\Omega \|G_s(w-v)\|dw^n) dv^n \\
&= (\frac{1}{\omega_n})^p C_1^p \cdot \int_\Omega \|f(v)\|^p dv^n \\
&= (\frac{1}{\omega_n})^p C_1^p \cdot \|f\|^p_{L^p}
\end{aligned}$$

where $p,\ q > 1$ are positive integers and $\frac{1}{p} + \frac{1}{q} = 1$, where

$$C_1 \leq \big| \int_{\mathbb{S}^n} \|G_s(w-v)\|dv^n \big| = \big| \int_{\mathbb{S}^n} \frac{1}{\|w-v\|^{n-1}} dv^n \big|.$$

Due to the symmetry we can choose any fixed point w, hence we choose $w = (1, 0, 0, ..., 0)$ and $v = (x_0, x_1, \cdots, x_n) \in \mathbb{S}^n$, i.e. $\sum_{i=0}^{n} \|x_i\|^2 = 1$. Let $v = \cos\theta e_0 + \sin\theta\zeta$, where ζ is a vector on $n-1$-sphere, then we have $dv^n = \sin^{n-1}\theta d\theta$,

$$\begin{aligned}
& \int_{\mathbb{S}^n} \frac{1}{[2(1-x_1)]^{\frac{n-1}{2}}} dv^n \\
= & \ 2^{-\frac{n-1}{2}} \int_0^\pi \frac{1}{(1-\cos\theta)^{\frac{n-1}{2}}} \sin^{n-1}\theta d\theta \\
= & \ 2^{-\frac{n-1}{2}} \int_0^\pi (2\sin^2\frac{\theta}{2})^{-\frac{n-1}{2}} (2\sin\frac{\theta}{2}\cos\frac{\theta}{2})^{n-1} d\theta \\
= & \int_0^\pi \cos^{n-1}\frac{\theta}{2} d\theta \\
= & \ 2 \cdot \frac{1}{2} \cdot \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2}+1)} \\
= & \ \sqrt{\pi}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}.
\end{aligned}$$

Since $\omega_n = \dfrac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$, we have $\|T\|_{L^p} \leq \dfrac{\omega_{n-1}}{4}$. □

Let G_0 be the operator defined by

$$G_0 g(w) = -\frac{1}{(n-1)\omega_n} \int_{\mathbb{S}^n} \frac{1}{\|w-v\|^{n-1}} g(v) dv, \ n \geq 3,$$

and $R_s = \overline{\Gamma_0} \circ G_0$ is a Riesz transformation of gradient type (see [3]). Then we have,

Proposition 5.13. *[3], The operator R_s is a L^p operator and the L^p norm is bounded by*

$$\frac{\pi^{1/2}}{2\sqrt{2}}(\frac{p}{p-1})^{1/2} B_p,$$

where $B_p = C_{M,p} + C_p$, $C_{M,p}$ is the L^p norm of the maximal truncated Hilbert transformation on $\mathbb{S}^1$, and $C_p = cot\frac{\pi}{2p^*}$, $\frac{1}{p} + \frac{1}{p^*} = 1$.

Hence,

$$
\begin{aligned}
& \|\overline{\Gamma_0}\frac{1}{\omega_n}\int_\Omega \frac{1}{\|w-v\|^{n-1}}\cdot\frac{\overline{w-v}}{\|w-v\|}f(v)dv\|_{L^p} \\
\leq & \quad (n-1)\frac{\pi^{1/2}}{2\sqrt{2}}(\frac{p}{p-1})^{1/2}B_p\|f(v)\|_{L^p} \\
= & \quad (n-1)\frac{\pi^{1/2}}{2\sqrt{2}}(\frac{p}{p-1})^{1/2}B_p\|f(v)\|_{L^p}. \qquad (1)
\end{aligned}
$$

Recall that $\Pi_{s,0}f = (\overline{D_s}+\overline{w})Tf = (\overline{w}(\overline{\Gamma_0}-\frac{n}{2})+\overline{w})Tf = \overline{w}\overline{\Gamma_0}Tf + (1-\frac{n}{2})\overline{w}Tf$, and by Theorem 5.12,

$$||(1-\frac{n}{2})\overline{w}Tf||_{L^p} = \|(1-\frac{n}{2})\frac{\overline{w}}{\omega_n}\int_\Omega \frac{\overline{w-v}}{\|w-v\|^n}f(v)dv\|_{L^p} \leq (\frac{n}{2}-1)\frac{\omega_{n-1}}{4}\|f\|_{L^p}. \qquad (2)$$

By inequalities (1) and (2), we show that $\Pi_{s,0}$ is a bounded operator mapping from L^p space to itself, and

$$\|\Pi_{s,0}\|_{L^p} \leq (n-1)\frac{\pi^{1/2}}{2\sqrt{2}}(\frac{p}{p-1})^{1/2}B_p + (\frac{n}{2}-1)\frac{\omega_{n-1}}{4}.$$

Remark: The spherical Π-type operator $\Pi_{s,0}$ preserves most properties of the Π operator in Euclidean space and more importantly, it is a singular integral operator which helps to solve the corresponding Beltrami equation. Unfortunately, it is also only an L^2 isometry up to isomorphism as shown in Theorem 5.10 . In the next section, we will use the spectrum theory of differential operators to claim that there is a spherical Π-type operator which is also an L^2 isometry.

5.3 Eigenvectors of Spherical Dirac Type Operators

In this section, we will investigate the spectrums of several spherical Dirac type operators and the spherical Laplacian. During the investigation, we will point out there is a spherical Π-type operator which is an L^2 isometry.

Since $\Gamma_0 = \overline{x}D_0 - E_r$, it is easy to verify the fact that if p_m is a monogenic polynomial and is homogeneous with degree m, that is $D_0 f_m = 0$ and $E_r f_m = m f_m$, then $\Gamma_0 f_m = -m f_m$, so f_m is an eigenvector of Γ_0 with eigenvalue $-m$. Similarly, if $\overline{D_0} g_m = 0$, g_m is an eigenvector of $\overline{\Gamma_0}$ with eigenvalue $-m$.

Let $\mathcal{H}_k$ be the space of $\mathcal{C}l_n$-valued harmonic polynomials homogeneous of degree k and $\mathcal{M}_k$ be the $\mathcal{C}l_n$-valued monogenic polynomials homogeneous of degree k, $\overline{\mathcal{M}_k}$ is the Clifford involution of $\mathcal{M}_k$. By an Almansi-Fischer decomposition [13] and [17], $\mathcal{H}_k = \mathcal{M}_k \bigoplus \bar{x}\overline{\mathcal{M}_{k-1}}$. Hence, for for all harmonic functions with homogeneity of degree k, there exist $p_k \in KerD_0$, and $p_{k-1} \in Ker\overline{D_0}$ such that $h_k = p_k + \bar{x}\overline{p_{k-1}}$. Then, it is easy to get that $\Gamma_0 p_k = -kp_k$ and $\Gamma_0 \bar{x}\overline{p_{k-1}} = (n+k)\bar{x}\overline{p_{k-1}}$.

Let H_m denote the restriction to $\mathbb{S}^n$ of the space of $\mathcal{C}l_n$-valued harmonic polynomials with homogeneity of degree m. P_m is the space of spherical $\mathcal{C}l_n$-valued left monogenic polynomials with homogeneity of degree $-m$ and Q_m is the space of spherical $\mathcal{C}l_n$-valued left monogenic polynomials with homogeneity of degree $n+m$, $m = 0, 1, 2, ...$ Then we have $H_m = P_m \bigoplus Q_m$ ([8]). It is well known that $L^2(\mathbb{S}^n) = \displaystyle\sum_{m=0}^{\infty} H_m$ ([5]), it follows $L^2(\mathbb{S}^n) = \displaystyle\sum_{m=0}^{\infty} P_m \bigoplus Q_m$. If $p_m \in P_m$, since $\Gamma_0 p_m = -mp_m$, it is an eigenvector of Γ_0 with eigenvalue $-m$, and for $q_m \in Q_m$, it is an eigenvector of Γ_0 with eigenvalue $n+m$. Therefore, the spectrum of Γ_0 is $\sigma(\Gamma_0) = \{-m, m = 1, 2, ...\} \cup \{m+n, m = 0, 1, 2, ...\}$,. Since $D_s = w(\Gamma_0 - \dfrac{n}{2})$, the spectrum of D_s is $\sigma(D_s) = \sigma(\Gamma_0) - \frac{n}{2}$, which is $\{-m - \frac{n}{2}, m = 0, 1, 2, ...\} \cup \{m + \frac{n}{2}, m = 0, 1, 2, ...\}$.

As mentioned in the previous section $D_s T = T D_s = I$, and we know that $D_s : P_m \longrightarrow Q_m$ ([8]). Hence, we have $T:\ Q_m \longrightarrow P_m$ and the spectrum of T is the reciprocal of the spectrum of D_s. It is $\sigma(T) = \{\frac{1}{m+\frac{n}{2}}, m = 0, 1, 2, ...\} \bigcup \{\frac{1}{-m-\frac{n}{2}}, m = 0, 1, 2, ...\}$. Similar arguments apply for $\overline{D_s}$ and $\overline{T}$, in fact $\sigma(\overline{D_s}) = \sigma(D_s)$ and $\sigma(\overline{T}) = \sigma(T)$.

Now with a similar strategy as in [8], we consider the operator $\overline{D_s}T$ which maps $L^2(\mathbb{S}^n)$

to $L^2(\mathbb{S}^n)$. If $u \in C^1(\mathbb{S}^n)$ then $u \in L^2(\mathbb{S}^n)$. It follows that

$$u = \sum_{m=0}^{\infty} \sum_{p_m \in P_m} p_m + \sum_{m=0}^{-\infty} \sum_{q_m \in Q_m} q_m,$$

where p_m and q_m are eigenvectors of Γ_0. Further the eigenvectors p_m and q_m can be chosen so that within P_m they are mutually orthogonal. The same can be done for the eigenvectors q_m. Moreover, as $u \in C^1(\mathbb{S}^n)$ then $\overline{D_s}Tu \in C^0(\mathbb{S}^n)$ and so $\overline{D_s}Tu \in L^2(\mathbb{S}^n)$. Consequently,

$$\begin{aligned}\overline{D_s}Tu &= \sum_{m=0}^{\infty} \sum_{p_m \in P_m} \overline{D_s}Tp_m + \sum_{m=0}^{\infty} \sum_{q_m \in Q_m} \overline{D_s}Tq_m \\ = & \sum_{m=0}^{\infty} \sum_{q_m \in Q_m} \overline{D_s}\frac{1}{m+\frac{n}{2}}q_m + \sum_{m=0}^{\infty} \sum_{p_m \in P_m} \overline{D_s}\frac{1}{-m-\frac{n}{2}}p_m\end{aligned}$$

and

$$\begin{aligned}||\overline{D_s}Tu||_{L^2}^2 &= \sum_{m=0}^{\infty}(\frac{1}{m+\frac{n}{2}})^2 \sum_{q_m \in Q_m} \|\overline{D_s}q_m\|_{L^2} + \sum_{m=0}^{\infty}(\frac{1}{-m-\frac{n}{2}})^2 \sum_{p_m \in P_m} \|\overline{D_s}p_m||_{L^2} \\ = & \sum_{m=0}^{\infty}(\frac{1}{m+\frac{n}{2}})^2(m+\frac{n}{2})^2 \sum_{p_m \in P_m} \|p_m||_{L^2} + \sum_{m=0}^{\infty}(\frac{1}{-m-\frac{n}{2}})^(- m - \frac{n}{2})^2 \sum_{q_m \in Q_m} \|q_m||_{L^2} \\ = & \sum_{m=0}^{\infty} \sum_{p_m \in P_m} ||p_m||_{L^2} + \sum_{m=0}^{\infty} \sum_{q_m \in Q_m} ||q_m||_{L^2} \\ = & ||u||_{L^2}.\end{aligned}$$

The above proof shows

Theorem 5.14. *$\overline{D_s}T$ is an $L^2(\mathbb{S}^n)$ isometry.*

By the help of the spectrum of T, we have the L^2 norm estimate of the $\Pi_{s,0}$, that is

$$\begin{aligned}
\|\Pi_{s,0}u\|_{L^2} &\leq \|\overline{D_s}Tu\|_{L^2} + \|\overline{w}\|_{L^2}\|Tu\|_{L^2} \\
&= \|u\|_{L^2} + (\frac{1}{m+\frac{n}{2}})^2(\sum_{m=0}^{\infty}\sum_{p_m\in P_m}\|p_m\|_{L^2} + \sum_{m=0}^{\infty}\sum_{q_m\in Q_m}\|q_m\|_{L^2}) \\
&\leq (1+\frac{4}{n^2})\|u\|_{L^2}.
\end{aligned}$$

Hence we have $\|\Pi_{s,0}\|_{L^2} \leq 1 + \dfrac{4}{n^2}$.

By Theorem 5.2, $\Delta_s = \overline{D_s}(D_s + w) = (\overline{D_s} - \overline{w})D_s = D_s(\overline{D_s} + \overline{w}) = (D_s - w)\overline{D_s}$.

Since $D_s = w(\Gamma_0 - \dfrac{n}{2})$, $\overline{D_s} = \overline{w}(\overline{\Gamma_0} - \dfrac{n}{2})$, a straightforward calculation shows us that

$$\begin{aligned}
\Delta_s &= -(\Gamma_0 - \frac{n}{2})^2 - \overline{w}w(\Gamma_0 - \frac{n}{2}) = -\Gamma_0^2 + (n-1)\Gamma_0 - (\frac{n^2}{4} - \frac{n}{2}) \\
&= -(\overline{\Gamma_0} - \frac{n}{2})^2 - \overline{w}w(\overline{\Gamma_0} - \frac{n}{2}) = -\overline{\Gamma_0}^2 + (n-1)\overline{\Gamma_0} - (\frac{n^2}{4} - \frac{n}{2}).
\end{aligned}$$

Since for $0 < r < 1$, any harmonic function $h_m \in B(0,r) = \{x \in \mathbb{R}^n : \ ||x|| < r\}$ with homogeneity degree m, we have $h_m = f_m + g_m$, where $f_m \in KerD_0$ and $g_m \in \overline{D_0}$, they are both homogeneous with degree m (see Lemma 3 [21]). Consequently,

$$\Delta_s f_m = (-\Gamma_0^2 + (n-1)\Gamma_0 - (\frac{n^2}{4} - \frac{n}{2}))f_m = (-m^2 - m(n-1) - (\frac{n^2}{4} - \frac{n}{2}))f_m,$$

and

$$\Delta_s g_m = (-\overline{\Gamma_0}^2 + (n-1)\overline{\Gamma_0} - (\frac{n^2}{4} - \frac{n}{2}))g_m = (-m^2 - m(n-1) - (\frac{n^2}{4} - \frac{n}{2}))g_m.$$

Hence

$$\begin{aligned}\Delta_s h_m &= \Delta_s(f_m+g_m) = (-m^2 - m(n-1) - (\frac{n^2}{4} - \frac{n}{2}))(f_m+g_m) \\ &= (-m^2 - m(n-1) - (\frac{n^2}{4} - \frac{n}{2}))h_m.\end{aligned}$$

Since for any function $u \in L^2(\mathbb{S}^n) : \Omega \mapsto \mathcal{C}l_n$, $u = \sum_{m=0}^{\infty} h_m$, where $h_m \in H_m$, it follows that Δ_s has spectrum $\sigma(\Delta_s) = \{-m^2 - m(n-1) - (\frac{n^2}{4} - \frac{n}{2}) : m = 0, 1, 2, ...\}$.

In order to preserve the property of isometry of the Π-operator on the sphere, we define the isometric spherical Π-operator as $\Pi_{s,1}$ as $\Pi_{s,1} = \overline{D_s}T$, which is an isometry in L^2 space. We can solve the Beltrami equation related to $\Pi_{s,1}$ as follows.

Let $\Omega \subseteq \mathbb{S}^n$ be a bounded, simply connected domain with sufficiently smooth boundary, and $q, f : \Omega \longrightarrow \mathcal{C}l_{n-1}$, q is a measurable function, and f is sufficiently smooth. The spherical Beltrami equation is as follows:

$$D_s f = q\overline{D_s}f.$$

Substitute $f = \phi + Th$ where ϕ is an arbitrary left-monogenic function such that $D_s\phi = 0$, we have

$$D_s(\phi + Th) = h = q\overline{D_s}(\phi + Th) = q(\overline{D_s}\phi + \Pi_{s,1}h).$$

Therefore, we transform the Beltrami equation to the integral equation

$$h = q(\overline{D_s}\phi + \Pi_{s,1}h).$$

If h is the unique solution of the previous equation, then $f = \phi + Th$ is the unique solution of the Beltrami equation. Similar argument could be found in Introduction. By the Banach fixed point theorem, the previous integral equation has a unique solution in the case of

$$\|q\| \leq q_0 < \frac{1}{\|\Pi_{s,1}\|}$$

with q_0 being a constant. Hence, we can use the estimate of the L^p norm of $\Pi_{s,1}$ with $p > 1$, where

$$\|\Pi_{s,1}\|_{L^p} \leq (n-1)\frac{\pi^{1/2}}{2\sqrt{2}}(\frac{p}{p-1})^{1/2}B_p + \frac{n}{2}\frac{\omega_{n-1}}{4}.$$

Notice that the spherical Π-operator constructed in this section does not satisfy some basic identities as it does in the Euclidean space, see Theorem 4.2. In Euclidean space, these identities rely on the fact that the Euclidean Dirac operator and its dual operator commute. However, this is not true for the spherical Dirac operator, and hence, our spherical Π-operator no longer satisfies these identities.

6 Π-Operators on Real Projective Space

Recall the construction of our Π-operator in the previous section, if we let X to be the real projective space $\mathbb{R}P^n$ with the measure η by pushing forward the Lebesgue measure on $\mathbb{S}^n$. Then, $H = L^2(\mathbb{R}P^n, \mathbb{R})$ becomes a real Hilbert space, and $H \otimes \mathcal{C}l_n$ is a Clifford-Hilbert module with the inner product

$$\langle f, g\rangle = \int_{V'} \overline{f} g d\eta(x),$$

where V' is a subset of real projective space with $\overline{V'}$ inclosed and $f, g : V' \longrightarrow \mathcal{C}l_n$. Therefore we can obtain the Π-operator theory on real projective space as a special case of Section 4. More details are as follows.

6.1 Dirac Operators on Real Projective Space

We know that the real projective space $\mathbb{R}P^n$ is defined as $\mathbb{S}^n/\Gamma$, where $\mathbb{S}^n$ is the n-dimensional unit sphere and $\Gamma = \{\pm 1\}$. This implies that Π-operator theory on real projective space should be generalized from the Π-operator theory on the unit sphere. Notice that there is a projection map $p : \mathbb{S}^n \longrightarrow \mathbb{R}P^n$, such that for each $x \in \mathbb{S}^n$, $p(\pm x) = x'$. If Q is a subset of $\mathbb{S}^n$, we denote $p(\pm Q) = Q'$. Firstly we consider the bundle E_1 by making the identification of (x, X) and $(-x, X)$ where $x \in \mathbb{S}^n$ and $X \in \mathcal{C}l_n$.

Now we change the generalized spherical Cauchy kernel $G_s(x,y) = -\dfrac{1}{\omega_n}\dfrac{\overline{x-y}}{\|x-y\|^n}$, $x, y \in \mathbb{S}^n$ into a kernel which is invariant with respect to $\{\pm 1\}$. We obtain a kernel $G_{\mathbb{R}P_1^n}(x,y) = G_s(x,y) + G_s(-x,y)$ for $\mathbb{R}P^n$ ([29]).

Suppose S is a suitably smooth hypersurface lying in the northern hemisphere of $\mathbb{S}^n$ and V is also a domain lying in the northern hemisphere sphere that S bounds a subdomain W of V. If $f : V \longrightarrow \mathcal{C}l_n$ is a left spherical monogenic function and $y \in W$, then

$$f(x) = \int_S \big(G_s(x,y) + G_s(-x,y)\big) n(y) f(y) d\sigma(y),$$

where ω_n is the surface area of $\mathbb{S}^n$ and $n(y)$ is the unit outer normal vector to S at x lying in the tangent space of $\mathbb{S}^n$ at y. Now we use the projection map $p : \mathbb{S}^n \longrightarrow \mathbb{R}P^n$ to note that this projection map induces a function $f' : V' \longrightarrow E_1$. We have ([29])

$$f'(x') = \int_{S'} G_{\mathbb{R}P_1^n}(x', y') dp(n(y)) f'(y') d\sigma'(y'),$$

where $x' = p(x)$, $y' = p(y)$, and $S' = p(S)$. This projection induces a measure σ' on S' from the measure σ on S. Now we will assume the domain V is such that $-x \in V$ for each $x \in V$, and the function f is two fold periodic, so that $f(x) = f(-x)$ and $S = -S$. Now the projection map p give rise to a well defined domain V' on $\mathbb{R}P^n$ and a well defined function $f'(x') : V' \longrightarrow E_1$ such that $f'(x') = f(\pm x)$. As the function is spherical monogenic, which is $D_s f(x) = 0$, we could induce a Dirac operator on $\mathbb{R}P^n$ and $D_{\mathbb{R}P_1^n} f'(x') = 0$. In this case ([29]),

$$2f'(x') = \int_{S'} G_{\mathbb{R}P_1^n}(x', y') dp(n(x)) f'(y') d\sigma'(y').$$

Similarly, we have the conjugate of the Dirac operator $\overline{D_{\mathbb{R}P_1^n}}$ induced by $\overline{D_s}$, and the kernel of $\overline{D_{\mathbb{R}P_1^n}}$ is $\overline{G_{\mathbb{R}P_1^n}(x,y)} = \overline{G_s(x,y)} + \overline{G_s(-x,y)}$.

Now we induce the Cauchy transform and its conjugate from $\mathbb{S}^n$ to $\mathbb{R}P^n$ as follows.

$$T_{V_1'} f'(x') = \int_{V'} G_{\mathbb{R}P_1^n}(x', y') f'(y') dy',$$
$$\overline{T_{V_1'}} f'(x') = \int_{V'} \overline{G_{\mathbb{R}P_1^n}(x', y')} f'(y') dy'.$$

Also, the non-singular boundary integral operator and its conjugate are given by

$$F_{S'} f'(x') = \int_{S'} G_{\mathbb{R}P_1^n}(x', y') dp(n(y')) f'(y') d\sigma'(y'),$$
$$\overline{F_{S'}} f'(x') = \int_{S'} \overline{G_{\mathbb{R}P_1^n}(x', y')} dp(n(y')) f'(y') d\sigma'(y').$$

Hence, the Borel-Pompeiu formula is stated as follows.

Theorem 6.1. *For $f' \in C^1(V', \mathcal{C}l_n) \cap C(\bar{V}')$, we have*

$$2f'(x') = \int_{S'} G_{\mathbb{R}P_1^n}(x', y') dp(n(y)) f'(y') d\sigma'(y') + \int_{V'} G_{\mathbb{R}P_1^n}(x', y') D_{\mathbb{R}P_1^n} f'(y') dy'.$$

In particular, if f' has compact support, then

$$2f'(x') = \int_{V'} G_{\mathbb{R}P_1^n}(x', y') D_{\mathbb{R}P_1^n} f'(y') dy',$$

from which we could obtain $TD_{\mathbb{R}P_1^n} = 2I$.

Since the domain $V = -V$, if we restrict it on the northern hemisphere, the Dirac operator $D_{\mathbb{R}P_1^n}$ is locally homeomorphic to D_s. We project it on the domain V' on $\mathbb{R}P^n$ and we have

$$D_{\mathbb{R}P_1^n} \int_V G_s(x, y) f(y) dy = f(x).$$

Now, for the whole domain V, after projection on the domain V'on $\mathbb{R}P^n$, we obtain

$$D_{\mathbb{R}P_1^n} \int_{V'} \big(G_s(x, y) + G_s(-x, y)\big) f'(y') dy' = 2f'(x),$$

that is $D_{\mathbb{R}P_1^n} T = 2I$. Similarly, we have $\overline{D_{\mathbb{R}P_1^n} T} = \overline{T D_{\mathbb{R}P_1^n}} = 2I$.

In the rest of this section, we will study the spectrums of our operators $\overline{D_{\mathbb{R}P_1^n}}$ and T. This will helps us to show that our Π-operator (defined in Section 3.2) also possesses the L^2 isometry property. The Dirac operator $D_{\mathbb{R}P_1^n}$ is induced by the spherical monogenic functions. Since $D_s D_s^* f = -D_s \overline{D_s} f \neq -\overline{D_s} D_s f = D_s^* D_s f$ (see [15]), we have $D_{\mathbb{R}P_1^n} D_{\mathbb{R}P_1^n}^* f' \neq D_{\mathbb{R}P_1^n}^* D_{\mathbb{R}P_1^n} f'$ also. In order to prove the property of L^2 isometry we are using the method of spectrum. Similar argument can be found on the spherical Π-operator, see [15].

Let H_m denote the space of $\mathcal{C}l_n$-valued harmonic polynomials with homogeneity of degree m restricted to $\mathbb{S}^n$. It is well known that $L^2(\mathbb{S}^n) = \sum_{m=0}^{\infty} H_m$, see [5]. Now we consider a function $f(x)$ which is defined on an open domain $V \subseteq \mathbb{S}^n$ such that $-x \in V$ for each $x \in V$ and $f(x) = f(-x)$. Such a function f could be projected on the real projective space $\mathbb{R}P^n$ by $p(\pm x) = x'$. Since $\sum_{m=0}^{\infty} h_{2m}(x) = f(x) = f(-x) = \sum_{m=0}^{\infty} h_{2m}(-x)$, we should have $f(x) = \sum_{m=0}^{\infty} h_{2m}(x)$. Then by the projection map we have $f'(x') = \sum_{m=0}^{\infty} h'_{2m}(x')$. Hence, $L^2(\mathbb{R}P^n) = \sum_{m=0}^{\infty} H'_{2m}$ with the spinor bundle E_1, where H'_{2m} is H_{2m} projected on the real projective space.

Assume that P_m is the space of spherical $\mathcal{C}l_n$-valued left monogenic polynomials with homogeneity of degree $-m$ and Q_m is the space of spherical $\mathcal{C}l_n$-valued left monogenic polynomials with homogeneity of degree $n+m$, $m=0,1,2,....$ We have already known that $H_m = P_m \oplus Q_m$ on $\mathbb{S}^n$ ([8]), that is for each $h_m(x) \in H_m(\mathbb{S}^n)$ there exists $p_m(x) \in P_m(\mathbb{S}^n)$ and $q_m(x) \in Q_m(\mathbb{S}^n)$ such that $h_m(x) = p_m(x) + q_m(x)$. Hence $h_m(-x) = p_m(-x) + q_m(-x)$. However, from the discussion given in the previous paragraph, there are only even degree polynomials in the decomposition of $L^2(\mathbb{R}P^n)$ when $\mathbb{R}P^n$ has the spinor bundle E_1. Therefore, by the projection map, we have a decomposition on the real projective space as $h'_{2m}(x') = p'_{2m}(x') + q'_{2m}(x')$. In other words, $L^2(\mathbb{R}P^n) = \sum_{m=0}^{\infty} P'_{2m} \oplus Q'_{2m}$. As we know that $D_s(P_{2m}) = Q_{2m}$ and $D_s(Q_{2m}) = P_{2m}$, we also have $D_{\mathbb{R}P_1^n}(P'_{2m}) = Q'_{2m}$ and $D_{\mathbb{R}P_1^n}(Q'_{2m}) = P'_{2m}$. Hence $D_{\mathbb{R}P_1^n}$ maps $L^2(\mathbb{R}P^n)$ to itself, similarly for $\overline{D_{\mathbb{R}P_1^n}}$. Similar as the case on the unit sphere, we have the spectrum of the real projective Dirac operator as follows.

$$\sigma(D_{\mathbb{R}P_1^n}) = \sigma(\overline{D_{\mathbb{R}P_1^n}}) = \{-2m - \frac{n}{2}, m = 0,1,2,...\} \cup \{2m + \frac{n}{2}, m = 0,1,2,...\}.$$

Since we previously mentioned that $\overline{D_{\mathbb{R}P_1^n} T} = \overline{T D_{\mathbb{R}P_1^n}} = 2I$, and $T: \ Q'_m \longrightarrow P'_m$ and $T: \ P'_m \longrightarrow Q'_m$. The spectrums of T and its conjugation $\overline{T}$ on the real projective space are

$$\sigma(\overline{T}) = \sigma(T) = \{\frac{2}{2m + \frac{n}{2}}, m = 0,1,2,...\} \cup \{\frac{2}{-2m - \frac{n}{2}}, m = 0,1,2,...\}.$$

6.2 Construction of a Π-Operator on the Real Projective Space

We first give the definition for the Π-operator on the real projective space as follows.

Definition 6.1. Define the Π-operator on the real projective space as

$$\Pi_{\mathbb{R}P_1^n} = \frac{1}{2}(\overline{D_{\mathbb{R}P_1^n}})T.$$

Here, the constant $\frac{1}{2}$ allows $\Pi_{\mathbb{R}P_1^n}$ to be L^2 isometric, we will see more details below. Notice that $\Pi_{\mathbb{R}P_1^n}$ maps $L^2(\mathbb{R}P^n)$ to $L^2(\mathbb{R}P^n)$. Further, we can prove that $\Pi_{\mathbb{R}P_1^n}$ is an

L^2-isometry on $\mathbb{R}P^n$ as follows.

Theorem 6.2. $\Pi_{\mathbb{R}P_1^n}$ *is an* $L^2(\mathbb{R}P^n)$ *isometry.*

Proof. Hence, we assume the function $u \in C^1(\mathbb{R}P^n) \subset L^2(\mathbb{R}P^n)$, since $C^1(\mathbb{R}P^n)$ is dense in $L^2(\mathbb{R}P^n)$. For such a function u, we have the decomposition

$$u = \sum_{m=0}^{\infty} \sum_{p'_{2m} \in P'_{2m}} p'_{2m} + \sum_{m=0}^{-\infty} \sum_{q'_{2m} \in Q'_{2m}} q'_{2m}.$$

Hence, with similar arguments as in [15], we have

$$\begin{aligned}
& ||\frac{1}{2}\overline{D_{\mathbb{R}P_1^n}} Tu||_{L^2}^2 \\
= & \sum_{m=0}^{\infty} (\frac{1}{2m+\frac{n}{2}})^2 \sum_{q'_{2m} \in Q'_{2m}} \|\overline{D_{\mathbb{R}P_1^n}} q'_{2m}\|_{L^2} + \sum_{m=0}^{\infty} (\frac{1}{-2m-\frac{n}{2}})^2 \sum_{p'_{2m} \in P'_{2m}} \|\overline{D_{\mathbb{R}P_1^n}} p'_{2m}||_{L^2} \\
= & \sum_{m=0}^{\infty} (\frac{1}{2m+\frac{n}{2}})^2 (2m+\frac{n}{2})^2 \sum_{p'_{2m} \in P'_{2m}} \|p'_{2m}||_{L^2} \\
& + \sum_{m=0}^{\infty} (\frac{1}{-2m-\frac{n}{2}})^2 (-2m-\frac{n}{2})^2 \sum_{q'_{2m} \in Q'_{2m}} \|q'_{2m}||_{L^2} \\
= & \sum_{m=0}^{\infty} \sum_{p'_{2m} \in P'_{2m}} ||p'_{2m}||_{L^2} + \sum_{m=0}^{\infty} \sum_{q'_{2m} \in Q'_{2m}} ||q'_{2m}||_{L^2} = ||u||_{L^2}.
\end{aligned}$$

□

We can also assign another bundle E_2 to $\mathbb{R}P^n$ by identifying the pair (x, X) with $(-x, -X)$, where $x \in \mathbb{S}^n$ and $X \in \mathcal{C}l_n$. In this circumstance, the projection map p induces a Cauchy kernel $G_{\mathbb{R}P_2^n}$ which is a antiperiodic with respect to $\Gamma = \{\pm 1\}$. Hence $G_{\mathbb{R}P_2^n}(x' - y') = G_s(x, y) - G_s(-x, y)$. In this case, a Clifford holomorphic function $f : V \longrightarrow \mathcal{C}l_n$ satisfying $f(x) = -f(-x)$ will give a Clifford holomorphic function $f : V' \longrightarrow E_2$. Similarly, we could induce another Cauchy transform and its conjugate from

$\mathbb{S}^n$ to $\mathbb{R}P^n$ as follows.

$$T_{V_2'}f'(x') = \int_{V'} G_{\mathbb{R}P_2^n}(x'-y')f'(y')dy',$$
$$\overline{T_{V_2'}}f'(x') = \int_{V'} \overline{G_{\mathbb{R}P_2^n}(x'-y')}f'(y')dy'.$$

With similar arguments as for $D_{\mathbb{R}P_1^n}$, we can define $D_{\mathbb{R}P_2^n}$ on $\mathbb{R}P^n$ with the bundle E_2, and the Π-operator is defined as $\Pi_{\mathbb{R}P_2^n} = \frac{1}{2}D_{\mathbb{R}P_2^n}T_{V_2'}$. Similar arguments for $\Pi_{\mathbb{R}P_1^n}$ shows that $\Pi_{\mathbb{R}P_2^n}$ also possesses the L^2 isometry property. It is worth pointing out that on the bundle E_2 we only have odd degree eigenvectors for $f(-x) = -f(x)$, hence the decomposition of f is $f = \sum_{m=0}^{\infty} H_{m+1} = \sum_{m=0}^{\infty} P'_{2m+1} \oplus Q'_{2m+1}$.

With similar arguments as for $D_{\mathbb{R}P_1^n}$, we can define $D_{\mathbb{R}P_2^n}$ on $\mathbb{R}P^n$ with the bundle E_2, and the Π-operator is defined as $\Pi_{\mathbb{R}P_2^n} = \frac{1}{2}D_{\mathbb{R}P_2^n}T_{V_2'}$, which is also induced from Π_s. Similar arguments for $\Pi_{\mathbb{R}P_1^n}$ shows that $\Pi_{\mathbb{R}P_2^n}$ also possesses the L^2 isometry property. It is worth pointing out that on the bundle E_2 we only have odd degree eigenvectors for $f(-x) = -f(x)$, hence the decomposition of f is $f = \sum_{m=0}^{\infty} H_{m+1} = \sum_{m=0}^{\infty} P'_{2m+1} \oplus Q'_{2m+1}$.

6.3 The Beltrami Equation on the Real Projective Space

In this section, we will demonstrate how to use our Π-operator $D_{\mathbb{R}P_1^n}$ to determine the existence of the solutions to the Beltrami equation on the real projective space.

Let $V' \subseteq \mathbb{R}P^n$ be a bounded, simply connected domain with sufficiently smooth boundary, and $q, f' : V' \longrightarrow E_1$, q is a measurable function, and f' is sufficiently smooth. The Beltrami equation on the real projective space is as follows:

$$D_{\mathbb{R}P_1^n}f' = q\overline{D_{\mathbb{R}P_1^n}}f'.$$

By substituting $f' = \phi + \frac{1}{2}Th$ where ϕ is an arbitrary left-monogenic function such that

$D_{\mathbb{R}P_1^n}\phi = 0$, we have

$$D_{\mathbb{R}P_1^n}(\phi + \frac{1}{2}Th) = h = q\phi + \frac{1}{2}T(\phi + \frac{1}{2}Th) = q(\overline{D_{\mathbb{R}P_1^n}}\phi + \Pi_{\mathbb{R}P_1^n}h).$$

Hence we transformed the Beltrami equation into an integral equation

$$h = q(\overline{D_{\mathbb{R}P_1^n}}\phi + \Pi_{\mathbb{R}P_1^n}h).$$

By the Banach fixed point theorem, the previous integral equation has a unique solution in the case of

$$\|q\| \leq q_0 < \frac{1}{\|\Pi_{\mathbb{R}P_1^n}\|}$$

with q_0 being a constant. Therefore, the problem of the existence of the solutions to the Beltrami equation becomes the estimation of the L^p norm of $\Pi_{\mathbb{R}P_1^n}$ with $p > 1$. Similar argument could be found in the Introduction.

Since the domain $V = -V$ on $\mathbb{S}^n$, this means if we restrict V to the northern hemisphere as V', $\overline{D_{\mathbb{R}P_1^n}}T$ is locally homeomorphic to Π_s. Hence, if we project V to V' on $\mathbb{R}P^n$, we have $\|\Pi_{\mathbb{R}P_1^n}\|_{L^p} = \frac{1}{2}\|\Pi_s\|_{L^p}$. This allows us to use the estimate of $\|\Pi_s\|_{L^p}$ to obtain

$$\|\Pi_{\mathbb{R}P_1^n}\|_{L^p} = \frac{1}{2}\|\Pi_s\|_{L^p} \leq (n-1)\frac{\pi^{1/2}}{2\sqrt{2}}(\frac{p}{p-1})^{1/2}B_p + \frac{n}{2}\frac{\omega_{n-1}}{4}$$

where $B_p = C_{M,p} + C_p$, $C_{M,p}$ is the L^p norm of the maximal truncated Hilbert transformation on $\mathbb{S}^1$, and $C_p = cot\frac{\pi}{2p^*}$, $\frac{1}{p} + \frac{1}{p^*} = 1$. For more details, see [3, 15].

7 Π-Operators on Cylinders and Hopf Manifolds

Π-operator theory on cylinders and Hopf manifolds are special cases of Section 4. We let X to be the cylinders C_k with the measure η by pushing forward the Lebesgue measure on $\mathbb{R}^{n+1}$ via the quotient map $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}/\mathbb{Z}^k$ given in Section 6.1.1. Meanwhile,

$H = L^2(\mathcal{C}_k, \mathbb{R})$ is a real Hilbert space, and $H \otimes \mathcal{C}l_n$ is a Clifford-Hilbert module with the inner product

$$\langle f, g\rangle = \int_{V'} \overline{f} g d\eta(x),$$

where V' is a subset of cylinder C_k with $\overline{V'}$ inclosed and $f, g : V' \longrightarrow \mathcal{C}l_n$. Therefore we can construct the Π-operator theory on cylinders as demonstrated in Section 4.

Similarly, if we let X to be Hopf manifolds $\mathbb{S}^1 \times \mathbb{S}^n$ with the pushforward measure obtained via the quotient map defined below in Section 6.2.1, and $H = L^2(\mathbb{S}^1 \times \mathbb{S}^n, \mathbb{R})$ is a Hilbert space. Then we can build the Π-operator theory on the Clifford-Hilbert module $H \otimes \mathcal{C}l_n$. More details are given below.

7.1 Π-Operators on Cylinders

7.1.1 Dirac Operator on the Cylinder

For integer k, $1 \leq k \leq n$, we define the k-cylinder C_k to be the k-dimensional manifold $\mathbb{R}^{n+1}/\mathbb{Z}^k$ where $\mathbb{Z}^k = \mathbb{Z}e_0 + \mathbb{Z}e_1 + ... + \mathbb{Z}e_{k-1}$. In particular, when $k = n$, C_k is the k-torus. Each element in C_k has the form $m_0e_0 + \cdots m_{k-1}e_{k-1}$ for $m_0, \cdots, m_{k-1} \in \mathbb{Z}$ and it is denoted by $\underline{t}$. For each k the space $\mathbb{R}^{n+1}$ is the universal covering space of the cylinder C_k. Hence, there is a projection map $p_k : \mathbb{R}^{n+1} \longrightarrow C_k$.

Let U be a open subset of $\mathbb{R}^{n+1}$. It is called k-fold periodic if for each $x \in U$ we also have $x + \underline{t} \in U$. Hence, $U' = p_k(U)$ is an open subset of C_k. Suppose that $U \subseteq \mathbb{R}^{n+1}$ is a k-periodic open set. $f(x)$ is a Clifford valued function defined on U. We say that $f(x)$ is a k-fold periodic function if we have $f(x) = f(x + \underline{t})$ for each $x \in U$. Hence, the projection p_k induces a well defined function $f' : U' \longrightarrow \mathcal{C}l_n$, where $f'(x') = f(x)$ for each $x' \in U'$ and x is arbitrary representative of $p_k^{-1}(x')$. Moreover, any function $f' : U' \longrightarrow \mathcal{C}l_n$ lifts to an k-fold periodic function $f : U \longrightarrow \mathcal{C}l_n$, where $U = p_k^{-1}(U')$.

In ([30]) the spinor bundle over C_k is trivial on $C_k \times \mathcal{C}l_n$. Other k spinor bundles $E^{(l)}$ over C_k are given rise by making the identification (x, X) with $(x + \underline{m} + \underline{n}, (-1)^{m_0+m_1+...+m_l} X)$, where l is an integer and $0 \leq l \leq k$, $\underline{m}$ is in the lattice $\mathbb{Z}^l = \mathbb{Z}e_0 + \mathbb{Z}e_1 + ... + \mathbb{Z}e_{l-1}$, and $\underline{n}$ is in the lattice $\mathbb{Z}^{k-l} = \mathbb{Z}e_l + \mathbb{Z}e_{l+1} + ... + \mathbb{Z}e_{k-1}$.

Let $G(x, y) = \dfrac{\overline{x-y}}{||x-y||^{n+1}}$ be the fundamental solution of the Euclidean Dirac operator. Consider the series

$$\cot_{k,0}(x, y) = \sum_{\underline{m} \in \mathbb{Z}^k} G(x - y + \underline{m})$$

which converges on $\mathbb{R}^{n+1} \setminus \mathbb{Z}^k$, for $k < n - 1$, see [29]. Then, the kernel of Dirac operator on the cylinder C_k with the trivial bundle has the form $\cot_{k,0}(x', y')$ which is defined on $(C_k \times C_k) \setminus diagonal(C_k)$, where $diagonal(C_k) = \{(x', x') : x' \in C_k\}$. More generally, For $k < n - 1$ and $l \leq k$, the kernel $\cot_{k,l}(x', y')$ of the Dirac operator on C_k with the bundle $E^{(l)}$ is given rise by applying p_k on

$$\cot_{k,l}(x, y) = \sum_{\underline{m} \in \mathbb{Z}^k, \underline{n} \in \mathbb{Z}^{k-l}} (-1)^{m_0+m_1+...m_{l-1}} G(x - y + \underline{m} + \underline{n}).$$

On the other hand, with the projection map p_k, we can induce the Dirac operator on $\mathbb{R}^{n+1}$ to C_k with the bundle $E^{(l)}$, which is denoted by D_l. Similar argument applies for the conjugation $\overline{D_l}$ and its fundamental solution $\overline{\cot_{k,l}(x', y')}$. Furthermore, $D_l \overline{D_l} = \overline{D_l} D_l = \Delta_l$, where Δ_l is a spinorial Laplacian, see [29].

Suppose $f : V \longrightarrow \mathbb{R}^{n+1}$ satisfying $f(x + \underline{m} + \underline{n}) = (-1)^{m_0+m_1+...m_{l-1}} f(x)$, where $\underline{m} \in \mathbb{Z}^l, \underline{n} \in \mathbb{Z}^{k-l}$. Then, f can be lifted by the projection map p_k to a function $f' : V' \longrightarrow E^{(l)}$, where $V' = p_k^{-1}(V)$. If $D_l f' = 0$, f' is called an $E^{(l)}$ left Clifford monogenic function.

Using the fundamental solutions of the Dirac operators, we can define the Cauchy transform on different bundles. If $f' : V' \longrightarrow E^{(l)}$, S' is a surface lying in V' and bounding a subdomain W'. Suppose $x' \in W'$, then

$$T_{V'}f'(x') = \frac{1}{\omega_n}\int_{V'} \cot_{k,l}(x',y')f'(y')dy',$$
$$\overline{T_{V'}}f'(x') = \frac{1}{\omega_n}\int_{V'} \overline{\cot_{k,l}(x',y')}f'(y')dy'.$$

Also, the non-singular boundary integral operator and its conjugate are given by

$$F_{S'}f'(x') = \frac{1}{\omega_n}\int_{S'} \cot_{k,l}(x',y')dp(n(y'))f'(y')d\sigma'(y'),$$
$$\overline{F_{S'}}f'(x') = \frac{1}{\omega_n}\int_{S'} \overline{\cot_{k,l}(x',y')}dp(n(y'))f'(y')d\sigma'(y').$$

Hence, the Borel-Pompeiu formula is stated as follows.

Theorem 7.1. *([30]) For $f' \in C^1(V', \mathcal{C}l_n) \cap C(\overline{V'})$, we have*

$$f'(x') = \frac{1}{\omega_n}\Big(\int_{S'} \cot_{k,l}(x',y')dp(n(y))f'(y')d\sigma'(y') + \frac{1}{\omega_n}\int_{V'} \cot_{k,l}(x',y')D_lf'(y')dy'\Big).$$

Similar as the case in Euclidean space, for a function f' with compact support, we have $D_lT_{V'} = T_{V'}D_l = I$, and $\overline{D_lT_{V'}} = \overline{T_{V'}D_l} = I$ as well.

7.1.2 Construction and Applications of the Π-Operator on Cylinders

Now we define the Π-operator on the cylinder as follows.

Definition 7.1. Define the Π-operator on the cylinder as

$$\Pi_l = \overline{D_l}T.$$

Since Π_l is induced from the Π-operator in Euclidean space, we expect similar results as in ([25]).

Theorem 7.2. Π_l *is an* $L^2(C_k)$ *isometry operator.*

Proof. The proof is similar to the proof of Proposition 5 in [25]. □

In this section, we will use the norm estimation of the Π-operator on the cylinder to determine existence of the solution of Beltrami equation on the cylinder. First, we define the Beltrami equation on the cylinder as follows.

Let $V' \subseteq C_k$ be a bounded, simply connected domain with sufficiently smooth boundary, and $q, f' : V' \longrightarrow E^{(l)}$, q is a measurable function, and f' is sufficiently smooth. The Beltrami equation on the cylinder is as follows:

$$D_l f' = q\overline{D_l} f'.$$

It could be transformed to an integral equation

$$h = q(\overline{D_l}\phi + \Pi_l h)$$

by $f' = \phi + Th$ where ϕ is an arbitrary left-monogenic function such that $D_l\phi = 0$. By the Banach fixed point theorem, the previous integral equation has a unique solution in the case of

$$\|q\| \leq q_0 < \frac{1}{\|\Pi_l\|}$$

with q_0 being a constant, we can use the estimate of the L^p norm of Π_l with $p > 1$.

Suppose $V = \bigcup_{i=1}^{\infty} V_i = p_k^{-1}(V')$, such that $p_k(V_i) = V',\ i = 1, 2, \cdots$. $f : V_i \longrightarrow \mathcal{C}l_n$ is a piecewise continuous function with compact support, and f can be induced to $f' : V' \longrightarrow E$. For the Π-operator on $\mathbb{R}^{n+1}$, we have $\|\Pi\|_{L^p(\mathbb{R}^{n+1})} \leq (n+1)(p^* - 1)$, where $p^* = max(p, p/(p-1))$, see [42].

Recall that $\Pi_l = \overline{D_l}\cot_{q,k,0} *$, where “ $*$ ” is the standard convolution. On each subdomain V_i, we have $\|\overline{D_l}\cot_{q,k,0} * f'(x')\|_{L^p(V_i)} = \|\overline{D}G * f(x)\|_{L^p(V_i)} \leq (n+1)(p^* - 1)$.

Hence for the domain $V = \bigcup_{i=1}^{\infty} V_i$, we have

$\|\overline{D_l}\cot_{q,k,0} * f(x)\|_{L^p(V)} = \|\overline{D}G * f(x)\|_{L^p(V)} \leq (n+1)(p^*-1)$. Applying the projection p_k on V, we could obtain

Theorem 7.3.

$$\|\overline{D'}\cot'_{q,k,0} * f(x')\|_{L^p(V')} \leq (n+1)(p^*-1),$$

which shows $\|\Pi_l\|_{L^p(C_k)} \leq (n+1)(p^*-1)$, where $p^* = max(p, p/(p-1))$.

7.2 Π-Operator on Hopf Manifolds

7.2.1 Dirac operators on the Hopf Manifolds

A Hopf manifold is diffeomorphic to the conformally flat spin manifold $U/\Gamma = \mathbb{S}^1 \times \mathbb{S}^n$, where $U = \mathbb{R}^{n+1} \setminus \{0\}$ and $\Gamma = \{2^k : k \in Z\}$. There exists a projection $p_k : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{S}^1 \times \mathbb{S}^n$, such that $p_k(2^k x) = x'$.

Let $V \subseteq \mathbb{R}^{n+1}$ be open, and if $x \in V$, $2^k x \in V$. Hence $p_k(V) = V' \subseteq \mathbb{S}^1 \times \mathbb{S}^n$, which is also open. A left Clifford holomorphic functions $f : V \longrightarrow \mathcal{C}l_n$ which satisfying $f(x) = f(2^k x)$ could be lifted to a well defined function $f' : V' \longrightarrow \mathcal{C}l_n$ by the projective map p_k, where $f'(x') = f(x)$ for each $x' \in V'$ and x is one of $p_k^{-1}(x')$.

The spinor bundle E over $\mathbb{S}^1 \times \mathbb{S}^n$ is constructed by identifying (x, X) with $(2^k x, X)$ for $k \in Z$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$, $X \in \mathcal{C}l_n$. In [30], the Cauchy kernel for $\mathbb{S}^1 \times \mathbb{S}^n$ is given as follows. Let $C(x-y) = C_1(x-y) + 2^{2-2n}C_2(x-y)$, where

$$C_1(x-y) = \sum_{k=0}^{\infty} G(2^k x - 2^k y),$$
$$C_2(x-y) = G(x) \sum_{k=-1}^{-\infty} G(2^{-k}x^{-1} - 2^{-k}y^{-1})G(y),$$

and $G(x,y) = \dfrac{\overline{x-y}}{||x-y||^{n+1}}$ is the fundamental solution of the Euclidean Dirac operator. Applying the projective map we obtain the Cauchy kernel $C'(x', y')$ for the Dirac operator on $(\mathbb{S}^1 \times \mathbb{S}^n) \times (\mathbb{S}^1 \times \mathbb{S}^n) \setminus diagonal(\mathbb{S}^1 \times \mathbb{S}^n)$, which is denoted by D'. A function f' defined on $V' \subseteq \mathbb{S}^1 \times \mathbb{S}^n$ is left monogenic if $D'f' = 0$.

Using the kernel of the Dirac operators D', we can define the Cauchy transform on $S^1 \times \mathbb{S}^n$. If $f' : V' \longrightarrow E$, S' is a surface lying in V' and bounding a subdomain W'. Suppose $x' \in W'$,

$$\begin{aligned} T_{V'}f'(x') &= \frac{1}{\omega_n}\int_{V'} C(x'-y')f'(y')dy', \\ \overline{T_{V'}}f'(x') &= \frac{1}{\omega_n}\int_{V'} \overline{C(x'-y')}f'(y')dy'. \end{aligned}$$

Also, the non-singular boundary integral operator and its conjugate are given by

$$\begin{aligned} F_{S'}f'(x') &= \frac{1}{\omega_n}\int_{S'} C(x'-y')dp(n(y'))f'(y')d\sigma'(y'), \\ \overline{F_{S'}}f'(x') &= \frac{1}{\omega_n}\int_{S'} \overline{C(x'-y')}dp(n(y'))f'(y')d\sigma'(y'). \end{aligned}$$

And the Borel-Pompeiu formula is stated as follows.

Theorem 7.4. *([30]) For $f' \in C^1(V', \mathcal{C}l_n) \cap C(\overline{V'})$, we have*

$$f'(x') = \frac{1}{\omega_n}\Big(\int_{S'} C(x'-y')dp(n(y))f'(y')d\sigma'(y') + \int_{V'} C(x'-y')D_l f'(y')dy'\Big).$$

7.2.2 Construction and Applications of the Π-Operator on Hopf Manifolds

Definition 7.2. Define the Π-operator on the Hopf manifold as

$$\Pi' f' = \overline{D'}Tf'.$$

Since Π' is induced from the Π-operator in Euclidean space, we expect similar results as in ([25]).

Theorem 7.5. Π' *is an* L^2 *isometry operator.*

Proof. The proof is similar to Proposition 5 in [25]. $\square$

Let $V' \subseteq \mathbb{S}^1 \times \mathbb{S}^n$ be a bounded, simply connected domain with sufficiently smooth boundary, and $q, f' : V' \longrightarrow E$, q is a measurable function, and f' is sufficiently smooth. The Beltrami equation on the Hopf manifold is as follows:

$$D'f' = q\overline{D'}f'.$$

Substitute $f' = \phi + Th$ we have

$$D'(\phi + Th) = h = q\overline{D'}(\phi + Th) = q(\overline{D'}\phi + \overline{D'}Th) = q(\overline{D'}\phi + \Pi'h).$$

Therefore, the Beltrami equation has a unique solution $f' = \phi + Th$ where ϕ is an arbitrary left-monogenic function such that $D'\phi = 0$ and h is the solution of an integral equation

$$h = q(\overline{D'}\phi + \Pi'h).$$

By the Banach fixed point theorem, the previous integral equation has a unique solution in the case of

$$\|q\| \leq q_0 < \frac{1}{\|\Pi'\|}$$

with q_0 being a constant, we can use the estimate of the L^p norm of Π_l with $p > 1$. Similar argument could be found in Introduction.

Suppose $V = \bigcup_{i=1}^{\infty} V_i$ is the inverse image of V' under p_k, such that $p_k(V_i) = V'$. $f : V_i \longrightarrow \mathcal{C}l_n$ is a piecewise continuous function with compact support, and f could be induced to $f' : V' \longrightarrow E$. For the Π-operator on $\mathbb{R}^{n+1}$, we have $\|\Pi\|_{L^p(\mathbb{R}^{n+1})} \leq (n+1)(p^* - 1)$, where $p^* = max(p, p/(p-1))$, see [42].

On each subdomain V_i, we have $\|\overline{D'}C * f(x)\|_{L^p(V_i)} = \|\overline{D}G * f(x)\|_{L^p(V_i)}$, hence for the domain $V = \sum_{i=1}^{\infty} V_i$, we have $\|\overline{D'}C * f(x)\|_{L^p(V)} = \|\overline{D}G * f(x)\|_{L^p(V)} \leq (n+1)(p^* - 1)$. Applying the projection p_k on V, we could obtain $\|\overline{D'}C' * f(x')\|_{L^p(V')} \leq (n+1)(p^* - 1)$, which shows that

Theorem 7.6.

$$\|\Pi'\|_{L^p(\mathbb{S}^1 \times \mathbb{S}^n)} \leq (n+1)(p^* - 1), \ \textit{where } p^* = max(p, p/(p-1)).$$

8 A Π-Operator on the Hyperbolic Upper Half Space

In this section, we let X to be the upper half space $\mathbb{R}^{n+1}_+$ with the hyperbolic measure. Then Hilbert space $H = L^2(\mathbb{R}^{n+1}_+, \mathbb{R})$ becomes a real Hilbert space, and $H \otimes \mathcal{C}l_n$ is a Clifford-Hilbert module with the inner product

$$\langle f, g \rangle = \int_\Omega \overline{f} g \frac{dx^n}{x_n^{n-1}},$$

where Ω is a subset of the upper half space with $\overline{\Omega}$ inclosed and $f, g : \Omega \longrightarrow \mathcal{C}l_n$. Then the Π-operator theory on the hyperbolic upper half space is actually a special case of Section 4, which is demonstrated as follows.

8.1 Hyperbolic Dirac Operator

Denote the upper half space $\mathbb{R}^{n+1}_+ = \{x_0e_0 + x_1e_1 \cdots + x_ne_n : x_n > 0\}$. The Poincaré half-space is a Riemannian manifold $(\mathbb{R}^{n+1}_+, ds^2)$ with the Riemannian metric

$$ds^2 = \frac{(dx_0^2 + dx_1^2 + ... + dx_n^2)}{x_n^2}.$$

The Clifford algebra $\mathcal{C}l_n$ could be expressed as $\mathcal{C}l_n = \mathcal{C}l_{n-1} + \mathcal{C}l_{n-1}e_n$. So if $A \in \mathcal{C}l_n$, there exist unique elements B and $C \in \mathcal{C}l_{n-1}$, such that $A = B + Ce_n$. This gives rise to a pair of projection maps P and Q, where

$$P : \mathcal{C}l_n \longrightarrow \mathcal{C}l_{n-1}, P(A) = B,$$

$$Q : \mathcal{C}l_n \longrightarrow \mathcal{C}l_{n-1}, Q(A) = C.$$

We denote $-e_n Q(A) e_n$ by $Q'(A) \in \mathcal{C}l_{n-1}$. The modified Dirac operator is defined as

$$Mf = D_0 f + \frac{n-1}{x_n} Q' f,$$

where $D_0 = \sum_{i=0}^{n} e_i \partial_{x_i}$ is the Dirac operator on $\mathbb{R}^{n+1}$. Let $\Omega \subset \mathbb{R}^{n+1}_+$, we say a function $f : \Omega \longrightarrow \mathcal{C}l_n$ is hypermonogenic if $Mf(x) = 0$ for each $x \in \Omega$.

The conjugate of the modified Dirac operator is defined by

$$\overline{M} f = \overline{D_0} f - \frac{n-1}{x_n} Q' f,$$

where $\overline{D_0} = e_0 \partial_{x_0} - \sum_{i=i}^{n} e_i \partial_{x_i}$, see [38].

Theorem 8.1. *$M^* = -\overline{M}$.*

Proof. Let $f, g \in L^2(\mathbb{R}^{n+1}_+, \mathcal{C}l_n)$ with compact support. From the decomposition that $A = P(A) + Q(A) e_n$, we notice that $||f||_h^2 = ||Pf||_h^2 + ||Qf||_h^2$, where

$$||f||_h^2 = \int_\Omega \overline{f(x)} f(x) \frac{dx^n}{x_n^{n-1}}$$

defines the norm of f in the upper half space with hyperbolic metric. If we replace f that in the previous identity with $f + g$, one can easily see that $P(f)$ is orthogonal to $Q(g) e_n$.

More specifically,

$$\int_\Omega \overline{P(f)} \cdot (Q(g)e_n) \frac{dx^n}{x_n^{n-1}} = 0. \tag{3}$$

On one hand, since we have

$$\langle Mf, g\rangle = \langle \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} + \frac{n-1}{x_n} Q' f, g\rangle = \langle \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} - \frac{n-1}{x_n} e_n Q f e_n, g\rangle,$$

then

$$\begin{aligned}
& \langle \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i}, g\rangle = \int_\Omega \overline{\sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i}} \cdot g \frac{dx^n}{x_n^{n-1}} = \int_\Omega \overline{\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}} \cdot \overline{e_i} g \frac{dx^n}{x_n^{n-1}} \\
= \; & -\int_\Omega \overline{f} \cdot \sum_{i=0}^{n} \frac{\partial}{\partial x_i} (\overline{e_i} g) \frac{dx^n}{x_n^{n-1}} = -\int_\Omega \overline{f} (\sum_{i=0}^{n} \overline{e_i} \frac{\partial g}{\partial x_i} \frac{dx^n}{x_n^{n-1}}) - \int_\Omega \overline{f}\,\overline{e_n} g \frac{-(n-1)}{x_n^n} dx^n \\
= \; & \langle f, -\overline{D_0} g\rangle - (n-1) \int_\Omega \overline{f} \cdot e_n g \frac{dx^n}{x_n^n}.
\end{aligned}$$

On the other hand,

$$\langle -\frac{n-1}{x_n} e_n Q f e_n, g\rangle \;=\; -(n-1) \int_\Omega \overline{e_n Q f e_n} g \frac{dx^n}{x_n^n} = (n-1) \int_\Omega \overline{Q f e_n} \cdot e_n g \frac{dx^n}{x_n^n}.$$

Hence,

$$\begin{aligned}
& \langle Mf, g\rangle = \langle \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} - \frac{n-1}{x_n} e_n Q f e_n, g\rangle \\
= \; & \langle f, -\overline{D_0} g\rangle - (n-1) \int_\Omega \overline{f} \cdot e_n g \frac{dx^n}{x_n^n} + (n-1) \int_\Omega \overline{Q f e_n} \cdot e_n g \frac{dx^n}{x_n^n} \\
= \; & \langle f, -\overline{D_0} g\rangle - (n-1) \int_\Omega \overline{Pf} \cdot e_n g \frac{dx^n}{x_n^n} \\
= \; & \langle f, -\overline{D_0} g\rangle - (n-1) \int_\Omega \overline{Pf} \cdot e_n (Pg + Qg e_n) \frac{dx^n}{x_n^n}.
\end{aligned}$$

Since $e_n P g$ can be rewritten as $\pm P g e_n$, where " $\pm$ " depends on that n is even or odd. This can also be considered as Qhe_n for some function $h \in L^2(\mathbb{R}^{n+1}_+, \mathcal{C}l_n)$. Hence, from (3), we can see that Pf is orthogonal to e_nPg. Thus, the previous equation becomes

$$= \langle f, -\overline{D_0}g\rangle - (n-1)\int_\Omega \overline{Pf} \cdot e_nQge_n \frac{dx^n}{x_n^n}.$$

With a similar argument as above, the previous equation is equal to

$$\begin{aligned}
&= \langle f, -\overline{D_0}g\rangle - (n-1)\int_\Omega \overline{Pf + Qfe_n} \cdot e_nQge_n \frac{dx^n}{x_n^n} \\
&= \langle f, -\overline{D_0}g\rangle - (n-1)\int_\Omega \overline{f} \cdot e_nQge_n \frac{dx^n}{x_n^n} \\
&= \langle f, -\overline{D_0}g + \frac{n-1}{x_n}Q'g\rangle = \langle f, -\overline{M}g\rangle.
\end{aligned}$$

Therefore, $M^* = -\overline{D_0} + \frac{n-1}{x_n}Q' = -\overline{M}$. Similarly, $\overline{M}^* = -M$. □

By straight forward calculation, we can obtain

$$M\overline{M}f = \overline{M}Mf = \Delta f - \frac{n-1}{x_n}\frac{\partial}{\partial_{x_n}}f + (n-1)\frac{Qfe_n}{x_n^2},$$

where Δ is the Laplace operator in $\mathbb{R}^{n+1}$. In the hyperbolic function theory, we define hyperbolic harmonic function $f : \Omega \longrightarrow \mathcal{C}l_n$ as a solution of the equation

$$\overline{M}Mf(x) = 0$$

for $x \in \Omega$. Let

$$E(x,y) = \frac{(x-y)^{-1}}{\|x-y\|^{n-1}\|x-\widehat{y}\|^{n-1}}, \ F(x,y) = \frac{(\widehat{x}-y)^{-1}}{\|x-y\|^{n-1}\|\widehat{x}-y\|^{n-1}},$$

where $\widehat{x}=\sum_{i=0}^{n-1}x_ie_i-x_ne_n$. Hence the Cauchy transform is defined as ([20])

$$T_\Omega f(y)=-\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\int_\Omega\big(E(x,y)f(x)-F(x,y)\widehat{f(x)}\big)dx^n.$$

Also, the non-singular boundary integral operator is given by

$$F_{\partial\Omega}f(y)=\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\int_{\partial\Omega}\big(E(x,y)n(x)f(x)-F(x,y)\widehat{n}(x)\widehat{f}(x)\big)d\sigma(x).$$

Hence, we have the Borel-Pompeiu Theorem as follows.

Theorem 8.2. *[20] Let $\Omega\subseteq\mathbb{R}^{n+1}_+$ be a bounded region with smooth boundary in $\mathbb{R}^{n+1}_+$. Suppose $f:\Omega\longrightarrow\mathcal{C}l_n$ is a C^1 function on Ω with a continuous extension to the closure of Ω. Then for $y\in\Omega$, we have*

$$\begin{aligned}f(y)&=\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\int_{\partial\Omega}\big(E(x,y)n(x)f(x)-F(x,y)\widehat{n}(x)\widehat{f}(x)\big)d\sigma(x)\\&\quad-\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\int_\Omega\big(E(x,y)Mf(x)-F(x,y)\widehat{Mf(x)}\big)dx^n.\end{aligned}$$

When f is a hypermonogenic function,

$$f(y)=\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\int_{\partial\Omega}\big(E(x,y)n(x)f(x)-F(x,y)\widehat{n}(x)\widehat{f}(x)\big)d\sigma(x).$$

In particular, if $f\in\overset{\circ}{W}{}_2^1(\Omega,\mathcal{C}l_n)$, then

$$f(y)=-\frac{2^{n-1}y_n^{n-1}}{\omega_{n+1}}\int_\Omega\big(E(x,y)Mf(x)-F(x,y)\widehat{Mf(x)}\big)dx^n,$$

in other words, $TM=I$. If we apply the hyperbolic Dirac operator M on both sides of the equation, we can easily obtain $MT = I$.

8.2 Construction of the Hyperbolic Π-Operator

It is well known that in complex analysis, the Π-operator can be realized as the composition of $\partial_{\bar{z}}$ and the Cauchy transform. As the generalization to higher dimension in Clifford algebra, we have the Π-operator in $\mathbb{R}^{n+1}_{+}$ defined as follows.

Definition 8.1. The hyperbolic Π-operator in $\mathbb{R}^{n+1}_{+}$ is defined as

$$\Pi_h = \overline{M}T.$$

The following are some well known properties for the Π_h-operator.

Theorem 8.3. *Suppose* $f \in \overset{\circ}{W}{}^{k}_{p}(\Omega)$ $(1 < p < \infty, k \geq 1)$, *then*

1. $M\Pi_h f = \overline{M}f$,
2. $\Pi_h M f = \overline{M}f - \overline{M}F_{\partial\Omega}f$,
3. $F_{\partial\Omega}\Pi_h f = (\Pi_h - T\overline{M})f$,
4. $M\Pi_h f - \Pi_h M f = \overline{M}F_{\partial\Omega}f$.

The proof is a straight forward calculation.

The following decomposition of $L^2(\Omega, \mathcal{C}l_n)$ helps us to observe that the Π-operator actually maps $L^2(\Omega, \mathcal{C}l_n)$ to $L^2(\Omega, \mathcal{C}l_n)$.

Theorem 8.4. ***(Decomposition of*** $L^2(\Omega, \mathcal{C}l_n)$***)***

$$L^2(\Omega, \mathcal{C}l_n) = L^2(\Omega, \mathcal{C}l_n) \ \cap \ Ker\overline{M} \oplus M(\overset{\circ}{W}{}^{1}_{2}(\Omega, \mathcal{C}l_n)),$$

and

$$L^2(\Omega, \mathcal{C}l_n) = L^2(\Omega, \mathcal{C}l_n) \ \cap \ KerM \oplus \overline{M}(\overset{\circ}{W}{}^{1}_{2}(\Omega, \mathcal{C}l_n)).$$

The proof is similar to Theorem 1 in [25]. Notice that

$$\Pi_h(L^2(\Omega,\mathcal{C}l_n)\cap Ker\overline{M}) = L^2(\Omega,\mathcal{C}l_n)\cap KerM,$$
$$\Pi_h(M(\overset{\circ}{W}{}_2^1(\Omega,\mathcal{C}l_n)) = \overline{M}(\overset{\circ}{W}{}_2^1(\Omega,\mathcal{C}l_n)).$$

Hence, Π_h maps $L^2(\Omega,\mathcal{C}l_n)$ to $L^2(\Omega,\mathcal{C}l_n)$.

One key property of the Π-operator is that it is an L^2 isometry, in other words,

Theorem 8.5. *For functions in $L^2(\Omega,\mathcal{C}l_n)$, we have*

$$\Pi^*\Pi = I.$$

Proof. Let $f\in L^2(\Omega,\mathcal{C}l_n)$ with compact support,

$$\begin{aligned}\langle \Pi_h f,\Pi_h f\rangle &= \langle \overline{M}Tf,\overline{M}Tf\rangle = -\langle Tf, M\overline{M}Tf\rangle = -\langle Tf,\overline{M}MTf\rangle \\ &= \langle MTf, MTf\rangle = \langle f,f\rangle.\end{aligned}$$

Here we use $\overline{M}^* = -M$. ☐

To complete this section, we give the example of the Π_h-operator solving the hyperbolic Beltrami equation. Let $\Omega\subseteq\mathbb{R}^{n+1}_+$, $q:\Omega\to\mathcal{C}l_n$ a bounded measurable function and $\omega:\Omega\to\mathcal{C}l_n$ be a sufficiently smooth function. The generalized Beltrami equation

$$M\omega = q\overline{M}\omega$$

could be transformed into an integral equation

$$h = q(\overline{M}\phi + \Pi_h h)$$

by substitute $\omega = Th + \phi$, where ϕ is an arbitrary hypermonogenic function as follows.

$$M(Th + \phi) = h = q\overline{M}(Th + \phi) = q(\overline{M}\phi + \overline{M}Th) = q(\overline{M}\phi + \Pi_h h).$$

By the Banach fixed point theory, this equation could have a unique solution if $\|q\| \leq q_0 < \dfrac{1}{\|\Pi_h\|}$, with q_0 being a constant. With such unique fuction h, $\omega = Th + \phi$ is the unique solution of the Hyperbolic Beltrami equation.

9 Π-Operator in Higher Spin Spaces

All our previous work is on classical Clifford analysis, which is centered around the study of functions on $\mathbb{R}^n$ taking values in Clifford numbers. Several authors have been studying generalizations of classical Clifford analysis techniques to the so-called *higher spin spaces*. This concerns the study of higher spin operators acting on functions on $\mathbb{R}^n$, taking values in arbitrary irreducible representations of $Spin(n)$. In Clifford analysis, these arbitrary irreducible representations are traditionally defined in terms of polynomial spaces satisfying certain differential equations. More specifically, the choices for the higher spin spaces are the following.

- $\mathcal{H}_k$: k-homogeneous harmonic polynomial space,
- $\mathcal{M}_k$: k-homogeneous monogenic polynomial space.

The generalization of the Euclidean Dirac operator in higher spin space is called *Rarita-Schwinger operator*. This was first studied systematically by Bures et al. ([6]) as the first order conformally invariant differential operator in 2002. It has the following analytic construction. Recall the Almansi-Fischer decomposition

$$\mathcal{H}_k = \mathcal{M}_k \oplus u\mathcal{M}_{k-1}.$$

We define P_k as the projection map

$$P_k : \mathcal{H}_k \longrightarrow \mathcal{M}_k.$$

Suppose U is a domain in $\mathbb{R}^n = span\{e_1, \cdots, e_n\}$. Consider $f : U \times \mathbb{R}^n \longrightarrow \mathcal{C}l_n$, such that for each $x \in U$, $f(x,u)$ is a left monogenic polynomial homogeneous of degree k in u. The Rarita-Schwinger operator is defined as follows

$$R_k := P_k D_x f(x,u) = (\frac{uD_u}{n+2k-2} + 1) D_x f(x,u),$$

where $D_x = \sum_{i=1}^n e_i \partial_{x_i}$ is the Dirac operator in variable x. We also have a right projection $P_{k,r}: \ \mathcal{H}_k \longrightarrow \overline{\mathcal{M}}_k$, and a right Rarita-Schwinger operator $R_{k,r} = D_x P_{k,r}$. See [6, 17].

Let $Z_k(u,v)$ be the reproducing kernel for $\mathcal{M}_k$ in the sense that

$$f(v) = \int_{\mathbb{S}^{n-1}} \overline{Z_k(u,v)} f(u) dS(u), \text{ for all } f(u) \in \mathcal{M}_k.$$

Then the fundamental solution for R_k is given by

$$E_k(x,u,v) = \frac{1}{\omega_n c_k} \frac{x}{||x||^n} Z_k(\frac{xux}{||x||^2}, v),$$

where $c_k = \frac{n-2}{n+2k-2}$ and ω_n is the surface area of the unit sphere $\mathbb{S}^{n-1}$, see [6, 17].

As the generalization of the Dirac operator, Rarita-Schwinger operator also has Stokes' theorem as follows.

Theorem 9.1 ([17]). ***(Stokes' theorem for** R_k)*

Let Ω' and Ω be domains in $\mathbb{R}^n$ and suppose the closure of Ω lies in Ω'. Further suppose *the closure of* Ω *is compact and* $\partial\Omega$ *is piecewise smooth. Let* $f, g \in C^1(\Omega', \mathcal{M}_k)$. *Then*

$$\begin{aligned}&\int_{\Omega}\left[(\overline{g(x,u)}R_k, f(x,u))_u + (g(x,u), R_k\overline{f(x,u)})_u\right]dx^n\\ =&\int_{\partial\Omega}(\overline{g(x,u)}, d\sigma_x f(x,u))_u,\end{aligned}$$

where $d\sigma_x = n(x)d\sigma(x)$, $d\sigma(x)$ *is the area element.* $(P(u), Q(u))_u = \int_{\mathbb{S}^{n-1}} P(u)Q(u)dS(u)$ *is the inner product for any pair of* Cl_n*-valued polynomials.*

9.1 Construction of the Higher Spin Π-Operator

The idea to construct a Π-operator in higher spin spaces is similar as in Section 4.5. Before we give the definition of our Π-operator in higher spin spaces, we need some preliminary work and technical lemmas.

Assume Ω is a domain in $\mathbb{R}^n$, with the fundamental solution for R_k, we can define an integral operator as follows.

$$Tf(y,v) = \int_{\Omega}(E_k(x-y,u,v)f(x,u))_u dx^n = \int_{\Omega}\int_{\mathbb{S}^{n-1}} E_k(x-y,u,v)f(x,u)dS(u)dx^n,$$

This integral operator has been shown as the inverse of R_k ($R_kT = TR_k = Id$) for $f(x,u) \in C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ with compact support in the variable x, see [17].

Let $f(x,u),\ g(x,u) \in C^\infty(\Omega, \mathcal{M}_k)$ with compact support in the variable x. The inner product is given by

$$\langle f, g\rangle = \int_{\Omega}(\overline{f(x,u)}, g(x,u))_u dx^n.$$

Then we claim that

Lemma 9.2.

$$R_k^* = -\overline{R_k} = -\overline{P_k D_x} = -\overline{D_x}\big(1 + \frac{\overline{uD_u}}{m+2k-2}\big).$$

Proof. Let $f(x,u),\ g(x,u) \in C^\infty(\Omega, \mathcal{M}_k)$ with compact support in the variable x, where $\Omega \subset \mathbb{R}^n$ is a domain. Then, from the Stokes' Theorem for R_k, we can see that

$$\int_\Omega (\overline{g(x,u)}R_k, f(x,u))_u dx^n = -\int_\Omega (\overline{g(x,u)}, R_k f(x,u))_u dx^n,$$

since the integral over the boundary vanishes because of the compact support of f and g in variable x. Next, we consider

$$\begin{aligned} & \langle f(x,u), R_k g(x,u)\rangle = \int_\Omega (\overline{f(x,u)}, R_k g(x,u))_u dx^n \\ = \quad & -\int_\Omega (\overline{f(x,u)}R_k, g(x,u))_u dx^n = -\int_\Omega (\overline{\overline{R_k} f(x,u)}, g(x,u))_u dx^n \\ = \quad & \langle -\overline{R_k} f, g\rangle. \end{aligned}$$

This completes the proof. $\square$

To prove our Π-operator is isometric for $f(x,u) \in L^2(\mathbb{R}^n, \mathcal{M}_k)$ with compact support in the variable x, we need the following theorems and technique lemmas.

Theorem 9.3. *[17]*

$$\iint_{\mathbb{R}^n} -(E_k(x-y,u,v), R_k f(x,u))_u dx^n = f(y,v)$$

for each $f \in C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ with compact support in varaible x.

Notice that $R_k f(x,u) \in C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ if $f \in C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ with compact support in the variable x. Hence, the theorem above tells us that $Tf \in C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ as well.

Theorem 9.4. *([17]) $R_kTf = f$ for $f \in C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ with compact support in the variable x. i.e.*

$$R_k \iint_{\mathbb{R}^n} (E_k(x-y,u,v), f(x,u))_u dx^n = f(y,v).$$

From the proof of the previous theorem in [17], we observed the following fact, which is critical in our argument for Theorem 9.6 below.

Lemma 9.5. *$D_yTf = f$ for $f \in C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ with compact support in the variable x. i.e.*

$$D_y \iint_{\mathbb{R}^n} (E_k(x-y,u,v), f(x,u))_u dx^n = f(y,v).$$

Now, we are ready to give our L^2 isometric Π-operator as follows.

Definition 9.1. The Π-*operator* in higher spin spaces is defined by $\Pi = \sqrt{\frac{n+2k-2}{3n+4k-6}}\overline{R_k}T$.

Note the constant in our definition allows the Π-operator to be isometric when acting on each $f(x,u) \in L^2(\mathbb{R}^n, \mathcal{M}_k)$ with compact support in the variable x.

Theorem 9.6. *Π is isometric for $f(x,u) \in L^2(\mathbb{R}^n, \mathcal{M}_k)$ with compact support in the variable x.*

Proof. Let $f(x,u), g(x,u) \in C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ with compact support in the variable x, then we have

$$\frac{3n+4k-6}{n+2k-2}\langle \Pi f, \Pi g\rangle = \langle \overline{R_k}Tf, \overline{R_k}Tg\rangle = -\langle Tf, R_k\overline{R_k}Tg\rangle.$$

The last identity comes from Lemma 9.2. Since $\overline{R_k} = \overline{P_kD_y} = \overline{D}_y\overline{P_k}$, the previous equation is equal to

$$-\int_{\mathbb{R}^n} (\overline{Tf}, P_kD_y\overline{D}_y\overline{P_k}Tg)_v dy^n = -\int_{\mathbb{R}^n} (\overline{Tf}, P_k\Delta_y\overline{P_k}Tg)_v dy^n = -\int_{\mathbb{R}^n} (\overline{Tf}, \Delta_yP_k\overline{P_k}Tg)_v dy^n. \quad (4)$$

Next, we will show the relation between $\overline{P_k}$ and P_k. Recall that

$$vD_v = -\mathbb{E}_v + \sum_{1\leq i<j\leq n} \Gamma_{i,j} e_i e_j$$

where $\mathbb{E}_v = \sum_{i=1}^n v_i \partial_{v_i}$ is the Euler operator and $\Gamma_{i,j} = v_i \partial_{v_j} - v_j \partial_{v_i}$ is the angular momentum operator. Hence,

$$\overline{vD_v} = -\mathbb{E}_v - \sum_{1\leq i<j\leq n} \Gamma_{i,j} e_i e_j = -vD_v - 2\mathbb{E}_v.$$

Thus,

$$\overline{P_k} = 1 + \frac{\overline{vD_v}}{m+2k-2} = 1 + \frac{-vD_v - 2\mathbb{E}_v}{m+2k-2} = -P_k + 2 - \frac{2\mathbb{E}_v}{m+2k-2}.$$

Now we consider the term $P_k \overline{P_k} Tg$ in (4). Since $g(x,u) \in C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ has compact support in the variable x, Theorem 9.3 tells us that $Tg \in C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ and $P_k Tg = Tg$. On the other hand, from the argument in page 5 in [17], we know that

$$Tg(y,v) = \int_{\mathbb{R}^n} (E_k(x-y,u,v), g(x,u))_u dx^n$$

has degree $2-k-m$ in the variable v. Hence

$$\begin{aligned} P_k \overline{P_k} Tg(y,v) &= P_k(-P_k + 2 - \frac{2\mathbb{E}_v}{n+2k-2}) Tg(y,v) \\ &= (1 - \frac{2\mathbb{E}_v}{n+2k-2}) Tg(y,v) = \frac{3n+4k-6}{n+2k-2} Tg(y,v). \end{aligned}$$

Therefore, equation (4) is equal to

$$
\begin{aligned}
& -\frac{3n+4k-6}{n+2k-2}\int_{\mathbb{R}^n}(\overline{Tf},\Delta_y Tg)_v dy^n = -\frac{3n+4k-6}{n+2k-2}\int_{\mathbb{R}^n}(\overline{Tf},\overline{D}_y D_y Tg)_v dy^n \\
= & \frac{3n+4k-6}{n+2k-2}\int_{\mathbb{R}^n}(\overline{Tf}\ \overline{D}_y, D_y Tg)_v dy^n = \frac{3n+4k-6}{n+2k-2}\langle D_y Tf, D_y Tg\rangle \\
= & \frac{3n+4k-6}{n+2k-2}\langle f, g\rangle.
\end{aligned}
$$

The last equation comes from Lemma 9.5. Since $C^\infty(\mathbb{R}^n, \mathcal{M}_k)$ with compact support is dense in $L^2(\mathbb{R}^n, \mathcal{M}_k)$. This completes the proof. $\square$

To conclude this section, we give the example of the Π-operator solving the higher spin Beltrami equation. Let $\Omega \subseteq \mathbb{R}^n$, $q : \Omega \to \mathcal{M}_k$ a bounded measurable function and $\omega : \Omega \to \mathcal{M}_k$ be a sufficiently smooth function. The generalized Beltrami equation

$$R_k\omega = q\overline{R_k}\omega$$

could be transformed into an integral equation by substitute $\omega = Th + \phi$. The integral equation is

$$h = q(\overline{R_k}\phi + \Pi_h h),$$

where ϕ is an arbitrary function such that $R_k\phi = 0$. By the Banach fixed point theory, this equation could have a unique solution if $\|q\| \leq q_0 < \frac{1}{\|\Pi\|}$, with q_0 being a constant. Hence, $\omega = \phi + Th$ is the solution to the Beltrami equation.

9.2 Ahlfors-Beurling Type Inequality

In 1950, Ahlfors and Buerling gave an inequality in [1], which states that

$$\left|\frac{1}{2\pi}\int_X \frac{d\lambda}{\zeta - a}\right| \leq \left(\frac{1}{4\pi}\cdot\lambda(X)\right)^{\frac{1}{2}},$$

where X is a compact subset of the complex plane $\mathbb{C}$ and λ is the two-dimensional Lebesgue measure. This inequality provides an important tool to study rational approximation [7, 22]. Putinar proved a generalization of the Ahlfors-Beurling Inequality in [37], which states that

$$|\frac{1}{2\pi}\int_{\mathbb{C}}\frac{\varphi(\zeta)}{\zeta - a}d\lambda| \leq \frac{1}{2\sqrt{\pi}}\|\varphi\|_1^{1/2}\|\varphi\|_\infty^{1/2},$$

where φ is a nonnegative function and $\varphi \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C})$. This inequality gives an estimate of Cauchy transformation in the complex plane. In 1998, Martin extended Putinar's result to higher dimensional spaces. His result gives a uniform estimates of higher-dimensional Cauchy transforms, which states as follows:

$$|G * f(x)| \leq \alpha_n \|f\|_1^{1/(n+1)}\|f\|_\infty^{n/(n+1)},$$

where $f \in L^1(\mathbb{R}^n, \mathbb{R}) \cap L^\infty(\mathbb{R}^n, \mathbb{R})$, $x \in \mathbb{R}^{n+1}$, and α_n is a constant depending on the subset $\mathbb{B}^{n+1} = \{x \in \mathbb{R}^n, |x| \leq 1\}$ and $\mathbb{K}^{n+1} = \{x \in \mathbb{R}^n, |x|^{n+1} \leq x_0\}$. For more details see [33].

Our goal is to generalize the Ahlfors-Beurling inequality to higher spin spaces, to give a uniform estimate of the convolution type operator, which is a generalization of the Cauchy transformation in higher spin spaces.

Theorem 9.7. *Let $f \in L^1(\mathbb{R}^n, \mathcal{M}_k) \cap L^\infty(\mathbb{R}^n, \mathcal{M}_k)$, then for each fixed $y \in \mathbb{R}^n$, we have $\|E_k * f(y, v)\|_v$ is bounded, where $\|f(x, v)\|_v = (\int_{\mathbb{S}^n} \overline{f(x, v)} f(x, v) ds(v))^{\frac{1}{2}}$, and E_k is the fundamental solution of the Rarita-Schwinger operator R_k.*

Proof. Since

$$
\begin{aligned}
&\|E_k * f(y,v)\|_{L^1} = \int_{\mathbb{R}^n} \|E_k * f\|_v dy^n \\
&\|E_k * f\|_v = \| \int_{\mathbb{R}^n}\int_{\mathbb{S}^n} \frac{x-y}{\|x-y\|^n} Z_k(\frac{(x-y)u(x-y)}{\|x-y\|^n}, v) f(x,u) ds(u) dx^n \|_v \\
\leq\ & \int_{\mathbb{R}^n}\int_{\mathbb{S}^n} \|\frac{x-y}{\|x-y\|^n} Z_k(\frac{(x-y)u(x-y)}{\|x-y\|^n}, v) f(x,u)\|_v ds(u) dx^n \\
=\ & \int_{\mathbb{R}^n}\int_{\mathbb{S}^n} \sqrt{\frac{(x-y)\overline{(x-y)}}{\|x-y\|^{2n}}} (f(x,u)\overline{f(x,u)})^{\frac{1}{2}} \int_{\mathbb{S}^n} (Z_k(u,v)\overline{Z_k(u,v)} ds(v))^{\frac{1}{2}} ds(u) dx^n \\
=\ & \int_{\mathbb{R}^n}\int_{\mathbb{S}^n} \sqrt{\frac{(x-y)\overline{(x-y)}}{\|x-y\|^{2n}}} (f(x,u)\overline{f(x,u)})^{\frac{1}{2}} \cdot ||Z_k|| ds(u) dx^n.
\end{aligned}
$$

Notice that $||Z_k|| \leq \dim\mathcal{M}_k(\mathbb{R}^{n+1})$, similar proof can be in Proposition 5.27 in [2]. Let $C = \dim\mathcal{M}_k(\mathbb{R}^{n+1})$, then $\int_{\mathbb{S}^n} Z_k(u,v)\overline{Z_k(u,v)} ds(v) \leq C \cdot \mathrm{Area}(\mathbb{S}^n)$. Then

$$
\begin{aligned}
& C\frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{1}{\|x-y\|^{n-1}} (\int_{\mathbb{S}^n} (f(x,u)\overline{f(x,u)})^{\frac{1}{2}} ds(u)) dx^n \\
\leq\ & C\frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{1}{\|x-y\|^{n-1}} (\int_{\mathbb{S}^n} f(x,u)\overline{f(x,u)} ds(u))^{\frac{1}{2}} (\int_{\mathbb{S}^n} 1 ds(u))^{\frac{1}{2}} dx^n \\
\leq\ & C_1 \int_{\mathbb{R}^n} \frac{1}{\|x-y\|^{n-1}} \|f(x,u)\|_u dx^n.
\end{aligned}
$$

Notice that $\int_{\mathbb{R}^n} \frac{1}{\|x-y\|^{n-1}} ||f(x,u)||_u dx^n = \int_{\mathbb{R}^n} \frac{1}{\|x\|^{n-1}} \|f(x-y,u)\|_u dx^n$, then let $K_\delta = \{x \in \mathbb{R}^n, \|x\| \leq \delta\}$. We have

$$
\begin{aligned}
& \int_{\mathbb{R}^n} \frac{1}{\|x\|^{n-1}} \|f(x-y,u)\|_u dx^n \\
=\ & \int_{K_\delta} (\frac{1}{\|x\|^{n-1}} - \frac{1}{\delta^{n-1}}) \|f(x-y,u)\|_u dx^n + \int_{K_\delta} \frac{1}{\delta^{n-1}} ||f(x,u)||_u dx^n \\
& + \int_{K_\delta^c} \frac{1}{\|x\|^{n-1}} \|f(x-y,u)\|_u dx^n.
\end{aligned}
$$

Let $||f(x,u)||_u = \psi(x)$, we can observe that

$$\int_{K_\delta} (\frac{1}{\|x\|^{n-1}} - \frac{1}{\delta^{n-1}})\|f(x-y,u)\|_u dx^n \leq \|\psi\|_\infty \cdot \int_{K_\delta} (\frac{1}{\|x\|^{n-1}} - \frac{1}{\delta^{n-1}}) dx^n. \qquad (\star)$$

Let $x = (\rho cos\theta)e_0 + (\rho sin\theta)\omega$, where $\omega \in \mathbb{S}^n, \rho \geq 0, 0 \leq \theta \leq \pi$, the $(\star)$ is equal to

$$\begin{aligned} & \|\psi\|_\infty \int_{\mathbb{S}^{n-1}} \int_0^\pi \int_0^\delta (\frac{1}{\rho^{n-1}} - \frac{1}{\delta^{n-1}})\rho^{n-1}(sin\theta)^{n-2} d\rho d\theta d\sigma(\omega) \\ = & \|\psi\|_\infty \frac{1}{\omega_{n-1}} \int_0^\pi (sin\theta)^{n-2} d\theta \int_0^\delta (1 - \frac{\rho^{n-1}}{\delta^{n-1}}) d\rho \\ = & \|\psi\|_\infty \frac{1}{\omega_{n-1}} \int_0^\pi (sin\theta)^{n-2} d\theta \cdot \frac{n-1}{n}\delta. \end{aligned}$$

Since

$$V(K_\delta) = \int_{\mathbb{S}^{n-1}} \int_0^\pi \int_0^\delta \rho^{n-1}(sin\theta)^{n-2} d\rho d\theta d\sigma(\omega) = \frac{1}{\omega_{n-1}} \int_0^\pi (sin\theta)^{n-2} d\theta \cdot \frac{1}{n}\delta^n,$$

the previous equation becomes

$$\int_{K_\delta} \frac{1}{\|x\|^{n-1}} \|f(x-y,u)\|_u dx^n \leq ||\psi||_\infty (n-1)\delta^{1-n} V(K_\delta),$$

and

$$\begin{aligned} & \frac{1}{\delta^{n-1}} \int_{K_\delta} ||f(x-y,u)||_u dx^n + \int_{K_\delta^c} \frac{1}{\|x\|^{n-1}} \|f(x-y,u)\|_u dx^n \\ \leq & \frac{1}{\delta^{n-1}} (\int_{K_\delta} ||f(x-y,u)||_u dx^n + \int_{K_\delta^c} \|f(x-y,u)\|_u dx^n) \\ = & \frac{1}{\delta^{n-1}} \int_{\mathbb{R}^n} \psi dx^n = \frac{1}{\delta^{n-1}} \|\psi\|_{L^1}. \end{aligned}$$

Combining both equalities we obtain the following:

$$\int_{\mathbb{R}^n} \frac{1}{\|x\|^{n-1}} \|f(x-y,u)\|_u dx^n \leq \delta^{1-n}(||\psi||_\infty (n-1)V(K_\delta) + ||\psi||_{L^1}).$$

Notice the volume of subset $V(K_\delta) = \delta^n V(\mathbb{B})$, where $\mathbb{B}$ is the unit ball in $\mathbb{R}^{n+1}$. Taking $\delta = (\frac{\|\psi\|_1}{V(\mathbb{B})\|\psi\|_\infty})^{\frac{1}{n}}$, we finally have

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{1}{\|x\|^{n-1}} \|f(x-y,u)\|_u dx^n \\ \leq \quad & nV(\mathbb{B})^{\frac{n-1}{n}} ||\psi||_1^{\frac{1}{n}} ||\psi||_\infty^{\frac{n-1}{n}} = nV(\mathbb{B})^{\frac{n-1}{n}} \|f\|_1^{\frac{1}{n}} \|f\|_\infty^{\frac{n-1}{n}}. \end{aligned}$$

Therefore, $\|E_k * f(y,v)\|_{L^1}$ is bounded by $\|f\|_1$ and $\|f\|_\infty$. □

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