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Closed Range Composition Operators on BMOA

Kevser Erdem
University of Arkansas, Fayetteville

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Closed range composition operators on BMOA

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Mathematics

by

Kevser Erdem
Nigde University
Bachelor of Science in Mathematics, 2006
University of Arkansas
Master of Science in Mathematics, 2015

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University of Arkansas

This dissertation is approved for recommendation to the Graduate Council.

_____
Maria Tjani, Ph.D.
Dissertation Director

_____   _____
Phil Harrington, Ph.D.   Daniel Luecking, Ph.D.
Committee Member   Committee Member

Abstract

Let φ be an analytic self-map of the unit disk $\mathbb{D}$. The composition operator with symbol φ is denoted by C_φ. Reverse Carleson type conditions, counting functions and sampling sets are important tools to give a complete characterization of closed range composition operators on $BMOA$ and on Q_p for all $p \in (0, \infty)$.

Let $\mathcal{B}$ denote the Bloch space, let H^2 denote the Hardy space. We show that if C_φ is closed range on $\mathcal{B}$ or on H^2 then it is also closed range on $BMOA$. Closed range composition operators $C_\varphi : \mathcal{B} \to BMOA$ are also characterized. Laitila found the isometries among composition operators on $BMOA$. We extend this to Q_p for all $p \in (0, \infty)$.

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Dedication

I dedicate this dissertation to my beloved daughter Afra and husband Burhan. They have a special place in my completion of this degree. A huge thank you for your support behind the scenes. You are nothing but a blessing. Thank you darlings, for always being with me and believing in success; without your positive energy and humongous smiles I would not be standing where I am now.

Table of Contents

1 Introduction

The *Hardy space H^2* is the collection of analytic functions on the open unit disk $\mathbb{D}$ whose power series have square summable coefficients, that is if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

then

$$||f||_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

The above defines a norm in H^2. Let φ be an analytic self-map of $\mathbb{D}$. *The composition operator,* denoted by C_φ, is the operator that maps f to $f \circ \varphi$, that is

$$C_\varphi f = f \circ \varphi$$

for f analytic on $\mathbb{D}$. Studies of composition operators start with the Hardy space setting. Boundedness, compactness as well as closed rangeness have been studied in this setting. Then, a very natural question is to study these properties on other function spaces.

Closed range composition operators were first studied by Cima, Thomson and Wogen in [9], in the context of H^2. Their results are in terms of the boundary behavior of the symbol φ. They posed the question of studying closed range composition operators in terms of properties of φ on $\mathbb{D}$ rather than on its boundary. The question was answered by Zorboska in [38] who studied the problem in H^2 and also in weighted Bergman spaces. In [17] Jovovic and MacCluer studied the problem in weighted Dirichlet spaces. Closed range composition operators in the Bloch space were investigated by Ghatage, Yan and Zheng in [14], by Ghatage, Zheng and Zorboska in [15], by Chen and Gauthier in [8] and by Akeroyd, Ghatage and Tjani in [3]. Akeroyd, Ghatage and Tjani revisited the problem in the context of the Bergman space, see [2] and [3]. The journey on closed range composition

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operators is continued on Besov type spaces $\mathcal{B}_{p,\alpha}, p > 1, \alpha > 1$ by Tjani, see [34].

Let X, Y be Banach spaces, let $T : X \to Y$ be a linear operator. The norms in X, Y are denoted by $||.||_X, ||.||_Y$ respectively. We say that a bounded operator T is *bounded below* if there exists a $\delta > 0$ such that for all $x \in X$

$$||Tx||_Y \geq \delta ||x||_X . \tag{1}$$

Moreover, if TX is a closed subspace of Y, then we say that T is *closed range.*

The Closed Graph Theorem, see [37, Theorem 1.5], says that T is a bounded operator if and only if its graph is closed in $X \times Y$. If T is one-to-one and is closed range then, by the Closed Graph Theorem and since $T^{-1} : T(X) \to X$ is a bounded operator, and we can easily see that T is bounded below. Moreover, if T is bounded below and (x_n) is a sequence in X such that $Tx_n \to y$ for some $y \in Y$, then by (1) we can easily see that (x_n) is a Cauchy sequence, hence it converges to some $x \in X$. Since T is a bounded operator $Tx_n \to Tx$ and so $Tx = y$. We conclude that T is closed range. Therefore if $T : X \to Y$ is a bounded linear operator that is one-to-one then T is closed range if and only if T is bounded below.

Here and thereafter, if A and B are two quantities that depend on an analytic function f on $\mathbb{D}$, we will use the notation $A \asymp B$ and say that A is equivalent to B; that is, there exist positive constants c_1, c_2 such that

$$c_1 A \leq B \leq c_2 A.$$

This thesis aims to contribute by giving a complete characterization of closed range composition operators on $BMOA$, the space of analytic functions on $\mathbb{D}$ with bounded mean oscillation and also on the spaces $Q_p, 0 < p < \infty$.

In Chapter 2, we give definitions of $BMOA$, $VMOA$, the Bloch Space, the Dirichlet space and the Bergman space. We also give the properties and connections of these spaces that we will use in later chapters.

In Chapter 3, we characterize closed range composition operators, C_φ, on $BMOA$. It is well known that composition operators in $BMOA$ are bounded for any symbol φ that is an analytic self-map of $\mathbb{D}$. A crucial tool to this work is a non univalent change of variables formula involving the counting function for $BMOA$ that is similar to the well known change of variables formula involving the Nevanlinna counting function. Let $q \in \mathbb{D}$, $\alpha_q = \frac{q-z}{1-\bar{q}z}$, and $\zeta \in \mathbb{D}$. If $\zeta \in \varphi(\mathbb{D})$ the $BMOA$ counting function is

$$N_{q,\varphi}(\zeta) = \sum_{\varphi(z)=\zeta} \left(1 - |\alpha_q(z)|^2\right),$$

it is understood that if ζ is not in the range of φ, then $N_{q,\varphi}(\zeta) = 0$.

Given $\varepsilon > 0$ and $q \in \mathbb{D}$ let $G_{\varepsilon,q} = \{\zeta : N_{q,\varphi}(\zeta) > \varepsilon (1 - |\alpha_q(\zeta)|^2)\}$ and let $G_\varepsilon = \cup_{q\in\mathbb{D}} G_{\varepsilon,q}$. We define the notion of sampling in $BMOA$ and prove the following.

Theorem 3.6 *Let φ be a non-constant analytic self map of $\mathbb{D}$. If C_φ is closed range on $BMOA$ then there exists $\varepsilon > 0$ such that G_ε is a sampling set for $BMOA$. Moreover, if $\cap_{q\in\mathbb{D}} G_{\varepsilon,q}$ is a sampling set for $BMOA$, then C_φ is closed range on $BMOA$.*

The above result provides necessary and in general different sufficient conditions for C_φ to have closed range on $BMOA$. Next we look for a single condition that is both necessary and sufficient.

Given $\varepsilon > 0$ and $q, q' \in \mathbb{D}$ let $G_{\varepsilon,q',q} = \{\zeta : N_{q',\varphi}(\zeta) > \varepsilon (1 - |\alpha_q(\zeta)|^2)\}$.

Theorem 3.8 *Let φ be an analytic self-map of $\mathbb{D}$. Then, there exists $k > 0$ such that for all $q \in \mathbb{D}$, $\|\alpha_q \circ \varphi\|_* \geq k$ if and only if there exists an $\varepsilon > 0$ and $r \in (0,1)$ such that for all $q \in \mathbb{D}$, there exists $q' \in \mathbb{D}$ such that*

$$\frac{|G_{\varepsilon,q',q} \cap D(q,r)|}{|D(q,r)|} \asymp 1.$$

Moreover, $G_\varepsilon = \cup_{q\in\mathbb{D}} G_{\varepsilon,q',q}$ satisfies the reverse Carleson condition.

By using reverse Carleson type sets we prove the following.

Theorem 3.11 *Let φ be an analytic self-map of $\mathbb{D}$. Then the composition operator*

$C_\varphi : \mathcal{B} \to BMOA$ *is closed range if and only if there exists an* $\varepsilon > 0$ *and* $r \in (0,1)$ *such that for all* $q \in \mathbb{D}$, *there exists* $q' \in \mathbb{D}$ *such that*

$$\frac{|G_{\varepsilon,q',q} \cap D(q,r)|}{|D(q,r)|} \asymp 1 \,.$$

In Theorem 3.14 we show that a set satisfies the reverse Carleson condition if and only if it is a sampling set on $BMOA$.

Next, we use another method and provide a full characterization of closed range composition operators on $BMOA$. We note that by Theorem 3.8 and Theorem 3.11, C_φ is closed range on $BMOA$ if and only if it is closed range as an operator from $\mathcal{B}$ to $BMOA$.

Theorem 3.12 *The composition operator* C_φ *is closed range on* $BMOA$ *if and only if there exists a constant* $k > 0$ *such that* $||\alpha_q \circ \varphi||_* \geq k$, *for all* $q \in \mathbb{D}$.

Since a composition operator C_φ is bounded on $VMOA$ if and only if $\varphi \in VMOA$, we have the following.

Corollary 3.13 *Let* $\varphi \in VMOA$. *Then* C_φ *is closed range on* $VMOA$ *if and only if there exists a constant* $k > 0$ *such that* $||\alpha_q \circ \varphi||_* \geq k$, *for all* $q \in \mathbb{D}$.

We also show that if C_φ is closed range on the Bloch space or the Hardy space then it is closed range on $BMOA$, see Corollary 3.16 and Corollary 3.18. Moreover, we show that for a univalent self-map of $\mathbb{D}$, C_φ is closed range on $\mathcal{B}$ if and only if C_φ closed range on $BMOA$, see Corollary 3.17.

Finally, in Theorem 3.19 we give another characterization of closed range composition operators on $BMOA$.

Theorem 3.19 *For each* φ *analytic self map of* $\mathbb{D}$, *the following conditions are equivalent:*

(a) *There exists* $k \in (0,1]$ *such that for every* $w \in \mathbb{D}$, $||\alpha_w \circ \varphi||_* \geq k$.

(b) *There exists* $k \in (0,1]$ *such that for every* $w \in \mathbb{D}$ *there exists* $w' \in \mathbb{D}$ *with*

$|\alpha_w(w')|^2 \leq 1 - k^2$, *there exists a sequence (q_n) in $\mathbb{D}$ such that $\varphi(q_n) \to w'$ and*

$$\lim_{n\to\infty} ||\varphi_{q_n}||_{H^2} \geq k\,,$$

where for all natural numbers n, $\varphi_{q_n} = \alpha_{\varphi(q_n)} \circ \varphi \circ \alpha_{q_n}$.

In Chapter 4 we define the spaces Q_p and study closed range composition operators on them.

Firstly, we show a Lipschitz continuity property that $||f \circ \alpha_q - f(q)||_{Q_p}$ has with respect to the pseudohyperbolic metric. The space $\mathcal{D}_p$ will be defined in Chapter 4.

Theorem 4.1 *Let $p \in (0, 1]$. Then, for each $f \in Q_p$ and for all $q, w \in \mathbb{D}$*

$$|\ ||f \circ \alpha_q - f(q)||^2_{\mathcal{D}_p} - ||f \circ \alpha_w - f(w)||^2_{\mathcal{D}_p}\ | \leq 2^{p+1}\rho(q,w)\,||f||^2_{Q_p}\,.$$

If $p \in (1, \infty)$ then Q_p is the Bloch space with and equivalent norm and we have the following.

Theorem 4.3 *Let $p \in (1, \infty)$. Then for each $f \in Q_p$ and all $q, w \in \mathbb{D}$*

$$|\ ||f \circ \alpha_q - f(q)||^2_{\mathcal{D}_p} - ||f \circ \alpha_w - f(w)||^2_{\mathcal{D}_p}\ | \leq 4p\,\rho(q,w)\,||f||^2_{Q_p}\,.$$

Let $p \in [0, \infty)$. The *counting function* for Q_p for each $q \in \mathbb{D}$, and if $\zeta \in \varphi(\mathbb{D})$ is defined by

$$N_{q,\varphi}(\zeta, p) = \sum_{\varphi(z)=\zeta} (1 - |\alpha_q(z)|^2)^p\,;$$

if $\zeta \notin \varphi(\mathbb{D})$ then $N_{q,\varphi}(\zeta, p) = 0$.

In Theorem 4.5 and Theorem 4.6 we give necessary and different sufficient conditions for C_φ to be closed range in Q_p, for $p \in (0, 1)$, in terms of sampling type measures.

Theorem 4.5 *Let $p \in (0, 1)$ and assume that C_φ is a bounded operator on Q_p. If there*

exists an $\epsilon > 0$ such that

$$\sup_{q \in \mathbb{D}} \int_{\cap_{q \in \mathbb{D}} G_{\epsilon,q}} |f'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) \asymp ||f||^2_{Q_p}$$

then the composition operator C_φ is closed range on Q_p.

Theorem 4.6 *Let $p \in (0,1)$ and assume that C_φ is a bounded operator on Q_p. If C_φ is closed range on Q_p then there exists an $\epsilon > 0$ such that for all $f \in Q_p$*

$$\sup_{q \in \mathbb{D}} \int_{G_\epsilon} |f'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) \asymp ||f||^2_{Q_p}.$$

The main results of the chapter are Theorem 4.7, Theorem 4.8 and Theorem 4.9. They provide a complete characterization of closed range composition operators on Q_p for all $p \in (0, \infty)$. We also note that Theorem 4.9 gives a characterization of all isometries on Q_p for all $p \in (0, \infty)$. The isometries in $\mathcal{B}$ with the standard norm are known. Theorem 4.9 gives the isometries in the norm $||.||_{Q_p}$, $p > 1$.

Theorem 4.7 *Let $p > 0$ and assume that C_φ is a bounded operator on Q_p. Then, there exists $k > 0$ such that for all $q \in \mathbb{D}$, $||\alpha_q \circ \varphi||_{Q_p} \geq k$ if and only if there exists an $\varepsilon > 0$ and $r \in (0,1)$ such that for all $q \in \mathbb{D}$, there exists $q' \in \mathbb{D}$ such that*

$$\frac{\int_{G_{\varepsilon,q'} \cap D(q,r)} N_{q',\varphi}(\zeta) \, dA(\zeta)}{|D(q,r)|} \asymp 1 \,.$$

Theorem 4.8 *Let $p > 0$ and assume that C_φ is a bounded operator on Q_p. The composition operator C_φ is closed range on Q_p if and only if there exists a constant $k > 0$ such that $||\alpha_q \circ \varphi||_{Q_p} \geq k$, for all $q \in \mathbb{D}$.*

Theorem 4.9 *Let $p > 0$, assume that C_φ is a bounded operator on Q_p. The following conditions are equivalent:*

(a) *There exists $k \in (0,1]$ such that for every $w \in \mathbb{D}$, $||\alpha_w \circ \varphi||_{Q_p} \geq k$.*

(b) *There exists $k \in (0,1]$ such that for every $w \in \mathbb{D}$ there exists $w' \in \mathbb{D}$ with*

$|\alpha_w(w')|^2 \leq 1 - k^2$, there exists a sequence (q_n) in $\mathbb{D}$ such that $\varphi(q_n) \to w'$ and

$$\lim_{n\to\infty} ||\varphi_{q_n}||_{D_p} \geq k \,,$$

where for all natural numbers n, $\varphi_{q_n} = \alpha_{\varphi(q_n)} \circ \varphi \circ \alpha_{q_n}$.

We conclude this chapter by providing the following corollary which summarizes our results on closed range composition operators on $\mathcal{B}$.

Corollary 4.10 *Let φ be a non-constant analytic self-map of $\mathbb{D}$, and $p > 1$. Then, the following are equivalent*

(a) *The composition operator C_φ is closed range on $\mathcal{B}$.*

(b) *For all $q \in \mathbb{D}$, $||\alpha_q \circ \varphi||_{\mathcal{B}} \asymp 1$.*

(c) *There exists an $\epsilon > 0$ and $r \in (0, 1)$ such that for all $q \in \mathbb{D}$, there exists $q' \in \mathbb{D}$ such that*

$$\frac{\int_{G_{\epsilon,q'} \cap D(q,r)} N_{q',\varphi}(\zeta)\, dA(\zeta)}{|D(q,r)|} \asymp 1 \,.$$

(d) *There exists $k \in (0, 1]$ such that for every $w \in \mathbb{D}$ there exists $w' \in \mathbb{D}$ with $|\alpha_w(w')|^2 \leq 1 - k^2$, there exists a sequence (q_n) in $\mathbb{D}$ such that $\varphi(q_n) \to w'$ and*

$$\lim_{n\to\infty} ||\varphi_{q_n}||_{D_p} \geq k \,,$$

where for all natural numbers n, $\varphi_{q_n} = \alpha_{\varphi(q_n)} \circ \varphi \circ \alpha_{q_n}$.

2 The Bloch space, $BMOA$ and $VMOA$

Let $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $\mathbb{T}$ be the unit circle. A *Möbius transformation* is a one-to-one analytic function that maps $\mathbb{D}$ onto itself. It has the form $e^{i\theta}\alpha_q(z)$ where

$$\alpha_q(z) = \frac{q-z}{1-\bar{q}z}$$

for $q \in \mathbb{D}$ and $z \in \mathbb{D}$. The set of all Möbius transformations is denoted by $Aut(\mathbb{D})$. Note that α_q exchanges the points 0 and q and its inverse under composition is itself. Moreover,

$$|\alpha_q'(z)| = \frac{1-|q|^2}{|1-\bar{q}z|^2}$$

and

$$1 - |\alpha_q(z)|^2 = \frac{(1-|q|^2)(1-|z|^2)}{|1-\bar{q}z|^2} = (1-|z|^2)|\alpha_q'(z)| \tag{2}$$

for $q, z \in \mathbb{D}$.

For each $p, q \in \mathbb{D}$

$$
\begin{aligned}
\alpha_q \circ \alpha_p(z) &= \alpha_q\Big(\frac{p-z}{1-\bar{p}z}\Big) \\
&= \frac{q - \frac{p-z}{1-\bar{p}z}}{1 - \bar{q}\frac{p-z}{1-\bar{p}z}} \\
&= \frac{p-q-z(1-q\bar{p})}{-(1-\bar{q}p)+z(\bar{p}-\bar{q})} \\
&= -\frac{1-q\bar{p}}{1-\bar{q}p}\frac{\alpha_p(q)-z}{1-\bar{\alpha}_p(q)z}
\end{aligned}
$$

Let $b = \alpha_p(q)$ and $\rho = -\frac{1-q\bar{p}}{1-\bar{q}p}$. Then, $|\rho| = 1$ and we obtain

$$\alpha_q \circ \alpha_p = \rho\, \alpha_{\alpha_p(q)}. \tag{3}$$

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The *pseudohyperbolic distance* ρ between two points $q, z \in \mathbb{D}$ is defined by

$$\rho(q, z) = |\alpha_q(z)| = \left|\frac{q - z}{1 - \bar{q}z}\right|, \tag{4}$$

is invariant under Möbius transformations, that is, for all $z, q \in \mathbb{D}$ and $\varphi \in Aut(\mathbb{D})$.

$$\rho(\varphi(q), \varphi(z)) = \rho(q, z)$$

and satisfies the following version of triangle inequality, see [12, Lemma 1.4]. Given $z, q, w \in \mathbb{D}$,

$$\frac{\rho(q, w) - \rho(w, z)}{1 - \rho(q, w)\rho(w, z)} \leq \rho(q, z) \leq \frac{\rho(q, w) + \rho(w, z)}{1 + \rho(q, w)\rho(w, z)}. \tag{5}$$

Let A denote area measure on $\mathbb{D}$ normalized by the condition $A(\mathbb{D}) = 1$, for each $H \subseteq \mathbb{D}$, abbreviate $A(H)$ with $|H|$. We denote the euclidean disk centered at q with radius $r \in (0, 1)$ with $\mathbb{D}(q, r)$. The *pseudohyperbolic disk* $D(q, r)$ centered at q with radius $r \in (0, 1)$ is

$$D(q, r) = \{z : \rho(z, q) < r\}. \tag{6}$$

If $r \in (0, 1)$ is fixed and $z \in D(q, r)$ then by [37, Proposition 4.5]

$$A(D(q, r)) \asymp (1 - |z|^2)^2 \asymp |D(q, r)| \asymp (1 - |q|^2)^2, \tag{7}$$

and

$$|1 - \bar{q}z| \asymp 1 - |z|^2 \asymp 1 - |q|^2. \tag{8}$$

Definition 2.1. *The Bloch space $\mathcal{B}$ of $\mathbb{D}$ is the space of analytic functions f on $\mathbb{D}$ such that*

$$||f||_{\mathcal{B}} = \sup_{z \in \mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

The Bloch space is a Banach space with norm $||f||_B = |f(0)| + ||f||_{\mathcal{B}}$. It is invariant under Möbius transformations, that is, if $f \in \mathcal{B}$ then $f \circ \varphi \in \mathcal{B}$, for all Möbius transformation φ. In fact, by (2) and for all $q, z \in \mathbb{D}$

$$
\begin{aligned}
||f \circ \alpha_q||_{\mathcal{B}} &= \sup_{z \in \mathbb{D}}(1 - |z|^2)|(f \circ \alpha_q)'(z)| \\
&= \sup_{z \in \mathbb{D}}(1 - |z|^2)|f'(\alpha_q(z))||\alpha_q'(z)| \\
&= \sup_{z \in \mathbb{D}}(1 - |\alpha_q(z)|^2)||f'(\alpha_q(z))| \\
&= \sup_{z \in \mathbb{D}}|(1 - |\zeta|^2)|f'(\zeta)| \\
&= ||f||_{\mathcal{B}}.
\end{aligned}
$$

The polynomials are not dense in the Bloch space. The closure of the polynomials in the Bloch norm is called the little Bloch space, denoted by $\mathcal{B}_0$. It is well known, see for example [37, page 84].

$$f \in \mathcal{B}_0 \Leftrightarrow \lim_{|z| \to 1}(1 - |z|^2)|f'(z)| = 0.$$

Moreover, as Rubel and Timoney showed in [25] the Bloch space is the largest Möbius invariant Banach space that has a non-zero linear functional that extends to a continuous linear functional on $H(\mathbb{D})$, the space of all analytic functions on $\mathbb{D}$.

Definition 2.2. *The Bergman space A^2 is the Hilbert space of analytic functions f on $\mathbb{D}$ that are square-integrable with respect to normalized area measure that is,*

$$||f||_{A^2}^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty.$$

An equivalent norm on A^2, see for example [37, Theorem 4.28] is given by

$$||f||_{A^2}^2 \asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^2 dA(z).$$

It is well known that the Bloch space can be regarded as the area version of $BMOA$. In particular, by [1, Theorem 5.5] and for all $f \in \mathcal{B}$,

$$||f||_{\mathcal{B}} \asymp \sup_{q \in \mathbb{D}} |||f \circ \alpha_q - f(q)||_{A^2},$$

and by [37, Theorem 4.28],

$$||f||_{\mathcal{B}}^2 \asymp \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\alpha_q(z)|^2)^2 dA(z). \tag{9}$$

Definition 2.3. *The Dirichlet space $\mathcal{D}$ is the space of analytic functions on $\mathbb{D}$ such that*

$$||f||_{\mathcal{D}} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

Next, we give the definition of analytic BMO, namely $BMOA$ space which is a Möbius invariant version of the Hardy Space H^2.

Definition 2.4. *An analytic function $f \in H(\mathbb{D})$ belongs to $BMOA$ if*

$$||f||_G = \sup_{q \in \mathbb{D}} |||f \circ \alpha_q - f(q)||_{H^2} < \infty. \tag{10}$$

We note that $||f||_G$ defines a seminorm and $|f(0)| + ||f||_G$ is a norm in $BMOA$ that makes it a Banach space.

Notation $S(h, \theta) = \{z \in \mathbb{D} : |z - e^{i\theta}| < h\}$, where $\theta \in [0, 2\pi), h \in (0, 1)$.

The notion of $BMOA$ first arose in the context of mean oscillation of a function over cubes with edges parallel to the coordinate axes or equivalently over sets of the form

$S(h, \theta)$ ([28, pages, 36-39]). That is,

$$||f||_*^2 \asymp \sup_{h \in (0,1), \theta \in [0,2\pi)} \frac{1}{h} \int_{S(h,\theta)} |f'(z)|^2 (1 - |z|^2) dA(z). \tag{11}$$

Given $h > 0$ and $\theta \in \mathbb{R}$, let

$$Q = \{re^{i\theta} : \theta_0 < \theta < \theta_0 + h, 1 - h < r < 1\}.$$

A positive measure μ in $\mathbb{D}$ is a *Carleson measure for the Hardy space* if there exists a constant $k > 0$ such that $\mu(Q) \leq kh$, for all $h > 0$. It is known, see for example [12, Lemma 3.3], that μ is a Carleson measure for the Hardy space if and only if

$$\sup_{q \in \mathbb{D}} \int_{\mathbb{D}} \frac{1 - |q|^2}{|1 - \bar{q}z|^2} d\mu(z) < \infty.$$

Then $f \in BMOA$ if and only if $|f'(z)|^2 (1 - |z|^2) dA(z)$ is a Carleson measure for the Hardy space.

Theorem 2.1. *[31, page 178] (The Littlewood-Paley Identity) If f is an analytic on $\mathbb{D}$, then*

$$||f||_{H^2}^2 = |f(0)|^2 + 2 \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z). \tag{12}$$

The following allows us to obtain a useful variation of the Littlewood-Paley identity, see [31, page 178] and [12, Lemma 3.2].

Proposition 2.2. *For every analytic function g on $\mathbb{D}$,*

$$\int_{\mathbb{D}} |g(z)| \log \frac{1}{|z|} dA(z) \asymp \int_{\mathbb{D}} |g(z)| (1 - |z|^2) dA(z).$$

Proof. We know that for all $z \in \mathbb{D} \setminus \{0\}$

$$1 - |z|^2 \leq 2 \log \frac{1}{|z|}, \tag{13}$$

and $|z| > 1/4$ implies that there exists a constant $c_1 > 0$ such that

$$\log \frac{1}{|z|} \leq c_1(1 - |z|^2). \tag{14}$$

Let g be an analytic function on $\mathbb{D}$. Then by (13)

$$\int_{\mathbb{D}} |g(z)|(1 - |z|^2)dA(z) \leq 2 \int_{\mathbb{D}} |g(z)| \log \frac{1}{|z|} dA(z), \tag{15}$$

and $|g|$ is subharmonic. Therefore, if $|z| \leq 1/4$ then

$$\begin{aligned}
|g(z)| &\leq 16 \int_{D(z,1/4)} |g(\zeta)|dA(\zeta) \\
&= 16 \int_{D(z,1/4)} \frac{|g(\zeta)|(1 - |\zeta|^2)}{1 - |\zeta|^2} dA(\zeta).
\end{aligned}$$

Notice that if $\zeta \in D(z, 1/4)$, then $|\zeta| \leq 1/2$ and

$$|g(z)| \leq 22 \int_{\mathbb{D}} |g(\zeta)|(1 - |\zeta|^2)dA(\zeta). \tag{16}$$

By (14)

$$\int_{\mathbb{D} \setminus D(0,1/4)} |g(z)| \log \frac{1}{|z|} dA(z) \leq c_1 \int_{\mathbb{D}} |g(z)|(1 - |z|^2) dA(z) \tag{17}$$

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and by (16)

$$\int_{D(0,1/4)} |g(z)| \log \frac{1}{|z|} \, dA(z) \leq 22 \int_{D(0,1/4)} \log \frac{1}{|z|} \int_{\mathbb{D}} |g(\zeta)|(1-|\zeta|^2) \, dA(z) \, dA(\zeta)$$

$$= 22 \left(\int_{D(0,1/4)} \log \frac{1}{|z|} dz \right) \int_{\mathbb{D}} |g(\zeta)|(1-|\zeta|^2) dA(\zeta).$$

Notice that

$$\int_{D(0,1/4)} \log \frac{1}{|z|} dA(z) = \int_0^{1/4} \int_0^{2\pi} \log \frac{1}{r} r \, dr \frac{d\theta}{\pi}$$

$$= 2 \int_0^{1/4} r \log \frac{1}{r} dr$$

$$\asymp .1178934 \, ,$$

and there exists $c > 0$ such that

$$\int_{D(0,1/4)} |g(z)| \log \frac{1}{|z|} dA(z) \leq c \int_{\mathbb{D}} |g(\zeta)|(1-|\zeta|^2) dA(\zeta) \, . \tag{18}$$

Therefore by (15), (17), and (18) we arrive at the conclusion. $\qquad\square$

By the Littlewood-Paley identity

$$||f||_G^2 = \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \alpha_q)'(z)|^2 \log \frac{1}{|z|} dA(z)$$

and by Proposition 2.2, $||f||_G \asymp ||f||_*$ where

$$||f||_*^2 = \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \alpha_q)'(z)|^2 (1-|z|^2) dA(z) \, . \tag{19}$$

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Note that by (19) and by making change of variables $\alpha_q(z) = w$ we obtain

$$\begin{aligned}
||f||_*^2 &= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |(f'(\alpha_q(z)))|^2 |\alpha_q'(z)|^2 (1 - |z|^2) dA(z) \\
&= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(w)|^2 (1 - |\alpha_q(w)|^2) dA(w) \, . \qquad (20)
\end{aligned}$$

There are a number of equivalent norms in $BMOA$. We will use

$$||f||_{BMOA} = |f(0)| + ||f||_* .$$

It is easy to see that $BMOA$ is a Möbius invariant Banach space. Let $f \in BMOA$, $p \in \mathbb{D}$. Then by (20)

$$\begin{aligned}
||f \circ \alpha_p||_*^2 &= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \alpha_p)'(z)|^2 (1 - |\alpha_q(z)|^2) dA(z) \\
&= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |(f'(\alpha_p(z)))|^2 |\alpha_p'(z)|^2 (1 - |\alpha_q(z)|^2) dA(z) \\
&= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(\zeta)|^2 (1 - |\alpha_q(\alpha_p(\zeta))|^2) dA(\zeta) \, .
\end{aligned}$$

For each $q \in \mathbb{D}$ and by (3), let $\alpha_q \circ \alpha_p = e^{i\theta} \alpha_Q$ for some $Q \in \mathbb{D}$, $\theta \in \mathbb{R}$; then

$$\begin{aligned}
||f \circ \alpha_p||_*^2 &= \sup_{Q \in \mathbb{D}} \int_{\mathbb{D}} |f'(w)|^2 (1 - |\alpha_Q(w)|^2) dA(w) \\
&= ||f||_*^2 .
\end{aligned}$$

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For each $q \in \mathbb{D}$ and since $BMOA$ is a Möbius invariant Banach space, $||\alpha_q||_* = ||z||_*$,

$$
\begin{aligned}
||\alpha_q||_* \asymp ||z||_G \\
= \sup_{q\in\mathbb{D}} ||\alpha_q - q||_{H^2} \\
= \sup_{q\in\mathbb{D}} ||\frac{z(1 - |q|^2)}{1 - \bar{q}z}||_{H^2} \\
= \sup_{q\in\mathbb{D}}(1 - |q|^2)||\frac{z}{1 - \bar{q}z}||_{H^2} \\
= \sup_{q\in\mathbb{D}}(1 - |q|^2)\,||z\sum_{n=0}^{\infty}(\bar{q}z)^n||_{H^2} \\
= \sup_{q\in\mathbb{D}}(1 - |q|^2)||\sum_{n=1}^{\infty}(\bar{q})^{n-1}z^n|| \\
= \sup_{q\in\mathbb{D}\backslash\{0\}}(1 - |q|^2)\frac{1}{|q|}\Big(\sum_{n=1}^{\infty}|q|^{2n}\Big)^{\frac{1}{2}} \\
= \sup_{q\in\mathbb{D}}(1 - |q|^2)\frac{1}{|q|}\Big(\frac{1}{1 - |q|^2} - 1\Big)^{\frac{1}{2}} \\
= 1.
\end{aligned}
\tag{21}
$$

One of the many similarities between the Bloch space and $BMOA$ is that polynomials are not dense in either space. The closure of the polynomials in the $BMOA$ norm is $VMOA$, the *space of analytic functions with vanishing mean oscillation*. The space $VMOA$ can be characterized as all those analytic functions f on $\mathbb{D}$ such that

$$
\lim_{|q|\to 1}\int_{\mathbb{D}}|f'(w)|^2(1 - |\alpha_q(w)|^2 dA(w) = 0
\tag{22}
$$

(the "little-oh" version of (20)).

We can see that $BMOA$ is a subspace of the Bloch space as follows:

$$
|f'(0)| \leq ||f||_{H^2}
$$

for any f analytic on $\mathbb{D}$; therefore there exists $c > 0$ such that for each $q \in \mathbb{D}$

$$
\begin{aligned}
|f'(q)|(1 - |q|^2) &= |(f \circ \alpha_q - f(q))'(0)| \\
&\leq ||f \circ \alpha_q - f(q)||_{H^2} \\
&\leq c||f||_* .
\end{aligned}
\tag{23}
$$

Hence, $||f||_{\mathcal{B}} \leq c||f||_*$ for all $f \in BMOA$ and $BMOA \subset \mathcal{B}$ and the inclusion map $i : BMOA \to \mathcal{B}$ is continuous.

Below we establish the growth of $BMOA$ functions. Let $f \in BMOA$. Then f is analytic on $\mathbb{D}$ and for all $z \in \mathbb{D}$,

$$
|f(z) - f(0)| = |z \int_0^1 f'(tz)dt\,|.
$$

Since the inclusion map $i : BMOA \to \mathcal{B}$ is continuous, there exists $c > 0$ such that for all $z \in \mathbb{D}$

$$
|f'(z)|(1 - |z|^2) \leq c||f||_*
$$

and

$$
\begin{aligned}
|f(z) - f(0)| &\leq |z| \int_0^1 |f'(tz)|dt \\
&\leq c|z|\,||f||_* \int_0^1 \frac{1}{1 - t^2|z|^2} dt \\
&= \frac{c}{2}||f||_* \log \frac{1 + |z|}{1 - |z|}.
\end{aligned}
\tag{24}
$$

Therefore,

$$
|f(z)| \leq |f(0)| + \frac{c}{2}||f||_* \log \frac{1 + |z|}{1 - |z|}.
\tag{25}
$$

Finally we note that

$$\mathcal{D} \subset VMOA \subset BMOA \subset \mathcal{B} \subset A^2.$$

Let H^∞ denote the set of all functions in $H(\mathbb{D})$ that are bounded. Given $f \in H^\infty$ let

$$||f||_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

By applying Schwarz Lemma we obtain that for all $z \in \mathbb{D}$, $|f'(z)|(1 - |z|^2) \le ||f||_\infty$, see [37, page 101]. By (10) it is now easy to see that $H^\infty \subset BMOA$. There are unbounded functions in $BMOA$.

The function $\log(1 - z) \in BMOA$. In fact, if f is any analytic, univalent, and zero free function then $\log f \in BMOA$ (This result first appeared in [5] and [10]). Below we give more examples of $BMOA$ functions.

If (a_n) is a bounded sequence then $\sum_{n=1}^\infty \frac{1}{n} a_n z^n \in BMOA$, and if $\sum_{n=1}^\infty |a_n|^2 < \infty$ then $\sum_{n=0}^\infty a_n z^{\lambda_n} \in BMOA$, where (λ_n) is a sequence of integers satisfying

$$\frac{\lambda_{n+1}}{\lambda_n} \ge \lambda > 1,$$

λ is a constant and n is a natural number.

3 Bounded and closed range composition operators on $BMOA$

Given φ an analytic self-map of $\mathbb{D}$, the *composition operator*, C_φ is defined for all $f \in H(\mathbb{D})$ as follows,

$$C_\varphi f = f \circ \varphi.$$

It is well known that all composition operators C_φ are bounded on the Hardy space H^2 and, if $\varphi(0) = 0$, then C_φ is a contraction.

Theorem 3.1. *[31, page 13]* **Littlewood's Subordination Principle (1925)**

    Suppose φ is an analytic self-map of $\mathbb{D}$, with $\varphi(0) = 0$. Then for each $f \in H^2$, $C_\varphi f \in H^2$ and $||C_\varphi f||_{H^2} \leq ||f||_{H^2}$.

    More generally, we have the following.

Theorem 3.2. *[31, page 13]* **Littlewood's Theorem (1925)**

    Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function. Then C_φ is bounded on $H^2(\mathbb{D})$, and

$$||C_\varphi f||_{H^2} \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}} \, ||f||_{H^2}.$$

Definition 3.1. *The Nevanlinna counting function of φ is*

$$N_\varphi(w) = \sum_{\varphi(z)=w} \log \frac{1}{|z|}$$

if $w \in \varphi(\mathbb{D})$; if $w \notin \varphi(\mathbb{D})$ then we set $N_\varphi(w) = 0$.

Theorem 3.3. *[31, page 187]* **Littlewood's Inequality** *If φ is an analytic self-map of $\mathbb{D}$, then for each $w \in \mathbb{D} \setminus \{\varphi(0)\}$,*

$$N_\varphi(w) \leq \log \left| \frac{1 - \bar{w}\varphi(0)}{w - \varphi(0)} \right|.$$

Definition 3.2. *For each $q \in \mathbb{D}$, and if $\zeta \in \varphi(\mathbb{D})$ the BMOA counting function is*

$$N_{q,\varphi}(\zeta) = \sum_{\varphi(z)=\zeta} \left(1 - |\alpha_q(z)|^2\right);$$

if $\zeta \notin \varphi(\mathbb{D})$ then we set $N_{q,\varphi}(\zeta) = 0$.

Counting functions have played an important role in the study of compact, bounded and closed range composition operators. The Nevanlinna counting function for composition operators in BMOA appears with a non-univalent change of variables as it is done in [31, page 186].

For each $f \in BMOA$,

$$
\begin{aligned}
||C_\varphi f||_*^2 &= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 (1 - |\alpha_q(z)|^2) dA(z) \\
&= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |(f'(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |\alpha_q(z)|^2) dA(z).
\end{aligned}
$$

Then we have

$$||C_\varphi f||_*^2 = \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(\zeta)|^2 N_{q,\varphi}(\zeta)\, dA(\zeta). \tag{26}$$

It is well known that C_φ is always bounded on $BMOA$, see [4, Theorem 12]. Littlewood's Subordination Principle provides a proof, see for example [33]. Smith points out in [32] that Littlewood's inequality provides an alternate proof, which we will describe below for our chosen norm.

Theorem 3.4. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is a bounded operator on $BMOA$.*

Proof. For all $z \in \mathbb{D} \setminus \{0\}$ we have,

$$1 - |z|^2 \leq \log \frac{1}{|z|^2}$$

therefore by (26)

$$\|C_\varphi f\|_*^2 \;\leq\; \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(\zeta)|^2 \sum_{\varphi(z)=\zeta} \log \frac{1}{|\alpha_q(z)|^2} dA(\zeta) . \qquad (27)$$

By Proposition 2.2 and Littlewood's Inequality, there exists $C > 0$ such that

$$\begin{aligned}
\|C_\varphi f\|_*^2 \;&\leq\; \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(\zeta)|^2 \log \frac{1}{|\alpha_{\varphi(q)}(\zeta)|^2} dA(\zeta) \\
&\leq\; \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(\zeta)|^2 \log \frac{1}{|\alpha_q(\zeta)|^2} dA(\zeta) \\
&\leq\; C\|f\|_*^2 , \qquad\qquad\qquad\qquad\qquad\qquad\qquad (28)
\end{aligned}$$

and C_φ is a bounded operator on $BMOA$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, let $X = \mathcal{B}$, $BMOA$ or $VMOA$ and $Y = BMOA$ or $VMOA$. We show that for C_φ to be bounded below, we can restrict on subspaces of functions that vanish at the origin. The canonical seminorm in X is denoted by $\|.\|_{sX}$.

Proposition 3.5. *Let $X = \mathcal{B}$, $BMOA$ or $VMOA$ and $Y = BMOA$ or $VMOA$. The composition operator $C_\varphi : X \to Y$ is bounded below if and only if there exists a $\delta > 0$ such that $\|C_\varphi f\|_* \geq \delta \|f\|_{sX}$, for all $f \in X$.*

Proof. First, assume that $C_\varphi : X \to BMOA$ is bounded below. Let $f \in X$ and $g(z) = f(z) - f(\varphi(0))$. Then, $g \in X$, $g(\varphi(0)) = 0$, $\|g\|_{sX} = \|f\|_{sX}$ and $\|g \circ \varphi\|_* = \|f \circ \varphi\|_*$. Thus, there exists $\delta > 0$ such that

$$\|C_\varphi f\|_* = \|f \circ \varphi\|_* = \|g \circ \varphi\|_* = \|g \circ \varphi\|_{BMOA} \geq\; \delta \|g\|_X \geq \delta \|g\|_{sX} = \delta \|f\|_{sX} .$$

Conversely, assume that there exists a $\delta > 0$ such that $\|f \circ \varphi\|_* \geq \delta \|f\|_{sX}$, for all $f \in X$. Then by (24) there exists $C > 0$ such that

$$|f(0)| \leq |f(\varphi(0))| + C \|f\|_{sX} \log \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} ,$$

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which implies that there exists $C > 0$ such that

$$
\begin{aligned}
||f||_X &= |f(0)| + ||f||_{sX} \\
&\leq |f(\varphi(0))| + C \, ||f||_{sX} \, \log \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} + \frac{1}{\delta} ||f \circ \varphi||_* \\
&\leq |f(\varphi(0))| + \frac{c}{\delta} \, ||f \circ \varphi||_* \, \log \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} + \frac{1}{\delta} ||f \circ \varphi||_* \\
&\leq C \, ||f \circ \varphi||_{BMOA}.
\end{aligned}
$$

Hence, C_φ is bounded below on $BMOA$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 3.3. *Let μ be a finite positive Borel measure on $\mathbb{D}$. We say that μ is a Carleson measure for the Bergman space if there exists constant $c > 0$ such that for all $f \in A^2$*

$$
\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq c \int_{\mathbb{D}} |f(z)|^2 dA(z).
$$

By [37, Theorem 7.4], given $r \in (0, 1)$, μ is a Carleson measure if and only if there exists $c_r > 0$ such that for all $w \in \mathbb{D}$,

$$
\mu(D(w, r)) \leq c_r |D(w, r)|.
$$

The *Berezin symbol of μ* is

$$
\tilde{\mu}(q) = \int_{\mathbb{D}} |\alpha_q'(z)|^2 \, d\mu(z), \quad q \in \mathbb{D}. \tag{29}
$$

It is well known that μ is a Carleson measure for the Bergman space if and only if $\tilde{\mu}$, is a bounded function on $\mathbb{D}$, that is,

$$
\sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |\alpha_q'(z)|^2 d\mu(z) \lesssim 1, \tag{30}
$$

see for example [37, Theorem 7.5]. The proof of (30) can be used to show that $\mu_{q'}$, $q' \in \mathbb{D}$,

is a collection of uniformly Carleson measures for the Bergman space if and only if

$$\sup_{w,q'\in\mathbb{D}} \int_{\mathbb{D}} |\alpha'_w(z)|^2 d\mu(z) \lesssim 1 \,. \tag{31}$$

For any symbol φ and by Theorem 3.4, C_φ is a bounded operator on $BMOA$. Then for all $q \in \mathbb{D}$, $||C_\varphi \alpha_q||_* \lesssim 1$ and by (26)

$$\sup_{q,q'\in\mathbb{D}} \int_{\mathbb{D}} |\alpha'_q(\zeta)|^2 N_{q',\varphi}(\zeta) dA(\zeta) \lesssim 1 \,.$$

By (31) and for all $q' \in \mathbb{D}$, $N_{q',\varphi}(\zeta) dA(\zeta)$ is a collection of uniformly Carleson measures for the Bergman space.

Definition 3.4. *Let μ be a finite positive Carleson measure on $\mathbb{D}$. We say that μ satisfies the reverse Carleson condition if there exists $r \in (0,1)$ such that for all $w \in \mathbb{D}$,*

$$|D(w,r)| \asymp \mu(D(w,r)).$$

It is shown in [16, Theorem 4.1] that μ satisfies the reverse Carleson condition if and only if

$$\sup_{q\in\mathbb{D}} \int_{\mathbb{D}} |\alpha'_q(z)|^2 d\mu \asymp 1 \,.$$

A set $H \subset \mathbb{D}$ satisfies the *reverse Carleson condition* if the Carleson measure $\chi_H(z) dA(z)$ satisfies the reverse Carleson condition; Luecking showed in [20] that this is equivalent to

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) \le C \int_H |f(z)|^2 dA(z),$$

for all $f \in A^2$. It is also equivalent to

$$\int_{\mathbb{D}} |f'(z)|^2 \left(1 - |z|^2\right) dA(z) \le c \int_H |f'(z)|^2 \left(1 - |z|^2\right) dA(z) \,,$$

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for all $f \in H^2$.

Definition 3.5. *We say that $H \subseteq \mathbb{D}$ is a sampling set for BMOA if for all $f \in BMOA$*

$$\sup_{q \in \mathbb{D}} \int_H |f'(z)|^2 \left(1 - |\alpha_q(z)|^2\right) dA(z) \asymp \|f\|_*^2.$$

For each $\varepsilon > 0$ and $q \in \mathbb{D}$ let $G_{\varepsilon,q}$ be

$$G_{\varepsilon,q} = \{\zeta : N_{q,\varphi}(\zeta) > \varepsilon \left(1 - |\alpha_q(\zeta)|^2\right)\} \tag{32}$$

and let G_ε be

$$G_\varepsilon = \cup_{q \in \mathbb{D}} G_{\varepsilon,q}. \tag{33}$$

Theorem 3.6. *Let φ be a non-constant analytic self map of $\mathbb{D}$. If C_φ is closed range on BMOA then there exists $\varepsilon > 0$ such that G_ε is a sampling set for BMOA. Moreover, if $\cap_{q \in \mathbb{D}} G_{\varepsilon,q}$ is a sampling set for BMOA, then C_φ is closed range on BMOA.*

Proof. First, suppose that C_φ is closed range on $BMOA$. If the conclusion fails, then for every $\varepsilon > 0$, G_ε is not a sampling set for $BMOA$ and we can find a sequence (f_n) in $BMOA$ with $f_n(0) = 0$ and $\|f_n\|_{BMOA} = 1$ for all n such that

$$\lim_{n \to \infty} \sup_{q \in \mathbb{D}} \int_{G_{\frac{1}{n}}} |f_n'(z)|^2 (1 - |\alpha_q(z)|^2) \, dA(z) = 0. \tag{34}$$

For each n we have,

$$\begin{aligned} \|f_n \circ \varphi\|_*^2 &= 2 \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f_n'(\zeta)|^2 N_{q,\varphi}(\zeta) \, dA(\zeta) \\ &\leq I_1 + I_2, \end{aligned} \tag{35}$$

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where

$$I_1 = 2 \sup_{q \in \mathbb{D}} \int_{G_{\frac{1}{n}}} |f_n'(\zeta)|^2 N_{q,\varphi}(\zeta) \, dA(\zeta) \tag{36}$$

and

$$I_2 = 2 \sup_{q \in \mathbb{D}} \int_{\mathbb{D} \backslash G_{\frac{1}{n}}} |f_n'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) \,. \tag{37}$$

By Littlewood's inequality, there exists $C > 0$ such that for all $\zeta \neq \varphi(q)$,

$$\sum_{\varphi(z)=\zeta} \log \frac{1}{|\alpha_q(z)|^2} \leq C \log \frac{1}{|\alpha_{\varphi(q)}(\zeta)|^2} \,. \tag{38}$$

For all n,

$$\begin{aligned}
I_1 &\leq C \sup_{q \in \mathbb{D}} \int_{G_{\frac{1}{n}}} |f_n'(\zeta)|^2 \sum_{\varphi(z)=\zeta} \log \frac{1}{|\alpha_q(z)|^2} \, dA(\zeta) \\
&\leq C \sup_{q \in \mathbb{D}} \int_{G_{\frac{1}{n}}} |f_n'(\zeta)|^2 \log \frac{1}{|\alpha_{\varphi(q)}(\zeta)|^2} \, dA(\zeta) \,.
\end{aligned}$$

Note that for any $q \in \mathbb{D}$ and for any natural number n,

$$\int_{G_{\frac{1}{n}}} |f_n'(\zeta)|^2 \log \frac{1}{|\alpha_q(\zeta)|^2} \, dA(\zeta) = A - B \,,$$

where

$$A = \int_{\mathbb{D}} |f_n'(\zeta)|^2 \log \frac{1}{|\alpha_q(\zeta)|^2} \, dA(\zeta)$$

and

$$B = \int_{\mathbb{D} \backslash G_{\frac{1}{n}}} |f_n'(\zeta)|^2 \log \frac{1}{|\alpha_q(\zeta)|^2} \, dA(\zeta) \,.$$

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Then by Proposition 2.2,

$$A \leq C \int_{\mathbb{D}} |f'_n(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2)\, dA(\zeta)$$

and by (13),

$$B \geq \frac{1}{2} \int_{\mathbb{D} \setminus G_{\frac{1}{n}}} |f'_n(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2)\, dA(\zeta).$$

We conclude that

$$I_1 \leq C \sup_{q \in \mathbb{D}} \int_{G_{\frac{1}{n}}} |f'_n(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2)\, dA(\zeta). \tag{39}$$

Next, for n a fixed natural number, $\zeta \notin G_{1/n}$, and $q \in \mathbb{D}$,

$$N_{q,\varphi}(\zeta) \leq \frac{1}{n}(1 - |\alpha_q(\zeta)|^2),$$

and

$$
\begin{aligned}
I_2 &\leq \frac{1}{n} \sup_{q \in \mathbb{D}} \int_{\mathbb{D} \setminus G_{\frac{1}{n}}} |f'_n(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2)\, dA(\zeta) \\
&\leq \frac{1}{n} \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'_n(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2)\, dA(\zeta) \\
&\leq \frac{1}{n} \|f_n\|_*^2 \\
&= \frac{1}{n}. \tag{40}
\end{aligned}
$$

By (34), (35), (39) and (40), $\|f_n \circ \varphi\|_* \to 0$, as $n \to \infty$ and C_φ is not closed range on $BMOA$ which contradicts our hypothesis and the conclusion holds.

Conversely, assume that there exists $\varepsilon > 0$ such that $\cap_{q \in \mathbb{D}} G_{\varepsilon,p}$ is a sampling set for

BMOA. Let $f \in BMOA$. Then, for all $q \in \mathbb{D}$

$$
\begin{aligned}
||f \circ \varphi||_*^2 &\geq \int_{G_{\varepsilon,q}} |f'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) \\
&\geq \varepsilon \int_{G_{\varepsilon,q}} |f'(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2) \, dA(\zeta) \\
&\geq \int_{\cap_{q\in\mathbb{D}} G_{\varepsilon,q}} |f'(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2) \, dA(\zeta) .
\end{aligned}
\tag{41}
$$

Since G_ε is a sampling set for $BMOA$ we conclude that C_φ is closed range on $BMOA$. $\qquad\square$

The following is a corollary of the proof of [16, Proposition 4.1].

Theorem 3.7. *Let μ_q, $q \in \mathbb{D}$ be a collection of positive uniformly Carleson measures, let α, β be such that $\alpha + \beta = 2$. Then*

$$
\limsup_{r\to 1} \sup_{q,w\in\mathbb{D}} \int_{\mathbb{D}\setminus D(w,r)} \frac{(1-|w|^2)^\alpha (1-|\zeta|^2)^\beta}{|1-\bar{w}\zeta|^4} d\mu_q(\zeta) = 0 .
\tag{42}
$$

For each $\varepsilon > 0$ and $q, q' \in \mathbb{D}$ let $G_{\varepsilon,q',q}$ be

$$
G_{\varepsilon,q',q} = \{ \zeta \, : \, N_{q',\varphi}(\zeta) \geq \varepsilon \, (1 - |\alpha_q(\zeta)|^2) \} .
\tag{43}
$$

Theorem 3.8. *Let φ be an analytic self-map of $\mathbb{D}$. Then, there exists $k > 0$ such that for all $q \in \mathbb{D}$, $||\alpha_q \circ \varphi||_* \geq k$ if and only if there exists an $\varepsilon > 0$ and $r \in (0,1)$ such that for all $q \in \mathbb{D}$, there exists $q' \in \mathbb{D}$ such that*

$$
\frac{|G_{\varepsilon,q',q} \cap D(q,r)|}{|D(q,r)|} \asymp 1 .
\tag{44}
$$

Moreover, if the above holds then $G_\varepsilon = \cup_{q\in\mathbb{D}} G_{\varepsilon,q',q}$ satisfies the reverse Carleson condition.

Proof. First, assume that there exists $k > 0$ such that for all q in $\mathbb{D}$, $||\alpha_q \circ \varphi||_* \geq k$. Then, for each $q \in \mathbb{D}$ there exists a $q' \in \mathbb{D}$ such that

$$\int_{\mathbb{D}} |\alpha_q'(\zeta)|^2 N_{q',\varphi}(\zeta)\, dA(\zeta) \geq \frac{k}{2}\,. \tag{45}$$

By Theorem 3.7 and since for all $q' \in \mathbb{D}$, $N_{q',\varphi}(\zeta)dA(\zeta)$ is a collection of uniformly Carleson measures for the Bergman space,

$$\lim_{r \to 1} \sup_{q,q' \in \mathbb{D}} \int_{\mathbb{D}\setminus D(q,r)} |\alpha_q'(\zeta)|^2 N_{q',\varphi}(\zeta)\, dA(\zeta) = 0\,.$$

We conclude that there exists $r \in (0,1)$ such that for all $q, q' \in \mathbb{D}$

$$\int_{\mathbb{D}\setminus D(q,r)} |\alpha_q'(\zeta)|^2 N_{q',\varphi}(\zeta)\, dA(\zeta) < \frac{k}{4}\,. \tag{46}$$

Now, for each $\varepsilon > 0$, $q \in \mathbb{D}$ and $q' \in \mathbb{D}$ as in (45), write

$$\int_{\mathbb{D}} |\alpha_q'(\zeta)|^2 N_{q',\varphi}(\zeta)\, dA(\zeta) \;=\; I_1 + I_2 + I_3\,, \tag{47}$$

where

$$I_1 \;=\; \int_{\mathbb{D}\setminus D(q,r)} |\alpha_q'(\zeta)|^2\, N_{q',\varphi}(\zeta)\, dA(\zeta)\,,$$

$$I_2 \;=\; \int_{D(q,r)\setminus G_{\varepsilon,q',q}} |\alpha_q'(\zeta)|^2\, N_{q',\varphi}(\zeta)\, dA(\zeta)\,,$$

and

$$I_3 \;=\; \int_{G_{\varepsilon,q',q}\cap D(q,r)} |\alpha_q'(\zeta)|^2\, N_{q',\varphi}(\zeta)\, dA(\zeta)\,.$$

By (46),

$$I_1 \;<\; \frac{k}{4}.$$
<div align="right">(48)</div>

Next,

$$
\begin{aligned}
I_2 \;&\leq\; \varepsilon \int_{D(q,r)\backslash G_{\varepsilon,q',q}} |\alpha_q'(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2)\, dA(\zeta) \\
&\leq\; \varepsilon \, \|\alpha_q\|_*^2 \\
&\leq\; C\,\varepsilon .
\end{aligned}
$$
<div align="right">(49)</div>

By (38)

$$
\begin{aligned}
I_3 \;&=\; \int_{G_{\varepsilon,q',q}\cap D(q,r)} |\alpha_q'(\zeta)|^2 N_{q',\varphi}(\zeta)\, dA(\zeta) \\
&\leq\; C \int_{G_{\varepsilon,q',q}\cap D(q,r)} |\alpha_q'(\zeta)|^2 (1 - |\alpha_{\varphi(q')}(\zeta)|^2)\, dA(\zeta) \\
&\leq\; C \int_{G_{\varepsilon,q',q}\cap D(q,r)} \frac{(1 - |q|^2)^2}{|1 - \bar{q}\zeta|^4} dA(\zeta) \\
&\asymp\; \int_{G_{\varepsilon,q',q}\cap D(q,r)} \frac{1}{|D(q,r)|} dA(\zeta) \\
&\asymp\; \frac{|G_{\varepsilon,q',q}\cap D(q,r)|}{|D(q,r)|} .
\end{aligned}
$$
<div align="right">(50)</div>

By (45), (48), (49), (50), we conclude that there exists $r \in (0,1)$ such that for all $w \in \mathbb{D}$,

$$\frac{k}{4} \;<\; \varepsilon + C\frac{|G_{\varepsilon,q',q}\cap D(w,r)|}{|D(q,r)|}$$

or

$$\frac{k}{4} - \varepsilon \;\leq\; \frac{|G_{\varepsilon,q',q}\cap D(q,r)|}{|D(q,r)|} .$$
<div align="right">(51)</div>

Hence, for any $\varepsilon > 0$ with $\varepsilon < k/4$, the set $G_{\varepsilon,q',q}$ satisfies (44).

Next, assume that there exists an $\varepsilon > 0$ and $r \in (0,1)$ such that for all $q \in \mathbb{D}$, there exists $q' \in \mathbb{D}$ such that (44) holds. Then, for each fixed $q \in \mathbb{D}$,

$$\begin{aligned}
||\alpha_q \circ \varphi||_*^2 &\geq \int_{G_{\varepsilon,q',q}} |\alpha_q'(\zeta)|^2 \, N_{q',\varphi}(\zeta) \, dA(\zeta) \\
&\geq \int_{G_{\varepsilon,q',q} \cap D(q,r)} |\alpha_q'(\zeta)|^2 \, (1 - |\alpha_q(\zeta)|^2) \, dA(\zeta) \, .
\end{aligned}$$

By (7), (8)

$$||\alpha_q \circ \varphi||_* \geq C \frac{|G_{\varepsilon,q',q} \cap D(q,r)|}{|D(q,r)|} \asymp 1$$

and the conclusion follows.

Lastly, since

$$\frac{|G_{\varepsilon,q',q} \cap D(w,r)|}{|D(w,r)|} \leq \frac{|G_\varepsilon \cap D(w,r)|}{|D(w,r)|} \, ,$$

and by (45) and (50), it is clear that G_ε satisfies the reverse Carleson condition. $\qquad\square$

In section III of [20], Luecking characterized when a measurable subset F of $\mathbb{D}$ satisfies the reverse Carleson condition by using pseudohyperbolic disks, disks whose centers lying on the unit circle $\mathbb{T}$ and also using a collection of disks whose centers lie on $\mathbb{D}$. All the arguments remain valid if we replace F with a collection of measurables subsets F_q, $q \in \mathbb{D}$ and we obtain the following proposition.

Proposition 3.9. *Given a collection of measurable sets F_q, $q \in \mathbb{D}$, the following are equivalent:*

(A) There exists a constant $\delta > 0$ and $r \in (0,1)$ such that for all $q \in \mathbb{D}$ and for all disks D with centers on $\mathbb{T}$, there exists $q' \in \mathbb{D}$ such that

$$|F_{q'} \cap D| > \delta \, |\mathbb{D} \cap D| \, .$$

(B) *There exists a constant $\delta_0 > 0$ and $\eta \in (0, 1)$ such that for all $q \in \mathbb{D}$ there exists $q' \in \mathbb{D}$ such that*

$$|F_{q'} \cap \mathbb{D}\,(q, \eta(1 - |q|))\,| > \delta_0\,|\mathbb{D}\,(q, \eta(1 - |q|))\,|\,.$$

(C) *There exists a constant $\delta_1 > 0$ and $r \in (0, 1)$ such that for all $q \in \mathbb{D}$ there exists $q' \in \mathbb{D}$ such that*

$$|F_{q'} \cap D(q, r)| > \delta_1\,|D(q, r)|\,.$$

Next, we discuss how the main result in [20] can be modified to allow for a reverse Carleson type condition for a collection of measurable sets F_q, $q \in \mathbb{D}$. Let δ_0 and η be fixed and assume (B) in Proposition 3.9 holds. Since η is fixed, as Luecking does in [20, page 6], we abbreviate the notation for the disk $\mathbb{D}(q, \eta(1 - |q|))$ with $D(q)$. Let A^1 denote the space of analytic and Lebesgue integrable functions on $\mathbb{D}$. Given an analytic function f and $\lambda \in (0, 1)$ define the set

$$E_\lambda(q) = \{z \in D(q) : |f(z)| > \lambda\,|f(q)|\}$$

and the operator

$$B_\lambda f(q) = \frac{1}{|E_\lambda(q)|} \int_{E_\lambda(q)} |f(z)|\,dA(z).$$

Note Lemma 1, Lemma 2 and Lemma 3 in [20]. Moreover as done in [20, page 9], if $\varepsilon \in (0, 1)$ is small enough and

$$F = \{q \in \mathbb{D} : |f(q)| \geq \varepsilon^3\,B_\lambda f(q)\} \tag{52}$$

then

$$\int_{\mathbb{D}} |f(z)|\,dA(z) < 2 \int_F |f(z)\,dA(z)\,,$$

and for all $q \in F$ there exists $q' \in \mathbb{D}$ such that

$$\frac{1}{|D(q)|} \int_{F_{q'} \cap D(q)} |f(z)| \, dA(z) \geq \frac{1}{2} \delta_0 \lambda |f(q)| \, .$$

Integrating the above over F we obtain

$$\int_F \frac{1}{|D(q)|} \int_{F_{q'} \cap D(q)} |f(z)| \, dA(z) \, dA(q) \;\geq\; \frac{1}{2} \delta_0 \lambda \int_F |f(q)| \, dA(q)$$

$$\geq\; \frac{1}{4} \delta_0 \lambda \int_{\mathbb{D}} |f(q)| \, dA(q) \, , \qquad\qquad (53)$$

for all $f \in A^1$. The theorem below is now an extension of one direction of the main theorem in [20].

Theorem 3.10. *Given a collection of measurable subsets of $\mathbb{D}$, F_q, $q \in \mathbb{D}$, and F as in (52), if there exists a constant $\delta_0 > 0$ and $\eta \in (0,1)$ such that for all $q \in \mathbb{D}$ there exists $q' \in \mathbb{D}$ such that*

$$|F_{q'} \cap \mathbb{D}(q, \eta(1 - |q|))| > \delta_0 \, |\mathbb{D}(q, \eta(1 - |q|))| \, ,$$

then there exists a constant $C > 0$ such that for all $q \in F$ there exists $q' \in \mathbb{D}$ such that for all $f \in A^1$,

$$\int_F \frac{1}{|D(q)|} \int_{F_{q'} \cap D(q)} |f(z)| \, dA(z) \, dA(q) \;\geq\; C \int_{\mathbb{D}} |f(q)| \, dA(q) \, .$$

Remark 3.1. The main theorem in [20] as well as Theorem 3.10 remain valid if $|f|$ is replaced with $|f|^2$, see [20, page 6]. Moreover, as Luecking indicates in [20, page 10], instead of $dA(z)$ one may use the measure $(1 - |z|^2)^2 \, dA(z)$, and the main theorem in [20] is valid. The two critical properties of the weight $w(z) = (1 - |z|^2)^2$ are $w(q) \leq C \inf\{w(z) : z \in D(q, \eta)\}$ and that $w(\alpha_q(z)) = |\alpha'_q(z)|^2 \, (1 - |z|^2)^2$. Consequently, Theorem 3.10 is also valid for the measure $(1 - |z|^2)^2 \, dA(z)$.

We are now ready to prove the following.

Theorem 3.11. *Let φ be an analytic self-map of $\mathbb{D}$. Then the composition operator $C_\varphi : \mathcal{B} \to BMOA$ is closed range if and only if there exists an $\varepsilon > 0$ and $r \in (0,1)$ such that for all $q \in \mathbb{D}$, there exists $q' \in \mathbb{D}$ such that*

$$\frac{|G_{\varepsilon,q',q} \cap D(q,r)|}{|D(q,r)|} \asymp 1. \tag{54}$$

Proof. First, assume that there exists $\varepsilon > 0$ and $r \in (0,1)$ such that for all $q \in \mathbb{D}$, there exits $q' \in \mathbb{D}$ such that (54) holds. By Proposition 3.9 this is equivalent to the existence of an $\varepsilon > 0$, a constant $\delta_0 > 0$ and $\eta \in (0,1)$ such that for all $q \in \mathbb{D}$ there exists $q' \in \mathbb{D}$ such that

$$|G_{\varepsilon,q',q} \cap D(q)| > \delta_0 |D(q)|. \tag{55}$$

Given such $q \in \mathbb{D}$ and $q' \in \mathbb{D}$, for each $f \in \mathcal{B}$

$$
\begin{aligned}
\|C_\varphi f\|_*^2 &\geq \int_{G_{\varepsilon,q',q}} |f'(\zeta)|^2 \, N_{q',\varphi}(\zeta) \, dA(\zeta) \\
&\geq \varepsilon \int_{G_{\varepsilon,q',q}} |f'(\zeta)|^2 \, (1 - |\alpha_q(\zeta)|^2) \, dA(\zeta). \\
&\geq \varepsilon \int_{G_{\varepsilon,q',q} \cap D(q)} |f'(\zeta)|^2 \, (1 - |\alpha_q(\zeta)|^2) \, dA(\zeta).
\end{aligned}
$$

If $\zeta \in D(q)$ then $\zeta \in D(q,r)$, for any fixed η and $r \in (0,1)$ with $r \geq 2\eta/1 + \eta^2$, and by (2), (7), (8), $(1 - |\alpha_q(\zeta)|^2) \asymp 1$; therefore

$$\|C_\varphi f\|_*^2 \geq C \int_{G_{\varepsilon,q',q} \cap D(q)} |f'(\zeta)|^2 \, dA(\zeta). \tag{56}$$

By (7), (55), Theorem 3.10, and Remark 3.1 , for all $f \in A^2$

$$\int_F \int_{G_{\varepsilon,q',q} \cap D(q)} |f'(\zeta)|^2 \, dA(\zeta) \, dA(q) \geq C \int_{\mathbb{D}} |f'(q)|^2 (1 - |q|^2)^2 \, dA(q)$$

Therefore by (56), for all $f \in A^2$

$$
\begin{aligned}
||C_\varphi f||_*^2 &\geq C \int_{\mathbb{D}} |f'(q)|^2 (1 - |q|^2)^2 \, dA(q) \\
&\asymp ||f||_{A^2}^2 .
\end{aligned}
$$

Since the $BMOA$ seminorm is invariant under Mobius automorphisms, for all $q \in \mathbb{D}$

$$
||C_\varphi f||_*^2 \geq C ||f \circ \alpha_q - f(q)||_{A^2}^2 . \tag{57}
$$

By (9) we conclude that for all $f \in \mathcal{B}$, $||C_\varphi f||_* \geq C ||f||_{\mathcal{B}}$ and by Proposition 3.5, $C_\varphi : \mathcal{B} \to BMOA$ is closed range. Conversely, if $C_\varphi : \mathcal{B} \to BMOA$ is closed range then for all $q \in \mathbb{D}$, $||\alpha_q \circ \varphi||_* \geq C ||\alpha_q||_{\mathcal{B}} \geq C$ and by Theorem 3.8 the conclusion follows. $\qquad\square$

Theorem 3.12. *The composition operator C_φ is closed range on $BMOA$ if and only if there exists a constant $k > 0$ such that $||\alpha_q \circ \varphi||_* \geq k$, for all $q \in \mathbb{D}$.*

Proof. First, assume that there exists a constant $k > 0$ such that $||\alpha_q \circ \varphi||_* \geq k$. Then given a $q \in \mathbb{D}$, there exists $q_n \in \mathbb{D}$ such that

$$
k^2 \leq \lim_{n \to \infty} \int_{\mathbb{D}} |\alpha_q'(\zeta)|^2 \, N_{q_n, \varphi} \, dA(\zeta)
$$

or

$$
\lim_{n \to \infty} \int_{\mathbb{D}} |\alpha_q'(\zeta)|^2 \left(k^2 - N_{q_n, \varphi}(\zeta) \right) \, dA(\zeta) \leq 0 .
$$

Therefore

$$
\lim_{n \to \infty} \int_{\mathbb{D} \backslash G_{k^2, q_n, q}} |\alpha_q'(\zeta)|^2 \left(k^2 (1 - |\alpha_q(\zeta)|^2) - N_{q_n, \varphi}(\zeta) \right) \, dA(\zeta) \leq 0 . \tag{58}
$$

If for each $q \in \mathbb{D}$ there exists a natural number n such that the Lebesgue area measure

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of $\mathbb{D} \setminus G_{k^2,q_n,q}$ is 0, then

$$
\begin{aligned}
||C_\varphi f||_*^2 & \geq \int_{G_{k^2,q_n,q}} |f'(\zeta)|^2 \, N_{q_n,\varphi}(\zeta) \, dA(\zeta) \\
& = \int_{\mathbb{D}} |f'(\zeta)|^2 \, N_{q_n,\varphi}(\zeta) \, dA(\zeta) \\
& \geq k^2 \int_{\mathbb{D}} |f'(\zeta)|^2 \, (1 - |\alpha_q(\zeta)|^2) \, dA(\zeta)
\end{aligned}
$$

and $||C_\varphi f||_* \geq k \, ||f||_*$ for all $f \in BMOA$. By Proposition 3.5, C_φ is closed range on $BMOA$.

Therefore from now on we may assume that for some $q \in \mathbb{D}$ and for all n the Lebesgue area measure of $\mathbb{D} \setminus G_{k^2,q_n,q}$ is positive. The integrand in (58) is non negative in $\mathbb{D} \setminus G_{k^2,q_n,q}$, so we must have that for almost every $\zeta \in \mathbb{D} \setminus G_{k^2,q_n,q}$,

$$
k^2(1 - |\alpha_q(\zeta)|^2) - N_{q_n,\varphi}(\zeta) \to 0 \,. \tag{59}
$$

For each $f \in BMOA$

$$
\int_{\mathbb{D} \setminus G_{k^2,q_n,q}} |f'(\zeta)|^2 \, k^2(1 - |\alpha_q(\zeta)|^2) - N_{q_n,\varphi}(\zeta) \, dA(\zeta) \leq k^2 \, ||f||_*^2
$$

therefore by (59) and the Lebesgue Dominated Convergence Theorem

$$
\lim_{n \to \infty} \int_{\mathbb{D} \setminus G_{k^2,q_n,q}} |f'(\zeta)|^2 \, k^2(1 - |\alpha_q(\zeta)|^2) - N_{q_n,\varphi}(\zeta) \, dA(\zeta) = 0 \,. \tag{60}
$$

Write

$$
\int_{\mathbb{D}} |f'(\zeta)|^2 k^2(1 - |\alpha_q(\zeta)|^2) \, dA(\zeta) = I + II \,, \tag{61}
$$

where

$$I = \int_{\mathbb{D}\backslash G_{k^2,qn,q}} |f'(\zeta)|^2 \, k^2 (1 - |\alpha_q(\zeta)|^2) \, dA(\zeta)$$

and

$$II = \int_{G_{k^2,qn,q}} |f'(\zeta)|^2 \, k^2 (1 - |\alpha_q(\zeta)|^2) \, dA(\zeta) \,.$$

Note that $I = I_1 + I_2$, where

$$I_1 = \int_{\mathbb{D}\backslash G_{k^2,qn,q}} |f'(\zeta)|^2 \, k^2 (1 - |\alpha_q(\zeta)|^2) - N_{q_n,\varphi}(\zeta) \, dA(\zeta)$$

and

$$I_2 = \int_{\mathbb{D}\backslash G_{k^2,qn,q}} |f'(\zeta)|^2 \, N_{q_n,\varphi}(\zeta) \, dA(\zeta) \,.$$

By (60), $I_1 \to 0$ as $n \to \infty$, and

$$II \leq \int_{G_{k^2,qn,q}} |f'(\zeta)|^2 \, N_{q_n,\varphi}(\zeta) \, dA(\zeta) \,.$$

Therefore by (61) and for all $q \in \mathbb{D}$

$$\int_{\mathbb{D}} |f'(\zeta)|^2 k^2 (1 - |\alpha_q(\zeta)|^2) \, dA(\zeta) \leq \int_{\mathbb{D}} |f'(\zeta)|^2 \, N_{q_n,\varphi}(\zeta) \, dA(\zeta)| + I_1$$
$$\leq \|C_\varphi f\|_*^2 + I_1 \,. \tag{62}$$

We conclude that $k\|f\|_* \leq \|C_\varphi f\|_*$ for all $f \in BMOA$, and by Proposition 3.5, C_φ is closed range on $BMOA$.

For the converse if C_φ is closed range on $BMOA$ then C_φ is bounded below on

$BMOA$. By Proposition 3.5 and by (21) there exists $\delta > 0$ such that for all $q \in \mathbb{D}$,

$$||\alpha_q \circ \varphi||_* \geq \delta ||\alpha_q||_*$$
$$\asymp 1,$$

and the conclusion follows. □

It is proved in [4, Theorem 12] that C_φ is bounded in $VMOA$ if and only if $\varphi \in VMOA$. The following now is an immediate corollary of Theorem 3.12.

Corollary 3.13. *Let $\varphi \in VMOA$. Then C_φ is closed range on $VMOA$ if and only if there exists a constant $k > 0$ such that $||\alpha_q \circ \varphi||_* \geq k$, for all $q \in \mathbb{D}$.*

Theorem 3.14. *A subset H of $\mathbb{D}$ satisfies the reverse Carleson condition if and only if H is a sampling set for $BMOA$.*

Proof. First, assume that H satisfies the reverse Carleson condition. If H is not a sampling set for $BMOA$, then there exists a sequence (f_n) in $BMOA$ such that $||f_n||_* = 1$ and

$$\lim_{n \to \infty} \sup_{q' \in \mathbb{D}} \int_H |f_n'(z)|^2 \left(1 - |\alpha_{q'}(z)|^2\right) dA(z) = 0,$$

and so for $q' = 0$,

$$\lim_{n \to \infty} \int_H |(f_n'(\zeta)|^2 \left(1 - |\zeta|^2\right) dA(\zeta) = 0. \tag{63}$$

Since $BMOA$ is continuously contained in H^2, there exists a constant $C > 0$ such that $||f_n||_{H^2} \leq C$ and by (63), H does not satisfy reverse Carleson condition. We conclude that H is a sampling set for the $BMOA$ space.

Conversely, suppose that H is a sampling set for the $BMOA$ space. Then for all $q \in \mathbb{D}$,

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$||\alpha_q||_* \asymp 1$ and

$$\sup_{q' \in \mathbb{D}} \int_H |\alpha_q'(z)|^2 \, (1 - |\alpha_{q'}(z)|^2) \, dA(z) \asymp 1 \, .$$

Then, there exists $C > 0$ such that for all $q \in \mathbb{D}$

$$\int_H |\alpha_q'(z)|^2 \, dA(z) \geq C \, . \tag{64}$$

By [16, Theorem 4.1] and (64) we conclude that H satisfies the reverse Carleson condition. $\qquad\square$

Proposition 3.15. *Let $H \subseteq \mathbb{D}$ and $q \in \mathbb{D}$. Then H satisfies the reverse Carleson condition if and only if $\alpha_q(H)$ satisfies the reverse Carleson condition.*

Proof. For each $H \subseteq \mathbb{D}$ be a Carleson set for the Bergman space, for fixed $q \in \mathbb{D}$, and by making below the change of variables $z = \alpha_q(\zeta)$ we obtain

$$\int_{\alpha_q(H)} |\alpha_p'(z)|^2 \, dA(\zeta) = \int_H |(\alpha_p \circ \alpha_q)'(\zeta)|^2 \, dA(\zeta) \, .$$

By [16, Theorem 4.1] the result follows. $\qquad\square$

Corollary 3.16. *Let φ be an analytic self-map of $\mathbb{D}$. If C_φ is closed range on the Bloch space, then it is also closed range on BMOA.*

Proof. By [3, Corollary 2.3], C_φ is closed range on the Bloch space if and only if for all $q \in \mathbb{D}$, $||\alpha_q \circ \varphi||_\mathcal{B} \asymp 1$. Since $BMOA$ is continuously contained in $\mathcal{B}$, for all $q \in \mathbb{D}$ we have that $||\alpha_q \circ \varphi||_* \asymp 1$. By Theorem 3.12 we conclude that C_φ is closed range on $BMOA$. $\quad\square$

Corollary 3.17. *Let φ be a univalent self-map of $\mathbb{D}$. Then, C_φ is closed range on $\mathcal{B}$, if and only if C_φ is closed range on BMOA.*

Proof. It is shown in [15, Corollary 2] that for φ a univalent self-map of $\mathbb{D}$, if C_φ is closed

range on $BMOA$, then it is also closed range on $\mathcal{B}$. By Corollary 3.16 the result is now clear. $\square$

Corollary 3.18. *Let φ be an analytic self-map of $\mathbb{D}$. If C_φ is closed range on H^2, then it is also closed range on $BMOA$.*

Proof. Given $q \in \mathbb{D}$, let k_q denote the normalized reproducing kernel in H^2, that is

$$k_q(z) = \frac{1 - |q|^2}{1 - \overline{q}z} \, .$$

It is known, see [22] and [16], that C_φ is closed range on H^2 if and only if for all $q \in \mathbb{D}$, $||k_q \circ \varphi||_{H^2} \asymp 1$. By (12) it is easy to see that $||k_q \circ \varphi||_{H^2} \asymp |q| \, ||\alpha_q \circ \varphi||_{H^2}$, if $q \neq 0$. By Theorem 3.12 and since the inclusion map $i : BMOA \to H^2$ is continuous, the conclusion easily follows. $\square$

Laitila characterized in [18] the isometries among composition operators on $BMOA$. Moreover he showed that if φ is an inner function then $||C_\varphi f||_* = ||f||_*$, for all $f \in BMOA$. Below we give another characterization of closed range composition operators on $BMOA$. Our proof extends and simplifies the proof in [18] for the isometries.

Theorem 3.19. *For each φ analytic self map of $\mathbb{D}$, the following conditions are equivalent:*

    (a) *There exists $k \in (0,1]$ such that for every $w \in \mathbb{D}$, $||\alpha_w \circ \varphi||_* \geq k$.*

    (b) *There exists $k \in (0,1]$ such that for every $w \in \mathbb{D}$ there exists $w' \in \mathbb{D}$ with $|\alpha_w(w')|^2 \leq 1 - k^2$, there exists a sequence (q_n) in $\mathbb{D}$ such that $\varphi(q_n) \to w'$ and*

$$\lim_{n \to \infty} ||\varphi_{q_n}||_{H^2} \geq k \, ,$$

where for all natural numbers n, $\varphi_{q_n} = \alpha_{\varphi(q_n)} \circ \varphi \circ \alpha_{q_n}$.

Proof. First, assume that (a) holds for some $k \in (0,1]$ and let $w \in \mathbb{D}$. Then, there exists a

sequence (q_n) in $\mathbb{D}$ such that for all n,

$$||\alpha_w \circ \varphi \circ \alpha_{q_n} - \alpha_w(\varphi(q_n))||^2_{H^2} \geq k^2 - \frac{1}{n} \,. \tag{65}$$

Since $(\varphi(q_n))$ is a bounded sequence in $\mathbb{D}$, it has a convergent subsequence in $\overline{\mathbb{D}}$. Without loss of generality, there exists $w' \in \overline{\mathbb{D}}$ such that $\varphi(q_n) \to w'$. By (65),

$$
\begin{aligned}
|\alpha_w(\varphi(q_n))|^2 &= ||\alpha_w \circ \varphi \circ \alpha_{q_n}||^2_{H^2} - ||\alpha_w \circ \varphi \circ \alpha_{q_n} - \alpha_w(\varphi(q_n))||^2_{H^2} \\
&\leq 1 - k^2 + \frac{1}{n} \,,
\end{aligned}
$$

therefore, $|\alpha_w(w')|^2 \leq 1 - k^2$ and $w' \in \mathbb{D}$.

Let $\alpha_w \circ \alpha_{\varphi(q_n)} = \lambda_n \alpha_{b_n}$ where $\lambda_n \, b_n = \alpha_w(\varphi(q_n))$ and $|\lambda_n| = 1$. Then for all $z \in \mathbb{D}$,

$$
\begin{aligned}
\alpha_w \circ \varphi \circ \alpha_{q_n}(z) - \alpha_w(\varphi(q_n)) &= \alpha_w \circ \alpha_{\varphi(q_n)} \circ \varphi_{q_n}(z) - \alpha_w(\varphi(q_n)) \\
&= \lambda_n \alpha_{b_n} \circ \varphi_{q_n}(z) - \lambda_n b_n \\
&= \lambda_n \frac{(|b_n|^2 - 1)\varphi_{q_n}(z)}{1 - \bar{b}_n \varphi_{q_n}(z)} \,. \tag{66}
\end{aligned}
$$

Therefore,

$$||\alpha_w \circ \varphi \circ \alpha_{q_n} - \alpha_w(\varphi(q_n))||_{H^2} \leq (1 + |b_n|)||\varphi_{q_n}||_{H^2} \,;$$

by (65) we conclude, by taking a further subsequence if necessary, that

$$
\begin{aligned}
\lim_{n \to \infty} ||\varphi_{q_n}||_{H^2} &\geq \frac{k}{1 + |a_w(w')|} \\
&\geq \frac{k}{2} \tag{67}
\end{aligned}
$$

and (b) holds. Note that if $k = 1$ above then, $w = w'$ and

$$\lim_{n \to \infty} ||\varphi_{q_n}||_{H^2} = 1 \, .$$

Next, assume that (b) holds for some $k \in (0, 1]$ and let $w \in \mathbb{D}$. Then there exists $w' \in \mathbb{D}$ with $|\alpha_w(w')|^2 \leq 1 - k^2$, there exists a sequence (q_n) in $\mathbb{D}$ such that $\varphi(q_n) \to w'$ and

$$\lim_{n \to \infty} ||\varphi_{q_n}||_{H^2} \geq k \, . \tag{68}$$

Let $|\lambda_n| = 1$ and (b_n) be as in the proof of $(a) \to (b)$. Then (66) is valid, and for every $n \in \mathbb{N}$

$$||\alpha_w \circ \varphi \circ \alpha_{q_n} - \alpha_w(\varphi(q_n))||_{H^2}^2 \ \geq \ (1 - |b_n|)^2 ||\varphi_{q_n}||_{H^2}^2 \, .$$

We conclude that

$$||\alpha_w \circ \varphi||_*^2 \ \geq \ ||\alpha_w \circ \varphi \circ \alpha_{q_n} - \alpha_w(\varphi(q_n))||_{H^2}^2$$
$$\geq \ (1 - |b_n|)^2 ||\varphi_{q_n}||_{H^2}^2 \, ,$$

and by (68)

$$||\alpha_w \circ \varphi||_*^2 \geq \frac{(1 - |\alpha_w(w')|^2)^2}{(1 + |\alpha_w(w')|)^2} \, k^2 \, . \tag{69}$$

If $k \in (0, 1)$ and since $|\alpha_w(w')|^2 \leq 1 - k^2$ we conclude by (69) that for all $w \in \mathbb{D}$

$$||\alpha_w \circ \varphi||_* \geq \frac{k^4}{4} \, .$$

If $k = 1$, then $w = w'$ and by (69) $||\alpha_w \circ \varphi||_*^2 = 1$. The proof is now complete. $\square$

4   Composition operators on Q_p

In this chapter we study the spaces Q_p and closed range composition operators on them.

Definition 4.1. *For $p \in [0, \infty)$ we say that a function f analytic on $\mathbb{D}$ belongs to Q_p if*

$$||f||^2_{Q_p} = \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\alpha_q(z)|^2)^p \, dA(z) < \infty,$$

where $||f||_{Q_p}$ is a seminorm and $|||f|||_{Q_p} = |f(0)| + ||f||_{Q_p}$ defines a norm in Q_p that makes it a Banach space.

The collection of all Q_p spaces includes many familiar spaces.

- if $p = 0$ then Q_0 is the Dirichlet space with an equivalent norm.

- if $p = 1$ then Q_1 is the space $BMOA$.

- if $p > 1$ then Q_p is the Bloch space with an equivalent norm, see [36, Corollary 1.2.1].

It is known, see [36], that if $0 \le p < q < \infty$ then $Q_p \subseteq Q_q$ and the inclusion map is continuous. Moreover, if $0 \le p \ne q \le 1$ then $Q_p \ne Q_q$.

Let $p \in (0, \infty)$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with a_n nonnegative and non increasing. It is shown in [36, Corollary 3.3.1] that $f \in Q_p$ if and only if $\sup_{n \in N} na_n < \infty$. For example, $f(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = \log(1 - z) \in Q_p$ for all p.

Now, we estimate $||\alpha_q||_{Q_p}$ and show that there exists $c > 0$ such that $||\alpha_q||_{Q_p} \le c$, for each given $p \ge 0$.

Fixing $p \ge 0$, $q \in \mathbb{D}$ and by making the change of variables $\alpha_q(z) = \zeta$ we obtain,

$$
\begin{aligned}
||\alpha_q||^2_{Q_p} &= \sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} |\alpha_q'(z)|^2 (1 - |\alpha_\lambda(z)|^2)^p \, dA(z) \\
&= \sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} (1 - |\alpha_\lambda \circ \alpha_q(\zeta)|^2)^p \, dA(\zeta) \quad (70)
\end{aligned}
$$

Since for a fixed $q \in \mathbb{D}$, each Möbius transformation can be written as $e^{i\theta}\alpha_\lambda \circ \alpha_q$, $\lambda \in \mathbb{D}$, $\theta \in \mathbb{R}$ we obtain,

$$\|\alpha_q\|^2_{Q_p} = \sup_{\lambda \in \mathbb{D}} \int_\mathbb{D} (1 - |\alpha_\lambda(\zeta)|^2)^p \, dA(\zeta).$$

Note that by (2),

$$\int_\mathbb{D} (1 - |\alpha_\lambda(\zeta)|^2)^p dA(\zeta) = (1 - |\lambda|^2)^p \int_\mathbb{D} \frac{(1 - |\zeta|^2)^p}{|1 - \bar{\lambda}\zeta|^{2p}} dA(\zeta).$$

Now we apply [37, Lemma 3.10] with $c = p - 2$. We have the following cases:

- If $c < 0$ then $0 \leq p < 2$ and

$$\|\alpha_q\|^2_{Q_p} \lesssim (1 - |\lambda|^2)^p \lesssim 1.$$

- If $c > 0$ then $p > 2$ and as $|\lambda| \to 1$

$$\|\alpha_q\|^2_{Q_p} \asymp (1 - |\lambda|^2)^p \frac{1}{(1 - |\lambda|^2)^{p-2}} = (1 - |\lambda|^2)^2 \lesssim 1.$$

- if $c = 0$ then $p = 2$ and as $|\lambda| \to 1$

$$\|\alpha_q\|^2_{Q_p} \asymp \log \frac{1}{1 - |\lambda|^2}(1 - |\lambda|^2)^2 \lesssim 1.$$

Hence, there exists $c > 0$ such that for all $q \in \mathbb{D}$, $\|\alpha_q\|^2_{Q_p} \leq c$.

For each $p \in [0, \infty)$ the inclusion map $i : Q_p \to \mathcal{B}$ is continuous. Similarly to the proof of (25) we obtain that there exists $c > 0$ such that

$$|f(z)| \leq |f(0)| + c\|f\|_{Q_p} \log \frac{1 + |z|}{1 - |z|}. \tag{71}$$

Let $p \in [0, \infty)$. The *weighted Dirichlet space* $\mathcal{D}_p$ in the space of all functions $f \in H(\mathbb{D})$

satisfying

$$||f||^2_{\mathcal{D}_p} = \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^p dA(z) < \infty .$$

The above is a seminorm in $\mathcal{D}_p$ and $|||f|||_{\mathcal{D}_p} = |f(0)| + ||f||_{\mathcal{D}_p}$ is a norm in $\mathcal{D}_p$ and $\mathcal{D}_p$ is a Banach space with this norm.

Note that given $q \in \mathbb{D}$ and by making the change of variables $\alpha_q(z) = \zeta$ we obtain

$$
\begin{aligned}
||f \circ \alpha_q - f(q)||^2_{\mathcal{D}_p} &= \int_{\mathbb{D}} |(f \circ \alpha_q)'(z)|^2 (1 - |z|^2)^p dA(z) \\
&= \int_{\mathbb{D}} |f'(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2)^p dA(\zeta)
\end{aligned}
$$

therefore, $f \in Q_p$ if and only if

$$\sup_{q \in \mathbb{D}} ||f \circ \alpha_q - f(q)||_{\mathcal{D}_p} < \infty.$$

Below we show a Lipschitz continuity property that $||f \circ \alpha_q - f(q)||_{Q_p}$ has with respect to the pseudo hyperbolic metric.

Theorem 4.1. *Let $p \in (0, 1]$. Then, for each $f \in Q_p$ and for all $q, w \in \mathbb{D}$*

$$| \, ||f \circ \alpha_q - f(q)||^2_{\mathcal{D}_p} - ||f \circ \alpha_w - f(w)||^2_{\mathcal{D}_p} \, | \leq 2^{p+1} \rho(q, w)^p ||f||^2_{Q_p} .$$

Proof. Given $p, q \in \mathbb{D}$ and by (5),

$$
\begin{aligned}
A_{p,q} &:= | \int_{\mathbb{D}} | f'(z)|^2 \Big((1 - |\alpha_q(z)|^2)^p - (1 - |\alpha_w(z)|^2)^p \Big) dA(z) | \\
&\leq \int_{\mathbb{D}} |f'(z)|^2 | \, |\alpha_q(z)|^2 - |\alpha_w(z)|^2 |^p dA(z) \\
&\leq 2^p \int_{\mathbb{D}} |f'(z)|^2 | \, |\alpha_w(z)| - |\alpha_q(z)| \, |^p dA(z) \\
&\leq 2^p \rho(w, q)^p \int_{\mathbb{D}} |f'(z)|^2 (1 - |\alpha_q(z)| \, |\alpha_w(z)|)^p dA(z). \qquad (72)
\end{aligned}
$$

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Let $\mathcal{A}_1 := \{z \in \mathbb{D} : |\alpha_w(z)| > |\alpha_q(z)|\}$ and $\mathcal{A}_2 := \{z \in \mathbb{D} : |\alpha_q(z)| > |\alpha_w(z)|\}$. Then we have

$$\int_{\mathcal{A}_1} |f'(z)|^2 \, (1 - |\alpha_w(z)| \, |\alpha_q(z)|)^p \, dA(z) \;\leq\; \int_{\mathcal{A}_1} |f'(z)|^2 (1 - |\alpha_q(z)|^2)^p \, dA(z)$$
$$\leq \; \|f\|_{Q_p}^2$$

and

$$\int_{\mathcal{A}_2} |f'(z)|^2 \, (1 - |\alpha_w(z)| \, |\alpha_q(z)|)^p \, dA(z) \;\leq\; \int_{\mathcal{A}_2} |f'(z)|^2 (1 - |\alpha_w(z)|^2)^p \, dA(z)$$
$$\leq \; \|f\|_{Q_p}^2 \, .$$

Therefore by (72),

$$A_{p,q} \leq 2^{p+1} \, \rho(w,q)^p \, \|f\|_{Q_p}^2 \, .$$

and the conclusion follows. $\qquad\square$

Remark 4.1. Let U be a measurable subset of $\mathbb{D}$, $0 < p \leq 1$, $f \in Q_p$, $w \in \mathbb{D}$ and

$$A_w = \int_U |f'(z)|^2 \, (1 - |\alpha_w(z)|^2)^p \, dA(z) \, .$$

It is a corollary of the proof of Theorem 4.1 that for each $f \in Q_p$ and $q, w \in \mathbb{D}$,

$|A_w - A_q| \leq 2^{p+1} \, \rho(w,q)^p \|f\|_{Q_p}^2 .$

Since $Q_1 = BMOA$ the following is part of the above theorem.

Theorem 4.2. *There exists $C > 0$ such that for each $f \in BMOA$ and $p, q \in \mathbb{D}$*

$$\big| \, \|f \circ \alpha_p - f(p)\|_{H^2}^2 - \|f \circ \alpha_q - f(q)\|_{H^2}^2 \big| \leq C \, \rho(p,q) \|f\|_*^2 \, .$$

If $p > 1$ and $z, \zeta \in \mathbb{D}$ let $\gamma_\zeta(z) = 1 - |\alpha_\zeta(z)|^2$. Then for each $q, w, z \in \mathbb{D}$

$$|\gamma_q(z)^p - \gamma_w(z)^p| \leq p\Big(|\alpha_q(z)|^2 - |\alpha_w(z)|^2\Big)(\gamma_q(z)^{p-1} - \gamma_w(z)^{p-1}).$$

Therefore, if $f \in Q_p$ then similarly to the proof of Theorem 4.1 we obtain the following

Theorem 4.3. *Let $p \in (1, \infty)$. Then for each $f \in Q_p$ and all $q, w \in \mathbb{D}$*

$$|\;||f \circ \alpha_q - f(q)||^2_{\mathcal{D}_p} - ||f \circ \alpha_w - f(w)||^2_{\mathcal{D}_p}\;| \leq 4p\,\rho(q, w)\,||f||^2_{Q_p}\;.$$

Remark 4.2. Let U be a measurable subset of $\mathbb{D}$, $p > 1$, $f \in Q_p$, $w \in \mathbb{D}$ and

$$A_w = \int_U |f'(z)|^2\,(1 - |\alpha_w(z)|^2)^p\,dA(z)\,.$$

It is a corollary of the proof of Theorem 4.3 that for each $f \in Q_p$ and $q, w \in \mathbb{D}$,

$|A_w - A_q| \leq 4p\,\rho(w, q)||f||^2_{Q_p}.$

Definition 4.2. *Let $p \in [0, \infty)$. The counting function for Q_p for each $q \in \mathbb{D}$, and if $\zeta \in \varphi(\mathbb{D})$ is defined by*

$$N_{q,\varphi}(\zeta, p) = \sum_{\varphi(z)=\zeta} (1 - |\alpha_q(z)|^2)^p\;;$$

if $\zeta \notin \varphi(\mathbb{D})$ then $N_{q,\varphi}(\zeta, p) = 0$.

From now on we abbreviate the notation for the counting function for Q_p to $N_{q,\varphi}(\zeta)$. The Nevanlinna counting function for composition operators in Q_p appears with a non-univalent change of variables as is done in [31, page 186].

For each $f \in Q_p$,

$$
\begin{aligned}
||f \circ \varphi||^2_{Q_p} &= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 (1 - |\alpha_q(z)|^2) dA(z) \\
&= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |(f'(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |\alpha_q(z)|^2) dA(z),
\end{aligned}
$$

and we have

$$
||f \circ \varphi||^2_{Q_p} = \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(\zeta)|^2 N_{q,\varphi}(\zeta) \, dA(\zeta). \tag{73}
$$

If C_φ is a bounded operator on Q_p for some $p \in (0, \infty)$ then for all $q \in \mathbb{D}$, $||C_\varphi \alpha_q||_{Q_p} \lesssim 1$ and by (73)

$$
\sup_{q' \in \mathbb{D}} \int_{\mathbb{D}} |\alpha'_q(\zeta)|^2 N_{q',\varphi}(\zeta) dA(\zeta) \lesssim 1.
$$

By (31) and for all $q' \in \mathbb{D}$, $N_{q',\varphi}(\zeta) dA(\zeta)$ is a collection of uniformly Carleson measures on the Bergman space. By (71) and similarly to the proof of Proposition 3.5 we obtain the following.

Proposition 4.4. *Let $p \in (0, \infty)$. The composition operator C_φ is bounded below on Q_p if and only if there exists a $\delta > 0$ such that for all $f \in Q_p$, $||C_\varphi f||_{Q_p} \geq \delta ||f||_{Q_p}$.*

For each $\epsilon > 0$ and $q \in \mathbb{D}$ let $G_{\epsilon,q}$ be

$$
G_{\epsilon,q} = \{\zeta : N_{q,\varphi}(\zeta) > \epsilon(1 - |\alpha_q(\zeta)|^2)^p\} \tag{74}
$$

and let G_ϵ be

$$
G_\epsilon = \cup_{q \in \mathbb{D}} G_{\epsilon,q}. \tag{75}
$$

On the next two theorems we study C_φ on Q_p with $p \in (0, 1)$.

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Theorem 4.5. *Let $p \in (0,1)$ and assume that C_φ is a bounded operator on Q_p. If there exists an $\epsilon > 0$ such that*

$$\sup_{q \in \mathbb{D}} \int_{\cap_{q \in \mathbb{D}} G_{\epsilon,q}} |f'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) \asymp ||f||_{Q_p}^2 \tag{76}$$

then the composition operator C_φ is closed range on Q_p.

Proof. Let $f \in Q_p$, assume that (76) holds. Then by (73) and for all $q \in \mathbb{D}$,

$$
\begin{aligned}
||f \circ \varphi||_{Q_p}^2 &\geq \sup_{q \in \mathbb{D}} \int_{G_{\varepsilon,q}} |f'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) \\
&\geq \sup_{q \in \mathbb{D}} \int_{\cap_{q \in \mathbb{D}} G_{\epsilon,q}} |f'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) \\
&\geq \epsilon ||f||_{Q_p}^2
\end{aligned}
$$

and C_φ is closed range on Q_p. $\qquad\square$

Theorem 4.6. *Let $p \in (0,1)$ and assume that C_φ is a bounded operator on Q_p. If C_φ is closed range on Q_p then there exists an $\epsilon > 0$ such that for all $f \in Q_p$*

$$\sup_{q \in \mathbb{D}} \int_{G_\epsilon} |f'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) \asymp ||f||_{Q_p}^2. \tag{77}$$

Proof. Suppose that C_φ is closed range on Q_p. If the conclusion fails, then for every $\varepsilon > 0$, G_ε does not satisfy the condition (77) for Q_p and we can find a sequence (f_n) in Q_p with $f_n(0) = 0$ and $||f_n||_{Q_p} = 1$ for all n such that

$$\lim_{n \to \infty} \sup_{q \in \mathbb{D}} \int_{G_{\frac{1}{n}}} |f_n'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) = 0. \tag{78}$$

For each n we have by (73),

$$
\begin{aligned}
||f_n \circ \varphi||_{Q_p}^2 &= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f_n'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) \\
&\leq I_1 + I_2,
\end{aligned}
\tag{79}
$$

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where

$$I_1 = \sup_{q \in \mathbb{D}} \int_{G_{\frac{1}{n}}} |f_n'(\zeta)|^2 \, N_{q,\varphi}(\zeta) dA(\zeta), \tag{80}$$

$$I_2 = \sup_{q \in \mathbb{D}} \int_{\mathbb{D} \setminus G_{\frac{1}{n}}} |f_n'(\zeta)|^2 N_{q,\varphi}(\zeta) dA(\zeta) \, . \tag{81}$$

For n a fixed natural number, if $\zeta \notin G_{1/n}$ then for all $q \in \mathbb{D}$,

$$N_{q,\varphi}(\zeta) \le \frac{1}{n}(1 - |\alpha_q(\zeta)|^2)^p \, ,$$

and

$$
\begin{aligned}
I_2 &\le \frac{1}{n} \sup_{q \in \mathbb{D}} \int_{\mathbb{D} \setminus G_{1/n}} |f_n'(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2)^p \, dA(\zeta) \\
&\le \frac{1}{n} \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f_n'(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2)^p \, dA(\zeta) \\
&= \frac{1}{n} ||f_n||_{Q_p}^2 \\
&= \frac{1}{n} \, .
\end{aligned}
\tag{82}
$$

By (78), (79), (80) and (82), $||f_n \circ \varphi||_{Q_p} \to 0$, as $n \to \infty$ and C_φ is not closed range on Q_p which contradicts our hypothesis and the conclusion holds. $\qquad \square$

Theorem 4.7. *Let $p > 0$ and assume that C_φ is a bounded operator on Q_p. Then, there exists $k > 0$ such that for all $q \in \mathbb{D}$, $||\alpha_q \circ \varphi||_{Q_p} \ge k$ if and only if there exists an $\varepsilon > 0$ and $r \in (0,1)$ such that for all $q \in \mathbb{D}$, there exists $q' \in \mathbb{D}$ such that*

$$\frac{\int_{G_{\varepsilon,q'} \cap D(q,r)} N_{q',\varphi}(\zeta) \, dA(\zeta)}{|D(q,r)|} \asymp 1 \, . \tag{83}$$

Proof. First, assume that there exists $k > 0$ such that for all q in $\mathbb{D}$, $||\alpha_q \circ \varphi||_{Q_p} \ge k$. Then,

for each $q \in \mathbb{D}$ there exists a $q' \in \mathbb{D}$ such that

$$\int_{\mathbb{D}} |\alpha_q'(\zeta)|^2 N_{q',\varphi}(\zeta) \, dA(\zeta) \geq \frac{k}{2} \, . \tag{84}$$

By Theorem 3.7 and since for all $q' \in \mathbb{D}$, $N_{q',\varphi}(\zeta) dA(\zeta)$ is a collection of uniformly Carleson measures for the Bergman space,

$$\lim_{r \to 1} \sup_{q,q' \in \mathbb{D}} \int_{\mathbb{D}\setminus D(q,r)} |\alpha_q'(\zeta)|^2 N_{q',\varphi}(\zeta) \, dA(\zeta) = 0 \, .$$

We conclude that there exists $r \in (0,1)$ such that for all $q, q' \in \mathbb{D}$

$$\int_{\mathbb{D}\setminus D(q,r)} |\alpha_q'(\zeta)|^2 N_{q',\varphi}(\zeta) \, dA(\zeta) < \frac{k}{4} \, . \tag{85}$$

Now, for each $\varepsilon > 0$, $q \in \mathbb{D}$ and $q' \in \mathbb{D}$ as in (84), write

$$\int_{\mathbb{D}} |\alpha_q'(\zeta)|^2 N_{q',\varphi}(\zeta) \, dA(\zeta) \;\; = \;\; I_1 + I_2 + I_3 \, , \tag{86}$$

where

$$I_1 \;\; = \;\; \int_{\mathbb{D}\setminus D(q,r)} |\alpha_q'(\zeta)|^2 \, N_{q',\varphi}(\zeta) \, dA(\zeta) \, .$$

$$I_2 \;\; = \;\; \int_{D(q,r)\setminus G_{\varepsilon,q'}} |\alpha_q'(\zeta)|^2 \, N_{q',\varphi}(\zeta) \, dA(\zeta) \, ,$$

and

$$I_3 \;\; = \;\; \int_{G_{\varepsilon,q'}\cap D(q,r)} |\alpha_q'(\zeta)|^2 \, N_{q',\varphi}(\zeta) \, dA(\zeta) \, .$$

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By (85),

$$I_1 \ < \ \frac{k}{4}.\tag{87}$$

Next,

$$
\begin{aligned}
I_2 \ &\leq \ \varepsilon \int_{D(q,r)\setminus G_{\varepsilon,q'}} |\alpha_q'(\zeta)|^2 (1 - |\alpha_q(\zeta)|^2)^p \, dA(\zeta) \\
&\leq \ \varepsilon \, \|\alpha_q\|_{Q_p}^2 \\
&\leq \ C\varepsilon
\end{aligned}\tag{88}
$$

By (38)

$$
\begin{aligned}
I_3 \ &= \ \int_{G_{\varepsilon,q'} \cap D(q,r)} |\alpha_q'(\zeta)|^2 N_{q',\varphi}(\zeta) \, dA(\zeta) \\
&\asymp \ \int_{G_{\varepsilon,q'} \cap D(q,r)} \frac{1}{|D(q,r)|} N_{q',\varphi}(\zeta) \, dA(\zeta) \\
&\asymp \ \frac{1}{|D(q,r)|} \int_{G_{\varepsilon,q'} \cap D(q,r)} N_{q',\varphi}(\zeta) \, dA(\zeta).
\end{aligned}\tag{89}
$$

By (84), (87), (88), (89), we conclude that there exists $r \in (0,1)$ such that for all $w \in \mathbb{D}$,

$$\frac{k}{4} \ < \ \varepsilon + C \frac{\int_{G_{\varepsilon,q'} \cap D(q,r)} N_{q',\varphi}(\zeta) \, dA(\zeta)}{|D(q,r)|}$$

or

$$\frac{k}{4} - \varepsilon \ \leq \ \frac{\int_{G_{\varepsilon,q'} \cap D(q,r)} N_{q',\varphi}(\zeta) \, dA(\zeta)}{|D(q,r)|}.\tag{90}$$

Hence, for any $\varepsilon > 0$ with $\varepsilon < k/4$, the set $G_{\varepsilon,q'}$ satisfies (83).

Next, assume that there exists an $\varepsilon > 0$ and $r \in (0,1)$ such that for all $q \in \mathbb{D}$, there

exists $q' \in \mathbb{D}$ such that (83) holds. Then, for each fixed $q \in \mathbb{D}$,

$$
\begin{aligned}
||\alpha_q \circ \varphi||^2_{Q_p} &\geq \int_{G_{\varepsilon,q'}} |\alpha'_q(\zeta)|^2 \, N_{q',\varphi}(\zeta) \, dA(\zeta) \\
&\geq \int_{G_{\varepsilon,q'} \cap D(q,r)} |\alpha'_q(\zeta)|^2 \, N_{q',\varphi}(\zeta) \, dA(\zeta) \, .
\end{aligned}
$$

By (7),(8)

$$
\begin{aligned}
||\alpha_q \circ \varphi||^2_{Q_p} &\geq \int_{G_{\varepsilon,q'} \cap D(q,r)} |\alpha'_q(\zeta)|^2 \, N_{q',\varphi}(\zeta) \, dA(\zeta) \\
&\asymp \frac{1}{|D(q,r)|} \int_{G_{\varepsilon,q'} \cap D(q,r)} N_{q',\varphi}(\zeta) \\
&\asymp 1. \tag{91}
\end{aligned}
$$

and the conclusion follows. $\qquad\square$

The following is an immediate corollary of the proof of Theorem 3.12.

Theorem 4.8. *Let $p > 0$ and assume that C_φ is a bounded operator on Q_p. The composition operator C_φ is closed range on Q_p if and only if there exists a constant $k > 0$ such that $||\alpha_q \circ \varphi||_{Q_p} \geq k$, for all $q \in \mathbb{D}$.*

The following is an immediate corollary of the proof of Theorem 3.19. The whole proof goes through with the Hardy space seminorm $||.||_{H^2}$ being replaced with the weighted Dirichlet space seminorm $||.||_{\mathcal{D}_p}$. Note that for isometries among composition operators on Q_p, $k = 1$ and $w = w'$ below. If C_φ is an isometry on Q_p then by [29, Thm 2.1] $\varphi(0) = 0$ and so we may only look at the seminorms.

Theorem 4.9. *Let $p > 0$, assume that C_φ is a bounded operator on Q_p. The following conditions are equivalent:*

    (a) *There exists $k \in (0, 1]$ such that for every $w \in \mathbb{D}$, $||\alpha_w \circ \varphi||_{Q_p} \geq k$.*

    (b) *There exists $k \in (0, 1]$ such that for every $w \in \mathbb{D}$ there exists $w' \in \mathbb{D}$ with*

$|\alpha_w(w')|^2 \leq 1 - k^2$, there exists a sequence (q_n) in $\mathbb{D}$ such that $\varphi(q_n) \to w'$ and

$$\lim_{n\to\infty} ||\varphi_{q_n}||_{D_p} \geq k,$$

where for all natural numbers n, $\varphi_{q_n} = \alpha_{\varphi(q_n)} \circ \varphi \circ \alpha_{q_n}$.

If $p \in (1, \infty)$, then Q_p is Bloch space $\mathcal{B}$ with an equivalent norm. Therefore, Theorem 4.7, Theorem 4.8 and Theorem 4.9 provide another approach to closed range composition operators on the $\mathcal{B}$. Moreover, note that Theorem 4.9 provides a characterization for isometries among composition operators on $\mathcal{B}$ with norm $||.||_{Q_p}$, $p > 1$. Recall that if $p > 0$ then for each $\epsilon > 0$ and $q \in \mathbb{D}$ let $G_{\epsilon,q}$ be

$$G_{\epsilon,q} = \{\zeta : N_{q,\varphi}(\zeta) > \epsilon(1 - |\alpha_q(\zeta)|^2)^p\}.$$

Corollary 4.10. *Let φ be a non-constant analytic self-map of $\mathbb{D}$, and $p > 1$. Then, the following are equivalent*

(a) *The composition operator C_φ is closed range on $\mathcal{B}$.*

(b) *For all $q \in \mathbb{D}$, $||\alpha_q \circ \varphi||_{\mathcal{B}} \asymp 1$.*

(c) *There exists an $\epsilon > 0$ and $r \in (0, 1)$ such that for all $q \in \mathbb{D}$, there exists $q' \in \mathbb{D}$ such that*

$$\frac{\int_{G_{\epsilon,q'} \cap D(q,r)} N_{q',\varphi}(\zeta)\, dA(\zeta)}{|D(q,r)|} \asymp 1.$$

(d) *There exists $k \in (0, 1]$ such that for every $w \in \mathbb{D}$ there exists $w' \in \mathbb{D}$ with $|\alpha_w(w')|^2 \leq 1 - k^2$, there exists a sequence (q_n) in $\mathbb{D}$ such that $\varphi(q_n) \to w'$ and*

$$\lim_{n\to\infty} ||\varphi_{q_n}||_{D_p} \geq k,$$

where for all natural numbers n, $\varphi_{q_n} = \alpha_{\varphi(q_n)} \circ \varphi \circ \alpha_{q_n}$.

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