Equations of multi-Rees Algebras

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Equations of multi-Rees Algebras

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by

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Abstract

In this thesis we describe the defining equations of certain multi-Rees algebras. First, we determine the defining equations of the multi-Rees algebra $R[I^a_1 t_1, \ldots, I^a_r t_r]$ over a Noetherian ring $R$ when $I$ is an ideal of linear type. This generalizes a result of Ribbe and recent work of Lin-Polini and Sosa. Second, we describe the equations defining the multi-Rees algebra $R[I^a_1 t_1, \ldots, I^a_r t_r]$, where $R$ is a Noetherian ring containing a field and the ideals are generated by a subset of a fixed regular sequence.
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I would also like to acknowledge my family for their support during these years, even though we have not been together. I would especially like to acknowledge my brother Masood for all kinds of support and help that he gave to me during these years. He gave me a lot of advice and encouragement during my studies in the United States.
Dedication

This thesis is wholeheartedly dedicated to my mother and father who have always loved me unconditionally, providing the moral, spiritual, and emotional support to make this possible.

I also want to recognize my brother and sisters who have always been a constant source of love, support and encouragement in my life.
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1 Introduction

The Rees algebra \( R[It] \) plays an important role in commutative algebra because it encodes the asymptotic behavior of the ideal \( I \). Also, in algebraic geometry the projective scheme \( \text{Proj} \, R[It] \) defines the blowup of the scheme \( \text{Spec}(R) \) along \( V(I) \). Let \( R \) be a Noetherian ring and \( I \subseteq R \) be an ideal of \( R \). An important problem in the theory of Rees algebras is to describe \( R[It] \) in terms of generators and relations: find an ideal \( \mathcal{L} \) in a polynomial ring \( S = R[T_1, \ldots, T_n] \) such that \( R[It] \cong S/\mathcal{L} \). Generators of the ideal \( \mathcal{L} \) are called equations of the Rees algebra. This is a tough problem which is open for most classes of ideals. Some papers about this problem are [24], [12], [23], [16], [17], [14].

More generally, given any ideals \( I_1, \ldots, I_r \) in a ring \( R \), one would like to describe equations of the multi-Rees algebra \( R[I_1t_1, I_2t_2, \ldots, I_rt_r] \). There is little work on the defining equations of the multi-Rees algebra compared to the ordinary Rees algebra. Another motivation for investigating the multi-Rees algebra is an illustration of the theory of Rees algebra of modules [4], [19]. Indeed, the multi-Rees algebra in question is simply the Rees algebra of the module \( I_1 \oplus I_2 \oplus \cdots \oplus I_r \). However, in our work, we make no serious use of this theory. Ribbe [18] describes equations of the multi-Rees algebra \( R[It_1, It_2, \ldots, It_r] \) when \( I \) is of linear type and he also determines the relation type of the multi-Rees algebra \( R[I^{a_1}t_1, \ldots, I^{a_r}t_r] \) when \( n_i \geq 2 \). Note that the ideal \( I \) is of linear type, if the Rees algebra \( R[It] \) is defined by linear equations. In the work of Lin and Polini [15] these equations are described for \( R[I^{a_1}t_1, \ldots, I^{a_r}t_r] \), when \( R = k[x_1, \ldots, x_n], k \) a field, and \( I = \langle x_1, \ldots, x_n \rangle \). Sosa [20] describes the equations of the multi-Rees algebras \( R[I_1t_1, I_2t_2, \ldots, I_rt_r] \) when \( R \) is a polynomial ring over a field and \( I_i \) are monomial ideals with some special properties.

We now describe the contents of this dissertation in more detail. We first give some background and definitions in the Chapter 1. Then in Chapter 2, the results of which appeared in [11], we describe the equations of the multi-Rees algebra \( R[I^{a_1}t_1, \ldots, I^{a_r}t_r] \), where \( I \) is of linear type. We use Ribbe’s [18] result, when all powers coincide with \( I \), together with a Veronese type argument.
Theorem A. Let $R$ be a Noetherian ring and let $I$ be an ideal of linear type, with presentation matrix $\Phi$. Then for any integers $a_i \geq 1$, there is an isomorphism

$$R[I^{a_1}t_1, \ldots, I^{a_r}t_r] \cong R[T_1, \ldots, T_N]/I_2(B) + I_1(B^t \cdot \Phi)$$

for some $N$ and some matrix $B$ with entries in the $T_i$’s.

We remark that the multi-Rees algebra $R[I^{a_1}t_1, \ldots, I^{a_r}t_r]$ is a Veronese subring. To prove the results of this thesis we map Veronese subrings of polynomial rings to these multi-Rees algebras.

In Chapter 3, we determine the equations of the multi-Rees algebra $R[I^{a_1}t_1, \ldots, I^{a_r}t_r]$, where $R$ is any Noetherian ring containing a field and ideals $I_i$ are generated by a subset of a fixed regular sequence. In general, these equations can have arbitrarily large degrees. First, applying Buchberger’s Criterion we prove the main result for $R[I_1t_1, I_2t_2, \ldots, I_rt_r]$. Next, we use the method given in Chapter 2 to prove the main result for $R[I^{a_1}t_1, I^{a_2}t_2, \ldots, I^{a_r}t_r]$.

To describe the equations in this case, we introduce the notion of a quasi-matrix and that of a binary quasi-minor, which serves as a generalization $2 \times 2$-minors.

Theorem B. Let $R$ be a Noetherian ring containing a field and suppose that ideals $I_i$ are generated by subsets of a fixed regular sequence $s_1, \ldots, s_n$ contained in $\operatorname{rad} R$. Then there is a quasi-matrix $D$, whose entries are certain indeterminates, such that the multi-Rees algebra $R[I^{a_1}t_1, \ldots, I^{a_r}t_r]$ is defined by the ideal generated by all binary quasi-minors of $[s \mid D]$.

To prove Theorem B, it is straightforward to reduce to the case of a sequence of variables in a polynomial ring. In general, we cannot directly use Buchberger’s Criterion to prove the main result for $R[I^{a_1}t_1, I^{a_2}t_2, \ldots, I^{a_r}t_r]$, such an erroneous attempt was made in [15]. We provide a counter example to show that the arguments in [15] are incomplete.
2 Preliminaries

In the beginning of this Chapter, we present the following notations that we will use in Chapters 2 and 3.

If \( A \) is an \( m \times n \) matrix, then for \( r \leq \min\{m, n\} \), by \( I_r(A) \), we mean the ideal generated by \( r \times r \) minors of \( A \).

Definition 2.1. Recall that a presentation of a module \( M \) is an exact sequence

\[
F \xrightarrow{f} G \xrightarrow{g} M \rightarrow 0
\]

where \( F \) and \( G \) are free modules. Note the image of the standard basis under \( g \) is a generating set of \( M \). If \( R \) is Noetherian and \( M \) is finitely generated, then the module \( M \) is finitely presented and we can represent the map \( f \) by a matrix \( \Phi \), which is called a presentation matrix of \( M \).

Throughout this thesis, when a generating set of the module \( M \) is chosen, by a presentation matrix, we always mean a presentation matrix that is compatible with the chosen generating set, in the sense that the basis of \( G \) maps to the given generating set of \( M \).

Definition 2.2. Let \( a \) be positive integer, \( \mathcal{T}_a \) and \( \mathcal{T}'_a \) denote the sets in \( (\mathbb{Z}_{\geq 0})^{n-1} \) and \( \mathbb{N}^{n-1} \) respectively, that are defined as follows:

\[
\mathcal{T}_a := \{ \underline{j} = (j_{n-1}, \ldots, j_1) \mid 0 = j_0 \leq j_1 \leq \cdots \leq j_{n-1} \leq j_n = a \}
\]

\[
\mathcal{T}'_a := \{ \underline{j} = (j_{n-1}, \ldots, j_1) \mid 1 = j_0 \leq j_1 \leq \cdots \leq j_{n-1} \leq j_n = a \}.
\]

Note that the cardinality of \( \mathcal{T}_a \) (resp. \( \mathcal{T}'_a \)) is \( \binom{a+n-1}{n-1} \) (resp. \( \binom{a+n-2}{n-1} \)).

Definition 2.3. For \( \underline{j} \in \mathcal{T}_a \) we define

\[
s_{\underline{j}} := \prod_{i=1}^{n} s_i^{j_i-j_{i-1}}.
\]
Definition 2.4. We define the function

\[ j^k : \mathcal{T}_a' \to \mathcal{T}_a \] by \[ j^k((j_{n-1}, \ldots, j_1)) = (j_{n-1}, \ldots, j_k, j_{k-1} - 1, \ldots, j_1 - 1) \]

where \( 1 \leq k \leq n - 1 \). For convenience instead of \( j^k((j_{n-1}, \ldots, j_1)) \) we write \( j^k \).

2.1 Rees algebras

Let \( R \) be a Noetherian ring and \( I \subseteq R \) be an ideal of \( R \). The Rees algebra of the ideal \( I \) is defined to be the graded ring

\[ \mathcal{R}(I) = R[It] = \bigoplus_{n=0}^{\infty} I^n t^n \subseteq R[t]. \]

One can generalize the concept of the Rees algebra and define the Rees algebra of modules (one may see [4] and [19]). The multi-Rees algebra \( R[I_1 t_1, I_2 t_2, \ldots, I_r t_r] \) of the ideals \( I_i \) is one example of this generalization. Indeed, the multi-Rees algebra is simply the Rees algebra of the module \( \bigoplus_{i=1}^{r} I_i \).

As is well-known (e.g. [7]), if all the ideals \( I_i \) have positive height, then

\[ \dim R[I_1 t_1, \ldots, I_r t_r] = \dim R + r. \]

Definition 2.5. Let \( M \) be an \( R \)-module, we define the tensor algebra of \( M \) denoted by \( T(M) \) as the following graded \( R \)-algebra

\[ T(M) = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \ldots \]

The symmetric algebra of \( M \) is the algebra \( S(M) \) which is the quotient of tensor algebra of \( M \) modulo the ideal generated by all \( x \otimes y - y \otimes x \) \( (x, y \in M) \).
Lemma 2.6. [22] If $\Phi$ is a presentation matrix of $I$ with $n$ rows, then

$$\mathcal{S}(I) \cong R[T_1, \ldots, T_n]/I_1([T_1, \ldots, T_n] \cdot \Phi).$$

There is a natural epimorphism

$$\alpha : \mathcal{S}(I) \to \mathcal{R}(I).$$

Thus one may view the linear equations defining the symmetric algebra as a first order approximation of the equations of the Rees algebra.

Definition 2.7. We say the ideal $I$ is of linear type if $\alpha$ is an isomorphism.

Examples of ideals of linear type will be given later in Chapter 2.

The following result which is due to J. Ribbe, describes explicit equations of multi-Rees algebra of an ideal of linear type.

Proposition 2.8. [18, Proposition 3.1] Let $I$ be an ideal of linear type in a Noetherian ring $R$ and let the set $s_1, \ldots, s_n$ generates the ideal $I$. Suppose $R^m \xrightarrow{\Phi} R^n \to I$ is any presentation of $I$. Consider the generic $r \times n$ matrix $B = (T_{ij})$. Then the kernel $\mathcal{L}$ of the map

$$\phi : R[[T_{ij}]]_{1 \leq i \leq r, 1 \leq j \leq n} \to R[It_1, \ldots, It_r], \quad \phi(T_{ij}) = s_j t_i,$$

is

$$\mathcal{L} = I_2(B) + I_1(B \cdot \Phi).$$

Before we give the proof of proposition above, we need the following lemma.

Lemma 2.9. [18, Lemma 2.2] Let $i_1, \ldots, i_r, j_1, \ldots, j_r \in \mathbb{Z}_{\geq 0}$ be given such that

$$\sum_{\mu=1}^{n} i_{\nu \mu} = \sum_{\mu=1}^{n} j_{\nu \mu}, \text{ for every } \nu = 1, \ldots, r$$

and

$$\text{and}$$
\[(ii) \sum_{\nu=1}^{r} i_{\nu\mu} = \sum_{\nu=1}^{r} j_{\nu\mu}, \text{ for every } \mu = 1, \ldots, n.\]

Suppose \(U_i = \{T_{i1}, \ldots, T_{in}\}\), for \(1 \leq i \leq r\), and \(U_{i\nu}^{ir} = T_{i1}^{ir_{1}} \cdots T_{in}^{ir_{n}}\). Then

\[U_{i1}^{ir} \cdots U_{ir}^{ir} \equiv U_{j1}^{jr} \cdots U_{jr}^{jr} \mod I_2(B).\]

**Proof.** By induction on \(r\) we may assume that \(r \geq 2\) and that the claim holds for any two sets of \(r-1\) vectors that satisfy (i) and (ii). We choose \(u\) from the set \(\{1, \ldots, n+1\}\) maximal with \(i_{rw} = j_{rw}\) for all \(w < u\). We prove the claim by a second induction on \(u\). If \(u = n+1\), then \(i_r = j_r\), and by the first induction we have

\[U_{i1}^{ir} \cdots U_{ir-1}^{ir-1} \equiv U_{j1}^{jr} \cdots U_{jr-1}^{jr-1} \mod I_2(B).\]

If we multiply it by \(U_{ir}^{ir}\), then our claim is proved in this case. Now we assume that \(u \leq n\), say \(i_{ru} > j_{ru}\). By (i), there is \(v > u\) with \(i_{rv} < j_{rv}\). Hence, by (ii), there is \(s < r\) with \(i_{sv} > j_{sv}\).

If \(e_i\) means \(i\)-th unit vector in \(\mathbb{Z}^n_{\geq 0}\), then we obtain the following congruence modulo \(I_2(B)\):

\[U_{i1}^{ir} \cdots U_{ir}^{ir} = U_{i1}^{ir} \cdots U_{is}^{ir-e_v} \cdots U_{ir}^{ir-e_v} T_{sv} T_{ru} \equiv U_{i1}^{ir} \cdots U_{is}^{ir-e_v} \cdots U_{ir}^{ir-e_v} T_{sv} T_{ru} \mod I_2(B) \]

\[\equiv U_{i1}^{ir} \cdots U_{is}^{ir-e_v+e_u} \cdots U_{ir}^{ir-e_v+e_u} \mod I_2(B).\]

We see that (i) and (ii) still hold if \(i_1, \ldots, i_r\) are replaced by new exponent vectors \(i'_1 = i_1, \ldots, i'_s = i_s - e_v + e_u, \ldots, i'_r = i_r - e_u + e_v\). We continue this procedure until we obtain exponent vectors \(i'_\nu, \nu = 1, \ldots, r\), with \(i'_{rw} = j'_{rw}\) for all \(w \leq u\). Then applying the induction hypothesis we obtain

\[U_{i1}^{ir'} \cdots U_{ir}^{ir'} \equiv U_{j1}^{jr} \cdots U_{jr}^{jr} \mod I_2(B).\]

This completes the proof. \(\square\)
Proof. (Proof of Proposition 2.8).

Let $F \in \mathcal{L}$ be homogeneous of multidegree $(d_1, \ldots, d_r)$. We set $d = d_1 + \cdots + d_r$. For every vector $u = (u_1, \ldots, u_n) \in \mathbb{Z}_{\geq 0}^n$ we define $|u| = u_1 + \cdots + u_n$.

Let $E = \{ k \in \mathbb{Z}_{\geq 0}^n | k = d \}$ and for every $k \in E$,

\[ E(k) = \{ i = (i_1, \ldots, i_r) \in (\mathbb{Z}_{\geq 0}^n)^r | i_1 + \cdots + i_r = k; |i_s| = d_s, s = 1, \ldots, r \}. \]

Then we may write

\[ F = \sum_{k \in E} \left( \sum_{i \in E(k)} f_i U_{i}^{r_1} \cdots U_{i}^{r_r} \right) \equiv \sum_{k \in E} \left( \sum_{i \in E(k)} f_i \right) U_{i}^{i_1(k)} \cdots U_{i}^{i_r(k)} \mod \mathcal{L}, \quad (2.1) \]

where $f_i \in R$ and $i_1(k), \ldots, i_r(k)$ are vectors in $\mathbb{Z}_{\geq 0}^n$ which depend only on $k$, fulfilling the congruences $U_{i}^{r_1} \cdots U_{i}^{r_r} \equiv U_{i}^{i_1(k)} \cdots U_{i}^{i_r(k)} \mod \mathcal{L}$ for every $(i_1, \ldots, i_r) \in E(k)$. We see that every $k \in E$ admits vectors $i_s(k) \in \mathbb{Z}_{\geq 0}^n, s = 1, \ldots, r$ such that $(i_1(k), \ldots, i_r(k)) \in E(k)$.

We can define these vectors by recursion as

\[ i_s(k)_\mu = \max \left\{ 0, \min \left\{ d_s - \sum_{\tau = \mu + 1}^{n} i_s(k)_\tau, k_\mu - \sum_{\sigma = s + 1}^{r} i_\sigma(k)_\mu \right\} \right\} \]

for $s = r, \ldots, 1$ and $\mu = n, \ldots, 1$. Then the congruence 2.1 holds by Lemma 2.9.

On the other hand we have the following commutative diagram

\[ \begin{array}{ccc}
R[T_{ij}]_{1 \leq i \leq r, 1 \leq j \leq n} & \xrightarrow{\alpha} & R[X_1, \ldots, X_n] \\
\downarrow \phi & & \downarrow \psi \\
R[It_1, \ldots, It_r] & \xrightarrow{\beta} & R[It],
\end{array} \]

where $\alpha(T_{ij}) = X_j$ and $\beta(s_t s_t) = s_t t$. Hence we see that $F(X, \ldots, X) \in \ker(\psi)$, where $X = \{X_1, \ldots, X_n\}$. Moreover $\deg (F(X, \ldots, X)) = d$. There are linear forms $L_\nu = \sum_{\mu=1}^{n} l_{\nu \mu} X_\mu$
and forms $G_\nu = \sum_{|k|=d-1} g_{\nu,k} X^k$ of degree $d-1$ in $R[X_1, \ldots, X_n]$, $\nu = 1, \ldots, m$, such that

$$F(X, \ldots, X) = \sum_{\nu=1}^m L_\nu G_\nu = \sum_{k \in E} \left( \sum_{\nu=1}^m \sum_{\mu=1}^n l_{\nu\mu} g_{\nu,k} - e_{\mu} \right) X^k.$$  \hspace{1cm} (2.2)

Comparing 2.1 and 2.2, we have

$$F \equiv \sum_{k \in E} \left( \sum_{\nu=1}^m \sum_{\mu=1}^n l_{\nu\mu} g_{\nu,k} - e_{\mu} \right) \prod_{\gamma=1}^r U_{i_{\gamma}(k)} \mod \mathcal{L},$$

or equivalently

$$F \equiv \sum_{|k|=d-1} \left( \sum_{\nu=1}^m \sum_{\mu=1}^n l_{\nu\mu} g_{\nu,k} \right) \prod_{\gamma=1}^r U_{i_{\gamma}(k)+e_{\mu}} \mod \mathcal{L}. \hspace{1cm} (2.3)$$

We fix $\mu \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}_{\geq 0}^n$ with $|k| = d - 1$. Then

$$\prod_{\gamma=1}^r U_{i_{\gamma}(k)+e_{\mu}} \equiv U_{1\mu} \prod_{\gamma=1}^r U_{j_{\gamma\mu}(k)} \mod \mathcal{L}, \hspace{1cm} (2.4)$$

where $j_{\gamma\mu} \in \mathbb{Z}_{\geq 0}^n$ are suitably chosen vectors with $|j_{1\mu}(k)| = d_1 - 1$, $|j_{\gamma\mu}(k)| = d_\gamma$ for $\gamma = 2, \ldots, r$, and $\sum_{\gamma=1}^r j_{\gamma\mu}(k) = k$.

To be precise, if $i_1(k+e_{\mu})_\mu \geq 1$ we choose $j_{1\mu}(k) := i_1(k+e_{\mu}) - e_{\mu}$ and $j_{\gamma\mu}(k) := i_\gamma(k+e_{\mu})$ for $\gamma = 2, \ldots, r$. While if $i_1(k+e_{\mu})_\mu = 0$ we first find $\sigma \geq 2$ and $\eta \neq \mu$ with $i_\sigma(k+e_{\mu})_\mu \geq 1$ and $i_1(k+e_{\mu})_\eta \geq 1$; now put $j_{1\mu}(k) := i_1(k+e_{\mu}) - e_{\eta}$, $j_{\sigma\mu}(k) := i_\sigma(k+e_{\mu}) - e_{\mu} + e_{\eta}$, and for $\gamma \neq 1, \sigma$, $j_{\gamma\mu}(k) := i_\gamma(k+e_{\mu})$. Using the relation $U_{1\eta} U_{\sigma\mu} \equiv U_{1\mu} U_{\sigma\eta} \mod \mathcal{L}$ one can verify relation 2.4.

Next, note that modulo $\mathcal{L}$, the monomial $\prod_{\gamma=1}^r U_{j_{\gamma\mu}(k)}$ in 2.4 does not depend on $\mu$, so we shall denote it by $P_k$ (given $\mu, \mu' \in \{1, \ldots, n\}$ we have $|j_{\gamma\mu}(k)| = |j_{\gamma\mu'}(k)|$ for $\gamma = 1, \ldots, r$ and $\sum_{\gamma=1}^r j_{\gamma\mu}(k) = k = \sum_{\gamma=1}^r j_{\gamma\mu'}(k)$, so the independence is due to Lemma 2.9).
We use these facts and rewrite 2.3 as below:

\[ F \equiv \sum_{|k|=d-1} \left( \sum_{\nu=1}^{m} \sum_{\mu=1}^{n} l_{\nu\mu} g_{\nu,k} \right) U_{1\mu} P_k \equiv \sum_{|k|=d-1} \sum_{\nu=1}^{m} \left( \sum_{\mu=1}^{n} l_{\nu\mu} U_{1\mu} \right) g_{\nu,k} P_k \equiv 0. \]

This completes the proof. \( \square \)

### 2.2 Equations of the Veronese

**Definition** 2.10. Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded ring, and \( d \) be a positive integer number. We define the Veronese subring of \( R \) of degree \( d \), denoted by \( R^{(d)} \), to be \( R_0 \)-algebra generated by all elements of \( R \) whose degrees are multiple of \( d \), i.e. \( R^{(d)} = \bigoplus_{i \geq 0} R_{id} \).

It is clear that if we consider the polynomial ring \( R[x_1, \ldots, x_n] \), then the Veronese subring of degree \( d \) is \( R[X(d)] \) which is the ring generated over \( R \) by the monomials of degree \( d \) in \( x_1, \ldots, x_n \).

**Theorem** 2.11. [13, Proposition 2.5] Let \( R \) be a commutative ring. For fixed \( n, d \geq 1 \) let \( N = \binom{d+n-1}{n-1} \) and let

\[ \Theta : R[V_1, \ldots, V_N] \to R[X(d)] \]

be the ring homomorphism taking \( V_i \) to the \( i \)-th monomial of degree \( d \) in \( x_1, \ldots, x_n \) in lexicographic order. Then there exists an \( n \times \binom{d+n-2}{n-1} \) matrix \( M(n, d) \), whose entries are the \( V_i \)'s, such that \( \ker(\Theta) = I_2 (M(n, d)) \).

To describe the matrix \( M(n, d) \), write \( N_1, \ldots, N_r \) for the monomials of degree \( d-1 \) in \( x_1, \ldots, x_n \) written in lexicographic order. Then \( M(n, d) \) is the matrix whose \( (i, j) \)-th entry is the variable \( V_k \) which satisfies \( \Theta(V_k) = x_i N_j \).

The following are the two most well-known cases of this result:

**Example** 2.12. \( n = 2 \). In this case,

\[
M(2, d) = \begin{bmatrix}
V_1 & V_2 & \ldots & V_d \\
V_2 & V_3 & \ldots & V_{d+1}
\end{bmatrix}.
\]
The ideal $I_2(M(2, d))$ defines the rational normal curve of degree $d$ in $\mathbb{P}^d$ over $R$.

Example 2.13. $n = 3$ and $d = 2$. In this case $M(3, 2)$ is the generic symmetric matrix

$$M(3, 2) = \begin{bmatrix}
V_1 & V_2 & V_3 \\
V_2 & V_4 & V_5 \\
V_3 & V_5 & V_6
\end{bmatrix}.$$

The ideal $I_2(M(3, 2))$ defines the Veronese surface in $\mathbb{P}^5$ over $R$. 
3 Powers of linear type ideals

3.1 Equations of $R[I^{a_1t_1}, \ldots, I^{a_rt_r}]$

We fix a generating set $s_1, \ldots, s_n$ of the ideal $I$. If the ideal $I = \langle s_1, \ldots, s_n \rangle$ is of linear type in a Noetherian ring $R$, then the $R$-algebra homomorphism

$$f : R[T_1, \ldots, T_n] \rightarrow R[I], \quad f(T_i) = s_it$$

has $\ker(f) = \langle [T_1, \ldots, T_n] \cdot \Phi \rangle$, where $R^m \xrightarrow{\Phi} R^n \xrightarrow{s} I$ is a presentation of $I$.

Let $R[I^{a_1t_1}, \ldots, I^{a_rt_r}]$ be the multi-Rees algebra of $I$. We denote $a = a_1, \ldots, a_r$.

One knows that $I^a$ is generated by $s_1^{k_1} \ldots s_n^{k_n}$, where $k_i \geq 0$ and $k_1 + \cdots + k_n = a$. We can rewrite every generator as the following

$$s_1^{k_1} \ldots s_n^{k_n} = s_1^{k_1 - 0} s_2^{(k_1+k_2)-k_1} \cdots s_l^{(k_1+\cdots+k_l)-(k_1+\cdots+k_{l-1})} \cdots s_n^{(k_1+\cdots+k_n)-(k_1+\cdots+k_{n-1})}.$$

If we set $j_0 = 0, j_1 = k_1, j_2 = k_1 + k_2, \ldots, j_l = k_1 + \cdots + k_l, \ldots, j_n = a$, then we have $s_1^{k_1} \ldots s_n^{k_n} = s_1^{j_1} \ldots s_n^{j_n}$. For convenience we will use this notation which is already used in [15].

Since $I^a$ is generated by all $s_1^{j_1}$ ($j \in T_a$), the multi-Rees algebra $R[I^{a_1t_1}, \ldots, I^{a_rt_r}]$ is the ring $R\left[\left\{s_1^{j_1}t_1\right\}_{1 \leq l \leq r, j \in T_{a_l}}\right]$.

Let $S := R\left[\left\{T_{i,j}^{l} \right\}_{1 \leq l \leq r, j \in T_{a_l}}\right]$. We fix the $R$-algebra epimorphism

$$\phi : S \rightarrow R[I^{a_1t_1}, \ldots, I^{a_rt_r}], \quad \text{by} \quad \phi(T_{i,j}^{l}) = s_1^{j_1}t_1.$$

**Definition 3.1.** Consider the set $\{(l, j) | 1 \leq l \leq r; j \in T_{a_l}'\}$ ordered increasing lexicographically as a subset of $\mathbb{N}^n$. We define the $n \times (\sum_{l=1}^{n} (a_l+n-2))$ matrix $B_a$, whose entry in row $k$ and column $(l, j)$ is $T_{l,j}^{k}$. 

**Example 3.2.** In the following we list some examples of various $B_a$. 

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Theorem 3.3. Let $R$ be a Noetherian ring and the ideal $I = \langle s_1, \ldots, s_n \rangle \subseteq R$ be of linear
type. Let \( \Phi \) be a presentation matrix with \( n \) rows. Then the kernel \( \mathcal{L} \) of the map \( \phi \) is

\[
\mathcal{L} = I_2(B_a) + I_1(B_a^t \cdot \Phi).
\]

**Proof.** We can easily see \( I_2(B_a) \subseteq \mathcal{L} \), because

\[
\phi \left( \begin{pmatrix}
T_{l', j', k} & T_{l, j}^{(k)} \\
T_{l', j', k'} & T_{l, j}^{(k')}
\end{pmatrix}
\right) = s_k s_{l'} s_{k'} s_{j'} t_l - s_{k'} s_{j'} s_k s_{l} t_l = 0.
\]

If \( L \) is a linear form in \( \ker(f) \) we have,

\[
\phi \left( L(T_{l, j_1}, T_{l, j_2}, \ldots, T_{l, j_n}) \right) = L(s_1 s_{j_1} t_l, s_2 s_{j_2} t_l, \ldots, s_n s_{j_n} t_l) = (s_l t_l) L(s_1, s_2, \ldots, s_n) = 0,
\]

therefore \( I_1(B_a^t \cdot \Phi) \subseteq \mathcal{L} \). Hence \( I_2(B_a) + I_1(B_a^t \cdot \Phi) \subseteq \mathcal{L} \). Thus it is enough to prove \( \mathcal{L} \subseteq I_2(B_a) + I_1(B_a^t \cdot \Phi) \).

Let \( X = (x_{i,l}) \) be a generic \( n \times r \) matrix. Let \( x_l \) denote the regular sequence \( x_{1,l}, x_{2,l}, \ldots, x_{n,l} \) of entries of the \( l \)-th column of \( X \). We define \( A_l \) to be the \( a_l \)-th Veronese subring \( R[x_{1,l}, x_{2,l}, \ldots, x_{n,l}]^{(a_l)} \).

Let \( X_l(a) \) be the family of monomials of degree \( a \) (\( a \in \mathbb{N} \)) in the variables \( x_{1,l}, \ldots, x_{n,l} \), we define \( A \) to be the \( \mathbf{a} \)-th Veronese subring \( R[X]^a = R[X_1(a_1), X_2(a_2), \ldots, X_r(a_r)] \).

For arbitrary \( l \), \( 1 \leq l \leq r \) we define

\[
\alpha_l : R \left[ \left\{ T_{l, j} \right\} \_j \in \mathbb{N} \_l \right] \to A_l, \quad \alpha_l(T_{l, j}) = x_{l,j}.
\]

These induce a map \( \alpha : S \to A \).

Let \( B_{ai} \) for \( 1 \leq l \leq r \), be the \( n \times \binom{a_l+n-2}{n-1} \) submatrix of \( B_a \) consisting of the \((l,j)\)-columns
of $B_a (j \in \mathcal{J}_a^t)$. By Theorem 2.11 we know that $\ker(\alpha_l) = I_2(B_{a_l})$. We have

$$A \cong A_1 \otimes_R A_2 \otimes_R \ldots \otimes_R A_r$$

$$\cong R[T_{1,l}] / I_2(B_{a_1}) \otimes_R R[T_{2,l}] / I_2(B_{a_2}) \otimes_R \ldots \otimes_R R[T_{r,l}] / I_2(B_{a_r})$$

$$\cong S / \langle I_2(B_{a_1}), I_2(B_{a_2}), \ldots, I_2(B_{a_r}) \rangle.$$ 

Since the isomorphism above is an $R$-algebra homomorphism and image of $T_{l,j}$ is the same as its image under $\alpha$, it follows that

$$\ker(\alpha) = \langle I_2(B_{a_1}), I_2(B_{a_2}), \ldots, I_2(B_{a_r}) \rangle \subseteq I_2(B_a).$$

If we consider

$$\psi : R[X] \to R[Iu_1, Iu_2, \ldots, Iu_r], \quad \psi(x_{i,l}) = s_iu_l$$

then by Proposition 2.8

$$\ker(\psi) = I_2(X) + I_1(X^t \cdot \Phi). \quad (3.1)$$

By the following commutative diagram we define the map $g$:

$$
\begin{array}{ccc}
S & \xrightarrow{\phi} & R[I^{a_1}t_1, \ldots, I^{a_r}t_r] \\
\downarrow{\alpha} & & \downarrow{\iota}
\end{array}
\begin{array}{ccc}
\alpha & \xrightarrow{g} & R[I^{a_1}u_1, \ldots, I^{a_r}u_r] \\
\downarrow{\psi(a)} & & \downarrow{\iota}
\end{array}
\begin{array}{ccc}
A & \xrightarrow{\psi} & R[Iu_1, \ldots, Iu_r] \\
\downarrow & & \downarrow
\end{array}
\begin{array}{ccc}
R[X] & \xrightarrow{} & R[Iu_1, \ldots, Iu_r]
\end{array}
$$

In this step we prove that the kernel of $g$ is generated by polynomials of the form

$L(x_1, t_1^{\hat{j}}, x_2, t_2^{\hat{j}}, \ldots, x_n, t_n^{\hat{j}})$ ($L \in \ker(f)$, $\hat{j} \in \mathcal{J}_{a_l-1}$) and

$x_{i,l}t_l^j x_{i,m} x_m^k - x_{h,l}t_l^j x_{i,m} x_m^k$ ($\hat{j} \in \mathcal{J}_{a_l-1}, \hat{k} \in \mathcal{J}_{a_m-1}$) in $A$.  

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Therefore we have two cases:

monomial of degree $a$

monomial of degree $p$ (resp. $\Gamma_2^{\iota}$ divides at least one $\Gamma_i$ and it is the monomial of degree $a$

we choose one in which $\mu$’s are $2 \times 2$-minors of $X$, or $q_k = L(x_{1,l}, x_{2,l}, \ldots, x_{n,l})$, $r_k \in R$ and $\mu_k \in R[X]$ is a monomial. Among all such expressions we choose one in which $m$ is minimal. Since $p \neq 0$, $p$ has either a monomial, say $\mu_1 x_{k,l}$ or $\mu_1 x_{k,l} x_{h,m}$ in its support. We claim that there is $\xi_1 \in A$ such that

$$
\mu_1 = \begin{cases} 
\xi_1 x_1^j & \text{(with } j_n = a_l - 1) \\
\xi_1 x_1^j x_{m}^k & \text{(with } j_n = a_l - 1, k_n = a_m - 1) 
\end{cases}
\quad \text{if } q_k = L(x_{1,l}, x_{2,l}, \ldots, x_{n,l})
\quad \text{if } q_k = x_{i,l} x_{h,m} - x_{h,i} x_{i,m}
$$

Since one of the terms in $p$ is $r \mu_1 x_{1,l}$ or $r \mu_1 x_{i,l} x_{h,m}$ ($r \in R$), then $\mu_1 x_{1,l}$ or $\mu_1 x_{i,l} x_{h,m}$ is in $A$. Therefore we have two cases:

case i) $\mu_1 x_{1,l}$ is in the support of $p$. Then $\mu_1 x_{1,l} = \Gamma_1 \Gamma_2 \ldots \Gamma_w$, where $\Gamma_j \in A_t$ and it is a monomial of degree $a_i$, therefore $x_{1,l}$ divides at least one $\Gamma_j$, say $x_{1,l}$ divides $\Gamma_1$, then $\Gamma_1$ is a monomial of degree $a_l$ in $A_t$ and $\Gamma_1 = x_{1,l} x_1^j (j_n = a_l - 1)$. Hence

$$
\mu_1 x_{1,l} = x_{1,l} x_1^j \Gamma_2 \ldots \Gamma_w \Rightarrow \mu_1 = \xi_1 x_1^j, \text{ with } \xi_1 = \Gamma_2 \ldots \Gamma_w \in A.
$$

Case ii) $\mu_1 x_{i,l} x_{h,m}$ is in the support of $p$. Then $\mu_1 x_{i,l} x_{h,m} = \Gamma_1 \Gamma_2 \ldots \Gamma_w$, where $\Gamma_j \in A_t$ and it is the monomial of degree $a_i$. Since $l \neq m$, then $x_{i,l}$ divides at least one $\Gamma_j$ and $x_{h,m}$ divides at least one $\Gamma_i$ and $i \neq j$, say $x_{i,l}$ divides $\Gamma_1$, and $x_{h,m}$ divides $\Gamma_2$, therefore $\Gamma_1$ (resp. $\Gamma_2$) is a monomial of degree $a_l$ in $A_t$ (resp. of degree $a_m$ in $A_m$) and $\Gamma_1 = x_{i,l} x_1^j (j_n = a_l - 1)$ (resp. $\Gamma_2 = x_{h,m} x_m^k (k_n = a_m - 1)$). Therefore

$$
\mu_1 x_{i,l} x_{h,m} = x_{i,l} x_1^j x_{h,m} x_m^k \Gamma_3 \ldots \Gamma_w
\quad \Rightarrow \mu_1 = \xi_1 x_1^j x_m^k, \text{ with } \xi_1 = \Gamma_3 \ldots \Gamma_w \in A.
$$
Hence either

\[ \mu_1 q_1 = \xi_1 x_1^j q_1 = \xi_1 x_1^j L(x_{1,l}, \ldots, x_{n,l}) = \xi_1 x_1^j \sum_i a_i^{(l)} x_{i,l} \]

or

\[ \mu_1 q_1 = \xi_1 x_1^j x_m^k q_1 = \xi_1 x_1^j x_m^k (x_{i,l} x_{h,m} - x_{h,l} x_{i,m}) \]

\[ = \xi_1 (x_{i,l} x_1^j x_{h,m} x_m^k - x_{h,l} x_1^j x_{i,m} x_m^k) \].

Since \( p \) and \( \mu_1 q_1 \in A \), we conclude that \( p - r_1 \mu_1 q_1 \in A \). Since the expression \( \sum_{i=2}^m r_i \mu_i q_i \) is still minimal, by induction of \( m \) (the base case \( m = 1 \) being clear by the above), we conclude that \( p - r_1 \mu_1 q_1 \), and hence \( p \), is a linear combination of the required polynomials. Therefore we have described the generators of \( \ker(g) \). On the other hand

\[ \alpha \left( L(T_{l,j_n-1+1,\ldots,j_2+1,j_1+1}, T_{l,j_n-1+1,\ldots,j_2+1,j_1,\ldots}, T_{l,j_n-1,\ldots,j_2,j_1}) \right) \]

\[ = L(x_1 x_1^j, x_2 x_1^j, \ldots, x_n x_1^j) \]

and under \( \alpha \)

\[ T_{l,j_n-1+1,\ldots,j_2+1,j_1-1,\ldots,j_2,1} T_{m,k_n-1+1,\ldots,k_2+1,k_1-1,\ldots,k_2,1} \]

\[ - T_{l,j_n-1+1,\ldots,j_2+1,j_1-1,\ldots,j_2,1} T_{m,k_n-1+1,\ldots,k_2+1,k_1-1,\ldots,k_2,1} \]

maps to

\[ x_{i,l} x_1^j x_{h,m} x_m^k - x_{h,l} x_1^j x_{i,m} x_m^k. \]

Now if \( p \in \mathcal{L} \), then \( g \circ \alpha(p) = 0 \) so that \( \alpha(p) \in \ker(g) \). Thus there is \( q \in I_2(B_a) + I_1(B_a^t \cdot \Phi) \) such that \( \alpha(p) = \alpha(q) \). Hence \( p - q \in \ker(\alpha) \subset I_2(B_a), \) hence \( p \in I_2(B_a) + I_1(B_a^t \cdot \Phi) \).

This completes the proof. \( \square \)
3.2 Examples

In this section we give some examples of ideals of linear type to illustrate the defining equations of their multi-Rees algebras. In the examples of ideals of linear type that follow, it is known that the Rees algebra is Cohen-Macaulay (e.g. [25]). It follows that, by a well-known result [7], the multi-Rees algebra $R[I_1^{a_1}, \ldots, I_r^{a_r}]$ is Cohen-Macaulay for any $a_i$.

**Example 3.4.** If $\{s_1, s_2, \ldots, s_n\}$ is a regular sequence, then $I = \langle s_1, \ldots, s_n \rangle$ is of linear type (e.g. [22], corollary 5.5.5). Therefore

$$L = I_2 \left( \begin{bmatrix} s_1 & | & B_a \\ \vdots & & \vdots \\ s_n & | & \end{bmatrix} \right).$$

This example generalizes a result of [15], where the result is claimed in the case when $I$ is the maximal ideal of the polynomial ring over a field. However, their proof of the result in this case is incomplete. See the end of section 4.2, for a discussion of this issue.

**Example 3.5.** Let $k$ be a field and let $X$ be a generic $n \times n - 1$ matrix over $k$, $R = k[X]$, and let $I = I_{n-1}(X)$. Then $I$ is of linear type [10]. Furthermore, by the Hilbert-Burch Theorem we may take $\Phi = X$ with respect to the $n$ signed minors of $X$ obtained by deleting the $i$-th row. Therefore $L = I_2(B_a) + I_1(B_a^t X)$.

**Example 3.6.** Let $k$ be a field, $n \geq 3$ be an odd integer, $X$ a generic $n \times n$ alternating matrix over $k$, $R = k[X]$, and let $I = Pf_{n-1}(X)$ denote the ideal of $(n - 1)$ sized Pfaffians. Then $I$ is of linear type [9]. Furthermore, by the Buchsbaum-Eisenbud structure Theorem [1], we may take $\Phi = X$ with respect to the $n$ signed Pfaffians of $X$ obtained by deleting the $i$-th row and column. Therefore $L = I_2(B_a) + I_1(B_a^t X)$.

**Example 3.7.** Let $Z = (z_{ij})$ be an $m \times m$ generic matrix over $\mathbb{Z}$. Let $T = \mathbb{Z}[z_{ij}]$ and $\Delta_{ij}$ denote the $m - 1 \times m - 1$ signed minors of $Z$ which are obtained by deleting the $j$th row
and \( i \)th column and

\[
I = I_{m-1}(Z) = \langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \ldots, \Delta_{mm} \rangle.
\]

The ideal \( I \) is of linear type [10]. The relations on the given generators of \( I_{m-1}(Z) \) are obtained from \( Z \det(Z) \mathbf{1} = \det(Z) \mathbf{1} = \det(Z) \mathbf{1} \).

If we define the \( m \times m \) matrix \( C_{(l,j)} \) by putting the first \( m \) entries of the corresponding column of \( B_a \) in the first row of \( C_{(l,j)} \), then the second \( m \) entries of this column in the second row \( C_{(l,j)} \), and so on, then the generators of \( \mathcal{L} \) are all \( 2 \times 2 \) minors of \( B_a \), off-diagonal entries of \( C_{(l,j)}Z \) and \( ZC_{(l,j)} \) and subtraction of each pair of entries in diagonals of this pair of matrices.
4 Complete intersections with common sequences

Suppose that $I_1, \ldots, I_r$ are monomial ideals in a polynomial ring $R = k[x_1, \ldots, x_n]$. Then the multi-Rees algebra $R[I_1 t_1, \ldots, I_r t_r]$ is defined by binomial equations (cf. [5]). We concentrate on a simple case where these binomial defining equations can be described effectively, generalizing the case of Example 3.5.

4.1 Gröbner basis of binary quasi-minors

Definition 4.1. An $n \times m$ quasi-matrix over $R$ is a rectangular array with $n$ rows and $m$ columns such that some entries may be empty.

A subquasi-matrix is a quasi-matrix that is obtained by deleting some rows, columns, or elements of a quasi-matrix.

Example 4.2.

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f & g \end{bmatrix}$$

is a quasi-matrix and $\begin{bmatrix} a & b \\ d \end{bmatrix}$ is a subquasi-matrix of $A$.

Definition 4.3. A binary quasi-matrix is a quasi-matrix having exactly two elements in each nonempty row and column.

Example 4.4. All $3 \times 3$ binary quasi-matrices are listed below:

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Note that a binary quasi-matrix is a square matrix, up to deleting an empty row or column. Since we usually identify a quasi-matrix canonically with the one obtained by
deleting any empty row or column, in the sequel we usually consider a binary quasi-matrix as a square matrix.

**Definition 4.5.** Let \( A = (a_{ij}) \) be an \( n \times n \) binary quasi-matrix over a ring \( R \). A binary quasi-determinant of \( A \) is an element

\[
a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)} - a_{1\tau(1)}a_{2\tau(2)} \cdots a_{n\tau(n)}
\]

where \( \sigma, \tau \) are permutations of \( \{1, 2, \ldots, n\} \) such that \( \sigma(l) \neq \tau(l) \) for all \( 1 \leq l \leq n \). A quasi-determinant of a binary subquasi-matrix \( A \) is called a binary quasi-minor of \( A \).

Note that by definition, if \( \delta \) is a binary quasi-determinant of a quasi-matrix, then so is \(-\delta\). In the sequel, we will usually consider a given binary quasi-minor up to sign.

**Remark 4.6.** (1) Note that the quasi-determinant of a \( 2 \times 2 \) binary quasi-matrix is equal to its determinant, up to sign. Hence all \( 2 \times 2 \) minors (which exist) of a quasi-matrix are binary quasi-minors.

(2) Note that a quasi-determinant of a \( 3 \times 3 \) binary quasi-matrix is uniquely determined up to sign. However, in general it is not equal to the determinant, even up to sign, of the matrix obtained by assigning value zero to all empty positions.

(3) For \( n \geq 4 \), a quasi-determinant of a binary \( n \times n \) quasi-matrix is not even unique, up to sign. For example consider the following binary quasi-matrix

\[
\begin{bmatrix}
a & b \\
c & d \\
e & f \\
g & h
\end{bmatrix}
\]

Then \( adeh - bcfg \) and \( adgf - bceh \) are both quasi-determinants.

**Notation 4.7.** If \( A \) is a quasi-matrix with entries in \( R \), then we denote the ideal generated by the binary quasi-minors of \( A \) by \( I_{\text{bin}}(A) \).
Example 4.8. Consider the quasi-matrix $A$ as below:

$$A = \begin{bmatrix} a & b \\ c & d & e \\ f & g \end{bmatrix},$$

then $adg - bef$ is a binary quasi-minor of $A$.

The next result shows that the ideal of binary quasi-minors generalizes to the quasi-matrices the classical ideal of $2 \times 2$ minors.

Proposition 4.9. Let $A$ be a matrix. Then $I_{\text{bin}}(A) = I_2(A)$.

Proof. It is enough to show that every binary quasi-minor in $A$ is an $R$-combination of $2 \times 2$ minors. Let $\delta = V_1V_2 \ldots V_n - W_1W_2 \ldots W_n$ be an arbitrary binary quasi-minor. We induct on $n \geq 2$. Since the result is clear for $n = 2$, we may assume $n \geq 3$ and that the result holds for binary quasi-minors of size $< n$.

We may assume $V_1$ is in the same row with $W_1$ and $V_2$ is in the same column with $W_1$. Let $U$ be the entry of $A$ in the same column as $V_1$ and same row as $V_2$. Then

$$\delta = \delta - UW_1V_3 \ldots V_n + UW_1V_3 \ldots V_n$$

$$= (V_1V_2 - UW_1)V_3 \ldots V_n + W_1(UV_3 \ldots V_n - W_2 \ldots W_n).$$

If $U$ is not one of the $W$’s, then the subquasi-matrix obtained by deleting the first row and column involving $W_1$ and $V_2$, and containing $U$ is binary quasi-matrix, with $UV_3 \ldots V_n - W_2 \ldots W_n$ as an $(n - 1)$-sized binary quasi-minor.

On the other hand, if $U$ is a $W_i$, say $W_2$ (which can only happen if $n \geq 4$), then $UV_3 \ldots V_n - W_2 \ldots W_n = W_2(V_3 \ldots V_n - W_3 \ldots W_n)$ and $V_3 \ldots V_n - W_3 \ldots W_n$ is a binary quasi-minor of $A$ of size $(n - 2)$. In either case, we are done by induction.

We say that a quasi-matrix is *generic* over a field $k$ if its entries are algebraically independent over $k$. 

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We recall a fact that, for a generic $m \times n$ matrix $X$, with $m \leq n$, the maximal minors of $X$ form a universal Gröbner basis for the ideal $I_m(X)$ [2]. It is well-known that this does not hold for lower order minors. The following result gives a corresponding statement for the ideal $I_2(X)$, and more generally in “quasi” situation.

**Proposition 4.10.** Let $A$ be a generic quasi-matrix over a polynomial ring over a field $k$. Then the set of binary quasi-minors is a universal Gröbner basis for the ideal $I_{\bin}(A)$.

**Proof.** By Buchberger’s Criterion [3, Theorem 6], it is enough to show that for each pair of binary quasi-minors $f$ and $g$, the $S$-polynomial $S(f, g)$ reduces to zero modulo the set of binary quasi-minors. Let

$$f = V_1 \ldots V_n - W_1 \ldots W_n, \quad g = Y_1 \ldots Y_m - Z_1 \ldots Z_m.$$ 

We may assume that $\text{in}(f) = V_1 \ldots V_n$ and $\text{in}(g) = Y_1 \ldots Y_m$. Then

$$h = S(f, g) = -Y_1 \ldots Y_t W_1 \ldots W_n + V_1 \ldots V_s Z_1 \ldots Z_m,$$

where we have reordered if necessary to assume that $Y_1, \ldots, Y_t$ are exactly the $Y$ variables that are not $V$’s, and $V_1, \ldots, V_s$ are exactly the $V$ variables that are not $Y$’s. We remark that $W_i, Y_i, Z_i, V_i$ are not necessarily distinct.

We consider the subquasi-matrix $B$ of $A$ consisting of all the elements $W_i, Y_i, Z_i$ and $V_i$ that appear in $h$. If two of these elements coincide we say that the entry has multiplicity 2 in the quasi-matrix $B$.

First of all, we show that each of $W_i$ and $Y_i$ in the first term of $h$, is in the same row with at least one of the $Z_i$ and $V_i$’s in the second term of $h$. Clearly every $Y_i$ is in the same row with a $Z_j$. On the other hand, for a $W_i$, it is in the same row as a $V_j$; if $j \leq s$ we are done. But if $j > s$, since $V_j$ is in the same row as a $Z_k$, it follows that $W_i$ is in the same row with this $Z_k$. The same argument can be made for elements in the second term and for columns.
Second, we show that in every row (column) of $B$ we have either exactly two of $W_i, Z_i, V_i, Y_i$ or exactly four of them. To verify this, it suffices to show that if three of them are in the same row (resp. column), then fourth one is also in this row (resp. column). If $W_i, Y_j, V_k$ are in the same row then some $Z_l$ is too. On the other hand, let $W_i, Y_j, Z_k$ be in the same row. If $W_i$ is in the same row (column) with a $V_l$, for $l > s$, then $Y_j$ and $Y_p$ ($p > t$) are in the same row, which is not possible. Hence $W_i$ is in the same row with a $V_l$ for $l \leq s$. The remaining cases follow by symmetry, and the case for columns is similar.

Now we see that for the binomial $h$, a row (resp. column) of $B$ contains either two factors (counting multiplicity), each appearing in separate terms, or four factors appearing in each term in two pairs. We say that $h$ and $B$ have the evenness property.

If the two terms in $h$ have a common factor, say $W = W_i = Z_j$, then we consider the polynomial $h' = h/W$, and the corresponding quasi-submatrix $B'$ obtained by deleting $W$. Then both still have the evenness property. To produce a standard expression of $h$, it is enough to produce one for $h'$. By factoring out all such common factors, we may reduce to the case that the binomial $h$ has relatively prime terms. We again denote the resulting binomial by $h$ and the corresponding quasi-submatrix by $B$.

Next, we associate to $h$ a (multi-)graph $H$ as follows: vertices of $H$ are the entries of $B$. If only two factors lie in a row (or column) and appear in $h$ in opposite terms, then they are joined by an edge in $H$. On the other hand, if there are four factors in a row (or column), $V, W, Y, Z$ (counting multiplicities), then we attach the vertices $W$ and $Z$, and the vertices $V$ and $Y$. This is an arbitrary choice. Then in $H$, the degree of any vertex is 2 or 4. In the graph $H$, we refer to edges as being either horizontal or vertical, depending on the positioning of the entries of the corresponding quasi-matrix.

In the next step, we reduce to the case that both terms in the binomial $h$ are squarefree. It is clear that the exponent of every variable in both terms of $h$ is at most 2. Suppose that $h$ has such a non-squarefree term. We choose a factor $W$ having multiplicity 2. We choose a circuit $H_1$ starting at $W$, whose edges are successively vertical and horizontal, and
starts initially vertically. We consider $H_1$ as a subgraph and let $H_2$ be the subgraph by removing all edges of $H_1$ and remaining isolated vertices. The vertices of $H_1$ and $H_2$ form two subquasi-matrices of $B$ and the element $W$ appears in both of them with one other element in its row and one other element in its column. Associated to the corresponding subquasi-matrices we have associated binomials

$$h' = m_1 - m_2, \ h'' = n_1 - n_2,$$

where $m_i$ (resp. $n_i$) is the product of the factors in $H_1$ (resp. $H_2$) in the $i$-th term of $h$. Each polynomial and its associated subquasi-matrix has the evenness property and the factor $W$ now has multiplicity one in each graph and polynomial. Moreover,

$$h = m_1 n_1 - m_2 n_2.$$ 

We claim that $\text{in}(h) \geq \min\{m_1 n_2, m_2 n_1\}$. If not, then

$$m_1 n_1 \leq \text{in}(h) < m_1 n_2 \Rightarrow n_1 < n_2$$

and

$$m_2 n_2 \leq \text{in}(h) < m_2 n_1 \Rightarrow n_2 < n_1$$

which is a contradiction.

Assume $\text{in}(h) \geq m_1 n_2$. Then

$$h = m_1 n_1 - m_2 n_2$$

$$= m_1 n_1 - m_2 n_2 - m_1 n_2 + m_1 n_2$$

$$= m_1 (n_1 - n_2) + n_2 (m_1 - m_2)$$

$$= m_1 h'' + n_2 h'.$$
Further, $\text{in}(m_1h'') = m_1\text{in}(h'') = m_1(n_1 - n_2)$, $\text{in}(n_2h') = n_2\text{in}(h') = n_2(m_1 - m_2)$ and both are $\leq \text{in}(h)$ by our assumption. This proves the claim.

Repeating this construction for every factor of multiplicity $> 1$, we eventually obtain a standard expression

$$h = \sum m_i h_i \; (\ast)$$

with $m_i$ monomial, $\text{in}(h) \geq \text{in}(m_i h_i)$ for all $i$, and $h_i$ is a binomial with relatively prime terms, all factors of multiplicity 1, and the $h_i$ and their associated subquasi-matrix has the evenness property.

To complete the proof, we claim that there is a standard expression ($\ast$) in which each $h_i$ is a binary quasi-minor. To accomplish this, for any binomial $b = h_i$ as above, we let $\tau(b)$ denote the number of rows and columns of the associated subquasi-matrix of $b$ having four (necessarily distinct) entries. We will show that if $\tau(b) > 0$, then there is a pair $(b', b'')$ as above, such that $\tau(b') < \tau(b)$ and $\tau(b'') < \tau(b)$. Applying this repeatedly, we obtain a standard expression ($\ast$) in which every binomial $h_i$ that appears satisfies $\tau(h_i) = 0$. But a binomial $b$ with $\tau(b) = 0$ is precisely a binary quasi-minor of $A$, which would prove the claim.

To verify that we may decrease $\tau$ in the prescribed manner, suppose that $b = h_i$ and $\tau(b) > 0$, with a row, say, with entries $W, Y, Z, V$. We again consider the graph $H$ of $b$, in this case every vertex now having degree 2. We note that the graph $H$ is not necessarily connected. Indeed, starting vertically from a vertex $W$ two cases occur. The path reaches $Z$ without passing through $V$ or $Y$, or before the path reaches $Z$ it reaches $V$ or $Y$ (actually the $Y$ cannot be reached vertically). In the second case we remove edges $WZ$ and $VY$ from $H$, and instead we add edges $VW$ and $YZ$ to $H$. In either case we have a circuit $H_1$ in which only two vertices lie in it. Again we let $H_2$ be the subgraph by removing all edges of $H_1$ and remaining isolated vertices. Now for the corresponding binomials $b'$ and $b''$ as above, we obtain that $\tau(b') < \tau(b)$ and $\tau(b'') < \tau(b)$. This completes the proof.

$\square$
4.2 Equations of the multi-Rees algebra

We fix a regular sequence \( s = s_1, \ldots, s_n \) in any ring \( R \) containing a field. Let \( R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r] \) be the multi-Rees algebra of powers of ideals \( I_i \), where \( a_i \)'s are positive integers and the ideals are generated by arbitrary subsets of \( s \), in the rest of this chapter by generators of \( I_i \) we mean these generators. We denote \( a = a_1, \ldots, a_r \).

**Definition 4.11.** Let \( a \) be a positive integer. We define \( \mathcal{F}_a^l \) as a subset of \( \mathcal{F}_a \) if and only if \( s^a \in I_i^a \).

Since \( I_i^a \) is generated by all \( s^a \)'s \( \langle j \rangle \in \mathcal{F}_a^l \), the multi-Rees algebra \( R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r] \) is the ring \( R \left[ \{s^a t_i\}_{1 \leq i \leq r, j \in \mathcal{F}_a^l} \right] \). Let \( S := R \left[ \{T_{l,j} \}_{1 \leq l \leq r, j \in \mathcal{F}_a^l} \right] \). We define and fix the \( R \)-algebra epimorphism

\[
\phi : S \to R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r], \quad \text{by} \quad \phi(T_{l,j}) = s^a t_i.
\]

We want to find generators of \( \mathcal{L} = \ker(\phi) \).

**Definition 4.12.** For a fixed \( l \), consider the set \( \{(l, j); \ j \in \mathcal{F}_a^l \} \) ordered lexicographically as a subset of \( \mathbb{N}^n \). We define the matrix \( B_{a_1} \), whose entry in row \( k \) and column \( (l, j) \) is \( T_{l,j}^{(k)} \).

We see that \( \phi(T_{l,j}^{(k)}) \) contains at least a factor of \( s_k \). Let \( I_l = \langle s_{k_1}, \ldots, s_{k_v} \rangle \), and \( k_1 < k_2 < \cdots < k_v \). Then the only possible \( T_{l,j}^{(k)} \)'s whose images under \( \phi \) are monomials in \( s_{k_1}, \ldots, s_{k_v} \) are in rows \( k_1, \ldots, k_v \) of \( B_{a_1} \). On the other hand for \( u < w \), if we compare images of \( T_{l,j}^{(u)} \) and \( T_{l,j}^{(w)} \),

\[
\phi(T_{l,j}^{(u)}) = s_{1}^{j_1-u+1} s_{2}^{j_2-1} \cdots s_{u-1}^{j_u-u+1} s_{u}^{j_u-u+1} \cdots s_{w}^{j_w-j_w-1} \cdots s_{n}^{j_n-j_n-1},
\]

\[
\phi(T_{l,j}^{(w)}) = s_{1}^{j_1-u+1} s_{2}^{j_2-1} \cdots s_{u-1}^{j_u-u+1} s_{u}^{j_u-u+1} \cdots s_{w}^{j_w-j_w-1} \cdots s_{n}^{j_n-j_n-1},
\]

then we see that when we move on the column \( (l, j) \) from row \( u \) to row \( w \) we lose one factor \( s_u \) and we get one factor \( s_w \).

**Definition 4.13.** In the matrix \( B_{a_1} \), in the row \( k_1 \), we choose \( T_{l,j}^{(k_1)} \)'s whose images under \( \phi \) are monomials in \( s_{k_1}, \ldots, s_{k_v} \). We define the quasi-matrix \( D_{a_1} \) to be the subquasi-matrix of
by choosing these columns and rows $k_i$, $1 \leq i \leq v$. The entries of the submatrix $D_a$ are all $T_{l,j^{(k_i)}}$s in $B_a$ whose images under $\phi$ are monomials in $s_{k_1}, \ldots, s_{k_v}$.

We define the matrix $B_a := (B_{a1}|B_{a2}| \ldots |B_{ar})$ and the matrix $C_a := (s|B_a)$. We also define the subquasi-matrices $D_a := (D_{a1}|D_{a2}| \ldots |D_{ar})$ and $E_a := (s|D_a)$ of $C_a$.

Then the theorem below describes the defining equations of the multi-Rees algebra $R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r]$.

**Theorem 4.14.** Let $R$ be a Noetherian ring containing a field and suppose that ideals $I_i$ are generated by subsets of a fixed regular sequence $s_1, \ldots, s_n$ contained in $\text{rad } R$. Then

$$R[I_1^{a_1}t_1, \ldots, I_r^{a_r}t_r] \cong S/I_{\text{bin}}(E_a).$$

**Remark 4.15.** We will show that the defining ideal is generated by the $2 \times 2$ minors of $E = E_a$ involving $s_1, \ldots, s_n$, the $2 \times 2$ minors of the $D_{ai}$, and the binary quasi-minors of $E$ (which are not minors) and have at most two entries from each $D_{ai}$.

**Proof.** First we show that $I_{\text{bin}}(E_a) \subseteq \mathcal{L}$. If $f$ is a $T$-binary quasi-minor, then

$$f = T_{l_1, j^{(1)}|^{(k_{i_1})}} T_{l_2, j^{(2)}|^{(k_{i_2})}} \cdots T_{l_{ij}, j^{(ij)}|^{(k_{ij})}} - T_{l_1, j^{(1)}|^{(k_{u_1})}} T_{l_2, j^{(2)}|^{(k_{u_2})}} \cdots T_{l_{ij}, j^{(ij)}|^{(k_{u_j})}}.$$

For arbitrary $T_{l_1, j^{(v)}|^{(k_{iv})}}$ we have

$$\phi(T_{l_1, j^{(v)}|^{(k_{iv})}}) = t_{iv}s_{k_{iv}} \prod_{\alpha=1}^{n} s_{j^{(v)} - j_{\alpha - 1}}^{(v)}, \text{ with } j^{(v)} \in \mathcal{T}_{a_{iv}},$$

and similarly

$$\phi(T_{l_1, j^{(v)}|^{(k_{uw})}}) = t_{uw}s_{k_{uw}} \prod_{\alpha=1}^{n} s_{j^{(v)} - j_{\alpha - 1}}^{(v)}, \text{ with } j^{(v)} \in \mathcal{T}_{a_{uw}}.$$
Hence we have

\[ \phi(f) = s_{k_1}s_{k_2}\cdots s_{k_\beta}t_1t_2\cdots t_{\beta} \prod_{\alpha=1}^{n} s_{\alpha}^{j_{\alpha}^{(1)} - j_{\alpha-1}^{(1)}} \prod_{\alpha=1}^{n} s_{\alpha}^{j_{\alpha}^{(2)} - j_{\alpha-1}^{(2)}} \prod_{\alpha=1}^{n} s_{\alpha}^{j_{\alpha}^{(\beta)} - j_{\alpha-1}^{(\beta)}}. \]

\[ -s_{k_u}s_{k_{u_2}}\cdots s_{k_{u_\beta}}t_1t_2\cdots t_{\beta} \prod_{\alpha=1}^{n} s_{\alpha}^{j_{\alpha}^{(1)} - j_{\alpha-1}^{(1)}} \prod_{\alpha=1}^{n} s_{\alpha}^{j_{\alpha}^{(2)} - j_{\alpha-1}^{(2)}} \prod_{\alpha=1}^{n} s_{\alpha}^{j_{\alpha}^{(\beta)} - j_{\alpha-1}^{(\beta)}} = 0. \]

If \( f \) is an \( s \)-binary quasi-minor, then

\[ f = s_{k_1}T_{l_2,j_{(1)}^{(2)|k_{l_2}}} \cdots T_{l_{\beta},j_{(\beta)}^{(2)|k_{l_\beta}}} - s_{k_{u_1}}T_{l_2,j_{(1)}^{(2)|k_{u_2}}} \cdots T_{l_{\beta},j_{(\beta)}^{(2)|k_{u_\beta}}}. \]

and similarly we see that \( \phi(f) = 0 \):

\[ \phi(f) = s_{k_1}s_{k_2}\cdots s_{k_\beta}t_1t_2\cdots t_{\beta} \prod_{\alpha=1}^{n} s_{\alpha}^{j_{\alpha}^{(2)} - j_{\alpha-1}^{(2)}} \prod_{\alpha=1}^{n} s_{\alpha}^{j_{\alpha}^{(\beta)} - j_{\alpha-1}^{(\beta)}}. \]

\[ -s_{k_u}s_{k_{u_2}}\cdots s_{k_{u_\beta}}t_1t_2\cdots t_{\beta} \prod_{\alpha=1}^{n} s_{\alpha}^{j_{\alpha}^{(2)} - j_{\alpha-1}^{(2)}} \prod_{\alpha=1}^{n} s_{\alpha}^{j_{\alpha}^{(\beta)} - j_{\alpha-1}^{(\beta)}} = 0. \]

Thus \( I_{bin}(E_a) \subseteq \mathcal{L} \). Now we prove that \( \mathcal{L} \subseteq I_{bin}(E_a) \). We prove this claim in two steps. First we prove the claim for the case that \( a = (1, \ldots, 1) \).

To prove this case, we reduce the problem to the case when \( s_1, \ldots, s_n \) is a sequence in variables in a polynomial ring \( R = k[s_1, \ldots, s_n] \) over a field \( k \). Suppose the Theorem is proved in this case.

Let \( k \subset R \) be a field. Since \( s_1, \ldots, s_n \) is a regular sequence contained in \( \text{rad } R \), \( R \) is flat over its subring \( k[s_1, \ldots, s_n] \), and the latter ring is a polynomial ring \([6]\). (Actually \([6]\) only proves the flatness for a local ring, but the proof only needs the local criterion of flatness, which holds as long as the ideal belongs to \( \text{rad } R \) \([5, \text{Theorem 22.3}]\).) Set \( A = k[s_1, \ldots, s_n] \).

Since \( I_i \) is generated by a subset of \( s_1, \ldots, s_n \), there is an ideal \( J_i \) be an ideal of \( A \) such that \( J_i R = I_i \). By hypothesis, the theorem holds for the multi-Rees algebra of the ideals \( J_i \) of \( A \).
However, since $R$ is flat over $A$,

$$A[J_1t_1, \ldots, J_rt_r] \otimes_A R \cong R[I_1t_1, \ldots, I_rt_r].$$

Indeed, the isomorphism holds essentially since $J_i \otimes_A R \cong J_iR = I_i$. From this isomorphism, it follows immediately that

$$R[I_1t_1, \ldots, I_rt_r] \cong (S_{A/I_{bin}(E_{(1,\ldots,1)})) \otimes_A R \cong S/I_{bin}(E_{(1,\ldots,1)}).$$

This completes the proof of the reduction to the case of a sequence of variables.

We consider the ideal $I = \langle s_1, \ldots, s_n \rangle$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
R \left[ \left\{ T_{l,j} \right\}_{1 \leq l \leq r, j \in J_l} \right] & \xrightarrow{\phi} & R[I_1t_1, \ldots, I_rt_r] \\
\downarrow{\vartheta_1} & & \downarrow{\vartheta_2} \\
R \left[ \left\{ T_{l,j} \right\}_{1 \leq l \leq r, j \in J_l} \right] & \xrightarrow{\phi'} & R[I_1t_1, \ldots, I_rt_r],
\end{array}
$$

where $\phi'(T_{l,j}) = s_{j}t_{l}$. For any $f \in \mathcal{L}$, we have

$$\phi'(f) = \phi'\vartheta_1(f) = \vartheta_2\phi(f) = 0 \Rightarrow f \in \ker(\phi').$$

By Example 3.4, $f \in I_2(C_{a})$. By Proposition 4.9, $f \in I_{bin}(C_{a})$.

We put a total order on variables of $C_{a}$ such that all variables which are not in $E_{a}$ are greater than all variables which are in $E_{a}$. We also put the lexicographic order on $R \left[ \left\{ T_{l,j} \right\}_{1 \leq l \leq r, j \in J_l} \right]$. By Proposition 4.10 the set of all binary quasi-minors of $C_{a}$ form a Gröbner basis for $I_{bin}(C_{a})$, and so $\text{in}(f) \in \text{in}(I_{bin}(C_{a}))$. We prove that $f \in I_{bin}(E_{a})$ by induction on $\text{in}(f)$, the base change being trivial. There is a binary quasi-minor such as $h$ such that $w\text{in}(h) = \text{in}(f)$. On the other hand variables in $w\text{in}(h)$ are some of variables in $f$ and so they are variables in $E_{a}$. Since we have the lexicographic order and all variables
out of $E_a$ are greater than these variables, variables in the non-initial term of $h$ are also in $E_a$. Hence $h$ is a binary quasi-minor in $E_a$. Therefore $f - wh \in I_{\bin}(C_a)$, and variables in $f - wh$ are in $E_a$. Moreover $in(f) >_{\text{lex}} in(f - wh)$ so by induction $f - wh \in I_{\bin}(E_a)$ and hence so is $f$.

Now we prove the theorem in the general case. We use the same method that we have used in Theorem 3.3.

Let $Y = (x_{i,l})$ be a generic $n \times r$ matrix and let $X$ be a subquasi-matrix of $Y$ defined as follows: an arbitrary $x_{i,l}$ is an entry of $X$ if $s_i$ is between generators of $I_l$. $x_l$ denotes the regular sequence $x_{1,l}, x_{2,l}, \ldots, x_{n,l}$ of entries of the $l$-th column of $Y$. We define $A_l$ to be the $a_l$-th Veronese subring $R[\{x_{i,l}\}_{x_{i,l} \text{ is in the } l\text{-column of } X}^{(a_l)}]$. Let $X_l(a)$ be the family of monomials of degree $a$ ($a \in \mathbb{N}$) in the variables which are in the $l$-column of $X$, we define $A$ to be the $a$-th Veronese subring $R[X]^a = R[X_1(a_1), X_2(a_2), \ldots, X_r(a_r)]$.

For arbitrary $l$, $1 \leq l \leq r$ we define

$$\alpha_l : R\left[\left\{T_{l,j}\right\}_{j \in \mathcal{F}_{a_l}}\right] \rightarrow A_l, \quad \alpha_l(T_{l,j}) = x_{l,j}^2.$$ 

These induce a map $\alpha : S \rightarrow A$. By the same argument given in Theorem 3.3, we have

$$\ker(\alpha) = \langle I_2(D_{a_1}), I_2(D_{a_2}), \ldots, I_2(D_{a_r}) \rangle \subseteq I_{\bin}(E_a).$$

We define the quasi-matrix $Z = (s|X)$. If we consider

$$\psi : R[X] \rightarrow R[I_1u_1, I_2u_2, \ldots, I_ru_r], \quad \psi(x_{i,l}) = s_iu_l,$$

then by first part of the proof $\ker(\psi)$ is generated by $2 \times 2 s$-minors and $x$-binary quasi-minors (binary quasi-minors which don’t contain $s_i$) of $Z$. 

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By the following commutative diagram we define the map $g$:

$$
\begin{array}{ccc}
S & \xrightarrow{\phi} & R[I_1^{a_1} t_1, \ldots, I_r^{a_r} t_r] \\
\downarrow{\alpha} & & \downarrow{\ell} \\
A & \xrightarrow{\psi(a)} & R[I_1^{a_1} u_1, \ldots, I_r^{a_r} u_r] \\
\downarrow & & \downarrow \\
R[X] & \xrightarrow{\psi} & R[I_1 u_1, \ldots, I_r u_r]
\end{array}
$$

In this step we prove that the kernel of $g$ is generated by polynomials of the form

$$s_\delta x_{\gamma,l} x_\ell^j - s_\gamma x_{\delta,l} x_\ell^j, \; j \in \mathcal{F}_{a_l-1}$$

and

$$\prod_{i=1}^{v} x_{\delta_i,\gamma_i} x_{\gamma_i}^{(i)} - \prod_{i=1}^{v} x_{\beta_i,\gamma_i} x_{\gamma_i}^{(i)}, \; j^{(i)} \in \mathcal{F}_{a_{\gamma_i}-1},$$

where $\gamma_i$ (resp. $\delta_i, \beta_i$) are distinct and $\{\delta_1, \ldots, \delta_v\} = \{\beta_1, \ldots, \beta_v\}$.

The proof of this fact is similar to the proof of Theorem 3.3, but because it is a different argument in many parts we include the details for completeness.

If $0 \neq p \in \ker(g)$, then $p \in \ker(\psi)$ and thus it is an $R[X]$-linear combination of $2 \times 2$ $\gamma$-minors or $x$-binary quasi-minors of $Z$. Thus we have an expression $p = \sum_{i=1}^{m} r_i \mu_i q_i$, where the $q_i$'s are $2 \times 2$ $\gamma$-minors or $x$-binary quasi-minors of $Z$, $r_k \in R$, and $\mu_k \in R[X]$ is a monomial. Among all such expressions we choose one in which $m$ is minimal. We claim that if $q_k$ is an $\gamma$-minor and its variables are chosen from $l + 1$ column of $Z$, then $\mu_k = \xi_k x_\ell^j$ ($j \in \mathcal{F}_{a_{\gamma_i}-1}^l$) and if $q_k = \prod_{i=1}^{v} x_{\delta_i,\gamma_i} - \prod_{i=1}^{v} x_{\beta_i,\gamma_i}$, then $\mu_k = \xi_k \prod_{i=1}^{v} x_{\gamma_i}^{(i)} (j^{(i)} \in \mathcal{F}_{a_{\gamma_i}-1}^l)$, in both cases $\xi_k \in A$. We may assume that $\mu_1 x_{\gamma,l}$ or $\mu_1 \prod_{i=1}^{v} x_{\delta_i,\gamma_i}$ appears in the support of $p$, where $q_1 = s_\delta x_{\gamma,l} - s_\gamma x_{\delta,l}$ or $q_1 = \prod_{i=1}^{v} x_{\delta_i,\gamma_i} - \prod_{i=1}^{v} x_{\beta_i,\gamma_i}$. Since one of terms in $p$ is $r \mu_1 x_{\gamma,l}$ or $r \mu_1 \prod_{i=1}^{v} x_{\delta_i,\gamma_i}$ ($r \in R$), $\mu_1 x_{\gamma,l}$ or $\mu_1 \prod_{i=1}^{v} x_{\delta_i,\gamma_i}$ is in $A$. Therefore we have two cases: case i) $\mu_1 x_{\gamma,l}$ is in the support of $p$. Then $\mu_1 x_{\gamma,l} = \Gamma_1 \Gamma_2 \ldots \Gamma_w$, where $\Gamma_j \in A_i$ and is a monomial
of degree \(a_i\), therefore \(x_{\gamma,l}\) divides at least one \(\Gamma_j\), say \(x_{\gamma,l}\) divides \(\Gamma_1\), then \(\Gamma_1\) is a monomial of degree \(a_l\) in \(A_l\) and \(\Gamma_1 = x_{\gamma,l}x_{l}^j\) (\(j \in \mathcal{F}_{a_l-1}^l\)). Hence

\[
\mu_1 x_{\gamma,l} = x_{\gamma,l}x_{l}^j \Gamma_2 \ldots \Gamma_w \Rightarrow \mu_1 = \xi_1 x_{l}^j, \text{ with } \xi_1 = \Gamma_2 \ldots \Gamma_w \in A.
\]

Case ii) \(\mu_1 \prod_{i=1}^v x_{\delta_i,\gamma_i}\) is in the support of \(p\). Then \(\mu_1 \prod_{i=1}^v x_{\delta_i,\gamma_i} = \Gamma_1 \Gamma_2 \ldots \Gamma_w\), where \(\Gamma_j \in A_i\) and is a monomial of degree \(a_i\). Since all \(\gamma_i\)'s are distinct, then \(x_{\delta_i,\gamma_i}\)'s divide distinct \(\Gamma_j\)'s say \(\Gamma_i\)'s, therefore \(\Gamma_i\)'s are monomials of degree \(a_{\gamma_i}\) in \(A_{\gamma_i}\), moreover \(\Gamma_i = x_{\delta_i,\gamma_i}x_{l}^{j(i)}\), \(j^{(i)} \in \mathcal{F}_{a_{\gamma_i}-1}^i\). Therefore

\[
\mu_1 \prod_{i=1}^v x_{\delta_i,\gamma_i} = \prod_{i=1}^v x_{\delta_i,\gamma_i}x_{l}^{j(i)} \Gamma_{v+1} \ldots \Gamma_w
\]

\[
\Rightarrow \mu_1 = \xi_1 \prod_{i=1}^v x_{l}^{j(i)}, \text{ with } \xi_1 = \Gamma_{v+1} \ldots \Gamma_w \in A.
\]

Hence either

\[
\mu_1 q_1 = \xi_1 x_{l}^j (s_{\delta} x_{\gamma,l} - s_{\gamma} x_{\delta,l}) = \xi_1 (s_{\delta} x_{\gamma,l}x_{l}^j - s_{\gamma} x_{\delta,l}x_{l}^j)
\]

or

\[
\mu_1 q_1 = \xi_1 \prod_{i=1}^v x_{l}^{j(i)} \left( \prod_{i=1}^v x_{\delta_i,\gamma_i} - \prod_{i=1}^v x_{\beta_i,\gamma_i} \right) = \xi_1 \left( \prod_{i=1}^v x_{\delta_i,\gamma_i}x_{l}^{j(i)} - \prod_{i=1}^v x_{\beta_i,\gamma_i}x_{l}^{j(i)} \right).
\]

Since \(p\) and \(\mu_1 q_1 \in A\), we conclude that \(p - r_1 \mu_1 q_1 \in A\). Then since the expression

\[
\sum_{i=2}^m r_i \mu_i q_i
\]

is still minimal, by induction of \(m\) (the base case \(m = 1\) being clear by the above), we conclude that \(p - r_1 \mu_1 q_1\), and hence \(p\), is a linear combination of the required polynomials. Therefore we have described the generators of \(\ker(g)\).

On the other hand if \(j' = j + (1, \ldots, 1)\), then \(s_{\delta} T_{l_{j'}^{(\gamma)}} - s_{\gamma} T_{l_{j'}^{(\delta)}}\) is an \(s\)-minor of \(E_a\) and

\[
\alpha \left( s_{\delta} T_{l_{j'}^{(\gamma)}} - s_{\gamma} T_{l_{j'}^{(\delta)}} \right) = s_{\delta} x_{\gamma,l}x_{l}^{j} - s_{\gamma} x_{\delta,l}x_{l}^{j}
\]
and if \( \bar{j}^{(k)} = j^{(k)} + (1, \ldots, 1) \), then

\[
T_{\gamma_1 \bar{j}^{(1)}|\delta_1} \cdots T_{\gamma_v \bar{j}^{(v)}|\delta_v} - T_{\gamma_1 j^{(1)}|\delta_1} \cdots T_{\gamma_v j^{(v)}|\delta_v}
\]

is a binary quasi-minor of \( E_a \) and its image under \( \alpha \) is

\[
\left( \prod_{i=1}^{v} x_{\delta_i \gamma_i} x_{\bar{j}^{(i)}_i} - \prod_{i=1}^{v} x_{\beta_i \gamma_i} x_{\bar{j}^{(i)}_i} \right).
\]

This completes the proof. \( \Box \)

**Remark 4.16.** We prove that every \( s \)-binary quasi-minor of \( E_a \) can be generated by \( 2 \times 2 \) \( s \)-minors and \( T \)-binary quasi-minors of \( E_a \). Let \( f = s_i V_1 V_2 \ldots V_n - s_j W_1 W_2 \ldots W_n \) (\( V_i \) and \( W_i \) are equal to \( T_{i, j^{(k)}} \)'s). Without loss of generality we may assume \( s_i \) and \( W_1 \) are in the same row and \( W_1 \) and \( V_1 \) are in the same column. If \( V_1 \) and \( s_j \) are in the same row, then we have

\[
f = s_i V_1 V_2 \ldots V_n - s_j W_1 W_2 \ldots W_n - s_j W_1 V_2 \ldots V_n + s_j W_1 V_2 \ldots V_n
= (s_i V_1 - s_j W_1)V_2 \ldots V_n + s_j W_1(V_2 \ldots V_n - W_2 \ldots W_n).
\]

If \( V_1 \) and \( s_j \) are not in the same row, then there is an \( s_k \) which is in the same row with \( V_1 \). We have

\[
f = s_i V_1 V_2 \ldots V_n - s_j W_1 W_2 \ldots W_n - s_k W_1 V_2 \ldots V_n + s_k W_1 V_2 \ldots V_n
= (s_i V_1 - s_k W_1)V_2 \ldots V_n + W_1(s_k V_2 \ldots V_n - s_j W_2 \ldots W_n).
\]

We can continue this procedure until all generators are either \( 2 \times 2 \) \( s \)-minors or \( T \)-binary quasi-minors of \( E_a \).

**Remark 4.17.** We can define \( D_{ai} \) in another way. If \( I_l = \langle s_{i_1}, \ldots, s_{i_u} \rangle \), then we define similar \( \mathcal{F}_{ai} \) and \( \mathcal{F}'_{ai} \) with members \( \bar{j} = (j_{u-1}, \ldots, j_1) \). We also define \( s^\bar{j} \) which is a monomial in
$s_i, \ldots, s_n$. Hence we define $T_{i,j}$ and so $D_{a_i}$ and finally $E_a$. We can see that image of corresponding entries in both $D_{a_i}$ is the same. We will have a new $R[T_{i,j}]$ and a new $\phi$, but this new one is isomorphic to the previous one and the image of corresponding variables under different $\phi$’s is the same. Then we have similar result about the generators of the kernel.

**Corollary 4.18.** Let $R$ be a local Cohen-Macaulay ring containing a field and suppose that ideals $I_i$ are generated by subsets of a fixed regular sequence. Then the multi-Rees algebra $R[I_{1}^{a_1}t_1, \ldots, I_{r}^{a_r}t_r]$ is Cohen-Macaulay.

**Proof.** We first verify the claim when $R$ is a polynomial ring in the fixed regular sequence $s_1, \ldots, s_n$. First we show that $R[I_{1}u_1, \ldots, I_{r}u_r]$ is normal. By Theorem 4.14, we have

$$R[I_{1}u_1, \ldots, I_{r}u_r] \cong R[T_{i,j}]/I_{bin}(E_a),$$

and by Lemma 4.10, $\text{in}(I_{bin}(E_a))$ is generated by initial terms of binary quasi-minors which are squarefree, hence by [21, Proposition 13.5, Proposition 13.15], $R[I_{1}u_1, \ldots, I_{r}u_r]$ is a normal domain. On the other hand

$$R[I_{1}^{a_1}t_1, \ldots, I_{r}^{a_r}t_r] \cong R[I_{1}^{a_1}u_{1}^{a_1}, \ldots, I_{r}^{a_r}u_{r}^{a_r}]$$

and $R[I_{1}^{a_1}u_{1}^{a_1}, \ldots, I_{r}^{a_r}u_{r}^{a_r}]$ is a direct summand of $R[I_{1}u_1, \ldots, I_{r}u_r]$, so $R[I_{1}^{a_1}t_1, \ldots, I_{r}^{a_r}t_r]$ is normal.

Now if we consider the semigroup $M$ generated by $x_1, \ldots, x_m$ and $ft_i$, where $f$ is a generator of $I_{1}^{a_1}$, then by [8, Proposition 1], $M$ is normal, hence by [8, Theorem 1], $k[M] \cong R[I_{1}^{a_1}t_1, \ldots, I_{r}^{a_r}t_r]$ is a Cohen-Macaulay domain.

To complete the proof, as in the proof of Theorem 4.14, we reduce to the latter case. Since the multi-Rees algebra is positively graded, it suffices to show that the special fiber ring $R/(s_1, \ldots, s_n)$ is Cohen-Macaulay, which is clear.
Remark 4.19. This result does not hold for arbitrary complete intersections, even of codimension 2. For example, if $R = k[[x, y]]$, and $I_1 = \langle x, y \rangle$, $I_2 = \langle x^2, y^2 \rangle$, then the multi-Rees algebra $R[I_1 t_1, I_2 t_2]$ is not Cohen-Macaulay (cf. [19, Example 4.9]).

Example 4.20. Let $k[s_1, s_2, s_3]$, where $k$ is a field. Let $I_1 = \langle s_1, s_2 \rangle$, $I_2 = \langle s_2, s_3 \rangle$, $I_3 = \langle s_1, s_3 \rangle$. Then we have the homomorphism

$$\phi : R[T_{1,1,1}, T_{1,1,0}, T_{2,1,0}, T_{2,0,0}, T_{3,1,1}, T_{3,0,0}] \to R[I_1 t_1, I_2 t_2, I_3 t_3],$$

$$T_{1,1,1} \mapsto s_1 t_1, \ T_{1,1,0} \mapsto s_2 t_1, \ T_{2,1,0} \mapsto s_2 t_2, \ T_{2,0,0} \mapsto s_3 t_2, \ T_{3,1,1} \mapsto s_1 t_3, \ T_{3,0,0} \mapsto s_3 t_3,$$

and its kernel is

$$\langle s_1 T_{1,1,0} - s_2 T_{1,1,1}, s_1 T_{3,0,0} - s_3 T_{3,1,1}, s_2 T_{2,0,0} - s_3 T_{2,1,0}, T_{1,1,1} T_{2,1,0} T_{2,0,0} - T_{3,1,1} T_{1,1,0} T_{2,0,0} \rangle.$$

We see that $E_a$ has the form below

$$\begin{bmatrix}
s_1 & T_{1,1,1} & T_{3,1,1} \\
s_2 & T_{1,1,0} & T_{2,1,0} \\
s_3 & T_{2,0,0} & T_{3,0,0}
\end{bmatrix}.$$  

This special example can also be recovered by using the theory of Rees algebras of modules, as follows. The module $M = I_1 \oplus I_2 \oplus I_3$ has a linear resolution

$$0 \to R^3 \xrightarrow{\phi} R^6 \to M \to 0.$$  

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where

\[
\Phi = \begin{bmatrix}
s_2 & 0 & 0 \\
-s_1 & 0 & 0 \\
0 & s_3 & 0 \\
0 & -s_2 & 0 \\
0 & 0 & s_3 \\
0 & 0 & -s_1 \\
\end{bmatrix}.
\]

Hence \( \text{pd} M = 1 \). Furthermore, since \( M \) is free in codimension 1, and 4-generated in codimension 2, by [19, Proposition 4.11] the Rees algebra of \( M \), which is the multi-Rees algebra in question, has the expected defining equations, in the sense that

\[
\mathcal{R}(M) \cong R[T_1, T_2, T_3, T_4, T_5, T_6]/\langle[s_1s_2s_3]B, \det B \rangle
\]

where

\[
B = \begin{bmatrix}
-T_2 & 0 & -T_6 \\
T_1 & -T_4 & 0 \\
0 & T_3 & T_5
\end{bmatrix}
\]

is the matrix defined by the equation

\[
[T] \Phi = [s] B.
\]

**Example 4.21.** Another simple case is for a pair of “transversal” ideals: Let \( R \) be a Noetherian local ring containing a field, let \( s_1, \ldots, s_n \) be a regular sequence and let \( I_1 = \langle s_1, \ldots, s_k \rangle \) and \( I_2 = \langle s_{k+1}, \ldots, s_n \rangle \). Then the multi-Rees algebra \( R[I_1t_1, I_2t_2] \) is defined by linear equations. Indeed, this follows from Theorem 4.14. (The result that \( M = I_1 \oplus I_2 \) is of linear type can be proved directly, at least in this case when \( R \) is a polynomial ring in \( s_1, \ldots, s_n \).)

**Remark 4.22.** One may wonder if we can use the method of proof of Proposition 4.10, to prove that for an arbitrary \( a \) the binary minors of \( E_a \) form a Gröbner basis so in the proof of
Theorem 4.14, we don’t need the Veronese type argument. Since in general elements of $E_a$ are not distinct we cannot use this method and doing this will cause the same mistake that is made in [15, Lemma 2.6], where the authors have proved that when $R = k[x_1, \ldots, x_n]$ ($k$ a field), $2 \times 2$-minors of $C_a$ form a Gröbner basis for $I_2(C_a)$ with reverse lexicographic order. Since the indeterminates of $C_a$ are not all distinct the proof of this lemma is incomplete. For example in their case 1, the authors assume that the $2 \times 2$-minors are

$$h_1 = \begin{vmatrix} T_{1,1,s} & T_{1,2,s} \\ T_{1,1,t} & T_{1,2,t} \end{vmatrix} = \begin{vmatrix} a & e \\ f & g \end{vmatrix}$$

$$h_2 = \begin{vmatrix} T_{1,1,s} & T_{1,2,s} \\ T_{1,1,u} & T_{1,2,u} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

but in the minor $h_2 = ad - bc$ we cannot always say $a = T_{1,1,s}$, because another $T_{1,2,u}$ may be equal to $a$, although in $h_1$ we have $a = T_{1,1,s}$. Thus we cannot always find $G$ such that $S(h_1, h_2) = -def + gbc = b(cg - fG) - f(de - Gb)$. For example over the ring $R = k[x_1, x_2, x_3, x_4]$ with the maximal ideal $m = \langle x_1, x_2, x_3, x_4 \rangle$, we consider the Rees algebra $R[m^3t]$. Since $r = 1$, when we show the matrix $B_a$, instead of $T_{1,2}^{(k)}$ we only write $T_{1,2}^k$. Now $B_a$, associated to $R[m^3t]$ is as below:

$$B_a = \begin{bmatrix} T_{1,1,1} & T_{2,1,1} & T_{2,2,1} & T_{3,1,1} & T_{3,2,1} & T_{3,3,1} & T_{3,3,2} & T_{3,3,3} \\
T_{1,1,0} & T_{2,1,0} & T_{2,2,0} & T_{3,1,0} & T_{3,2,0} & T_{3,3,0} & T_{3,3,1} & T_{3,3,2} \\
T_{1,0,0} & T_{2,0,0} & T_{2,1,0} & T_{3,0,0} & T_{3,1,0} & T_{3,2,0} & T_{3,2,1} & T_{3,2,2} \\
T_{0,0,0} & T_{1,0,0} & T_{1,1,0} & T_{2,0,0} & T_{2,1,0} & T_{2,2,0} & T_{2,2,1} & T_{2,2,2} \end{bmatrix}.$$ 

In this matrix we consider minors $h_1$ and $h_2$ as follows

$$h_1 = \begin{vmatrix} T_{2,1,1} & T_{2,2,1} \\ T_{2,1,0} & T_{2,2,0} \end{vmatrix} = \begin{vmatrix} a & e \\ f & g \end{vmatrix}, \quad h_2 = \begin{vmatrix} T_{2,1,1} & T_{3,0,0} \end{vmatrix} = \begin{vmatrix} a & b \\ T_{1,1,1} & T_{2,0,0} \end{vmatrix} = \begin{vmatrix} c & d \end{vmatrix},$$
we see that there is no $G$ such that $-def + gbc = b(cg - fG) - f(de - Gb)$, because there is no $G$ such that $cg - fG$ becomes a minor.
References


