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Equations of multi-Rees Algebras

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Abstract

In this thesis we describe the defining equations of certain multi-Rees algebras. First, we determine the defining equations of the multi-Rees algebra $R[I^{a_1}t_1, \ldots, I^{a_r}t_r]$ over a Noetherian ring R when I is an ideal of linear type. This generalizes a result of Ribbe and recent work of Lin-Polini and Sosa. Second, we describe the equations defining the multi-Rees algebra $R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r]$, where R is a Noetherian ring containing a field and the ideals are generated by a subset of a fixed regular sequence.

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Dedication

This thesis is wholeheartedly dedicated to my mother and father who have always loved me unconditionally, providing the moral, spiritual, and emotional support to make this possible.

I also want to recognize my brother and sisters who have always been a constant source of love, support and encouragement in my life.

Contents

1 Introduction

The Rees algebra $R[It]$ plays an important role in commutative algebra because it encodes the asymptotic behavior of the ideal I. Also, in algebraic geometry the projective scheme $\operatorname{Proj} R[It]$ defines the blowup of the scheme $\operatorname{Spec}(R)$ along $V(I)$. Let R be a Noetherian ring and $I \subseteq R$ be an ideal of R. An important problem in the theory of Rees algebras is to describe $R[It]$ in terms of generators and relations: find an ideal $\mathscr{L}$ in a polynomial ring $S = R[T_1, \ldots, T_n]$ such that $R[It] \cong S/\mathscr{L}$. Generators of the ideal $\mathscr{L}$ are called equations of the Rees algebra. This is a tough problem which is open for most classes of ideals. Some papers about this problem are [24], [12], [23], [16], [17], [14].

More generally, given any ideals $I_1, \ldots, I_r$ in a ring R, one would like to describe equations of the multi-Rees algebra $R[I_1t_1, I_2t_2, \ldots, I_rt_r]$. There is little work on the defining equations of the multi-Rees algebra compared to the ordinary Rees algebra. Another motivation for investigating the multi-Rees algebra is an illustration of the theory of Rees algebra of modules [4], [19]. Indeed, the multi-Rees algebra in question is simply the Rees algebra of the module $I_1 \oplus I_2 \oplus \cdots \oplus I_r$. However, in our work, we make no serious use of this theory. Ribbe [18] describes equations of the multi-Rees algebra $R[It_1, It_2, \ldots, It_r]$ when I is of linear type and he also determines the relation type of the multi-Rees algebra $R[I^{a_1}t_1, \ldots, I^{a_r}t_r]$ when $n_i \geq 2$. Note that the ideal I is of linear type, if the Rees algebra $R[It]$ is defined by linear equations. In the work of Lin and Polini [15] these equations are described for $R[I^{a_1}t_1, \ldots, I^{a_r}t_r]$, when $R = k[x_1, \ldots, x_n]$, k a field, and $I = \langle x_1, \ldots, x_n \rangle$. Sosa [20] describes the equations of the multi-Rees algebras $R[I_1t_1, I_2t_2, \ldots, I_rt_r]$ when R is a polynomial ring over a field and I_i are monomial ideals with some special properties.

We now describe the contents of this dissertation in more detail. We first give some background and definitions in the Chapter 1. Then in Chapter 2, the results of which appeared in [11], we describe the equations of the multi-Rees algebra $R[I^{a_1}t_1, \ldots, I^{a_r}t_r]$, where I is of linear type. We use Ribbe's [18] result, when all powers coincide with I, together with a Veronese type argument.

Theorem A. Let R be a Noetherian ring and let I be an ideal of linear type, with presentation matrix Φ. Then for any integers $a_i \geq 1$, there is an isomorphism

$$R[I^{a_1}t_1, \dots, I^{a_r}t_r] \cong R[T_1, \dots, T_N]/I_2(B) + I_1(B^t \cdot \Phi)$$

for some N and some matrix B with entries in the T_i's.

We remark that the multi-Rees algebra $R[I_1^{a_1}t_1, \dots, I_r^{a_r}t_r]$ is a Veronese subring. To prove the results of this thesis we map Veronese subrings of polynomial rings to these multi-Rees algebras.

In Chapter 3, we determine the equations of the multi-Rees algebra $R[I_1^{a_1}t_1, I_2^{a_2}t_2, \dots, I_r^{a_r}t_r]$, where R is any Noetherian ring containing a field and ideals I_i are generated by a subset of a fixed regular sequence. In general, these equations can have arbitrarily large degrees. First, applying Buchberger's Criterion we prove the main result for $R[I_1t_1, I_2t_2, \dots, I_rt_r]$. Next, we use the method given in Chapter 2 to prove the main result for $R[I_1^{a_1}t_1, I_2^{a_2}t_2, \dots, I_r^{a_r}t_r]$.

To describe the equations in this case, we introduce the notion of a quasi-matrix and that of a binary quasi-minor, which serves as a generalization 2×2-minors.

Theorem B. Let R be a Noetherian ring containing a field and suppose that ideals I_i are generated by subsets of a fixed regular sequence $s_1, \dots, s_n$ contained in $\operatorname{rad} R$. Then there is a quasi-matrix D, whose entries are certain indeterminates, such that the multi-Rees algebra $R[I_1^{a_1}t_1, \dots, I_r^{a_r}t_r]$ is defined by the ideal generated by all binary quasi-minors of $[\underline{s}|D]$.

To prove Theorem B, it is straightforward to reduce to the case of a sequence of variables in a polynomial ring. In general, we cannot directly use Buchberger's Criterion to prove the main result for $R[I_1^{a_1}t_1, I_2^{a_2}t_2, \dots, I_r^{a_r}t_r]$, such an erroneous attempt was made in [15]. We provide a counter example to show that the arguments in [15] are incomplete.

2 Preliminaries

In the beginning of this Chapter, we present the following notations that we will use in Chapters 2 and 3.

If A is an $m \times n$ matrix, then for $r \leq \min\{m, n\}$, by $I_r(A)$, we mean the ideal generated by $r \times r$ minors of A.

Definition 2.1. Recall that a presentation of a module M is an exact sequence

$$F \xrightarrow{f} G \xrightarrow{g} M \to 0$$

where F and G are free modules. Note the image of the standard basis under g is a generating set of M. If R is Noetherian and M is finitely generated, then the module M is finitely presented and we can represent the map f by a matrix Φ, which is called a presentation matrix of M.

Throughout this thesis, when a generating set of the module M is chosen, by a presentation matrix, we always mean a presentation matrix that is compatible with the chosen generating set, in the sense that the basis of G maps to the given generating set of M.

Definition 2.2. Let a be positive integer, $\mathscr{T}_a$ and $\mathscr{T}_a^{'}$ denote the sets in $(\mathbb{Z}_{\geq 0})^{n-1}$ and $\mathbb{N}^{n-1}$ respectively, that are defined as follows:

$$\mathscr{T}_a := \{\underline{j} = (j_{n-1}, ..., j_1) \mid 0 = j_0 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-1} \leq j_n = a\}$$

$$\mathscr{T}_a^{'} := \{\underline{j} = (j_{n-1}, ..., j_1) \mid 1 = j_0 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-1} \leq j_n = a\}.$$

Note that the cardinality of $\mathscr{T}_a$ (resp. $\mathscr{T}_a^{'}$) is $\binom{a+n-1}{n-1}$ (resp. $\binom{a+n-2}{n-1}$).

Definition 2.3. For $\underline{j} \in \mathscr{T}_a$ we define

$$s^{\underline{j}} := \prod_{i=1}^{n} s_i^{j_i - j_{i-1}}.$$

Definition 2.4. We define the function

$$\underline{j}^{|k\rangle} : \mathscr{T}_a' \to \mathscr{T}_a \text{ by } \underline{j}^{|k\rangle}((j_{n-1}, ..., j_1)) = (j_{n-1}, ..., j_k, j_{k-1} - 1, ..., j_1 - 1)$$

where $1 \leq k \leq n-1$. For convenience instead of $\underline{j}^{|k\rangle}((j_{n-1}, ..., j_1))$ we write $\underline{j}^{|k\rangle}$.

2.1 Rees algebras

Let R be a Noetherian ring and $I \subseteq R$ be an ideal of R. The Rees algebra of the ideal I is defined to be the graded ring

$$\mathcal{R}(I) = R[It] = \bigoplus_{n=0}^{\infty} I^n t^n \subseteq R[t].$$

One can generalize the concept of the Rees algebra and define the Rees algebra of modules (one may see [4] and [19]). The multi-Rees algebra $R[I_1t_1, I_2t_2, \ldots, I_rt_r]$ of the ideals I_i is one example of this generalization. Indeed, the multi-Rees algebra is simply the Rees algebra of the module $\oplus_{i=1}^r I_i$.

As is well-known (e.g. [7]), if all the ideals I_i have positive height, then

$$\dim R[I_1t_1, \ldots, I_rt_r] = \dim R + r.$$

Definition 2.5. Let M be an R-module, we define the tensor algebra of M denoted by $T(M)$ as the following graded R-algebra

$$T(M) = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \ldots$$

The symmetric algebra of M is the algebra $\mathcal{S}(M)$ which is the quotient of tensor algebra of M modulo the ideal generated by all $x \otimes y - y \otimes x$ $(x, y \in M)$.

Lemma 2.6. [22] If Φ is a presentation matrix of I with n rows, then

$$\mathcal{S}(I) \cong R[T_1, \ldots, T_n]/I_1([T_1, \ldots, T_n] \cdot \Phi).$$

There is a natural epimorphism

$$\alpha : \mathcal{S}(I) \to \mathcal{R}(I).$$

Thus one may view the linear equations defining the symmetric algebra as a first order approximation of the equations of the Rees algebra.

Definition 2.7. We say the ideal I is of linear type if α is an isomorphism.

Examples of ideals of linear type will be given later in Chapter 2.

The following result which is due to J. Ribbe, describes explicit equations of multi-Rees algebra of an ideal of linear type.

Proposition 2.8. [18, Proposition 3.1] Let I be an ideal of linear type in a Noetherian ring R and let the set $s_1, \ldots, s_n$ generates the ideal I. Suppose $R^m \xrightarrow{\Phi} R^n \to I$ is any presentation of I. Consider the generic $r \times n$ matrix $B = (T_{ij})$. Then the kernel $\mathscr{L}$ of the map

$$\phi : R[\{T_{ij}\}_{1 \le i \le r, 1 \le j \le n}] \to R[It_1, \ldots, It_r], \quad \phi(T_{ij}) = s_j t_i,$$

is

$$\mathscr{L} = I_2(B) + I_1(B \cdot \Phi).$$

Before we give the proof of proposition above, we need the following lemma.

Lemma 2.9. [18, Lemma 2.2] Let $i_1, \ldots, i_r, j_1, \ldots, j_r \in \mathbb{Z}^n_{\ge 0}$ be given such that

$$(i) \sum_{\mu=1}^{n} i_{\nu\mu} = \sum_{\mu=1}^{n} j_{\nu\mu}, \text{ for every } \nu = 1, \ldots, r$$

and

$$(ii) \sum_{\nu=1}^{r} i_{\nu\mu} = \sum_{\nu=1}^{r} j_{\nu\mu}, \text{ for every } \mu = 1, \dots, n.$$

Suppose $U_i = \{T_{i1}, \dots, T_{in}\}$, for $1 \leq i \leq r$, and $U_\nu^{i_\nu} = T_{\nu 1}^{i_{\nu 1}} \dots T_{\nu n}^{i_{\nu n}}$. Then

$$U_1^{i_1} \dots U_r^{i_r} \equiv U_1^{j_1} \dots U_r^{j_r} \mod I_2(B).$$

Proof. By induction on r we may assume that $r \geq 2$ and that the claim holds for any two sets of $r-1$ vectors that satisfy (i) and (ii). We choose u from the set $\{1, \dots, n+1\}$ maximal with $i_{rw} = j_{rw}$ for all $w < u$. We prove the claim by a second induction on u. If $u = n+1$, then $i_r = j_r$, and by the first induction we have

$$U_1^{i_1} \dots U_{r-1}^{i_{r-1}} \equiv U_1^{j_1} \dots U_{r-1}^{j_{r-1}} \mod I_2(B).$$

If we multiply it by $U_r^{i_r}$, then our claim is proved in this case. Now we assume that $u \leq n$, say $i_{ru} > j_{ru}$. By (i), there is $v > u$ with $i_{rv} < j_{rv}$. Hence, by (ii), there is $s < r$ with $i_{sv} > j_{sv}$. If e_i means i-th unit vector in $\mathbb{Z}_{\geq 0}^n$, then we obtain the following congruence modulo $I_2(B)$:

$$\begin{aligned} U_1^{i_1} \dots U_r^{i_r} &= U_1^{i_1} \dots U_s^{i_s - e_v} \dots U_r^{i_r - e_u} T_{sv} T_{ru} \\ &\equiv U_1^{i_1} \dots U_s^{i_s - e_v} \dots U_r^{i_r - e_u} T_{su} T_{rv} \mod I_2(B) \\ &\equiv U_1^{i_1} \dots U_s^{i_s - e_v + e_u} \dots U_r^{i_r - e_u + e_v} \mod I_2(B). \end{aligned}$$

We see that (i) and (ii) still hold if $i_1, \dots, i_r$ are replaced by new exponent vectors $i_1^{'} = i_1, \dots, i_s^{'} = i_s - e_v + e_u, \dots, i_r^{'} = i_r - e_u + e_v$. We continue this procedure until we obtain exponent vectors $i_\nu^{'}$, $\nu = 1, \dots, r$, with $i_{rw}^{'} = j_{rw}^{'}$ for all $w \leq u$. Then applying the induction hypothesis we obtain

$$U_1^{i_1^{'}} \dots U_r^{i_r^{'}} \equiv U_1^{j_1} \dots U_r^{j_r} \mod I_2(B).$$

This completes the proof. $\square$

Proof. (Proof of Proposition 2.8).

Let $F \in \mathscr{L}$ be homogeneous of multidegree $(d_1, \dots, d_r)$. We set $d = d_1 + \cdots + d_r$. For every vector $u = (u_1, \dots, u_n) \in \mathbb{Z}^n_{\geq 0}$ we define $|u| = u_1 + \cdots + u_n$.

Let

$$E = \{k \in \mathbb{Z}^n_{\geq 0} |\ |k| = d\}$$

and for every $k \in E$,

$$E(k) = \left\{i = (i_1, \dots, i_r) \in (\mathbb{Z}^n_{\geq 0})^r |\ i_1 + \cdots + i_r = k;\ |i_s| = d_s,\ s = 1, \dots, r\right\}.$$

Then we may write

$$F = \sum_{k \in E} \left(\sum_{i \in E(k)} f_i U_1^{i_1} \dots U_r^{i_r} \right) \equiv \sum_{k \in E} \left(\sum_{i \in E(k)} f_i \right) U_1^{i_1(k)} \dots U_r^{i_r(k)} \mod \mathscr{L}, \tag{2.1}$$

where $f_i \in R$ and $i_1(k), \dots, i_r(k)$ are vectors in $\mathbb{Z}^n_{\geq 0}$ which depend only on k, fulfilling the congruences $U_1^{i_1} \dots U_r^{i_r} \equiv U_1^{i_1(k)} \dots U_r^{i_r(k)} \mod \mathscr{L}$ for every $(i_1, \dots, i_r) \in E(k)$. We see that every $k \in E$ admits vectors $i_s(k) \in \mathbb{Z}^n_{\geq 0}$, $s = 1, \dots, r$ such that $(i_1(k), \dots, i_r(k)) \in E(k)$.

We can define these vectors by recursion as

$$i_s(k)_\mu = \max\left\{0, \min\left\{d_s - \sum_{\tau=\mu+1}^{n} i_s(k)_\tau, k_\mu - \sum_{\sigma=s+1}^{r} i_\sigma(k)_\mu\right\}\right\}$$

for $s = r, \dots, 1$ and $\mu = n, \dots, 1$. Then the congruence 2.1 holds by Lemma 2.9.

On the other hand we have the following commutative diagram

$$\begin{tikzcd}
R[\{T_{ij}\}_{1 \leq i \leq r, 1 \leq j \leq n}] \arrow[r, two heads, "\alpha"] \arrow[d, two heads, "\phi"] & R[X_1, \dots, X_n] \arrow[d, two heads, "\psi"] \\
R[It_1, \dots, It_r] \arrow[r, two heads, "\beta"] & R[It],
\end{tikzcd}$$

where $\alpha(T_{ij}) = X_j$ and $\beta(s_j t_i) = s_j t$. Hence we see that $F(X, \dots, X) \in \ker(\psi)$, where $X = \{X_1, \dots, X_n\}$. Moreover $\deg(F(X, \dots, X)) = d$. There are linear forms $L_\nu = \sum_{\mu=1}^{n} l_{\nu\mu} X_\mu$

and forms $G_\nu = \sum_{|k|=d-1} g_{\nu,k} X^k$ of degree $d-1$ in $R[X_1, \ldots, X_n]$, $\nu = 1, \ldots, m$, such that

$$F(X, \ldots, X) = \sum_{\nu=1}^{m} L_\nu G_\nu = \sum_{k \in E} \left(\sum_{\nu=1}^{m} \sum_{\mu=1}^{n} l_{\nu\mu} g_{\nu,k-e_\mu} \right) X^k. \tag{2.2}$$

Comparing 2.1 and 2.2, we have

$$F \equiv \sum_{k \in E} \left(\sum_{\nu=1}^{m} \sum_{\mu=1}^{n} l_{\nu\mu} g_{\nu,k-e_\mu} \right) \prod_{\gamma=1}^{r} U_\gamma^{i_\gamma(k)} \mod \mathscr{L},$$

or equivalently

$$F \equiv \sum_{|k|=d-1} \left(\sum_{\nu=1}^{m} \sum_{\mu=1}^{n} l_{\nu\mu} g_{\nu,k} \right) \prod_{\gamma=1}^{r} U_\gamma^{i_\gamma(k+e_\mu)} \mod \mathscr{L}. \tag{2.3}$$

We fix $\mu \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}^n_{\geq 0}$ with $|k| = d-1$. Then

$$\prod_{\gamma=1}^{r} U_\gamma^{i_\gamma(k+e_\mu)} \equiv U_{1\mu} \prod_{\gamma=1}^{r} U_\gamma^{j_{\gamma\mu}(k)} \mod \mathscr{L}, \tag{2.4}$$

where $j_{\gamma\mu} \in \mathbb{Z}^n_{\geq 0}$ are suitably chosen vectors with $|j_{1\mu}(k)| = d_1 - 1$, $|j_{\gamma\mu}(k)| = d_\gamma$ for $\gamma = 2, \ldots, r$, and $\sum_{\gamma=1}^{r} j_{\gamma\mu}(k) = k$.

To be precise, if $i_1(k+e_\mu)_\mu \geq 1$ we choose $j_{1\mu}(k) := i_1(k+e_\mu) - e_\mu$ and $j_{\gamma\mu}(k) := i_\gamma(k+e_\mu)$ for $\gamma = 2, \ldots, r$. While if $i_1(k+e_\mu)_\mu = 0$ we first find $\sigma \geq 2$ and $\eta \neq \mu$ with $i_\sigma(k+e_\mu)_\mu \geq 1$ and $i_1(k+e_\mu)_\eta \geq 1$; now put $j_{1\mu}(k) := i_1(k+e_\mu) - e_\eta$, $j_{\sigma\mu}(k) := i_\sigma(k+e_\mu) - e_\mu + e_\eta$, and for $\gamma \neq 1, \sigma$, $j_{\sigma\mu}(k) := i_\gamma(k+e_\mu)$. Using the relation $U_{1\eta} U_{\sigma\mu} \equiv U_{1\mu} U_{\sigma\eta} \mod \mathscr{L}$ one can verify relation 2.4.

Next, note that modulo $\mathscr{L}$, the monomial $\prod_{\gamma=1}^{r} U_\gamma^{j_{\gamma\mu}(k)}$ in 2.4 does not depend on μ, so we shall denote it by P_k (given $\mu, \mu' \in \{1, \ldots, n\}$ we have $|j_{\gamma\mu}(k)| = |j_{\gamma\mu'}(k)|$ for $\gamma = 1, \ldots, r$ and $\sum_{\gamma=1}^{r} j_{\gamma\mu}(k) = k = \sum_{\gamma=1}^{r} j_{\gamma\mu'}(k)$, so the independence is due to Lemma 2.9).

We use these facts and rewrite 2.3 as below:

$$F \equiv \sum_{|k|=d-1} \left(\sum_{\nu=1}^{m} \sum_{\mu=1}^{n} l_{\nu\mu} g_{\nu,k} \right) U_{1\mu} P_k \equiv \sum_{|k|=d-1} \sum_{\nu=1}^{m} \left(\sum_{\mu=1}^{n} l_{\nu\mu} U_{1\mu} \right) g_{\nu,k} P_k \equiv 0.$$

This completes the proof. □

2.2 Equations of the Veronese

Definition 2.10. Let $R = \oplus_{i \geq 0} R_i$ be a graded ring, and d be a positive integer number. We define the Veronese subring of R of degree d, denoted by $R^{(d)}$, to be R_0-algebra generated by all elements of R whose degrees are multiple of d, i.e. $R^{(d)} = \oplus_{i \geq 0} R_{id}$.

It is clear that if we consider the polynomial ring $R[x_1, \ldots, x_n]$, then the Veronese subring of degree d is $R[X(d)]$ which is the ring generated over R by the monomials of degree d in $x_1, \ldots, x_n$.

Theorem 2.11. [13, Proposition 2.5] Let R be a commutative ring. For fixed $n, d \geq 1$ let $N = \binom{d+n-1}{n-1}$ and let

$$\Theta : R[V_1, \ldots, V_N] \to R[X(d)]$$

be the ring homomorphism taking V_i to the i-th monomial of degree d in $x_1, \ldots, x_n$ in lexicographic order. Then there exists an $n \times \binom{d+n-2}{n-1}$ matrix $M(n, d)$, whose entries are the V_i's, such that $\ker(\Theta) = I_2(M(n, d))$.

To describe the matrix $M(n, d)$, write $N_1, \ldots, N_r$ for the monomials of degree $d-1$ in $x_1, \ldots, x_n$ written in lexicographic order. Then $M(n, d)$ is the matrix whose (i, j)-th entry is the variable V_k which satisfies $\Theta(V_k) = x_i N_j$.

The following are the two most well-known cases of this result:

Example 2.12. $n = 2$. In this case,

$$M(2, d) = \begin{bmatrix} V_1 & V_2 & \ldots & V_d \\ V_2 & V_3 & \ldots & V_{d+1} \end{bmatrix}.$$

The ideal $I_2(M(2,d))$ defines the rational normal curve of degree d in $\mathbb{P}^d$ over R.

Example 2.13. $n = 3$ and $d = 2$. In this case $M(3,2)$ is the generic symmetric matrix

$$M(3,2) = \begin{bmatrix} V_1 & V_2 & V_3 \\ V_2 & V_4 & V_5 \\ V_3 & V_5 & V_6 \end{bmatrix}.$$

The ideal $I_2(M(3,2))$ defines the Veronese surface in $\mathbb{P}^5$ over R.

3 Powers of linear type ideals

3.1 Equations of $R[I^{a_1}t_1,\ldots,I^{a_r}t_r]$

We fix a generating set $s_1,\ldots,s_n$ of the ideal I. If the ideal $I=\langle s_1,\ldots,s_n\rangle$ is of linear type in a Noetherian ring R, then the R-algebra homomorphism

$$f : R[T_1,\ldots,T_n] \to R[It], \quad f(T_i) = s_i t$$

has $\ker(f) = \langle [T_1,\ldots,T_n]\cdot\Phi\rangle$, where $R^m \xrightarrow{\Phi} R^n \xrightarrow{s} I$ is a presentation of I.

Let $R[I^{a_1}t_1,\ldots,I^{a_r}t_r]$ be the multi-Rees algebra of I. We denote $\boldsymbol{a} = a_1,...,a_r$.

One knows that I^a is generated by $s_1^{k_1}\ldots s_n^{k_n}$, where $k_i \geq 0$ and $k_1+\cdots+k_n = a$. We can rewrite every generator as the following

$$s_1^{k_1}\ldots s_n^{k_n} = s_1^{k_1-0}s_2^{(k_1+k_2)-k_1}\ldots s_l^{(k_1+\cdots+k_l)-(k_1+\cdots+k_{l-1})}\ldots s_n^{(k_1+\cdots+k_n)-(k_1+\cdots+k_{n-1})}.$$

If we set $j_0 = 0, j_1 = k_1, j_2 = k_1+k_2,\ldots,j_l = k_1+\cdots+k_l,\ldots,j_n = a$, then we have $s_1^{k_1}\ldots s_n^{k_n} = s^{\underline{j}}$. For convenience we will use this notation which is already used in [15].

Since I^a is generated by all $s^{\underline{j}}$ ($\underline{j}\in\mathscr{T}_a$), the multi-Rees algebra $R[I^{a_1}t_1,\ldots,I^{a_r}t_r]$ is the ring $R\left[\left\{s^{\underline{j}}t_l\right\}_{1\leq l\leq r,\underline{j}\in\mathscr{T}_{a_l}}\right]$.

Let $S := R\left[\left\{T_{l,\underline{j}}\right\}_{1\leq l\leq r,\underline{j}\in\mathscr{T}_{a_l}}\right]$. We fix the R-algebra epimorphism

$$\phi : S \to R[I^{a_1}t_1,\ldots,I^{a_r}t_r], \text{ by } \phi(T_{l,\underline{j}}) = s^{\underline{j}}t_l.$$

Definition 3.1. Consider the set $\left\{(l,\underline{j})|1\leq l\leq r;\ \underline{j}\in\mathscr{T}'_{a_l}\right\}$ ordered increasing lexicographically as a subset of $\mathbb{N}^n$. We define the $n\times\left(\sum_{l=1}^n\binom{a_l+n-2}{n-1}\right)$ matrix $B_{\boldsymbol{a}}$, whose entry in row k and column $(l,\underline{j})$ is $T_{l,\underline{j}^{|k\rangle}}$.

Example 3.2. In the following we list some examples of various $B_{\boldsymbol{a}}$.

1) $n=3, \boldsymbol{a}=2$

$$B_2 = \begin{bmatrix} T_{1,1,1} & T_{1,2,1} & T_{1,2,2} \\ T_{1,1,0} & T_{1,2,0} & T_{1,2,1} \\ T_{1,0,0} & T_{1,1,0} & T_{1,1,1} \end{bmatrix}.$$

2) $n=3, \boldsymbol{a}=3,2$

$$B_{3,2} = \begin{bmatrix} T_{1,1,1} & T_{1,2,1} & T_{1,2,2} & T_{1,3,1} & T_{1,3,2} & T_{1,3,3} & T_{2,1,1} & T_{2,2,1} & T_{2,2,2} \\ T_{1,1,1} & T_{1,2,1} & T_{1,2,2} & T_{1,3,1} & T_{1,3,2} & T_{1,3,3} & T_{2,1,0} & T_{2,2,0} & T_{2,2,1} \\ T_{1,1,0} & T_{1,2,0} & T_{1,2,1} & T_{1,3,0} & T_{1,3,1} & T_{1,3,2} & T_{2,0,0} & T_{2,1,0} & T_{2,1,1} \end{bmatrix}.$$

3) $n=5, \boldsymbol{a}=2$

$$B_2 = \begin{bmatrix} T_{1,1,1,1,1} & T_{1,2,1,1,1} & T_{1,2,2,1,1} & T_{1,2,2,2,1} & T_{1,2,2,2,2} \\ T_{1,1,1,1,0} & T_{1,2,1,1,0} & T_{1,2,2,1,0} & T_{1,2,2,2,0} & T_{1,2,2,2,1} \\ T_{1,1,1,0,0} & T_{1,2,1,0,0} & T_{1,2,2,0,0} & T_{1,2,2,1,0} & T_{1,2,2,1,1} \\ T_{1,1,0,0,0} & T_{1,2,0,0,0} & T_{1,2,1,0,0} & T_{1,2,1,1,0} & T_{1,2,1,1,1} \\ T_{1,0,0,0,0} & T_{1,1,0,0,0} & T_{1,1,1,0,0} & T_{1,1,1,1,0} & T_{1,1,1,1,1} \end{bmatrix}.$$

4) $n=5, \boldsymbol{a}=1,1,\ldots,1$

$$B_{1,1,\ldots,1} = \begin{bmatrix} T_{1,1,1,1,1} & T_{2,1,1,1,1} & \ldots & T_{r,1,1,1,1} \\ T_{1,1,1,1,0} & T_{2,1,1,1,0} & \ldots & T_{r,1,1,1,0} \\ T_{1,1,1,0,0} & T_{2,1,1,0,0} & \ldots & T_{r,1,1,0,0} \\ T_{1,1,0,0,0} & T_{2,1,0,0,0} & \ldots & T_{r,1,0,0,0} \\ T_{1,0,0,0,0} & T_{2,0,0,0,0} & \ldots & T_{r,0,0,0,0} \end{bmatrix}.$$

The theorem below describes the defining equations of the multi-Rees algebra.

Theorem 3.3. Let R be a Noetherian ring and the ideal $I = \langle s_1, \ldots, s_n \rangle \subseteq R$ be of linear type. Let Φ be a presentation matrix with n rows. Then the kernel $\mathscr{L}$ of the map ϕ is

$$\mathscr{L} = I_2(B_{\boldsymbol{a}}) + I_1({B_{\boldsymbol{a}}}^t \cdot \Phi).$$

Proof. We can easily see $I_2(B_{\boldsymbol{a}}) \subseteq \mathscr{L}$, because

$$\phi\left(\begin{vmatrix} T_{l',\underline{j}',k} & T_{l,\underline{j}|k\rangle} \\ T_{l',\underline{j}',k'} & T_{l,\underline{j},k'} \end{vmatrix}\right) = s_k s^{\underline{j}'} t_{l'} s_{k'} s^{\underline{j}} t_l - s_{k'} s^{\underline{j}'} t_{l'} s_k s^{\underline{j}} t_l = 0.$$

If L is a linear form in $\ker(f)$ we have,

$$\begin{aligned} \phi\left(L(T_{l,\underline{j},1}, T_{l,\underline{j},2}, \ldots, T_{l,\underline{j},n})\right) &= L\left(s_1 s^{\underline{j}} t_l, s_2 s^{\underline{j}} t_l, \ldots, s_n s^{\underline{j}} t_l\right) \\ &= \left(s^{\underline{j}} t_l\right) L(s_1, s_2, \ldots, s_n) = 0, \end{aligned}$$

therefore $I_1(B^t_{\boldsymbol{a}} \cdot \Phi) \subseteq \mathscr{L}$. Hence $I_2(B_{\boldsymbol{a}}) + I_1(B^t_{\boldsymbol{a}} \cdot \Phi) \subseteq \mathscr{L}$. Thus it is enough to prove $\mathscr{L} \subseteq I_2(B_{\boldsymbol{a}}) + I_1(B^t_{\boldsymbol{a}} \cdot \Phi)$.

Let $X = (x_{i,l})$ be a generic $n \times r$ matrix. Let x_l denote the regular sequence $x_{1,l}, x_{2,l}, \ldots, x_{n,l}$ of entries of the l-th column of X. We define A_l to be the a_l-th Veronese subring $R[x_{1,l}, ..., x_{n,l}]^{(a_l)}$. Let $X_l(a)$ be the family of monomials of degree a ($a \in \mathbb{N}$) in the variables $x_{1,l}, ..., x_{n,l}$, we define A to be the $\boldsymbol{a}$-th Veronese subring $R[X]^{\boldsymbol{a}} = R[X_1(a_1), X_2(a_2), ..., X_r(a_r)]$.

For arbitrary l, $1 \leq l \leq r$ we define

$$\alpha_l : R\left[\left\{T_{l,\underline{j}}\right\}_{\underline{j} \in \mathscr{T}_{a_l}}\right] \to A_l, \quad \alpha_l(T_{l,\underline{j}}) = x_l^{\underline{j}}.$$

These induce a map $\alpha : S \to A$.

Let B_{a_l} for $1 \leq l \leq r$, be the $n \times \binom{a_l+n-2}{n-1}$ submatrix of $B_{\boldsymbol{a}}$ consisting of the $(l, \underline{j})$-columns of $B_{\boldsymbol{a}}$ ($\underline{j} \in \mathscr{T}'_{a_l}$). By Theorem 2.11 we know that $\ker(\alpha_l) = I_2(B_{a_l})$. We have

$$\begin{aligned}
A &\cong A_1 \otimes_R A_2 \otimes_R \ldots \otimes_R A_r \\
&\cong R[T_{1,\underline{j}}]/I_2(B_{a_1}) \otimes_R R[T_{2,\underline{j}}]/I_2(B_{a_2}) \otimes_R \ldots \otimes_R R[T_{r,\underline{j}}]/I_2(B_{a_r}) \\
&\cong S/\langle I_2(B_{a_1}), I_2(B_{a_2}), \ldots, I_2(B_{a_r})\rangle.
\end{aligned}$$

Since the isomorphism above is an R-algebra homomorphism and image of $T_{l,\underline{j}}$ is the same as its image under α, it follows that

$$\ker(\alpha) = \langle I_2(B_{a_1}), I_2(B_{a_2}), \ldots, I_2(B_{a_r})\rangle \subseteq I_2(B_{\boldsymbol{a}}).$$

If we consider

$$\psi : R[X] \to R[Iu_1, Iu_2, \ldots, Iu_r], \quad \psi(x_{i,l}) = s_i u_l$$

then by Proposition 2.8

$$\ker(\psi) = I_2(X) + I_1(X^t \cdot \Phi). \tag{3.1}$$

By the following commutative diagram we define the map g:

$$\begin{tikzcd}
S \arrow[r, two heads, "\phi"] \arrow[d, two heads, "\alpha"] & R[I^{a_1}t_1, \ldots, I^{a_r}t_r] \arrow[d, "\wr"] \\
A \arrow[ur, "g"] \arrow[r, "\psi^{(\boldsymbol{a})}"] \arrow[d, hook] & R[I^{a_1}u_1^{a_1}, \ldots, I^{a_r}u_r^{a_r}] \arrow[d, hook] \\
R[X] \arrow[r, two heads, "\psi"] & R[Iu_1, \ldots, Iu_r]
\end{tikzcd}$$

In this step we prove that the kernel of g is generated by polynomials of the form $L(x_{1,l}x_l^{\underline{j}}, x_{2,l}x_l^{\underline{j}}, \ldots, x_{n,l}x_l^{\underline{j}})$ ($L \in \ker(f)$, $\underline{j} \in \mathscr{T}_{a_l-1}$) and $x_{i,l}x_l^{\underline{j}}x_{h,m}x_m^{\underline{k}} - x_{h,l}x_l^{\underline{j}}x_{i,m}x_m^{\underline{k}}$ ($\underline{j} \in \mathscr{T}_{a_l-1}$, $\underline{k} \in \mathscr{T}_{a_m-1}$) in A.

If $0 \neq p \in \ker(g)$, then $p \in \ker(\psi)$ and thus by (3.1) it is an $R[X]$-linear combination of 2×2-minors of X and $L(x_{1,l}, x_{2,l}, \ldots, x_{n,l})$'s, where $1 \leq l \leq r$ and L's are linear relations on $s_1, \ldots, s_n$. Thus we have an expression $p = \sum_{i=1}^{m} r_i \mu_i q_i$, where q_k's are 2×2-minors of X, or $q_k = L(x_{1,l}, x_{2,l}, \ldots, x_{n,l})$, $r_k \in R$ and $\mu_k \in R[X]$ is a monomial. Among all such expressions we choose one in which m is minimal. Since $p \neq 0$, p has either a monomial, say $\mu_1 x_{k,l}$ or $\mu_1 x_{k,l} x_{h,m}$ in its support. We claim that there is $\xi_1 \in A$ such that

$$\mu_1 = \begin{cases} \xi_1 x_l^{\underline{j}} & (\text{with } j_n = a_l - 1) \qquad\qquad \text{if } q_k = L(x_{1,l}, x_{2,l}, \ldots, x_{n,l}) \\ \xi_1 x_l^{\underline{j}} x_m^{\underline{k}} & (\text{with } j_n = a_l - 1, k_n = a_m - 1) \quad \text{if } q_k = x_{i,l} x_{h,m} - x_{h,l} x_{i,m} \end{cases}$$

Since one of the terms in p is $r\mu_1 x_{1,l}$ or $r\mu_1 x_{i,l} x_{h,m}$ ($r \in R$), then $\mu_1 x_{1,l}$ or $\mu_1 x_{i,l} x_{h,m}$ is in A. Therefore we have two cases:

case i) $\mu_1 x_{1,l}$ is in the support of p. Then $\mu_1 x_{1,l} = \Gamma_1 \Gamma_2 \ldots \Gamma_w$, where $\Gamma_j \in A_i$ and it is a monomial of degree a_i, therefore $x_{1,l}$ divides at least one Γ_j, say $x_{1,l}$ divides Γ_1, then Γ_1 is a monomial of degree a_l in A_l and $\Gamma_1 = x_{1,l} x_l^{\underline{j}}$ ($j_n = a_l - 1$). Hence

$$\mu_1 x_{1,l} = x_{1,l} x_l^{\underline{j}} \Gamma_2 \ldots \Gamma_w \Rightarrow \mu_1 = \xi_1 x_l^{\underline{j}}, \text{ with } \xi_1 = \Gamma_2 \ldots \Gamma_w \in A.$$

Case ii) $\mu_1 x_{i,l} x_{h,m}$ is in the support of p. Then $\mu_1 x_{i,l} x_{h,m} = \Gamma_1 \Gamma_2 \ldots \Gamma_w$, where $\Gamma_j \in A_i$ and it is the monomial of degree a_i. Since $l \neq m$, then $x_{i,l}$ divides at least one Γ_j and $x_{h,m}$ divides at least one Γ_i and $i \neq j$, say $x_{i,l}$ divides Γ_1, and $x_{h,m}$ divides Γ_2, therefore Γ_1 (resp. Γ_2) is a monomial of degree a_l in A_l (resp. of degree a_m in A_m) and $\Gamma_1 = x_{i,l} x_l^{\underline{j}}$ ($j_n = a_l - 1$) (resp. $\Gamma_2 = x_{h,m} x_m^{\underline{k}}$ ($k_n = a_m - 1$)). Therefore

$$\mu_1 x_{i,l} x_{h,m} = x_{i,l} x_l^{\underline{j}} x_{h,m} x_m^{\underline{k}} \Gamma_3 \ldots \Gamma_w$$
$$\Rightarrow \mu_1 = \xi_1 x_l^{\underline{j}} x_m^{\underline{k}}, \text{ with } \xi_1 = \Gamma_3 \ldots \Gamma_w \in A.$$

Hence either

$$\mu_1 q_1 = \xi_1 x_l^j q_1 = \xi_1 x_l^j L(x_{1,l}, \dots, x_{n,l}) = \xi_1 x_l^j \sum_i a_i^{(l)} x_{i,l}$$
$$= \xi_1 \sum_i a_i^{(l)} (x_{i,l} x_l^j) = \xi_1 L(x_{1,l} x_l^j, \dots, x_{n,l} x_l^j)$$

or

$$\mu_1 q_1 = \xi_1 x_l^j x_m^k q_1 = \xi_1 x_l^j x_m^k (x_{i,l} x_{h,m} - x_{h,l} x_{i,m})$$
$$= \xi_1 (x_{i,l} x_l^j x_{h,m} x_m^k - x_{h,l} x_l^j x_{i,m} x_m^k).$$

Since p and $\mu_1 q_1 \in A$, we conclude that $p - r_1 \mu_1 q_1 \in A$. Since the expression $\sum_{i=2}^m r_i \mu_i q_i$ is still minimal, by induction of m (the base case $m = 1$ being clear by the above), we conclude that $p - r_1 \mu_1 q_1$, and hence p, is a linear combination of the required polynomials. Therefore we have described the generators of $\ker(g)$. On the other hand

$$\alpha \left(L(T_{l,j_{n-1}+1,\dots,j_2+1,j_1+1}, T_{l,j_{n-1}+1,\dots,j_2+1,j_1}, \dots, T_{l,j_{n-1},\dots,j_2,j_1}) \right)$$
$$= L(x_{1,l} x_l^j, x_{2,l} x_l^j, \dots, x_{n,l} x_l^j)$$

and under α

$$T_{l,j_{n-1}+1,\dots,j_i+1,j_{i-1},\dots,j_1} T_{m,k_{n-1}+1,\dots,k_h+1,k_{h-1},\dots,k_1} - T_{l,j_{n-1}+1,\dots,j_h+1,j_{h-1},\dots,j_1} T_{m,k_{n-1}+1,\dots,k_i+1,k_{i-1},\dots,k_1}$$

maps to

$$x_{i,l} x_l^j x_{h,m} x_m^k - x_{h,l} x_l^j x_{i,m} x_m^k.$$

Now if $p \in \mathscr{L}$, then $g \circ \alpha(p) = 0$ so that $\alpha(p) \in \ker(g)$. Thus there is $q \in I_2(B_{\boldsymbol{a}}) + I_1(B_{\boldsymbol{a}}^t \cdot \Phi)$ such that $\alpha(p) = \alpha(q)$. Hence $p - q \in \ker(\alpha) \subset I_2(B_{\boldsymbol{a}})$, hence $p \in I_2(B_{\boldsymbol{a}}) + I_1(B_{\boldsymbol{a}}^t \cdot \Phi)$.

This completes the proof. $\square$

3.2 Examples

In this section we give some examples of ideals of linear type to illustrate the defining equations of their multi-Rees algebras. In the examples of ideals of linear type that follow, it is known that the Rees algebra is Cohen-Macaulay (e.g. [25]). It follows that, by a well-known result [7], the multi-Rees algebra $R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r]$ is Cohen-Macaulay for any $\boldsymbol{a}$.

Example 3.4. If $\{s_1, s_2, \ldots, s_n\}$ is a regular sequence, then $I = \langle s_1, \ldots, s_n \rangle$ is of linear type (e.g. [22], corollary 5.5.5). Therefore

$$\mathscr{L} = I_2 \left(\left[\begin{array}{c|c} s_1 & \\ \vdots & B_{\boldsymbol{a}} \\ s_n & \end{array} \right] \right).$$

This example generalizes a result of [15], where the result is claimed in the case when I is the maximal ideal of the polynomial ring over a field. However, their proof of the result in this case is incomplete. See the end of section 4.2, for a discussion of this issue.

Example 3.5. Let k be a field and let X be a generic $n \times n-1$ matrix over k, $R = k[X]$, and let $I = I_{n-1}(X)$. Then I is of linear type [10]. Furthermore, by the Hilbert-Burch Theorem we may take $\Phi = X$ with respect to the n signed minors of X obtained by deleting the i-th row. Therefore $\mathscr{L} = I_2(B_{\boldsymbol{a}}) + I_1({B_{\boldsymbol{a}}}^t X)$.

Example 3.6. Let k be a field, $n \geq 3$ be an odd integer, X a generic $n \times n$ alternating matrix over k, $R = k[X]$, and let $I = Pf_{n-1}(X)$ denote the ideal of $(n-1)$ sized Pfaffians. Then I is of linear type [9]. Furthermore, by the Buchsbaum-Eisenbud structure Theorem [1], we may take $\Phi = X$ with respect to the n signed Pfaffians of X obtained by deleting the i-th row and column. Therefore $\mathscr{L} = I_2(B_{\boldsymbol{a}}) + I_1({B_{\boldsymbol{a}}}^t X)$.

Example 3.7. Let $Z = (z_{ij})$ be an $m \times m$ generic matrix over $\mathbb{Z}$. Let $T = \mathbb{Z}[z_{ij}]$ and Δ_{ij} denote the $m-1 \times m-1$ signed minors of Z which are obtained by deleting the jth row and ith column and

$$I = I_{m-1}(Z) = \langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \ldots, \Delta_{mm} \rangle.$$

The ideal I is of linear type [10]. The relations on the given generators of $I_{m-1}(Z)$ are obtained from $Z \operatorname{adj}(Z) = \operatorname{adj}(Z) Z = \det(Z) \operatorname{Id}$.

If we define the $m \times m$ matrix $C_{(l,\underline{j})}$ by putting the first m entries of the corresponding column of $B_{\boldsymbol{a}}$ in the first row of $C_{(l,\underline{j})}$, then the second m entries of this column in the second row $C_{(l,\underline{j})}$, and so on, then the generators of $\mathscr{L}$ are all 2×2 minors of $B_{\boldsymbol{a}}$, off-diagonal entries of $C_{(l,\underline{j})} Z$ and $Z C_{(l,\underline{j})}$ and subtraction of each pair of entries in diagonals of this pair of matrices.

4 Complete intersections with common sequences

Suppose that $I_1, \dots, I_r$ are monomial ideals in a polynomial ring $R = k[x_1, \dots, x_n]$. Then the multi-Rees algebra $R[I_1t_1, \dots, I_rt_r]$ is defined by binomial equations (cf. [5]). We concentrate on a simple case where these binomial defining equations can be described effectively, generalizing the case of Example 3.5.

4.1 Gröbner basis of binary quasi-minors

Definition 4.1. An $n \times m$ quasi-matrix over R is a rectangular array with n rows and m columns such that some entries may be empty.

A subquasi-matrix is a quasi-matrix that is obtained by deleting some rows, columns, or elements of a quasi-matrix.

Example 4.2.

$$A = \begin{bmatrix} a & & b \\ c & d & \\ e & f & g \end{bmatrix}$$

is a quasi-matrix and $\begin{bmatrix} a & & b \\ & d & \end{bmatrix}$ is a subquasi-matrix of A.

Definition 4.3. A binary quasi-matrix is a quasi-matrix having exactly two elements in each nonempty row and column.

Example 4.4. All 3×3 binary quasi-matrices are listed below:

$$\begin{bmatrix} a & b & \\ & c & d \\ e & & f \end{bmatrix}, \begin{bmatrix} a & b & \\ c & & d \\ & e & f \end{bmatrix}, \begin{bmatrix} a & & b \\ & c & d \\ e & f & \end{bmatrix}, \begin{bmatrix} a & & b \\ c & d & \\ & e & f \end{bmatrix}, \begin{bmatrix} & a & b \\ c & d & \\ e & & f \end{bmatrix}, \begin{bmatrix} & a & b \\ c & & d \\ e & f & \end{bmatrix}$$

Note that a binary quasi-matrix is a square matrix, up to deleting an empty row or column. Since we usually identify a quasi-matrix canonically with the one obtained by deleting any empty row or column, in the sequel we usually consider a binary quasi-matrix as a square matrix.

Definition 4.5. Let $A = (a_{ij})$ be an $n \times n$ binary quasi-matrix over a ring R. A binary quasi-determinant of A is an element

$$a_{1\sigma(1)}a_{2\sigma(2)}\dots a_{n\sigma(n)} - a_{1\tau(1)}a_{2\tau(2)}\dots a_{n\tau(n)}$$

where σ, τ are permutations of $\{1, 2, \dots, n\}$ such that $\sigma(l) \neq \tau(l)$ for all $1 \leq l \leq n$. A quasi-determinant of a binary subquasi-matrix A is called a binary quasi-minor of A.

Note that by definition, if δ is a binary quasi-determinant of a quasi-matrix, then so is $-\delta$. In the sequel, we will usually consider a given binary quasi-minor up to sign.

Remark 4.6. (1) Note that the quasi-determinant of a 2×2 binary quasi-matrix is equal to its determinant, up to sign. Hence all 2×2 minors (which exist) of a quasi-matrix are binary quasi-minors.

(2) Note that a quasi-determinant of a 3×3 binary quasi-matrix is uniquely determined up to sign. However, in general it is not equal to the determinant, even up to sign, of the matrix obtained by assigning value zero to all empty positions.

(3) For $n \geq 4$, a quasi-determinant of a binary $n \times n$ quasi-matrix is not even unique, up to sign. For example consider the following binary quasi-matrix

$$\begin{bmatrix} a & b & & \\ c & d & & \\ & & e & f \\ & & g & h \end{bmatrix}.$$

Then $adeh - bcgf$ and $adgf - bceh$ are both quasi-determinants.

Notation 4.7. If A is a quasi-matrix with entries in R, then we denote the ideal generated by the binary quasi-minors of A by $I_{bin}(A)$.

Example 4.8. Consider the quasi-matrix A as below:

$$A = \begin{bmatrix} a & b & \\ c & d & e \\ f & & g \end{bmatrix},$$

then $adg - bef$ is a binary quasi-minor of A.

The next result shows that the ideal of binary quasi-minors generalizes to the quasi-matrices the classical ideal of 2×2 minors.

Proposition 4.9. Let A be a matrix. Then $I_{bin}(A) = I_2(A)$.

Proof. It is enough to show that every binary quasi-minor in A is an R-combination of 2×2 minors. Let $\delta = V_1 V_2 \dots V_n - W_1 W_2 \dots W_n$ be an arbitrary binary quasi-minor. We induct on $n \geq 2$. Since the result is clear for $n = 2$, we may assume $n \geq 3$ and that the result holds for binary quasi-minors of size $< n$.

We may assume V_1 is in the same row with W_1 and V_2 is in the same column with W_1. Let U be the entry of A in the same column as V_1 and same row as V_2. Then

$$\begin{aligned} \delta &= \delta - UW_1V_3 \dots V_n + UW_1V_3 \dots Vn \\ &= (V_1V_2 - UW_1)V_3 \dots V_n + W_1(UV_3 \dots V_n - W_2 \dots W_n). \end{aligned}$$

If U is not one of the W's, then the subquasi-matrix obtained by deleting the first row and column involving W_1 and V_2, and containing U is binary quasi-matrix, with $UV_3 \dots V_n - W_2 \dots W_n$ as an $(n-1)$-sized binary quasi-minor.

On the other hand, if U is a W_i, say W_2 (which can only happen if $n \geq 4$), then $UV_3 \dots V_n - W_2 \dots W_n = W_2(V_3 \dots V_n - W_3 \dots W_n)$ and $V_3 \dots V_n - W_3 \dots W_n$ is a binary quasi-minor of A of size $(n-2)$. In either case, we are done by induction. $\square$

We say that a quasi-matrix is *generic* over a field k if its entries are algebraically independent over k.

We recall a fact that, for a generic $m \times n$ matrix X, with $m \leq n$, the maximal minors of X form a universal Gröbner basis for the ideal $I_m(X)$ [2]. It is well-known that this does not hold for lower order minors. The following result gives a corresponding statement for the ideal $I_2(X)$, and more generally in "quasi" situation.

Proposition 4.10. Let A be a generic quasi-matrix over a polynomial ring over a field k. Then the set of binary quasi-minors is a universal Gröbner basis for the ideal $I_{bin}(A)$.

Proof. By Buchberger's Criterion [3, Theorem 6], it is enough to show that for each pair of binary quasi-minors f and g, the S-polynomial $S(f,g)$ reduces to zero modulo the set of binary quasi-minors. Let

$$f = V_1 \dots V_n - W_1 \dots W_n, \text{ and } g = Y_1 \dots Y_m - Z_1 \dots Z_m.$$

We may assume that $\text{in}(f) = V_1 \dots V_n$ and $\text{in}(g) = Y_1 \dots Y_m$. Then

$$h = S(f,g) = -Y_1 \dots Y_t W_1 \dots W_n + V_1 \dots V_s Z_1 \dots Z_m,$$

where we have reordered if necessary to assume that $Y_1, \dots, Y_t$ are exactly the Y variables that are not V's, and $V_1, \dots, V_s$ are exactly the V variables that are not Y's. We remark that W_i, Y_i, Z_i, V_i are not necessarily distinct.

We consider the subquasi-matrix B of A consisting of all the elements W_i, Y_i, Z_i and V_i that appear in h. If two of these elements coincide we say that the entry has multiplicity 2 in the quasi-matrix B.

First of all, we show that each of W_i and Y_i in the first term of h, is in the same row with at least one of the Z_i and V_i's in the second term of h. Clearly every Y_i is in the same row with a Z_j. On the other hand, for a W_i, it is in the same row as a V_j; if $j \leq s$ we are done. But if $j > s$, since V_j is in the same row as a Z_k, it follows that W_i is in the same row with this Z_k. The same argument can be made for elements in the second term and for columns.

Second, we show that in every row (column) of B we have either exactly two of W_i, Z_i, V_i, Y_i or exactly four of them. To verify this, it suffices to show that if three of them are in the same row (resp. column), then fourth one is also in this row (resp. column). If W_i, Y_j, V_k are in the same row then some Z_l is too. On the other hand, let W_i, Y_j, Z_k be in the same row. If W_i is in the same row (column) with a V_l, for $l > s$, then Y_j and Y_p $(p > t)$ are in the same row, which is not possible. Hence W_i is in the same row with a V_l for $l \leq s$. The remaining cases follow by symmetry, and the case for columns is similar.

Now we see that for the binomial h, a row (resp. column) of B contains either two factors (counting multiplicity), each appearing in separate terms, or four factors appearing in each term in two pairs. We say that h and B have the evenness property.

If the two terms in h have a common factor, say $W = W_i = Z_j$, then we consider the polynomial $h' = h/W$, and the corresponding quasi-submatrix B' obtained by deleting W. Then both still have the evenness property. To produce a standard expression of h, it is enough to produce one for h'. By factoring out all such common factors, we may reduce to the case that the binomial h has relatively prime terms. We again denote the resulting binomial by h and the corresponding quasi-submatrix by B.

Next, we associate to h a (multi-)graph H as follows: vertices of H are the entries of B. If only two factors lie in a row (or column) and appear in h in opposite terms, then they are joined by an edge in H. On the other hand, if there are four factors in a row (or column), V, W, Y, Z (counting multiplicities), then we attach the vertices W and Z, and the vertices V and Y. This is an arbitrary choice. Then in H, the degree of any vertex is 2 or 4. In the graph H, we refer to edges as being either horizontal or vertical, depending on the positioning of the entries of the corresponding quasi-matrix.

In the next step, we reduce to the case that both terms in the binomial h are squarefree. It is clear that the exponent of every variable in both terms of h is at most 2. Suppose that h has such a non-squarefree term. We choose a factor W having multiplicity 2. We choose a circuit H_1 starting at W, whose edges are successively vertical and horizontal, and starts initially vertically. We consider H_1 as a subgraph and let H_2 be the subgraph by removing all edges of H_1 and remaining isolated vertices. The vertices of H_1 and H_2 form two subquasi-matrices of B and the element W appears in both of them with one other element in its row and one other element in its column. Associated to the corresponding subquasi-matrices we have associated binomials

$$h^{'} = m_1 - m_2, \ h^{''} = n_1 - n_2,$$

where m_i (resp. n_i) is the product of the factors in H_1 (resp. H_2) in the i-th term of h. Each polynomial and its associated subquasi-matrix has the evenness property and the factor W now has multiplicity one in each graph and polynomial. Moreover,

$$h = m_1 n_1 - m_2 n_2.$$

We claim that $\mathrm{in}(h) \geq \min\{m_1 n_2, m_2 n_1\}$. If not, then

$$m_1 n_1 \leq \mathrm{in}(h) < m_1 n_2 \Rightarrow n_1 < n_2$$

and

$$m_2 n_2 \leq \mathrm{in}(h) < m_2 n_1 \Rightarrow n_2 < n_1$$

which is a contradiction.

Assume $\mathrm{in}(h) \geq m_1 n_2$. Then

$$\begin{aligned} h &= m_1 n_1 - m_2 n_2 \\ &= m_1 n_1 - m_2 n_2 - m_1 n_2 + m_1 n_2 \\ &= m_1(n_1 - n_2) + n_2(m_1 - m_2) \\ &= m_1 h^{''} + n_2 h^{'}. \end{aligned}$$

Further, $\mathrm{in}(m_1 h^{''}) = m_1 \,\mathrm{in}(h^{''}) = m_1 \,\mathrm{in}(n_1 - n_2)$, $\mathrm{in}(n_2 h^{'}) = n_2 \,\mathrm{in}(h^{'}) = n_2 \,\mathrm{in}(m_1 - m_2)$ and both are $\leq \mathrm{in}(h)$ by our assumption. This proves the claim.

Repeating this construction for every factor of multiplicity > 1, we eventually obtain a standard expression

$$h = \sum m_i h_i \ (*)$$

with m_i monomial, $\mathrm{in}(h) \geq \mathrm{in}(m_i h_i)$ for all i, and h_i is a binomial with relatively prime terms, all factors of multiplicity 1, and the h_i and their associated subquasi-matrix has the evenness property.

To complete the proof, we claim that there is a standard expression $(*)$ in which each h_i is a binary quasi-minor. To accomplish this, for any binomial $b = h_i$ as above, we let $\tau(b)$ denote the number of rows and columns of the associated subquasi-matrix of b having four (necessarily distinct) entries. We will show that if $\tau(b) > 0$, then there is a pair $(b^{'}, b^{''})$ as above, such that $\tau(b^{'}) < \tau(b)$ and $\tau(b^{''}) < \tau(b)$. Applying this repeatedly, we obtain a standard expression $(*)$ in which every binomial h_i that appears satisfies $\tau(h_i) = 0$. But a binomial b with $\tau(b) = 0$ is precisely a binary quasi-minor of A, which would prove the claim.

To verify that we may decrease τ in the prescribed manner, suppose that $b = h_i$ and $\tau(b) > 0$, with a row, say, with entries W, Y, Z, V. We again consider the graph H of b, in this case every vertex now having degree 2. We note that the graph H is not necessarily connected. Indeed, starting vertically from a vertex W two cases occur. The path reaches Z without passing through V or Y, or before the path reaches Z it reaches V or Y (actually the Y cannot be reached vertically). In the second case we remove edges WZ and VY from H, and instead we add edges VW and YZ to H. In either case we have a circuit H_1 in which only two vertices lie in it. Again we let H_2 be the subgraph by removing all edges of H_1 and remaining isolated vertices. Now for the corresponding binomials $b^{'}$ and $b^{''}$ as above, we obtain that $\tau(b^{'}) < \tau(b)$ and $\tau(b'') < \tau(b)$. This completes the proof. $\square$

4.2 Equations of the multi-Rees algebra

We fix a regular sequence $\underline{s} = s_1, \ldots, s_n$ in any ring R containing a field. Let $R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r]$ be the multi-Rees algebra of powers of ideals I_i, where a_i's are positive integers and the ideals are generated by arbitrary subsets of $\underline{s}$, in the rest of this chapter by generators of I_i we mean these generators. We denote $\boldsymbol{a} = a_1, \ldots, a_r$.

Definition 4.11. Let a be a positive integer. We define $\mathscr{F}_a^l$ as a subset of $\mathscr{T}_a$ such that $\underline{j} \in \mathscr{F}_a^l$ if and only if $s^{\underline{j}} \in I_l^a$.

Since $I_l^{a_l}$ is generated by all $s^{\underline{j}}$'s ($\underline{j} \in \mathscr{F}_{a_l}^l$), the multi-Rees algebra $R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r]$ is the ring $R\left[\{s^{\underline{j}}t_l\}_{1\leq l\leq r, \underline{j}\in\mathscr{F}_{a_l}^l}\right]$. Let $S := R\left[\{T_{l,\underline{j}}\}_{1\leq l\leq r, \underline{j}\in\mathscr{F}_{a_l}^l}\right]$. We define and fix the R-algebra epimorphism

$$\phi : S \to R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r], \text{ by } \phi(T_{l,\underline{j}}) = s^{\underline{j}}t_l.$$

We want to find generators of $\mathscr{L} = \ker(\phi)$.

Definition 4.12. For a fixed l, consider the set $\{(l, \underline{j});\ \underline{j} \in \mathscr{T}_{a_l}'\}$ ordered lexicographically as a subset of $\mathbb{N}^n$. We define the matrix B_{a_l}, whose entry in row k and column $(l, \underline{j})$ is $T_{l,\underline{j}^{|k\rangle}}$.

We see that $\phi(T_{l,\underline{j}^{|k\rangle}})$ contains at least a factor of s_k. Let $I_l = \langle s_{k_1}, \ldots, s_{k_v}\rangle$, and $k_1 < k_2 < \cdots < k_v$. Then the only possible $T_{l,\underline{j}^{|k\rangle}}$'s whose images under ϕ are monomials in $s_{k_1}, \ldots, s_{k_v}$ are in rows $k_1, \ldots, k_v$ of B_{a_l}. On the other hand for $u < w$, if we compare images of $T_{l,\underline{j}^{|u\rangle}}$ and $T_{l,\underline{j}^{|w\rangle}}$,

$$\phi(T_{l,\underline{j}^{|u\rangle}}) = s_1^{j_1-1-0}s_2^{j_2-j_1}\ldots s_{u-1}^{j_{u-1}-j_{u-2}}s_u^{j_u-j_{u-1}+1}\ldots s_w^{j_w-j_{w-1}}\ldots s_n^{j_n-j_{n-1}},$$

$$\phi(T_{l,\underline{j}^{|w\rangle}}) = s_1^{j_1-1-0}s_2^{j_2-j_1}\ldots s_u^{j_u-j_{u-1}}\ldots s_{w-1}^{j_{w-1}-j_{w-2}}s_w^{j_w-j_{w-1}+1}\ldots s_n^{j_n-j_{n-1}},$$

then we see that when we move on the column $(l, \underline{j})$ from row u to row w we lose one factor s_u and we get one factor s_w.

Definition 4.13. In the matrix B_{a_l}, in the row k_1, we choose $T_{l,\underline{j}^{|k\rangle}}$'s whose images under ϕ are monomials in $s_{k_1}, \ldots, s_{k_v}$. We define the quasi-matrix D_{a_l} to be the subquasi-matrix of

B_{a_l} by choosing these columns and rows k_i, $1 \leq i \leq v$. The entries of the submatrix D_{a_l} are all $T_{l,\underline{j}^{|k\rangle}}$'s in B_{a_l} whose images under ϕ are monomials in $s_{k_1}, \ldots, s_{k_v}$.

We define the matrix $B_{\boldsymbol{a}} \coloneqq (B_{a_1}|B_{a_2}|\ldots|B_{a_r})$ and the matrix $C_{\boldsymbol{a}} \coloneqq (\underline{s}|B_{\boldsymbol{a}})$. We also define the subquasi-matrices $D_{\boldsymbol{a}} \coloneqq (D_{a_1}|D_{a_2}|\ldots|D_{a_r})$ and $E_{\boldsymbol{a}} \coloneqq (\underline{s}|D_{\boldsymbol{a}})$ of $C_{\boldsymbol{a}}$.

Then the theorem below describes the defining equations of the multi-Rees algebra $R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r]$.

Theorem 4.14. Let R be a Noetherian ring containing a field and suppose that ideals I_i are generated by subsets of a fixed regular sequence $s_1, \ldots, s_n$ contained in $\operatorname{rad} R$. Then

$$R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r] \cong S/I_{bin}(E_{\boldsymbol{a}}).$$

Remark 4.15. We will show that the defining ideal is generated by the 2×2 minors of $E = E_{\boldsymbol{a}}$ involving $s_1, \ldots, s_n$, the 2×2 minors of the D_{a_l}, and the binary quasi-minors of E (which are not minors) and have at most two entries from each D_{a_l}.

Proof. First we show that $I_{bin}(E_{\boldsymbol{a}}) \subseteq \mathscr{L}$. If f is a T-binary quasi-minor, then

$$f = T_{l_1,\underline{j}^{(1)}|k_{i_1}\rangle} T_{l_2,\underline{j}^{(2)}|k_{i_2}\rangle} \ldots T_{l_\beta,\underline{j}^{(\beta)}|k_{i_\beta}\rangle} - T_{l_1,\underline{j}^{(1)}|k_{u_1}\rangle} T_{l_2,\underline{j}^{(2)}|k_{u_2}\rangle} \ldots T_{l_\beta,\underline{j}^{(\beta)}|k_{u_\beta}\rangle}.$$

For arbitrary $T_{l_v,\underline{j}^{(v)}|k_{i_v}\rangle}$ we have

$$\phi(T_{l_v,\underline{j}^{(v)}|k_{i_v}\rangle}) = t_{l_v} s_{k_{i_v}} \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(v)} - j_{\alpha-1}^{(v)}}, \text{ with } \underline{j}^{(v)} \in \mathscr{T}'_{a_{l_v}},$$

and similarly

$$\phi(T_{l_v,\underline{j}^{(v)}|k_{u_v}\rangle}) = t_{l_v} s_{k_{u_v}} \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(v)} - j_{\alpha-1}^{(v)}}, \text{ with } \underline{j}^{(v)} \in \mathscr{T}'_{a_{l_v}}.$$

Hence we have

$$\phi(f) = s_{k_{i_1}} s_{k_{i_2}} \dots s_{k_{i_\beta}} t_{l_1} t_{l_2} \dots t_{l_\beta} \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(1)} - j_{\alpha-1}^{(1)}} \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(2)} - j_{\alpha-1}^{(2)}} \cdots \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(\beta)} - j_{\alpha-1}^{(\beta)}}$$
$$- s_{k_{u_1}} s_{k_{u_2}} \dots s_{k_{u_\beta}} t_{l_1} t_{l_2} \dots t_{l_\beta} \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(1)} - j_{\alpha-1}^{(1)}} \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(2)} - j_{\alpha-1}^{(2)}} \cdots \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(\beta)} - j_{\alpha-1}^{(\beta)}} = 0.$$

If f is an $\underline{s}$-binary quasi-minor, then

$$f = s_{k_{i_1}} T_{l_2, \underline{j}^{(2)|k_{i_2}\rangle}} \dots T_{l_\beta, \underline{j}^{(\beta)|k_{i_\beta}\rangle}} - s_{k_{u_1}} T_{l_2, \underline{j}^{(2)|k_{u_2}\rangle}} \dots T_{l_\beta, \underline{j}^{(\beta)|k_{u_\beta}\rangle}},$$

and similarly we see that $\phi(f) = 0$:

$$\phi(f) = s_{k_{i_1}} s_{k_{i_2}} \dots s_{k_{i_\beta}} t_{l_2} \dots t_{l_\beta} \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(2)} - j_{\alpha-1}^{(2)}} \cdots \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(\beta)} - j_{\alpha-1}^{(\beta)}}$$
$$- s_{k_{u_1}} s_{k_{u_2}} \dots s_{k_{u_\beta}} t_{l_2} \dots t_{l_\beta} \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(2)} - j_{\alpha-1}^{(2)}} \cdots \prod_{\alpha=1}^{n} s_\alpha^{j_\alpha^{(\beta)} - j_{\alpha-1}^{(\beta)}} = 0.$$

Thus $I_{bin}(E_{\boldsymbol{a}}) \subseteq \mathscr{L}$. Now we prove that $\mathscr{L} \subseteq I_{bin}(E_{\boldsymbol{a}})$. We prove this claim in two steps. First we prove the claim for the case that $\boldsymbol{a} = (1, \dots, 1)$.

To prove this case, we reduce the problem to the case when $s_1, \dots, s_n$ is a sequence in variables in a polynomial ring $R = k[s_1, \dots, s_n]$ over a field k. Suppose the Theorem is proved in this case.

Let $k \subset R$ be a field. Since $s_1, \dots, s_n$ is a regular sequence contained in $\operatorname{rad} R$, R is flat over its subring $k[s_1, \dots, s_n]$, and the latter ring is a polynomial ring [6]. (Actually [6] only proves the flatness for a local ring, but the proof only needs the local criterion of flatness, which holds as long as the ideal belongs to $\operatorname{rad} R$ [5,Theorem 22.3].) Set $A = k[s_1, \dots, s_n]$. Since I_i is generated by a subset of $s_1, \dots, s_n$, there is an ideal J_i be an ideal of A such that $J_i R = I_i$. By hypothesis, the theorem holds for the multi-Rees algebra of the ideals J_i of A.

However, since R is flat over A,

$$A[J_1t_1,\ldots,J_rt_r]\otimes_A R\cong R[I_1t_1,\ldots,I_rt_r].$$

Indeed, the isomorphism holds essentially since $J_i\otimes_A R\cong J_iR=I_i$. From this isomorphism, it follows immediately that

$$R[I_1t_1,\ldots,I_rt_r]\cong (S_A/I_{bin}(E_{(1,\ldots,1)}))\otimes_A R\cong S/I_{bin}(E_{(1,\ldots,1)}).$$

This completes the proof of the reduction to the case of a sequence of variables.

We consider the ideal $I=\langle s_1\ldots,s_n\rangle$. Then we have the following commutative diagram:

$$\begin{tikzcd} R\left[\{T_{l,\underline{j}}\}_{1\le l\le r,\underline{j}\in\mathscr{F}_1^l}\right] \arrow[r, two heads, "\phi"] \arrow[d, hook, "\theta_1"] & R[I_1t_1,\ldots,I_rt_r] \arrow[d, hook, "\theta_2"] \\ R\left[\{T_{l,\underline{j}}\}_{1\le l\le r,\underline{j}\in\mathscr{F}_1}\right] \arrow[r, two heads, "\phi'"] & R[It_1,\ldots,It_r], \end{tikzcd}$$

where $\phi'(T_{l,\underline{j}})=s^{\underline{j}}t_l$. For any $f\in\mathscr{L}$, we have

$$\phi'(f)=\phi'\theta_1(f)=\theta_2\phi(f)=0\Rightarrow f\in\ker(\phi').$$

By Example 3.4, $f\in I_2(C_{\boldsymbol{a}})$. By Proposition 4.9, $f\in I_{bin}(C_{\boldsymbol{a}})$.

We put a total order on variables of $C_{\boldsymbol{a}}$ such that all variables which are not in $E_{\boldsymbol{a}}$ are greater than all variables which are in $E_{\boldsymbol{a}}$. We also put the lexicographic order on $R\left[\{T_{l,\underline{j}}\}_{1\le l\le r,\underline{j}\in\mathscr{F}_1}\right]$. By Proposition 4.10 the set of all binary quasi-minors of $C_{\boldsymbol{a}}$ form a Gröbner basis for $I_{bin}(C_{\boldsymbol{a}})$, and so $\mathrm{in}(f)\in\mathrm{in}(I_{bin}(C_{\boldsymbol{a}}))$. We prove that $f\in I_{bin}(E_{\boldsymbol{a}})$ by induction on $\mathrm{in}(f)$, the base change being trivial. There is a binary quasi-minor such as h such that $w\,\mathrm{in}(h)=\mathrm{in}(f)$. On the other hand variables in $w\,\mathrm{in}(h)$ are some of variables in f and so they are variables in $E_{\boldsymbol{a}}$. Since we have the lexicographic order and all variables out of $E_{\boldsymbol{a}}$ are greater than these variables, variables in the non-initial term of h are also in $E_{\boldsymbol{a}}$. Hence h is a binary quasi-minor in $E_{\boldsymbol{a}}$. Therefore $f - wh \in I_{bin}(C_{\boldsymbol{a}})$, and variables in $f - wh$ are in $E_{\boldsymbol{a}}$. Moreover $\mathrm{in}(f) >_{lex} \mathrm{in}(f - wh)$ so by induction $f - wh \in I_{bin}(E_{\boldsymbol{a}})$ and hence so is f.

Now we prove the theorem in the general case. We use the same method that we have used in Theorem 3.3.

Let $Y = (x_{i,l})$ be a generic $n \times r$ matrix and let X be a subquasi-matrix of Y defined as follows: an arbitrary $x_{i,l}$ is an entry of X if s_i is between generators of I_l. x_l denotes the regular sequence $x_{1,l}, x_{2,l}, \ldots, x_{n,l}$ of entries of the l-th column of Y. We define A_l to be the a_l-th Veronese subring $R\left[\{x_{i,l}\}_{x_{i,l} \text{ is in the } l\text{-column of } X}\right]^{(a_l)}$. Let $X_l(a)$ be the family of monomials of degree a ($a \in \mathbb{N}$) in the variables which are in the l-column of X, we define A to be the $\boldsymbol{a}$-th Veronese subring $R[X]^{\boldsymbol{a}} = R[X_1(a_1), X_2(a_2), ..., X_r(a_r)]$.

For arbitrary l, $1 \leq l \leq r$ we define

$$\alpha_l : R\left[\left\{T_{l,\underline{j}}\right\}_{\underline{j} \in \mathscr{F}^l_{a_l}}\right] \to A_l, \quad \alpha_l(T_{l,\underline{j}}) = x_l^{\underline{j}}.$$

These induce a map $\alpha : S \to A$. By the same argument given in Theorem 3.3, we have

$$\ker(\alpha) = \langle I_2(D_{a_1}), I_2(D_{a_2}), ..., I_2(D_{a_r}) \rangle \subseteq I_{bin}(E_{\boldsymbol{a}}).$$

We define the quasi-matrix $Z = (\underline{s}|X)$. If we consider

$$\psi : R[X] \to R[I_1u_1, I_2u_2, \ldots, I_ru_r], \quad \psi(x_{i,l}) = s_iu_l,$$

then by first part of the proof $\ker(\psi)$ is generated by 2×2 $\underline{s}$-minors and x-binary quasi-minors (binary quasi-minors which don't contain s_i) of Z.

By the following commutative diagram we define the map g:

$$\begin{tikzcd}
S \arrow[r, two heads, "\phi"] \arrow[d, two heads, "\alpha"] & R[I_1^{a_1}t_1,\dots,I_r^{a_r}t_r] \arrow[d, "\wr"] \\
A \arrow[ur, "g"] \arrow[r, "\psi^{(a)}"] \arrow[d, hook] & R[I_1^{a_1}u_1^{a_1},\dots,I_r^{a_r}u_r^{a_r}] \arrow[d, hook] \\
R[X] \arrow[r, two heads, "\psi"] & R[I_1u_1,\dots,I_ru_r]
\end{tikzcd}$$

In this step we prove that the kernel of g is generated by polynomials of the form

$$s_\delta x_{\gamma,l}x_l^{\underline{j}} - s_\gamma x_{\delta,l}x_l^{\underline{j}},\ \underline{j} \in \mathscr{F}_{a_l-1}^{l}$$

and

$$\prod_{i=1}^{v} x_{\delta_i,\gamma_i}x_{\gamma_i}^{\underline{j}^{(i)}} - \prod_{i=1}^{v} x_{\beta_i,\gamma_i}x_{\gamma_i}^{\underline{j}^{(i)}},\ \underline{j}^{(i)} \in \mathscr{F}_{a_{\gamma_i}-1}^{\gamma_i},$$

where γ_i (resp. δ_i, β_i) are distinct and $\{\delta_1,\dots,\delta_v\} = \{\beta_1,\dots\beta_v\}$.

The proof of this fact is similar to the proof of Theorem 3.3, but because it is a different argument in many parts we include the details for completeness.

If $0 \neq p \in \ker(g)$, then $p \in \ker(\psi)$ and thus it is an $R[X]$-linear combination of 2×2 $\underline{s}$-minors or x-binary quasi-minors of Z. Thus we have an expression $p = \sum_{i=1}^{m} r_i\mu_i q_i$, where the q_k's are 2×2 $\underline{s}$-minors or x-binary quasi-minors of Z, $r_k \in R$, and $\mu_k \in R[X]$ is a monomial. Among all such expressions we choose one in which m is minimal. We claim that if q_k is an $\underline{s}$-minor and its variables are chosen from $l+1$ column of Z, then $\mu_k = \xi_k x_l^{\underline{j}}$ ($\underline{j} \in \mathscr{F}_{a_l-1}^{l}$) and if $q_k = \prod_{i=1}^{v} x_{\delta_i,\gamma_i} - \prod_{i=1}^{v} x_{\beta_i,\gamma_i}$, then $\mu_k = \xi_k \prod_{i=1}^{v} x_{\gamma_i}^{\underline{j}^{(i)}}$ ($\underline{j}^{(i)} \in \mathscr{F}_{a_{\gamma_i}-1}^{\gamma_i}$), in both cases $\xi_k \in A$. We may assume that $\mu_1 x_{\gamma,l}$ or $\mu_1\prod_{i=1}^{v} x_{\delta_i,\gamma_i}$ appears in the support of p, where $q_1 = s_\delta x_{\gamma,l} - s_\gamma x_{\delta,l}$ or $q_1 = \prod_{i=1}^{v} x_{\delta_i,\gamma_i} - \prod_{i=1}^{v} x_{\beta_i,\gamma_i}$. Since one of terms in p is $r\mu_1 x_{\gamma,l}$ or $r\mu_1\prod_{i=1}^{v} x_{\delta_i,\gamma_i}$ ($r \in R$), $\mu_1 x_{\gamma,l}$ or $\mu_1\prod_{i=1}^{v} x_{\delta_i,\gamma_i}$ is in A. Therefore we have two cases: case i) $\mu_1 x_{\gamma,l}$ is in the support of p. Then $\mu_1 x_{\gamma,l} = \Gamma_1\Gamma_2\dots\Gamma_w$, where $\Gamma_j \in A_i$ and is a monomial of degree a_i, therefore $x_{\gamma,l}$ divides at least one Γ_j, say $x_{\gamma,l}$ divides Γ_1, then Γ_1 is a monomial of degree a_l in A_l and $\Gamma_1 = x_{\gamma,l} x_l^{\underline{j}}$ ($\underline{j} \in \mathscr{F}^l_{a_l-1}$). Hence

$$\mu_1 x_{\gamma,l} = x_{\gamma,l} x_l^{\underline{j}} \Gamma_2 \dots \Gamma_w \Rightarrow \mu_1 = \xi_1 x_l^{\underline{j}}, \text{ with } \xi_1 = \Gamma_2 \dots \Gamma_w \in A.$$

Case ii) $\mu_1 \prod_{i=1}^{v} x_{\delta_i,\gamma_i}$ is in the support of p. Then $\mu_1 \prod_{i=1}^{v} x_{\delta_i,\gamma_i} = \Gamma_1 \Gamma_2 \dots \Gamma_w$, where $\Gamma_j \in A_i$ and is a monomial of degree a_i. Since all γ_i's are distinct, then x_{δ_i,γ_i}'s divide distinct Γ_j's say Γ_i's, therefore Γ_i's are monomials of degree a_{γ_i} in A_{γ_i}, moreover $\Gamma_i = x_{\delta_i,\gamma_i} x_{\gamma_i}^{\underline{j}^{(i)}}$, $\underline{j}^{(i)} \in \mathscr{F}^{\gamma_i}_{a_{\gamma_i}-1}$. Therefore

$$\begin{aligned} \mu_1 \prod_{i=1}^{v} x_{\delta_i,\gamma_i} &= \prod_{i=1}^{v} x_{\delta_i,\gamma_i} x_{\gamma_i}^{\underline{j}^{(i)}} \Gamma_{v+1} \dots \Gamma_w \\ \Rightarrow \mu_1 &= \xi_1 \prod_{i=1}^{v} x_{\gamma_i}^{\underline{j}^{(i)}}, \text{ with } \xi_1 = \Gamma_{v+1} \dots \Gamma_w \in A. \end{aligned}$$

Hence either

$$\mu_1 q_1 = \xi_1 x_l^{\underline{j}} (s_\delta x_{\gamma,l} - s_\gamma x_{\delta,l}) = \xi_1 (s_\delta x_{\gamma,l} x_l^{\underline{j}} - s_\gamma x_{\delta,l} x_l^{\underline{j}})$$

or

$$\mu_1 q_1 = \xi_1 \prod_{i=1}^{v} x_{\gamma_i}^{\underline{j}^{(i)}} \left(\prod_{i=1}^{v} x_{\delta_i,\gamma_i} - \prod_{i=1}^{v} x_{\beta_i,\gamma_i} \right) = \xi_1 \left(\prod_{i=1}^{v} x_{\delta_i,\gamma_i} x_{\gamma_i}^{\underline{j}^{(i)}} - \prod_{i=1}^{v} x_{\beta_i,\gamma_i} x_{\gamma_i}^{\underline{j}^{(i)}} \right).$$

Since p and $\mu_1 q_1 \in A$, we conclude that $p - r_1 \mu_1 q_1 \in A$. Then since the expression $\sum_{i=2}^{m} r_i \mu_i q_i$ is still minimal, by induction of m (the base case $m = 1$ being clear by the above), we conclude that $p - r_1 \mu_1 q_1$, and hence p, is a linear combination of the required polynomials. Therefore we have described the generators of $\ker(g)$.

On the other hand if $\underline{j}' = \underline{j} + (1, \dots, 1)$, then $s_\delta T_{l,\underline{j}'|\gamma\rangle} - s_\gamma T_{l,\underline{j}'|\delta\rangle}$ is an s-minor of $E_{\boldsymbol{a}}$ and

$$\alpha \left(s_\delta T_{l,\underline{j}'|\gamma\rangle} - s_\gamma T_{l,\underline{j}'|\delta\rangle} \right) = s_\delta x_{\gamma,l} x_l^{\underline{j}} - s_\gamma x_{\delta,l} x_l^{\underline{j}}$$

and if $\underline{i}^{(k)} = \underline{j}^{(k)} + (1, \ldots, 1)$, then

$$T_{\gamma_1, \underline{i}^{(1)}|\delta_1\rangle} \ldots T_{\gamma_v, \underline{i}^{(v)}|\delta_v\rangle} - T_{\gamma_1, \underline{i}^{(1)}|\beta_1\rangle} \ldots T_{\gamma_v, \underline{i}^{(v)}|\beta_v\rangle}$$

is a binary quasi-minor of $E_{\boldsymbol{a}}$ and its image under α is

$$\left(\prod_{i=1}^{v} x_{\delta_i, \gamma_i} x_{\gamma_i}^{\underline{j}^{(i)}} - \prod_{i=1}^{v} x_{\beta_i, \gamma_i} x_{\gamma_i}^{\underline{j}^{(i)}} \right).$$

This completes the proof. $\square$

Remark 4.16. We prove that every $\underline{s}$-binary quasi-minor of $E_{\boldsymbol{a}}$ can be generated by 2×2 $\underline{s}$-minors and T-binary quasi-minors of $E_{\boldsymbol{a}}$. Let $f = s_i V_1 V_2 \ldots V_n - s_j W_1 W_2 \ldots W_n$ (V_i and W_i are equal to $T_{l, \underline{j}|k\rangle}$'s). Without loss of generality we may assume s_i and W_1 are in the same row and W_1 and V_1 are in the same column. If V_1 and s_j are in the same row, then we have

$$\begin{aligned} f &= s_i V_1 V_2 \ldots V_n - s_j W_1 W_2 \ldots W_n - s_j W_1 V_2 \ldots V_n + s_j W_1 V_2 \ldots V_n \\ &= (s_i V_1 - s_j W_1) V_2 \ldots V_n + s_j W_1 (V_2 \ldots V_n - W_2 \ldots W_n). \end{aligned}$$

If V_1 and s_j are not in the same row, then there is an s_k which is in the same row with V_1. We have

$$\begin{aligned} f &= s_i V_1 V_2 \ldots V_n - s_j W_1 W_2 \ldots W_n - s_k W_1 V_2 \ldots V_n + s_k W_1 V_2 \ldots V_n \\ &= (s_i V_1 - s_k W_1) V_2 \ldots V_n + W_1 (s_k V_2 \ldots V_n - s_j W_2 \ldots W_n). \end{aligned}$$

We can continue this procedure until all generators are either 2×2 $\underline{s}$-minors or T-binary quasi-minors of $E_{\boldsymbol{a}}$.

Remark 4.17. We can define D_{a_l} in another way. If $I_l = \langle s_{i_1}, \ldots, s_{i_u} \rangle$, then we define similar $\mathscr{T}_{a_l}$ and $\mathscr{T}'_{a_l}$ with members $\underline{j} = (j_{u-1}, \ldots, j_1)$. We also define $s^{\underline{j}}$ which is a monomial in

$s_{i_1}, \ldots, s_{i_u}$. Hence we define $T_{l,\underline{j}}$ and so D_{a_l} and finally $E_{\boldsymbol{a}}$. We can see that image of corresponding entries in both D_{a_l} is the same. We will have a new $R[T_{l,\underline{j}}]$ and a new ϕ, but this new one is isomorphic to the previous one and the image of corresponding variables under different ϕ's is the same. Then we have similar result about the generators of the kernel.

Corollary 4.18. Let R be a local Cohen-Macaulay ring containing a field and suppose that ideals I_i are generated by subsets of a fixed regular sequence. Then the multi-Rees algebra $R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r]$ is Cohen-Macaulay.

Proof. We first verify the claim when R is a polynomial ring in the fixed regular sequence $s_1, \ldots, s_n$. First we show that $R[I_1u_1, \ldots, I_ru_r]$ is normal. By Theorem 4.14, we have

$$R[I_1u_1, \ldots, I_ru_r] \cong R[T_{l,\underline{j}}]/I_{bin}(E_{\boldsymbol{a}}),$$

and by Lemma 4.10, $\mathrm{in}(I_{bin}(E_{\boldsymbol{a}}))$ is generated by initial terms of binary quasi-minors which are squarefree, hence by [21, Proposition 13.5, Proposition 13.15], $R[I_1u_1, \ldots, I_ru_r]$ is a normal domain. On the other hand

$$R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r] \cong R[I_1^{a_1}u_1^{a_1}, \ldots, I_r^{a_r}u_r^{a_r}]$$

and $R[I_1^{a_1}u_1^{a_1}, \ldots, I_r^{a_r}u_r^{a_r}]$ is a direct summand of $R[I_1u_1, \ldots, I_ru_r]$, so $R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r]$ is normal.

Now if we consider the semigroup M generated by $x_1, \ldots, x_m$ and ft_i, where f is a generator of $I_i^{a_i}$, then by [8, Proposition 1], M is normal, hence by [8, Theorem 1], $k[M] \cong R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r]$ is a Cohen-Macaulay domain.

To complete the proof, as in the proof of Theorem 4.14, we reduce to the latter case. Since the multi-Rees algebra is positively graded, it suffices to show that the special fiber ring $R/\langle s_1, \ldots, s_n\rangle$ is Cohen-Macaulay, which is clear. $\square$

Remark 4.19. This result does not hold for arbitrary complete intersections, even of codimension 2. For example, if $R = k[[x, y]]$, and $I_1 = \langle x, y\rangle$, $I_2 = \langle x^2, y^2\rangle$, then the multi-Rees algebra $R[I_1t_1, I_2t_2]$ is not Cohen-Macaulay (cf. [19, Example 4.9]).

Example 4.20. Let $k[s_1, s_2, s_3]$, where k is a field. Let $I_1 = \langle s_1, s_2\rangle$, $I_2 = \langle s_2, s_3\rangle$, $I_3 = \langle s_1, s_3\rangle$. Then we have the homomorphism

$$\phi : R[T_{1,1,1}, T_{1,1,0}, T_{2,1,0}, T_{2,0,0}, T_{3,1,1}, T_{3,0,0}] \to R[I_1t_1, I_2t_2, I_3t_3],$$
$$T_{1,1,1} \mapsto s_1t_1,\ T_{1,1,0} \mapsto s_2t_1,\ T_{2,1,0} \mapsto s_2t_2,\ T_{2,0,0} \mapsto s_3t_2,\ T_{3,1,1} \mapsto s_1t_3,\ T_{3,0,0} \mapsto s_3t_3,$$

and its kernel is

$$\langle s_1T_{1,1,0} - s_2T_{1,1,1}, s_1T_{3,0,0} - s_3T_{3,1,1}, s_2T_{2,0,0} - s_3T_{2,1,0}, T_{1,1,1}T_{2,1,0}T_{3,0,0} - T_{3,1,1}T_{1,1,0}T_{2,0,0}\rangle.$$

We see that $E_{\boldsymbol{a}}$ has the form below

$$\begin{bmatrix} s_1 & T_{1,1,1} & & T_{3,1,1} \\ s_2 & T_{1,1,0} & T_{2,1,0} & \\ s_3 & & T_{2,0,0} & T_{3,0,0} \end{bmatrix}.$$

This special example can also be recovered by using the theory of Rees algebras of modules, as follows. The module $M = I_1 \oplus I_2 \oplus I_3$ has a linear resolution

$$0 \to R^3 \xrightarrow{\Phi} R^6 \to M \to 0$$

where

$$\Phi = \begin{bmatrix} s_2 & 0 & 0 \\ -s_1 & 0 & 0 \\ 0 & s_3 & 0 \\ 0 & -s_2 & 0 \\ 0 & 0 & s_3 \\ 0 & 0 & -s_1 \end{bmatrix}.$$

Hence $\operatorname{pd} M = 1$. Furthermore, since M is free in codimension 1, and 4-generated in codimension 2, by [19, Proposition 4.11] the Rees algebra of M, which is the multi-Rees algebra in question, has the expected defining equations, in the sense that

$$\mathcal{R}(M) \cong R[T_1, T_2, T_3, T_4, T_5, T_6]/\langle [s_1 s_2 s_3]B, \det B\rangle$$

where

$$B = \begin{bmatrix} -T_2 & 0 & -T_6 \\ T_1 & -T_4 & 0 \\ 0 & T_3 & T_5 \end{bmatrix}$$

is the matrix defined by the equation

$$[T]\Phi = [s]B.$$

Example 4.21. Another simple case is for a pair of "transversal" ideals: Let R be a Noetherian local ring containing a field, let $s_1, \ldots, s_n$ be a regular sequence and let $I_1 = \langle s_1, \ldots, s_k\rangle$ and $I_2 = \langle s_{k+1}, \ldots s_n\rangle$. Then the multi-Rees algebra $R[I_1t_1, I_2t_2]$ is defined by linear equations. Indeed, this follows from Theorem 4.14. (The result that $M = I_1 \oplus I_2$ is of linear type can be proved directly, at least in this case when R is a polynomial ring in $s_1, \ldots, s_n$.)

Remark 4.22. One may wonder if we can use the method of proof of Proposition 4.10, to prove that for an arbitrary $\boldsymbol{a}$ the binary minors of $E_{\boldsymbol{a}}$ form a Gröbner basis so in the proof of

Theorem 4.14, we don't need the Veronese type argument. Since in general elements of $E_{\boldsymbol{a}}$ are not distinct we cannot use this method and doing this will cause the same mistake that is made in [15, Lemma 2.6], where the authors have proved that when $R = k[x_1, \ldots, x_n]$ (k a field), 2×2-minors of $C_{\boldsymbol{a}}$ form a Gröbner basis for $I_2(C_{\boldsymbol{a}})$ with reverse lexicographic order. Since the indeterminates of $C_{\boldsymbol{a}}$ are not all distinct the proof of this lemma is incomplete. For example in their case 1, the authors assume that the 2×2-minors are

$$h_1 = \begin{vmatrix} T_{l_1,\underline{i},s} & T_{l_2,\underline{j},s} \\ T_{l_1,\underline{i},t} & T_{l_2,\underline{j},t} \end{vmatrix} = \begin{vmatrix} a & e \\ f & g \end{vmatrix}$$

$$h_2 = \begin{vmatrix} T_{l_1,\underline{i},s} & T_{l_3,\underline{p},s} \\ T_{l_1,\underline{i},u} & T_{l_3,\underline{p},u} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

but in the minor $h_2 = ad - bc$ we cannot always say $a = T_{l_1,\underline{i},s}$, because another $T_{l_1,\underline{q},v}$ may be equal to a, although in h_1 we have $a = T_{l_1,\underline{i},s}$. Thus we cannot always find G such that $S(h_1, h_2) = -def + gbc = b(cg - fG) - f(de - Gb)$. For example over the ring $R = k[x_1, x_2, x_3, x_4]$ with the maximal ideal $\mathfrak{m} = \langle x_1, x_2, x_3, x_4 \rangle$, we consider the Rees algebra $R[\mathfrak{m}^3 t]$. Since $r = 1$, when we show the matrix $B_{\boldsymbol{a}}$, instead of $T_{l,\underline{j}^{|k\rangle}}$ we only write $T_{\underline{j},k}$. Now $B_{\boldsymbol{a}}$, associated to $R[\mathfrak{m}^3 t]$ is as below:

$$B_{\boldsymbol{a}} = \begin{bmatrix} T_{1,1,1} & T_{2,1,1} & T_{2,2,1} & T_{2,2,2} & T_{3,1,1} & T_{3,2,1} & T_{3,2,2} & T_{3,3,1} & T_{3,3,2} & T_{3,3,3} \\ T_{1,1,0} & T_{2,1,0} & T_{2,2,0} & T_{2,2,1} & T_{3,1,0} & T_{3,2,0} & T_{3,2,1} & T_{3,3,0} & T_{3,3,1} & T_{3,3,2} \\ T_{1,0,0} & T_{2,0,0} & T_{2,1,0} & T_{2,1,1} & T_{3,0,0} & T_{3,1,0} & T_{3,1,1} & T_{3,2,0} & T_{3,2,1} & T_{3,2,2} \\ T_{0,0,0} & T_{1,0,0} & T_{1,1,0} & T_{1,1,1} & T_{2,0,0} & T_{2,1,0} & T_{2,1,1} & T_{2,2,0} & T_{2,2,1} & T_{2,2,2} \end{bmatrix}.$$

In this matrix we consider minors h_1 and h_2 as follows

$$h_1 = \begin{vmatrix} T_{2,1,1} & T_{2,2,1} \\ T_{2,1,0} & T_{2,2,0} \end{vmatrix} = \begin{vmatrix} a & e \\ f & g \end{vmatrix}, \ h_2 = \begin{vmatrix} T_{2,1,1} & T_{3,0,0} \\ T_{1,1,1} & T_{2,0,0} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

we see that there is no G such that $-def + gbc = b(cg - fG) - f(de - Gb)$, because there is no G such that $cg - fG$ becomes a minor.

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