12-2018

Hartogs Domains and the Diederich-Fornæss Index

Muhenned Abdulameer Abdulsahib
University of Arkansas, Fayetteville

Hartogs Domains and the Diederich-Fornæss Index

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Mathematics

by

Muhenned A. Abdulsahib
University of Baghdad
Bachelor of Science in Mathematics, 1999
University of Arkansas
Master of Science in Mathematics, 2014

December 2018
University of Arkansas

This dissertation is approved for recommendation to the Graduate Council.

Phillip S. Harrington, PhD
Dissertation Director

Daniel H. Luecking, PhD
Committee Member

Andrew Raich, PhD
Committee Member

Abstract

The Diederich-Fornæss Index has played a crucial role in studying regularity of the Bergman projection on pseudoconvex domains in Sobolov spaces as is shown by Kohn, Harrington, Pinton and Zampieri and others. In this work, we discuss the Diederich-Fornæss Index on Hartogs domains, and its relation to other properties connected to regularity of the Bergman projection. An upper and lower bound for the Diederich-Fornæss Index is calculated for Hartogs domains and computed sharply for worm domains. Related conditions for the existence of a strong Stein neighborhood basis for Hartogs domains are introduced.

Acknowledgments

First, I would like to thank my advisor Dr. Phillip Harrington for his unlimited patience, constant support and invaluable guidance through my academic research. I owe a special thanks to Dr. Daniel H. Luecking and Dr. Andrew S. Raich for agreeing to be committee members. I appreciate the help of the staff of mathematics department who made my stay comfortable at University of Arkansas.

I would like to thank the chair of mathematics department Dr. Mark Johnson for his support during my stay at the department.

Finally, I am indebted forever to my parents Abdulameer and Kidega, my brothers and sisters, and my wife and daughters for their unconditional love.

Dedication

I dedicate this dissertation to the memory of my father for his constant support. I also dedicate it to my mother for her unconditional love. Lastly, I dedicate it to to my wife Asmaa, and my daughters Zeinib, Rukayyah, and Maryam.

Contents

Chapter 1

Background Review

1.1 Introduction

A holomorphic function is a solution for the homogeneous Cauchy-Riemann equation $\bar{\partial} u = 0$. A central problem in the study of partial differential equations in several complex variables is solving the inhomogeneous Cauchy-Riemann equation $\bar{\partial} u = f$. If a solution exists, then we will have many solutions. For any solution $u \in L^2$, if we subtract from u its orthogonal projection onto the kernel of $\bar{\partial}$ we get the canonical solution. The orthogonal projection from $L^2_{(0,q)}(\Omega)$ onto ker $\bar{\partial}$ is denoted by P_q, and it is called the Bergman projection. A domain of holomorphy in $\mathbb{C}^n$ is a domain on which holomorphic functions cannot extend to larger domain, for example a convex domain. It is known that $\bar{\partial} u = f$ admits a solution on a domain of holomorphy whenever $\bar{\partial} f = 0$. On a domain of holomorphy with smooth boundary, if derivatives of all order of f extend continuously to the boundary, it is not necessary for the canonical solution to have the same regularity, as we will see in our discussion of the worm domain.

The Riemann mapping theorem says if $U \subset \mathbb{C}$ is a simply connected, proper, open subset, then U is biholomorphic to the interior of the unit disc, i.e., there exists a bijective holomorphic mapping with a holomorphic inverse defined from U to the interior of the unit disc. This theorem is not correct in higher dimensions in general. For example, the unit ball in n dimensions and the cross product of n unit disks are not biholomorphic for $n > 1$ (see for example [7]).

Fefferman [14] proved that if we have two smooth bounded strictly pseudoconvex domains D_1 and D_2 in $\mathbb{C}^n$, and $f : D_1 \to D_2$ is biholomorphic, then f extends smoothly to the boundary. Bell and Ligocka [3] showed that the Fefferman theorem is also a consequence of subelliptic estimates for the solution operator to $\bar{\partial}$, which are known for strictly pseudoconvex domains.

A smooth bounded pseudoconvex domain is said to satisfy Condition R if the Bergman projection P associated with Ω maps the set of smooth functions on $\bar{\Omega}$ into a set of functions that is smooth on $\bar{\Omega}$ and holomorphic on Ω. If we have two smooth bounded pseudoconvex domains D_1 and D_2, at least one of them satisfying Condition R in $\mathbb{C}^n, n \geq 2$, and $f : D_1 \to D_2$ is biholomorphic, then f extends smoothly to the boundary [3]. This result is a nuanced substitute for the Riemann mapping theorem in higher dimensions.

The importance of worm domains (see Definition 2.1.7), a class of smoothly bounded pseudoconvex domains in $\mathbb{C}^2$, is due to the fact that they represent a counterexample to Condition R, i.e., the Bergman projection operator P_0 fails to be continuous on $C^\infty(\bar{\Omega})$, as is shown in Christ [8] based on work of Barrett [1].

In 1977, Diederich and Fornæss [11] proved that for any bounded pseudoconvex domain Ω with C^2 boundary in a Stein manifold there exists a C^2 defining function ρ on a neighborhood U of $\bar{\Omega}$ such that $\hat{\rho} = -(-\rho)^\tau$ is strictly plurisubharmonic on Ω for some $0 < \tau < 1$, where τ is called a Diederich-Fornæss exponent. The Diederich-Fornæss Index of Ω is the supremum of the Diederich-Fornæss exponents taken over all defining functions of Ω. Our interest in Diederich-Fornæss exponents is due to the fact that Kohn [22], Harrington [18], and Pinton and Zampieri [25] have shown that if the Diederich-Fornæss Index is equal to one with some additional hypotheses, we get Condition R (global regularity).

Diederich and Fornæss [12] have shown that for the worm domain the Diederich-Fornæss Index approaches zero when the winding number approaches infinity. Fornaess and Herbig [15] have shown that the Diederich-Fornæss Index is equal to one for a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ that admits a plurisubharmonic defining function on the boundary. Condition R is known to hold on such domains by a result of Boas and Straube [4]. Since the worm domain is a special case of a larger class of domains called Hartogs domains, we will focus on studying Hartogs domains.

The presence of an annulus in the boundary of a Hartogs domain plays a crucial rule in the study of the Diederich-Fornæss Index. Our primary innovation is to use the curvature term $\frac{\partial^2\tilde{\delta}}{\partial w\partial\bar{z}}$ on the annulus in the boundary to compute an upper and lower bound for the Diederich-Fornæss Index for Hartogs domains.

We denote the distance function from z to the boundary of Ω by $\delta(z)$ and the signed distance function by $\tilde{\delta}$ which is defined as follows

$$\tilde{\delta}(z) = \begin{cases} -\delta(z) & \text{on } \bar{\Omega} \\ \delta(z) & \text{outside of } \bar{\Omega}. \end{cases}$$

Theorem 1.1.1. *Let $\Omega \subset \mathbb{C}^2$ be a smooth Hartogs domain, and suppose that for some $w \in \mathbb{C}$, $B > A > 0$ and $\mathcal{C} > 0$ the set $M = \{(z,w) :\ A < |z|^2 < B\}$ is in $b\Omega$ and $\left|\frac{\partial^2\tilde{\delta}}{\partial w\partial\bar{z}}\right| > \frac{\mathcal{C}}{|z|}$ on M. Assume there exists a smooth strictly positive function h on $\bar{\Omega}$ such that*

$$\sigma = -h(-\tilde{\delta})^\tau$$

is plurisubharmonic on Ω, for some $0 < \tau < 1$. Then $\tau < \frac{\pi}{2\mathcal{C}\ln\frac{B}{A}+\pi}$.

We also get a lower bound for the Diederich-Fornæss Index if we place an additional hypothesis on the set of weakly pseudoconvex points.

Theorem 1.1.2. *Let Ω be a smooth Hartogs domain, and suppose that for some $B > A > 0$ and $\mathcal{C} > 0$ whenever the set $M = \{(z,w) :\ A \leq |z|^2 \leq B\}$ is in $b\Omega$ for some $w \in \mathbb{C}$, then $\left|\frac{\partial^2\tilde{\delta}}{\partial w\partial\bar{z}}\right| \leq \frac{\mathcal{C}}{|z|}$ on M. Moreover, let $M_1 = \{$weakly pseudoconvex point, such that $\frac{\partial\tilde{\delta}}{\partial z} = 0\}$, $M_2 = \{$weakly pseudoconvex points, such that $\frac{\partial\tilde{\delta}}{\partial z} \neq 0\}$, and assume $M_1 \cap \bar{M}_2 = \varnothing$ and M_1 has only finitely many connected components. Also, assume that for some τ, $0 < \tau < 1$ and $\tau < \frac{\pi}{2\mathcal{C}\ln\frac{B}{A}+\pi}$. Then there exists a smooth strictly positive function h on $\bar{\Omega}$ such that*

$$\sigma = -h(-\tilde{\delta})^\tau$$

is plurisubharmonic on Ω.

If Ω is a worm domain, this allows us to recover a result of Liu [24].

Corollary 1.1.1. *Let Ω_r be a worm domain defined as $\Omega_r = \{(z,w) : \rho_r(z,w) < 0\}$, and the set of weakly pseudoconvex in $b\Omega$ is given by the annulus $M_r = \{(z,w) : 1 < |z| < r, w = 0\}$ for some $r > 1$. Then the Diederich-Fornæss Index is equal to $\frac{\pi}{\ln r^2 + \pi}$.*

Remark 1.1.1. *We use the original definition of the worm domain. Some recent papers, including* [24]*, choose a parametrization in which the annulus in the boundary is given by*

$$M_\beta = \{(z,w) : -\beta + \frac{\pi}{2} \leq \ln |z|^2 \leq \beta - \frac{\pi}{2}, w = 0\}$$

for $\beta > \frac{\pi}{2}$. Under a rescaling with $r = exp(\beta - \frac{\pi}{2})$, these two definitions are equivalent. Therefore, the Diederich-Fornæss Index for the worm domain using the second parametrization is $\pi/2\beta$, which is the same value computed by Liu [24].

Another sufficient condition for Condition R was introduced by Boas and Straube in 1991. If a domain has a good vector field for some defining function ρ, then the Bergman projections $P_q,\ 0 \leq q \leq n$ are continuous on the Sobolev space W^s, $s \geq 0$. See Chapter 2 for the definition of Condition R.

On Hartogs domains, we have the following relationship between the Diederich-Fornæss Index and the Boas and Straube condition.

Theorem 1.1.3. *Let Ω be a smooth Hartogs domain in $\mathbb{C}^2$, and $M_1 = \{$weakly pseudoconvex points, such that $\frac{\partial \tilde{\delta}}{\partial z} = 0\}$, and $M_2 = \{$weakly pseudoconvex points, such that $\frac{\partial \tilde{\delta}}{\partial z} \neq 0\}$. Assume $M_1 \cap \bar{M}_2 = \varnothing$, and M_1 has only finitely many connected components. Then the Diederich-Fornæss Index equals one if and only if there exists a family of good vector fields on $b\Omega$.*

Our technique also sheds the light on the study of Stein neighborhood basis.

Definition 1.1.1. *Let Ω be a domain in $\mathbb{C}^n$. If for any open domain U containing $\bar{\Omega}$, there exists a pseudoconvex domain Ω_u such that $\bar{\Omega} \subset \Omega_u \subset U$, then we say $\bar{\Omega}$ has a Stein neighborhood basis.*

Zeytuncu [29] has shown that if the canonical solution to $\bar{\partial}u = f$ is regular on $C^{\infty}_{(0,1)}(\Omega)$, where $\Omega = \{(z_1, z_2) \in \mathbb{C}^2, z_1 \in D; |z_2| < e^{-\psi(z_1)}\}$, D is a unit disc in $\mathbb{C}$, and ψ is a smooth bounded below subharmonic function on D, then Ω has a Stein neighborhood basis. A worm domain does not have a Stein neighborhood basis when it has a sufficiently large winding number. The following results introduce conditions under which the existence of a Stein neighborhood basis is granted.

Theorem 1.1.4. *(Bedford and Fornæss[2]). Let Ω be a smooth Hartogs domain, and suppose that for some $B > A > 0$ and $\mathcal{C} > 0$ the set $M = \{(z, w) : \ A < |z|^2 < B, \ w \in \mathbb{C}\}$ is in $b\Omega$ and*

$$\left|\frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{w}}\right| < \frac{\pi}{2\sqrt{A}\left|log \frac{A}{B}\right|}$$

when $|z| = \sqrt{A}$. Then a Stein neighborhood basis for $\bar{\Omega}$ exists. If $\left|\frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{w}}\right| > \frac{\pi}{2\sqrt{A}}\left|\log \frac{A}{B}\right|$ when $|z| = \sqrt{A}$ then no Stein neighborhood basis exists.

Zeytuncu [30] has shown that the existence of a Stein neighborhood basis grants Condition R. Combining Theorem 1.1.4 with our earlier result, we obtain:

Corollary 1.1.2. *Let Ω be a smooth Hartogs domain in $\mathbb{C}^2$, and $M_1 = \{$weakly pseudoconvex points, such that $\frac{\partial \tilde{\delta}}{\partial z} = 0\}$, and $M_2 = \{$weakly pseudoconvex points, such that $\frac{\partial \tilde{\delta}}{\partial z} \neq 0\}$. Assume $M_1 \cap \bar{M}_2 = \varnothing$, and M_1 has finitely many connected components. If the Diederich-Fornæss Index for Ω is equal 1, then Ω admits a Stein Neighborhood basis.*

We can also generalize our results on the Diederich-Fornæss Index to a Stein neighborhood basis:

Theorem 1.1.5. *Let Ω be a smooth Hartogs domain, and suppose that for some $w \in \mathbb{C}, B > A > 0$ and $\mathcal{C} > 0$ the set $M = \{(z, w) : \ A < |z|^2 < B\}$ is in $b\Omega$ and $\left|\frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{z}}\right| > \frac{\mathcal{C}}{|z|}$ on M. Assume there exists a strictly positive function h such that*

$$\sigma = h\tilde{\delta}^{\tau}$$

is plurisubharmonic on Ω*, for some* $\tau > 1$*. Then* $\mathcal{C} < \frac{\pi}{2\ln\frac{B}{A}}$ *and* $\tau > \frac{\pi}{\pi - 2\mathcal{C}\ln\frac{B}{A}}$.

Using Theorem 1.1.5, we describe a sufficient and a necessary condition for the existence of the Stein neighborhood basis on the worm domain. This characterization is consistent with the recent work by Yum [28].

Corollary 1.1.3. *Let* Ω_r *be a worm domain defined as* $\Omega_r = \{(z,w) : \rho_r(z,w) < 0\}$ *with weakly pseudoconvex points given by the annulus* $M_r = \{(z,w) : 1 < |z| < r, w = 0\}$ *in* $b\Omega$ *for some* $r > 1$*. Then a Stein neighborhood basis exists if* $\left|\log\frac{1}{r^2}\right| < \pi$*, but if there exists* $\tau > 1$ *and a smooth function* $h > 0$ *such that* $\rho = h\tilde{\delta}^\tau$ *is plurisubharmonic outside* Ω*, then* $\tau \geq \frac{\pi}{\pi - \ln r^2}$ *and no Stein neighborhood basis exists if* $\left|\log\frac{1}{r^2}\right| > \pi$.

Chapter 2

Preliminaries

2.1 Functions and domains in $\mathbb{C}^n$

In this section, we will define tools that we use in our research.

Definition 2.1.1. *Let $z_j = x_j + iy_j$, for $1 \leq j \leq n$ and let f be a $\mathcal{C}^1$ function defined on $\mathbb{C}^n$. We use the following notation:*

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right),$$

$$\frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right).$$

Furthermore, $dz_j = dx_j + idy_j$ and $d\overline{z}_j = dx_j - idy_j$, and

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j$$

and

$$\bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

The following is the definition of a holomorphic function.

Definition 2.1.2. *Let $f(z)$ be a $\mathcal{C}^1$ function defined on an open subset D of $\mathbb{C}^n$.Then f is called a holomorphic function if $f(z)$ is holomorphic with respect to each of its variables z_j. That is*

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \qquad \forall j = 1, 2, ..., n.$$

The defining function is defined as follows.

Definition 2.1.3. *Let $D \subset \mathbb{R}^n, n \geq 2$ and $p \in bD$. We say D has a $\mathcal{C}^k$ boundary at p if there exists a $\mathcal{C}^k$ real valued function r defined in some open neighborhood U of p such that*

$D \cap U = \{x\colon r(x) < 0\}$ and $bD \cap U = \{x \in U : r(x) = 0, dr(x) \neq 0\}$. The function r is called a local defining function for D near p, and it is called a global defining function for D, or simply a defining function for D, if U is an open neighborhood of $\bar{D}$.

Using a defining function r and the operators ∂ and $\bar{\partial}$, we can define pseudoconvex domain as follows:

Definition 2.1.4. *Let $\Omega \subset \mathbb{C}^n, n \geq 2$, be a bounded C^2 domain. Then Ω is called pseudoconvex or Levi pseudoconvex at a point p on the boundary if the Levi form*

$$\langle Lr(z)t, t\rangle = \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{p}_j}(p) t_i \bar{t}_j \geq 0,$$

for all $t = (t_1, \cdots, t_n) \in \mathbb{C}^n$ with $\sum_{j=1}^{n} t_j(\frac{\partial r}{\partial z_j})(p) = 0$, where r is a C^2 defining function for Ω. In other words, the Levi form of ρ, restricted to the boundary, is positive semidefinite on vectors that are orthogonal to the complex normal.

An exhaustion function for a domain Ω is defined as follows.

Definition 2.1.5. *A function $\varphi : \Omega \to \mathbb{R}$, Ω be an open domain, is called an exhaustion function for Ω if the closure of $\{x \in \Omega | \varphi(x) < c\}$ is compact for all real numbers c.*

Definition 2.1.6. *Let ρ be a C^2 function defined on Ω. Then ρ is called plurisubharmonic if and only if for all $z \in \Omega$*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) t_j \bar{t}_k \geq 0$$

for all $t = (t_1, \cdots, t_n) \in \mathbb{C}^n$.

The following is the definition of a worm domain, a smooth pseudoconvex domain. It was proposed by Diederich and Fornæss [10].

Definition 2.1.7. *Suppose we have a smooth function $\lambda : \mathbb{R} \mapsto \mathbb{R}$ satisfying the following properties:*

1. $\lambda(x) = 0$ *if* $x \leq 0$

2. $\lambda(x) > 1$ *if* $x > 1$

3. $\lambda''(x) \geq 100\lambda'(x)$ *for all* x

4. $\lambda''(x) > 0$ *if* $x > 0$

5. $\lambda'(x) > 100$ *if* $\lambda(x) > \frac{1}{2}$.

Let the function $\rho_r : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$, *for any* $r > 1$, *be defined as follows:*

$$\rho_r(z, w) = \left|w + e^{(i \ln z\bar{z})}\right|^2 - 1 + \lambda(\frac{1}{|z|^2} - 1) + \lambda(|z|^2 - r^2).$$

Then $\Omega_r = \{(z, w) \in \mathbb{C} \times \mathbb{C} | \rho_r(z, w) < 0\}$ *is called a worm domain.*

The definition of a (p, q) differential form is given by.

Definition 2.1.8. *We can write* (p, q) *form* f *as:*

$$f = \sum_{|I|=p,|J|=q}{}' f_{I,J} dz^I \wedge d\overline{z}^J,$$

where $I = (i_1, \cdots, i_p)$ *and* $J = (j_1, \cdots, j_q)$ *are increasing multiindices and the superscript of the summation refers to the ascending order of the multiindices. Also,* $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ *and* $d\overline{z}^I = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$. *Note that* $\bar{\partial} f$ *is*

$$\bar{\partial} f = \sum_{I,J}{}' \bar{\partial} f_{I,J} \wedge dz^I \wedge d\overline{z}^J$$

which is $(p, q + 1)$ *form.*

Definition 2.1.9. *Let* $L^2(b\Omega)$ *denote the space of square integrable functions on* $b\Omega$, *and* $L^2_{(p,q)}(b\Omega)$ *denote the space of* (p, q) *forms whose coefficients are in* $L^2(b\Omega)$. *Let*

$$f = \sum_{I,J}{}' f_{I,J} dz^I \wedge d\overline{z}^J$$

and

$$g = \sum_{I,J}{}' g_{I,J} dz^I \wedge d\overline{z}^J$$

be (p,q) *forms in* $L^2_{(p,q)}(b\Omega)$ *where* $\sum'$ *is the summation over strictly increasing multiindices. Then we define*

$$\langle f, g\rangle = \sum_{I,J} \langle f_{I,J}, g_{I,J}\rangle \text{ and } \|f\| = \sum_{I,J}{}' \int_{b\Omega} |f_{I,J}|^2 dV$$

where the volume element $dV = i^n dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$.

For the sake of completeness, we will present the following definitions, which will be used in the proof of Theorem 1.1.3.

Definition 2.1.10. *A smooth r-form* ω *is closed if* $d\omega = 0$.

Definition 2.1.11. *A smooth r-form* ω *is exact if there exists an* $(r-1)$ *form* η *such that* $d\eta = \omega$.

Since $d \circ d = 0$, every exact form is closed, but the reverse is not valid in general.

Definition 2.1.12. *Let* $H^r(M)$ *be the r- th de Rham cohomology group of M (a vector space over* $\mathbb{R}$*). It is defined as follows*

$$H^r(M) = \{\text{closed r-forms on } M\}/\{\text{exact r-forms on } M\}$$

For a connected M, we have

$$H^0(M) = \{\text{the set of constant functions}\} \cong \mathbb{R}.$$

Moreover, for a simply connected M, we have $H^1(M) = 0$.

We will focus on studying the Diederich-Fornæss exponents for Hartogs domains in $\mathbb{C}^2$.

Definition 2.1.13 (Hartogs domain)**.** *We say that* $\Omega \in \mathbb{C}^2$ *is a Hartogs domain if it is invariant under rotation in one of the coordinates. That is,* Ω *contains* $(e^{i\theta}z, w)$ *whenever* $(z,w) \in \Omega$ *and* $\theta \in \mathbb{R}$.

A stronger notion of Stein neighborhood basis for smooth domains is introduced by Şahutoğlu [9].

Definition 2.1.14. *Let Ω be a smooth bounded pseudoconvex domain. We say that $\bar{\Omega}$ has a strong Stein neighborhood basis if there exists a defining function ρ and a parameter ε_0 such that the $\Omega_\varepsilon = \{z \in \mathbb{C}^n \colon \rho(z) < \varepsilon_0\}$ is a pseudoconvex domain for every $0 \leq \varepsilon \leq \varepsilon_0$.*

2.1.1 The Distance Function

The following result can be found in [20] and [27].

Theorem 2.1.1. *For any smooth bounded domain $\Omega \subseteq \mathbb{R}^n$, there exists a neighborhood U of $b\Omega$ such that for all $x \in U$*

$$\nabla^2\tilde{\delta}(x) = \nabla^2\tilde{\delta}(\pi(x)) \cdot \left(I + \tilde{\delta}(x)\nabla^2\tilde{\delta}(\pi(x))\right)^{-1}, \tag{2.1.1}$$

where I denotes the identity matrix and $\nabla^2\tilde{\delta}$ denotes the real Hessian of $\tilde{\delta}$.

Proof. From Federer [13], we have for all $x \in \Omega$, sufficiently close to bΩ, $\pi(x)$ is the unique nearest point to x on the boundary of Ω, where $\pi(x)$ is given by

$$\pi(x) = x - \tilde{\delta}(x)\nabla\tilde{\delta}(x).$$

We also have

$$\tilde{\delta}_{x_\ell}(x) = \tilde{\delta}_{x_\ell}(\pi(x)) \qquad \ell = 1, ..., n,$$

see for example Theorem 4.8 in [13]. Differentiating both sides with respect to x_j, $j = 1...n$, we get

$$\tilde{\delta}_{x_\ell x_j}(x) = \sum_{k=1}^{n} \tilde{\delta}_{x_\ell x_k}(\pi(x))\frac{\partial \pi_k}{\partial x_j}(x)$$

where

$$\pi_k(x) = x_k - \tilde{\delta}(x)\tilde{\delta}_{x_k}(x).$$

Here x_k means the k-th component of x. Then for all $k, j \in \{1, 2, ..., n\}$

$$\frac{\partial \pi_k(x)}{\partial x_j} = \frac{\partial x_k}{\partial x_j} - \tilde{\delta}_{x_j}(x)\tilde{\delta}_{x_k}(x) - \tilde{\delta}(x)\tilde{\delta}_{x_k x_j} \tag{2.1.2}$$

Therefore,

$$\tilde{\delta}_{x_\ell x_j}(x) = \tilde{\delta}_{x_\ell x_j}(\pi(x)) - \sum_{k=1}^{n} \tilde{\delta}_{x_\ell x_k}(\pi(x))\tilde{\delta}_{x_k}(x)\tilde{\delta}_{xj}(x) - \tilde{\delta}(x) \sum_{k=1}^{n} \tilde{\delta}_{x_\ell x_k}(\pi(x))\tilde{\delta}_{x_k x_j}(x) \tag{2.1.3}$$

Recall that

$$\left|\nabla \tilde{\delta}(x)\right|^2 = 1 \ \ \forall x \in U,$$

where U is some open neighborhood of the boundary of Ω. Differentiating both sides in a direction $u \in \mathbb{R}^n$, we get

$$\sum_{k=1}^{n} u_k \frac{\partial}{\partial x_k} \left(\left|\nabla \tilde{\delta}(x)\right|^2 \right) = 2 \sum_{k,l}^{n} \tilde{\delta}_{x_\ell x_k}(x)\tilde{\delta}_{x_\ell}(x) u_k = 0. \tag{2.1.4}$$

This implies the second term in (2.1.3) is zero. Hence we get the result.

□

With more calculation, we can reformulate the previous result in complex notation. let

$$z_j = x_j + i y_j.$$

Since

$$\tfrac{\partial}{\partial z_j} = \tfrac{1}{2}\left(\tfrac{\partial}{\partial x_j} - i\tfrac{\partial}{\partial y_j}\right), \quad \tfrac{\partial}{\partial \bar{z}_j} = \tfrac{1}{2}\left(\tfrac{\partial}{\partial x_j} + i\tfrac{\partial}{\partial y_j}\right),$$

we have

$$\frac{\partial^2 \tilde{\delta}}{\partial z_j \partial \bar{z}_k} = \frac{1}{4}\frac{\partial^2 \tilde{\delta}}{\partial x_j \partial x_k} + \frac{1}{4}\frac{\partial^2 \tilde{\delta}}{\partial y_j \partial y_k} + \frac{i}{4}\frac{\partial^2 \tilde{\delta}}{\partial x_j \partial y_k} - \frac{i}{4}\frac{\partial^2 \tilde{\delta}}{\partial y_j \partial x_k}. \tag{2.1.5}$$

Equation (2.1.1) can be approximated for small value of $\tilde{\delta}(z)$ by

$$\begin{aligned}\nabla^2\tilde{\delta}(z) &= \nabla^2\tilde{\delta}(\pi(z))\left(I - \tilde{\delta}(z)\nabla^2\tilde{\delta}(\pi(z))\right) + O((\tilde{\delta}(z))^2)\\ &= \nabla^2\tilde{\delta}(\pi(z)) - \tilde{\delta}(z)\left(\nabla^2\tilde{\delta}(\pi(z))\right)^2 + O((\tilde{\delta}(z))^2).\end{aligned}$$

By using formula (2.1.5), we get

$$\begin{aligned}\frac{\partial^2\tilde{\delta}(z)}{\partial z_j\partial\bar{z}_k} &= \frac{\partial^2\tilde{\delta}(\pi(z))}{\partial z_j\partial\bar{z}_k}\\ &\quad - \tilde{\delta}(z)\left(\sum_{\ell=1}^{n}\frac{\partial^2\tilde{\delta}(\pi(z))}{\partial z_j\partial x_\ell}\cdot\frac{\partial^2\tilde{\delta}(\pi(z))}{\partial x_\ell\partial\bar{z}_k} + \frac{\partial^2\tilde{\delta}(\pi(z))}{\partial z_j\partial y_\ell}\cdot\frac{\partial^2\tilde{\delta}(\pi(z))}{\partial y_\ell\partial\bar{z}_k}\right) + O((\tilde{\delta}(z))^2).\end{aligned}$$

After we plug in $\frac{\partial}{\partial x_j} = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial\bar{z}_j}$ and $\frac{\partial}{\partial y_j} = i\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial\bar{z}_j}\right)$, we get

$$\begin{aligned}\frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_k}(z) = \frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_k}(\pi(z)) - \tilde{\delta}(z)\sum_{\ell}^{n}\Bigg(&\frac{\partial^2\tilde{\delta}}{\partial z_j\partial z_\ell}\cdot\frac{\partial^2\tilde{\delta}}{\partial z_\ell\partial\bar{z}_k} + \frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_\ell}\cdot\frac{\partial^2\tilde{\delta}}{\partial z_\ell\partial\bar{z}_k}\\ &+ \frac{\partial^2\tilde{\delta}}{\partial z_j\partial z_\ell}\cdot\frac{\partial^2\tilde{\delta}}{\partial\bar{z}_\ell\partial\bar{z}_k} + \frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_\ell}\cdot\frac{\partial^2\tilde{\delta}}{\partial z_\ell\partial\bar{z}_k} - \frac{\partial^2\tilde{\delta}}{\partial z_j\partial z_\ell}\cdot\frac{\partial^2\tilde{\delta}}{\partial z_\ell\partial\bar{z}_k}\\ &+ \frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_\ell}\cdot\frac{\partial^2\tilde{\delta}}{\partial z_\ell\partial\bar{z}_k} + \frac{\partial^2\tilde{\delta}}{\partial z_j\partial z_\ell}\cdot\frac{\partial^2\tilde{\delta}}{\partial\bar{z}_\ell\partial\bar{z}_k} - \frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_\ell}\cdot\frac{\partial^2\tilde{\delta}}{\partial\bar{z}_\ell\partial\bar{z}_k}\Bigg)\pi(z) + O((\tilde{\delta}(z))^2).\end{aligned} \tag{2.1.6}$$

So,

$$\begin{aligned}\frac{\partial^2\tilde{\delta}(z)}{\partial z_j\partial\bar{z}_k} = \frac{\partial^2\tilde{\delta}(\pi(z)}{\partial z_j\partial\bar{z}_k} - \tilde{\delta}(z)\sum_{\ell=1}^{n}\Big(&2\frac{\partial^2\tilde{\delta}(\pi(z))}{\partial z_j\partial\bar{z}_\ell}\cdot\frac{\partial^2\tilde{\delta}(\pi(z))}{\partial z_\ell\partial\bar{z}_k}\\ &+2\frac{\partial^2\tilde{\delta}(\pi(z))}{\partial z_j\partial z_\ell}\cdot\frac{\partial^2\tilde{\delta}(\pi(z))}{\partial\bar{z}_\ell\partial\bar{z}_k}\Big) + O((\tilde{\delta}(z))^2).\end{aligned}$$

The following lemma is from [7].

Lemma 2.1.1. *For any two local defining functions r_1 and r_2 of a domain Ω of class C^k , $1 \leq k \leq \infty$, in a neighborhood U of $p \in b\Omega$, there exists a strictly positive function $h \in C^{k-1}$ on U satisfying the following:*

$$\begin{aligned} &(1)\ r_1 = hr_2 \quad \text{on } U, \\ &(2)\ dr_1(x) = h(x)dr_2(x) \quad \forall x \in U \cap b\Omega. \end{aligned} \tag{2.1.7}$$

2.1.2 $\bar{\partial}$-Neumann Regularity

Although this research is not directly connected with the $\bar{\partial}$-Neumann problem, some of our results will be relevant to this study. Therefore, we provide a brief introduction to the key terminology and results. The $\bar{\partial}$-Neumann problem represents an archetypal example of a boundary value problem with an elliptic operator but with non-coercive boundary conditions. We start by solving $\bar{\partial}u = f$, where $\bar{\partial}f = 0$. Since $\bar{\partial}^2 = 0$, $\bar{\partial}f = 0$ is a necessary condition for solvability. Define the inner product $(f, g) = \int_\Omega \langle f, g\rangle dV$. The adjoint operator $\bar{\partial}^*$ is defined by the relation $(\bar{\partial}f, g) = (f, \bar{\partial}^*g)$. Integration by parts will introduce a boundary term. In order for $\bar{\partial}^*$ to be a properly defined Hilbert space adjoint operator, we must restrict the domain of $\bar{\partial}^*$ to those forms where the complex normal component vanishes on the boundary.

The $\bar{\partial}$-Neumann problem is to find the inverse N_q of the complex Laplacian $\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$ on $(0, q)$ forms when $0 \leq q \leq n$ subject to two boundary conditions $u \in \mathrm{Dom}\bar{\partial}^*$ and $\bar{\partial}u \in \mathrm{Dom}\bar{\partial}^*$. Hörmander [21] has introduced the fundamental L^2 existence theorem for the $\bar{\partial}$-Neumann problem which states that for any $1 \leq q \leq n$ the solution $u \in L^2_{(0,q-1)}(\Omega)$ for $\bar{\partial}u = f$ exists under the condition that f is a $\bar{\partial}$-closed form in $L^2_{(0,q)}(\Omega)$, and Ω is pseudoconvex. Kohn [23] introduced the canonical solution $u = \bar{\partial}^*N_qf$. This solution is orthogonal to the kernel of $\bar{\partial}$.

Kohn [23] discussed the global regularity of the $\bar{\partial}$-Neumann problem in the L^2-Sobolev spaces $W^s(\Omega)$ for all nonnegative s on strictly pseudoconvex domains. He discovered an explicit relationship between the $\bar{\partial}$-Neumann operator and the Bergman projection: $P =$

$Id - \bar{\partial}^* N \bar{\partial}$.

The basic estimate $\|u\|^2 \leq C(\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2)$ for all $u \in$ Dom$(\bar{\partial}) \cap$ Dom $(\bar{\partial}^*)$ plays an important rule in proving regularity of the $\bar{\partial}$-Neumann problem in L^2. Our next question is can we generalize this estimate to other Sobolev spaces, i.e, $\|u\|_s^2 \leq \|\bar{\partial}u\|_s^2 + \|\bar{\partial}^* u\|_s^2$. This estimate does not hold for all $s \geq 0$, as shown by Barrett [1] on the worm domain. If we cover the boundary of Ω by special boundary charts such that tangential vector fields preserve the domain of $\bar{\partial}^*$, those vector fields have variable coefficients, so they do not commute with either $\bar{\partial}$ or $\bar{\partial}^*$ in general. The error terms that come from the commutator need to be handled. For T any tangential vector field, we have the a priori estimate

$$\|Tu\|^2 \leq \|\bar{\partial}Tu\|^2 + \|\bar{\partial}^* Tu\|^2 .$$

After some integration by parts, see Chen-Shaw [7, p.131] , we get

$$\|\bar{\partial}Tu\|^2 + \|\bar{\partial}^* Tu\|^2 \leq \|f\|_1^2 + (sc)\|Tu\|^2 + (sc)\|u\|_1^2 + \|[\bar{\partial}, T]u\|^2 + \|[\bar{\partial}^*, T]u\|^2 .$$

This motivates another condition that has been used to define sufficient condition for condition R: the existence of a family of good vector fields introduced by Boas and Straube [6]. Suppose there exists a positive constant $C > 0$ so that for every $\varepsilon > 0$, there exists a vector field X_ε of type $(1,0)$ such that the coefficients of X_ε are smooth in a neighborhood U_ε of the set of weakly pseudoconvex boundary points of Ω. Moreover, the following conditions are satisfied

1- $|arg X_\varepsilon \rho| < \varepsilon$ on U_ε, $C^{-1} < |X_\varepsilon \rho| < C$, and

2- $\partial\rho[X_\varepsilon, \frac{\partial}{\partial \bar{z}_j}] < \varepsilon$, for all $1 \leq j \leq n,$ on U_ε,

for some defining function ρ. Then the domain Ω satisfies condition R. The first condition grants that this family of vector field is transverse to $b\Omega$, and the second condition ensures the vector field commutes approximately with $\bar{\partial}$. We will study this condition further in Chapter Four.

Chapter 3

Proof of Main Result

3.1 Upper Bound for Diederich-Fornæss Index

In this section, we will calculate an upper bound for the Diederich-Fornæss Index on a Hartogs domain. In our calculations, we will use the signed distance function as a defining function for the domain.

Proof of Theorem 1.1.1. Let $t = |z|^2$. Since Ω is Hartogs, we may assume

$$\tilde{\delta}(z, w) = \rho(|z|^2, w),$$

for some smooth function ρ. Furthermore, we pick $p \in \Omega$ such that $\pi(p) \in M$. For such p, the signed distance function $\tilde{\delta}$ satisfies

$$\frac{\partial \tilde{\delta}}{\partial z}(\pi(p)) = 0, \tag{3.1.1}$$

and

$$\frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{z}}(\pi(p)) = 0. \tag{3.1.2}$$

From (2.1.2),

$$\frac{\partial \tilde{\delta}}{\partial z}(p) = 0$$

as well. We write

$$\frac{\partial \tilde{\delta}}{\partial w} = \frac{1}{2}(\frac{\partial \tilde{\delta}}{\partial x} - i\frac{\partial \tilde{\delta}}{\partial y}),$$

so at p

$$|\rho_w|^2 = \left|\frac{\partial \tilde{\delta}}{\partial w}\right|^2 = \frac{1}{4}\Big\{(\frac{\partial \tilde{\delta}}{\partial x})^2 + (\frac{\partial \tilde{\delta}}{\partial y})^2\Big\}. \tag{3.1.3}$$

But because $\left|\nabla\tilde{\delta}\right| = 1$, and $\frac{\partial\tilde{\delta}}{\partial z}(p) = 0$, we get

$$\left|\rho_w(p)\right|^2 = \left|\frac{\partial\tilde{\delta}}{\partial w}(p)\right|^2 = \frac{1}{4}. \tag{3.1.4}$$

Since this is true for all p such that $\pi(p) \in M$ and M has nonempty interior, we can differentiate in p. Take the derivative of both sides of (3.1.4) with respect to t, we have

$$\rho_{t\bar{w}}\rho_w + \rho_{tw}\rho_{\bar{w}} = 0. \tag{3.1.5}$$

Since

$$\frac{\partial\tilde{\delta}}{\partial z} = \rho_t(\left|z\right|^2, w)\overline{z},$$

we have

$$\rho_t(\pi(p)) = 0.$$

Then

$$\frac{\partial^2\tilde{\delta}}{\partial z\partial\overline{z}} = \rho_{tt}(\left|z\right|^2, w)\left|z\right|^2 + \rho_t(\left|z\right|^2, w) \Longrightarrow \rho_{tt}(\pi(p)) = 0.$$

We also compute

$$\frac{\partial^2\tilde{\delta}}{\partial w\partial z} = \rho_{tw}(\left|z\right|^2, w)\overline{z}, \tag{3.1.6}$$

and

$$\frac{\partial^2\tilde{\delta}}{\partial\overline{w}\partial z} = \rho_{t\overline{w}}(\left|z\right|^2, w)\overline{z}. \tag{3.1.7}$$

Also, on the boundary we have

$$\frac{\partial^2\tilde{\delta}}{\partial z\partial z} = \rho_{tt}(\left|z\right|^2, w)\overline{z}^2 \Longrightarrow \frac{\partial^2\tilde{\delta}}{\partial z\partial z}(\pi(p)) = 0.$$

Using (2.1.6), we compute

$$\frac{\partial^2\tilde{\delta}}{\partial z\partial\overline{z}}(p) = -2\tilde{\delta}(p)\Big(\frac{\partial^2\tilde{\delta}}{\partial z\partial\overline{w}}(\pi(p))\cdot\frac{\partial^2\tilde{\delta}}{\partial w\partial\overline{z}}(\pi(p))$$
$$+\frac{\partial^2\tilde{\delta}}{\partial z\partial\overline{z}}(\pi(p))\cdot\frac{\partial^2\tilde{\delta}}{\partial z\partial\overline{z}}(\pi(p))+\frac{\partial^2\tilde{\delta}}{\partial z\partial w}(\pi(p))\cdot\frac{\partial^2\tilde{\delta}}{\partial\overline{w}\partial\overline{z}}(\pi(p))\Big)+O((-\tilde{\delta}(p))^2).$$

So, using (3.1.7) and (3.1.6), we get

$$\frac{\partial^2\tilde{\delta}}{\partial z\partial\overline{z}}(p) = 4(-\tilde{\delta})(p)\left|\rho_{tw}(\pi(p))\right|^2|z|^2+O((-\tilde{\delta}(p))^2). \tag{3.1.8}$$

By our hypothesis, there exists some positive C^2 function h, such that $h(-\tilde{\delta})^\tau$ is a plurisubharmonic function. Since Ω is a Hartogs domain, using the symmetry property of Ω , we may define $\hat{h}$ from h as

$$\hat{h}(z,w)=\int_0^{2\pi}h(e^{i\theta}z,w)d\theta.$$

We use $\hat{h}$ to define another rotationally symmetric plurisubharmonic function in z, which is $\sigma=\hat{h}(-\tilde{\delta})^\tau$ on Ω. Now, we denote the complex Hessian of σ at the point $p=(z,w)$ by

$$H_\sigma\left(|z|^2,w,\xi,\eta\right)=\frac{\partial^2\sigma}{\partial z\partial\overline{z}}|\xi|^2+2Re\left(\frac{\partial^2\sigma}{\partial\bar{z}\partial w}\eta\overline{\xi}\right)+\frac{\partial^2\sigma}{\partial w\partial\overline{w}}|\eta|^2.$$

We compute

$$\frac{\partial^2\sigma}{\partial z\partial\bar{z}}(p)=-\hat{h}\tau\Big((-\tilde{\delta})^{\tau-1}\frac{\partial^2(-\tilde{\delta})}{\partial z\partial\overline{z}}+(\tau-1)(-\tilde{\delta})^{\tau-2}\frac{\partial\tilde{\delta}}{\partial\bar{z}}\frac{\partial\tilde{\delta}}{\partial z}\Big)$$
$$+\tau(-\tilde{\delta})^{\tau-1}\frac{\partial\hat{h}}{\partial\bar{z}}\frac{\partial\tilde{\delta}}{\partial z}-\frac{\partial^2\hat{h}}{\partial z\partial\overline{z}}(-\tilde{\delta})^\tau+\tau(-\tilde{\delta})^{\tau-1}\frac{\partial\hat{h}}{\partial z}\frac{\partial\tilde{\delta}}{\partial\bar{z}}.$$

Substituting (3.1.8), we get

$$\frac{\partial^2\sigma}{\partial z\partial\bar{z}}(p)=(-\tilde{\delta})^\tau\left(4\hat{h}\tau\left|\rho_{tw}\left(\pi\left(p\right)\right)\right|^2|z|^2-\frac{\partial^2\hat{h}}{\partial z\partial\overline{z}}\right)+O((-\tilde{\delta}(p))^{\tau-1}).$$

Also,

$$\begin{aligned}\frac{\partial^2\sigma}{\partial z\partial\bar{w}}(p) &= -\hat{h}\tau\Big((-\tilde{\delta})^{\tau-1}\frac{\partial^2(-\tilde{\delta})}{\partial z\partial\bar{w}} + (\tau-1)(-\tilde{\delta})^{(\tau-2)}\frac{\partial(-\tilde{\delta})}{\partial\bar{w}}\cancelto{0}{\frac{\partial(-\tilde{\delta})}{\partial z}}\Big)\\ &\quad + \tau(-\tilde{\delta}))^{(\tau-1)}\frac{\partial(-\hat{h})}{\partial\bar{w}}\cancelto{0}{\frac{\partial(-\tilde{\delta})}{\partial z}} - (-\tilde{\delta})^{\tau}\frac{\partial^2\hat{h}}{\partial z\partial\bar{w}} + \tau(-\tilde{\delta})^{(\tau-1)}\frac{\partial\hat{h}}{\partial z}\frac{\partial\tilde{\delta}}{\partial\bar{w}}.\end{aligned}$$

So,

$$2Re(\frac{\partial^2\sigma}{\partial z\partial\bar{w}}(p)\xi\bar{\eta}) = 2Re\Big(\big(-\hat{h}\tau(-\tilde{\delta})^{\tau-1}\big(-\rho_{t\bar{w}}(p)\bar{z} - (-\tilde{\delta})^{\tau}\frac{\partial^2 h}{\partial z\partial\overline{w}} + \tau(-\tilde{\delta})^{(\tau-1)}\frac{\partial h}{\partial z}\frac{\partial\tilde{\delta}}{\partial\overline{w}}\big)\xi\overline{\eta}\Big)$$

Using $|\rho_{t\bar{w}}(\pi(p)) - \rho_{t\bar{w}}(p)| < O(-\tilde{\delta}(p))$, we get

$$2Re(\frac{\partial^2\sigma}{\partial z\partial\bar{w}}(p)\xi\bar{\eta}) = 2Re\Big(\big(\hat{h}\tau(-\tilde{\delta})^{(\tau-1)}\rho_{t\bar{w}}\bar{z}(\pi(p)) + \tau(-\tilde{\delta})^{(\tau-1)}\frac{\partial h}{\partial z}\frac{\partial\tilde{\delta}}{\partial\bar{w}} + O((-\tilde{\delta})^{\tau})\big)\xi\bar{\eta}\Big).$$

Furthermore, the third term is given by

$$\begin{aligned}\frac{\partial^2\sigma}{\partial w\partial\bar{w}}(p) &= -\hat{h}\tau\Big((-\tilde{\delta})^{\tau-1}\frac{\partial^2(-\tilde{\delta})}{\partial w\partial\bar{w}} + (\tau-1)(-\tilde{\delta})^{\tau-2}\frac{\partial\tilde{\delta}}{\partial\bar{w}}\frac{\partial\tilde{\delta}}{\partial w}\Big)\\ &\quad + \tau(-\tilde{\delta})^{\tau-1}\frac{\partial\hat{h}}{\partial\bar{w}}\frac{\partial\tilde{\delta}}{\partial w} - (-\tilde{\delta})^{\tau}\frac{\partial^2\hat{h}}{\partial w\partial\bar{w}} + \tau(-\tilde{\delta})^{\tau-1}\frac{\partial\hat{h}}{\partial w}\frac{\partial\tilde{\delta}}{\partial\bar{w}},\end{aligned}$$

which can be further simplified to

$$\frac{\partial^2\sigma}{\partial w\partial\overline{w}}(p) = -\tau(\tau-1)(-\tilde{\delta})^{\tau-2}\hat{h}\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial\tilde{\delta}}{\partial\bar{w}} + O((-\tilde{\delta})^{\tau-1}).$$

Combining these, we have

$$\begin{aligned}H_{\sigma}(|z|^2, w, \xi, \eta) &= \left((-\tilde{\delta})^{\tau}\left(4\hat{h}\tau\left|\rho_{tw}\left(\pi\left(p\right)\right)\right|^2|z|^2 - \frac{\partial^2\hat{h}}{\partial z\partial\bar{z}}\right) + O\left((-\tilde{\delta})^{(\tau+1)}\right)\right)|\xi|^2\\ &\quad + 2Re\Big(\big(\hat{h}\tau(-\tilde{\delta})^{\tau-1}\rho_{t\bar{w}}(\pi(p))\bar{z} + \tau(-\tilde{\delta})^{\tau-1}\frac{\partial\hat{h}}{\partial z}\frac{\partial\tilde{\delta}}{\partial\bar{w}} + O((-\tilde{\delta})^{\tau})\big)\xi\overline{\eta}\Big)\\ &\quad + \left((-\hat{h})\tau(\tau-1)(-\tilde{\delta})^{\tau-2}\frac{\partial\tilde{\delta}}{\partial w}\cdot\frac{\partial\tilde{\delta}}{\partial\bar{w}} + O(-\tilde{\delta})^{\tau-1}\right)|\eta|^2.\end{aligned}$$

Due to the different order of vanishing in each term, we substitute $\eta = (-\tilde{\delta})\hat{\eta}$:

$$
\begin{aligned}
H_\sigma(|z|^2, w, \xi, \hat{\eta}) = & \Big(4\hat{h}\tau \left|\rho_{tw}(\pi(p))\right|^2 t - \frac{\partial^2 \hat{h}}{\partial z \partial \overline{z}}\Big)(-\tilde{\delta})^\tau \left|\xi\right|^2 \\
& + 2Re\Big\{\tau \left(\hat{h}\rho_{t\overline{w}}(\pi(p))\overline{z} + \frac{\partial \hat{h}}{\partial z}\frac{\partial \tilde{\delta}}{\partial \overline{w}}\right)(-\tilde{\delta})^\tau \xi \hat{\eta}\Big\} - \frac{1}{4}(-\tilde{\delta})^\tau \hat{h}\tau(\tau - 1)\left|\hat{\eta}\right|^2 \\
& + O((-\tilde{\delta})^{\tau+1}).
\end{aligned}
$$

Note that we have used (3.1.4) here. Next, dividing by $(-\tilde{\delta})^\tau$, then letting $(-\tilde{\delta}) \to 0$, we have

$$
\lim_{\tilde{\delta}\to 0} \frac{H_\sigma(|z|^2, w, \xi, \hat{\eta})}{(-\tilde{\delta})^\tau} \geq 0 \Longrightarrow
$$

$$
\Big(4\hat{h}\tau \left|\rho_{tw}(\pi(p))\right|^2 t - \frac{\partial^2 \hat{h}}{\partial z \partial \overline{z}}\Big) \left|\xi\right|^2 + 2Re\Big\{\tau \left(\hat{h}\rho_{t\overline{w}}(\pi(p))\overline{z} + \frac{\partial \hat{h}}{\partial z}\frac{\partial \tilde{\delta}}{\partial \bar{w}}\right)\xi\hat{\eta}\Big\} - \frac{1}{4}\hat{h}\tau(\tau - 1)\left|\hat{\eta}\right|^2 \geq 0.
$$

Let

$$
a = \hat{h}\tau\left(4\left|\rho_{tw}(\pi(p))\right|^2 t\right) - \frac{\partial^2 \hat{h}}{\partial z \partial \overline{z}},
$$

$$
b = -\frac{1}{4}\hat{h}\tau(\tau - 1)
$$

and

$$
c = \tau\left(\hat{h}\rho_{t\bar{w}}(\pi(p))\overline{z} + \frac{\partial \hat{h}}{\partial z}\frac{\partial \tilde{\delta}}{\partial \bar{w}}\right).
$$

Note that

$$
\lim_{\tilde{\delta}\to 0} \frac{H_\sigma(|z|^2, w, \xi, \hat{\eta})}{(-\tilde{\delta})^\tau} \geq 0 \Longleftrightarrow \begin{pmatrix} \xi & \eta \end{pmatrix} \begin{pmatrix} a & c \\ \bar{c} & b \end{pmatrix} \begin{pmatrix} \bar{\xi} \\ \bar{\eta} \end{pmatrix} \geq 0, \quad \forall \begin{pmatrix} \xi & \eta \end{pmatrix}
$$

Therefore, the matrix

$$
\begin{pmatrix} a & c \\ \bar{c} & b \end{pmatrix}
$$

must be positive semidefinite. Since $\hat{h} > 0$ and $0 < \tau < 1, b > 0$ on M.

So, we are left with showing the determinant $ab - |c|^2 \geq 0$.

We compute

$$\frac{\partial \hat{h}}{\partial z}(|z|^2, w) = \frac{\partial \hat{h}}{\partial t}(|z|^2, w) \cdot \bar{z},$$

and

$$\frac{\partial^2 \hat{h}}{\partial z \partial \bar{z}}(|z|^2, w) = \frac{\partial \hat{h}}{\partial t}(|z|^2, w) + \frac{\partial^2 \hat{h}}{\partial t^2}(|z|^2, w)\, |z|^2 .$$

We can simplify $|c|^2$ by

$$\begin{aligned} |c|^2 = c\bar{c} &= \tau \bar{z} \left(\hat{h}\rho_{t\overline{w}} + \frac{\partial \hat{h}}{\partial t}\rho_{\overline{w}} \right) \cdot \tau z \left(\hat{h}\rho_{tw} + \frac{\partial \hat{h}}{\partial t}\rho_{w} \right) \\ &= \tau^2 t \left(\left|\hat{h}\right|^2 |\rho_{t\overline{w}}|^2 + \frac{\partial \hat{h}}{\partial t}\left(\rho_{t\overline{w}}\rho_w + \rho_{tw}\rho_w\right) + \left(\frac{\partial \hat{h}}{\partial t}\right)^2 |\rho_w|^2 \right). \end{aligned}$$

Using (3.1.4) and (3.1.5), we get

$$|c|^2 = \tau^2 t \left(\left|\hat{h}\right|^2 |\rho_{t\overline{w}}|^2 + \frac{1}{4}\left(\frac{\partial \hat{h}}{\partial t}\right)^2 \right).$$

Here we will calculate the value for ab:

$$ab = \frac{-\hat{h}\tau(\tau-1)}{4} \left(4\hat{h}\tau\, |\rho_{tw}(\pi(p))|^2\, |z|^2 - \frac{\partial \hat{h}}{\partial t} - \frac{\partial^2 \hat{h}}{\partial t^2} t \right).$$

Compute

$$ab = -\hat{h}^2\tau^2(\tau-1)\, |\rho_{tw}(\pi(p))|^2\, t + \tau(\tau-1) \left(\frac{\partial \hat{h}}{\partial t}\frac{\hat{h}}{4} + \frac{\hat{h}}{4}\frac{\partial^2 \hat{h}}{\partial t^2} t \right).$$

Therefore,

$$ab - |c|^2 = -\hat{h}^2\tau^3\, |\rho_{tw}(\pi(p))|^2\, t + \tau(\tau-1) \left(\frac{\partial \hat{h}}{\partial t}\frac{\hat{h}}{4} + \frac{\hat{h}}{4}\frac{\partial^2 \hat{h}}{\partial t^2} t \right) - \tau^2 t \frac{1}{4}\left(\frac{\partial \hat{h}}{\partial t}\right)^2 .$$

Let $g = \hat{h}^{1/(1-\tau)}$. Since $\hat{h}$ is positive, the value of g is real and positive. To linearize the previous equation, we plug in $\hat{h} = g^{(1-\tau)}$.

So,

$$ab - |c|^2 = -g^{(2-2\tau)}\tau^2(\tau-1)\left|\rho_{tw}\right|^2 t + \frac{1}{4}\tau(\tau-1)g^{(1-\tau)} \cdot \left((1-\tau)g^{-\tau}\frac{\partial g}{\partial t} + (1-\tau)t\big(g^{-\tau}\frac{\partial^2 g}{\partial t^2} - \tau g^{-\tau-1}\left(\frac{\partial g}{\partial t}\right)^2\right) - \tau^2 t\left(g^{2(1-\tau)}\left|\rho_{t\overline{w}}\right|^2 + \frac{1}{4}\left((1-\tau)g^{-\tau}\right)^2\left(\frac{\partial g}{\partial t}\right)^2\right).$$

Dividing by $g^{1-2\tau}$ we get

$$\frac{ab-|c|^2}{g^{1-2\tau}} = -\tau^2(\tau-1)\left|\rho_{tw}\right|^2 tg - \frac{1}{4}\tau(1-\tau)^2\left(\frac{\partial g}{\partial t} + t\left(\frac{\partial^2 g}{\partial t^2} - \tau g^{-1}\left(\frac{\partial g}{\partial t}\right)^2\right)\right) - \tau^2 t\left(g\left|\rho_{t\overline{w}}\right|^2 + \frac{1}{4}\frac{1}{g}(1-\tau)^2\left(\frac{\partial g}{\partial t}\right)^2\right).$$

We compute

$$\frac{ab-|c|^2}{g^{1-2\tau}} = -\tau^3 tg\left|\rho_{tw}\right|^2 + \tau^2 tg\left|\rho_{tw}\right|^2 - \tau^2 tg\left|\rho_{t\overline{w}}\right|^2 - \frac{1}{4}\tau(1-\tau)^2\left(\frac{\partial g}{\partial t} + t\frac{\partial^2 g}{\partial t^2}\right),$$

so, $ab - |c|^2 \geq 0$ gives us

$$-\frac{1}{4}\tau(1-\tau)^2\left(\frac{\partial g}{\partial t} + t\frac{\partial^2 g}{\partial t^2}\right) - \tau^3 tg\left|\rho_{tw}\right|^2 \geq 0. \tag{3.1.9}$$

Our hypothesis on $\left|\frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{z}}\right|$ implies

$$\left|\rho_{tw}\right| > \frac{\mathcal{C}}{t}.$$

Then, (3.1.9) implies

$$-\frac{1}{4}\tau(1-\tau)^2\left(\frac{\partial g}{\partial t} + t\frac{\partial^2 g}{\partial t^2}\right) - \frac{\mathcal{C}^2}{t^2}\tau^3 tg > 0 \tag{3.1.10}$$

when $A \leq t \leq B$.

We will show that this implies a contradiction unless $\tau < \frac{\pi}{2\mathcal{C}Ln\frac{B}{A}+\pi}$. Let $\tilde{g}(s,w) = g(e^s,w)$. We will plug in $t = e^s$, so that $\ln A < s < \ln B$ on M.

Then

$$\frac{d\tilde{g}}{ds} = e^s \frac{dg}{dt}$$

and

$$\frac{d^2\tilde{g}}{ds^2} = e^s \frac{dg}{dt} + e^{2s}\frac{d^2 g}{dt^2}.$$

Substituting in (3.1.10) gives us

$$\frac{1}{4}\tau(1-\tau)^2 \left(\frac{d^2\tilde{g}}{ds^2}\right) + \tau^3\mathcal{C}^2\tilde{g} < 0. \tag{3.1.11}$$

Let us assume we have a strictly positive function $\tilde{g}$ on some interval I satisfying (3.1.11). We make the substitution $u = \sqrt{\frac{\tau^3\mathcal{C}^2}{\frac{1}{4}\tau(1-\tau)^2}}s = \frac{2\tau\mathcal{C}}{1-\tau}s$.

For u in the range

$$\frac{2\tau\mathcal{C}}{(1-\tau)}\ln A < u < \frac{2\tau\mathcal{C}}{(1-\tau)}\ln B,$$

we get

$$\frac{d^2\tilde{g}}{du^2} + \tilde{g} < 0. \tag{3.1.12}$$

Assume $\tau \geq \frac{\pi}{2\mathcal{C}\ln\frac{B}{A}+\pi}$. This implies

$$\frac{2\tau\mathcal{C}}{(1-\tau)}\ln\frac{B}{A} \geq \pi.$$

So (3.1.12) holds on an interval of length π, which implies $\frac{d^2\tilde{g}}{du^2} < 0$. So $\tilde{g}$ is strictly concave. Therefore there should be an interval of length at least $\frac{\pi}{2}$ on which $\tilde{g}$ is strictly increasing or strictly decreasing. If it is strictly increasing we can flip it by a reflection to make it strictly decreasing. So, after a translation $\frac{d\tilde{g}}{du} < 0$ on $[0, \frac{\pi}{2}]$. In [10], Diederich and Fornæss have shown in the proof of Theorem 6 that this differential inequality (3.1.12) has no positive solution $\tilde{g}$, which is contradicts the assumption. Hence, the conclusion of the theorem follows.

□

3.2 Lower Bound for Diederich-Fornæss Index

In this section, we will prove technical lemmas that will help us to deal with each connected component of the set of weakly pseudoconvex points separately. Then we patch the end results together to prove Theorem 1.1.2.

Lemma 3.2.1. *Let Ω be a Hartogs domain in $\mathbb{C}^2$, and for some $w \in \mathbb{C}, M = \{(z,w) : |z|^2 \leq B\} \subset b\Omega$ be a disk in $b\Omega$. Then there exists a real valued function h such that*

$$\frac{\partial h}{\partial z} = -\frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{w}} / \frac{\partial \tilde{\delta}}{\partial \bar{w}} \qquad \text{on } M \tag{3.2.1}$$

and $\rho = \tilde{\delta} e^h$ is plurisubharmonic on M.

Proof. Let

$$L = \frac{\partial \tilde{\delta}}{\partial w}\frac{\partial}{\partial z} - \frac{\partial \tilde{\delta}}{\partial z}\frac{\partial}{\partial w},$$

and

$$N = \frac{\partial \tilde{\delta}}{\partial \bar{z}}\frac{\partial}{\partial z} + \frac{\partial \tilde{\delta}}{\partial \bar{w}}\frac{\partial}{\partial w}.$$

On M we can assume $\frac{\partial \tilde{\delta}}{\partial \bar{w}} \neq 0$, because $\frac{\partial \tilde{\delta}}{\partial z} = 0$.

Let

$$\alpha = -\frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{w}} / \frac{\partial \tilde{\delta}}{\partial \bar{w}} dz - \frac{\partial^2 \tilde{\delta}}{\partial \bar{z} \partial w} / \frac{\partial \tilde{\delta}}{\partial w} d\bar{z} \tag{3.2.2}$$

be a one-form. This form is a scalar multiple of D'Angelo's one form (see [6] for its detailed analysis).

Next, we will show that α is closed on M.

$$d\alpha = \frac{\partial}{\partial \bar{z}}\{-\frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{w}} / \frac{\partial \tilde{\delta}}{\partial \bar{w}}\} d\bar{z} \wedge dz + \frac{\partial}{\partial z}\{-\frac{\partial^2 \tilde{\delta}}{\partial \bar{z} \partial w} / \frac{\partial \tilde{\delta}}{\partial w}\} dz \wedge d\bar{z}.$$

Compute

$$d\alpha = \Big\{\{-\frac{\partial\tilde{\delta}}{\partial\bar{w}}\frac{\partial^3\tilde{\delta}}{\partial\bar{z}\partial z\partial\bar{w}} + \frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial\bar{w}}\}/(\frac{\partial\tilde{\delta}}{\partial\bar{w}})^2\Big\}d\bar{z}\wedge dz$$
$$+ \Big\{\{-\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^3\tilde{\delta}}{\partial z\partial\bar{z}\partial w} + \frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\frac{\partial^2\tilde{\delta}}{\partial z\partial w}\}/(\frac{\partial\tilde{\delta}}{\partial w})^2\Big\}dz\wedge d\bar{z}. \qquad (3.2.3)$$

On M

$$\left|\frac{\partial\tilde{\delta}}{\partial w}\right|^2 = \frac{1}{4} \implies \frac{\partial^2}{\partial z\partial\bar{z}}\left|\frac{\partial\tilde{\delta}}{\partial w}\right|^2 = 0.$$

So,

$$\frac{\partial^2}{\partial z\partial\bar{z}}(\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial\tilde{\delta}}{\partial\bar{w}}) = \frac{\partial^3\tilde{\delta}}{\partial z\partial\bar{z}\partial w}\frac{\partial\tilde{\delta}}{\partial\bar{w}} + \frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}} + \frac{\partial^2\tilde{\delta}}{\partial z\partial w}\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial\bar{w}} + \frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^3\tilde{\delta}}{\partial z\partial\bar{z}\partial\bar{w}} = 0. \qquad (3.2.4)$$

Since $H_{\tilde{\delta}}(L,L) = 0$ on M, and $H_{\tilde{\delta}}(L,L) \geq 0$ on $b\Omega$, tangential derivatives of $H_{\tilde{\delta}}(L,L)$ must vanish on M. Consider the tangential derivative $\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial}{\partial\bar{w}} - \frac{\partial\tilde{\delta}}{\partial\bar{w}}\frac{\partial}{\partial w}$, so we have

$$\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial H_{\tilde{\delta}}(L,L)}{\partial\bar{w}} - \frac{\partial\tilde{\delta}}{\partial\bar{w}}\frac{\partial H_{\tilde{\delta}}(L,L)}{\partial w} = 0 \quad \text{ on } M. \qquad (3.2.5)$$

We compute

$$\frac{\partial H_{\tilde{\delta}}(L,L)}{\partial w} = \frac{1}{4}\frac{\partial^3\tilde{\delta}}{\partial w\partial z\partial\bar{z}} - \frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\frac{\partial^2\tilde{\delta}}{\partial z\partial w}\frac{\partial\tilde{\delta}}{\partial\bar{w}} - \frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}.$$

So, adding four times (3.2.5) to (3.2.4), we have

$$2\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^3\tilde{\delta}}{\partial\bar{w}\partial z\partial\bar{z}} - 4\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}(\frac{\partial\tilde{\delta}}{\partial w})^2\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial\bar{w}} - \frac{\partial\tilde{\delta}}{\partial\bar{w}}\frac{\partial^3\tilde{\delta}}{\partial w\partial z\partial\bar{z}} + 4\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\frac{\partial^2\tilde{\delta}}{\partial z\partial w}(\frac{\partial\tilde{\delta}}{\partial\bar{w}})^2 + \frac{\partial^3\tilde{\delta}}{\partial z\partial\bar{z}\partial w}\frac{\partial\tilde{\delta}}{\partial\bar{w}}$$
$$+ \left|\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\right|^2 + \left|\frac{\partial^2\tilde{\delta}}{\partial z\partial w}\right|^2 + \frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^3\tilde{\delta}}{\partial\bar{z}\partial\bar{w}\partial z} = 0.$$

Hence,

$$\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^3\tilde{\delta}}{\partial\bar{z}\partial\bar{w}\partial z} = 2\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}(\frac{\partial\tilde{\delta}}{\partial w})^2\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial\bar{w}} - 2\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}(\frac{\partial\tilde{\delta}}{\partial\bar{w}})^2\frac{\partial^2\tilde{\delta}}{\partial z\partial w} - \frac{1}{2}\left|\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\right|^2 - \frac{1}{2}\left|\frac{\partial^2\tilde{\delta}}{\partial z\partial w}\right|^2.$$

On M, we have

$$\frac{\partial}{\partial z}\left|\frac{\partial\tilde{\delta}}{\partial w}\right|^2 = 0 \Longrightarrow \frac{\partial^2\tilde{\delta}}{\partial z\partial w}\frac{\partial\tilde{\delta}}{\partial\bar{w}} + \frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^2\tilde{\delta}}{\partial\bar{w}\partial z} = 0.$$

By using this observation, and the property of the distance function that $\frac{1}{(\frac{\partial\tilde{\delta}}{\partial\bar{w}})^2} = 16(\frac{\partial\tilde{\delta}}{\partial w})^2$ we get

$$\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^3\tilde{\delta}}{\partial\bar{z}\partial\bar{w}\partial z} = -\frac{1}{2}\left|\frac{\partial^2\tilde{\delta}}{\partial\bar{w}\partial z}\right|^2 + \frac{1}{2}\left|\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\right|^2 - \frac{1}{2}\left|\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\right|^2 - \frac{1}{2}\left|\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\right|^2 = -\left|\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\right|^2. \quad (3.2.6)$$

So, (3.2.3) can be written as

$$\begin{aligned} d\alpha = &\Big\{\{-\frac{\partial\tilde{\delta}}{\partial\bar{w}}\frac{\partial^3\tilde{\delta}}{\partial\bar{z}\partial z\partial\bar{w}} \\ &+ \frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial\bar{w}}\}/(\frac{\partial\tilde{\delta}}{\partial\bar{w}})^2\Big\}d\bar{z}\wedge dz + \Big\{\{-\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^3\tilde{\delta}}{\partial z\partial\bar{z}\partial w} + \frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\frac{\partial^2\tilde{\delta}}{\partial z\partial w}\}/(\frac{\partial\tilde{\delta}}{\partial w})^2\Big\}dz\wedge d\bar{z}. \end{aligned}$$

Compute

$$\begin{aligned} d\alpha &= -4\left|\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\right|^2 d\bar{z}\wedge dz - 4\left|\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\right|^2 dz\wedge d\bar{z} \\ &+ \frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial\bar{w}}(\frac{\partial\tilde{\delta}}{\partial w})^2 d\bar{z}\wedge dz + \frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\frac{\partial^2\tilde{\delta}}{\partial z\partial w}(\frac{\partial\tilde{\delta}}{\partial\bar{w}})^2 dz\wedge d\bar{z} \\ &= 0. \end{aligned}$$

Since α is closed and M is simply connected, there exists $\tilde{h}$ such that $d\tilde{h} = \alpha$ on M. Next, we extend $\tilde{h}$ smoothly to a neighborhood of M. Let $h = \tilde{h} + s\tilde{\delta}$, for some number $s > 0$. Since $\frac{\partial\tilde{\delta}}{\partial z} = 0$, we have $\frac{\partial\tilde{\delta}}{\partial z} = \frac{\partial\delta}{\partial z}$. Hence, on M, we have (3.2.1).

For $\rho = \tilde{\delta} e^h$, we compute

$$\frac{\partial \rho}{\partial z} = \tilde{\delta} e^h \frac{\partial h}{\partial z} + e^h \frac{\partial \tilde{\delta}}{\partial z} = e^h \frac{\partial \tilde{\delta}}{\partial z},$$

$$\frac{\partial \rho}{\partial w} = \tilde{\delta} e^h \frac{\partial h}{\partial w} + \frac{\partial \tilde{\delta}}{\partial w} e^h = e^h \frac{\partial \tilde{\delta}}{\partial w},$$

$$\frac{\partial \rho}{\partial \bar{w}} = \frac{\partial \tilde{\delta}}{\partial \bar{w}} e^h,$$

and

$$\frac{\partial^2 \rho}{\partial z \partial \bar{z}}(q) = \frac{\partial}{\partial z}(e^h \frac{\partial \tilde{\delta}}{\partial \bar{z}} + \tilde{\delta} e^h \frac{\partial h}{\partial z}) = e^h \frac{\partial h}{\partial z} \frac{\partial \tilde{\delta}}{\partial \bar{z}} + e^h \frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{z}} + \frac{\partial \tilde{\delta}}{\partial z} e^h \frac{\partial h}{\partial z} + \tilde{\delta}(e^h (\frac{\partial h}{\partial z})^2) + \tilde{\delta} e^h \frac{\partial^2 h}{\partial z \partial \bar{z}} = 0$$

on M. Also,

$$\begin{aligned}
\frac{\partial^2 \rho}{\partial z \partial \bar{w}} &= \frac{\partial}{\partial z}(e^h \frac{\partial \tilde{\delta}}{\partial \bar{w}} + \tilde{\delta} e^h \frac{\partial h}{\partial \bar{w}}) \\
&= e^h \frac{\partial h}{\partial z} \frac{\partial \tilde{\delta}}{\partial \bar{w}} + e^h \frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{w}} + \frac{\partial \tilde{\delta}}{\partial z} e^h \frac{\partial h}{\partial \bar{w}} + \tilde{\delta} e^h \frac{\partial h}{\partial z} \frac{\partial h}{\partial \bar{w}} + \tilde{\delta} e^h \frac{\partial^2 h}{\partial z \partial \bar{w}} \\
&= e^h \left(\frac{\partial h}{\partial z} \frac{\partial \tilde{\delta}}{\partial \bar{w}} + \frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{w}} \right).
\end{aligned}$$

So, $\frac{\partial^2 \rho}{\partial z \partial \bar{w}} = 0$ on M because $\frac{\partial h}{\partial z} = \frac{\partial \tilde{h}}{\partial \bar{z}} = -\frac{\frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{w}}}{\frac{\partial \tilde{\delta}}{\partial \bar{w}}}$. Furthermore,

$$\frac{\partial^2 \rho}{\partial w \partial \bar{w}} = \frac{\partial}{\partial w}(e^h \frac{\partial \tilde{\delta}}{\partial \bar{w}} + \tilde{\delta} e^h \frac{\partial h}{\partial \bar{w}}). \tag{3.2.7}$$

Substitute $h = \tilde{h} + s\tilde{\delta}$, in (3.2.7), we get

$$\frac{\partial^2 \rho}{\partial w \partial \bar{w}} = e^{\tilde{h}} \{ \frac{\partial \tilde{h}}{\partial w} \frac{\partial \tilde{\delta}}{\partial \bar{w}} + \frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{w}} + \frac{\partial \tilde{\delta}}{\partial w} \frac{\partial \tilde{h}}{\partial \bar{w}} + 2s \left| \frac{\partial \tilde{\delta}}{\partial w} \right|^2 \}.$$

Using (3.1.4) and $\tilde{\delta} = 0$ on M, we get

$$\frac{\partial^2 \rho}{\partial w \partial \bar{w}} = e^{\tilde{h}} \{ \frac{\partial \tilde{h}}{\partial w} \frac{\partial \tilde{\delta}}{\partial \bar{w}} + \frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{w}} + \frac{\partial \tilde{\delta}}{\partial w} \frac{\partial \tilde{h}}{\partial \bar{w}} + \frac{s}{2} \}.$$

For a large enough number s, $\frac{\partial^2 \rho}{\partial w \partial \bar{w}} > 0$. Hence, ρ will be plurisubharmonic on M.

□

Lemma 3.2.2. *Let Ω be a Hartogs domain in $\mathbb{C}^2$. If $K \subset \partial\Omega$ is compact, and $\frac{\partial \tilde{\delta}}{\partial z} \neq 0$ on K, then there exists a plurisubharmonic defining function on a neighborhood of K.*

Proof. First, we will prove that for every $(z, w) \in K$, we have $z \neq 0$. Assume $p = (0, w) \in K, w \in \mathbb{C}$, and let $\{(z_j, w_j)\}$ be any sequence in $b\Omega$ converging to p. Let $U = (u_1, u_2)$ be a unit length vector tangential to $b\Omega$ at p. We may assume the restricted subsequence $\{\frac{(z_j, w_j - w)}{\sqrt{|z_j|^2 + |w_j - w|^2}}\}$ converges to U. Federer [13] considers this as the definition of the tangent vector, which coincides with the usual definition on a domain with $\mathcal{C}^1$ boundary. Since Ω is a Hartogs domain, $(e^{i\theta} z_j, w_j) \in b\Omega$ for any $\theta \in \mathbb{R}$. This implies $(e^{i\theta} u_1, u_2)$ is tangential to $b\Omega$ at p for any $\theta \in \mathbb{R}$, that is,

$$e^{i\theta} u_1 \frac{\partial \tilde{\delta}}{\partial \bar{z}}(p) + u_2 \frac{\partial \tilde{\delta}}{\partial \bar{w}}(p) = 0$$

for any $\theta \in \mathbb{R}$.

From the assumption $\frac{\partial \tilde{\delta}}{\partial \bar{z}} \neq 0$. Therefore, we must have $u_1 = 0$. So, every tangent vector at p must be in the form $(0, u_2)$, which does not fulfill the requirement to span a tangent space of three real dimensions. Hence, we are allowed to assume that $z \neq 0$ for every $(z, w) \in K$.

From the Implicit Function Theorem, locally there exists $f(w)$ such that $t = f(w)$, where $t = |z|^2$ for (z, w) in a neighborhood of $b\Omega$, and $\frac{\partial \tilde{\delta}}{\partial t} \neq 0$ on a neighborhood of $b\Omega$. So,

$$\tilde{\delta}(t, w) = \tilde{\delta}(f(w), w) = 0.$$

Our defining function will be

$$\rho(t,w) = e^{-\log f}(t-f) = (f(w))^{-1}t - 1.$$

Since f is uniquely determined, ρ is globally defined on K. Computing the first and the second derivatives of $(f(w))^{-1}$, we get

$$\frac{\partial (f(w))^{-1}}{\partial \bar{w}} = -(f(w))^{-2}\frac{\partial f}{\partial \bar{w}},$$
$$\frac{\partial^2 (f(w))^{-1}}{\partial w \partial \bar{w}} = -(f(w))^{-2}\frac{\partial^2 f}{\partial w \partial \bar{w}} + 2(f(w))^{-3}\left|\frac{\partial f}{\partial w}\right|^2.$$

From the pseudoconvexity of Ω, $H_\rho(L,L)(p) \geq 0$. This implies

$$\left|\frac{\partial f}{\partial w}\right|^2 \geq f(w)\frac{\partial^2 f}{\partial w \partial \bar{w}}.$$

$$\begin{aligned}\frac{\partial^2 (f(w))^{-1}}{\partial w \partial \bar{w}} &\geq -(f(w))^{-3}\left|\frac{\partial f}{\partial w}\right|^2 + 2(f(w))^{-3}\left|\frac{\partial f}{\partial w}\right|^2 \\ &= (f(w))^{-3}\left|\frac{\partial f}{\partial w}\right|^2.\end{aligned}$$

Next, we show that ρ is a plurisubharmonic defining function near K. We may compute the second derivatives with respect to z and $\bar{z}$

$$\frac{\partial^2 \rho}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}((f(w))^{-1}z) = (f(w))^{-1},$$

and with respect to w and $\bar{z}$

$$\frac{\partial^2 \rho}{\partial w \partial \bar{z}} = \frac{\partial}{\partial w}((f(w))^{-1}z) = -z(f(w))^{-2}\frac{\partial f}{\partial w},$$

which is equal to

$$\frac{\partial^2\rho}{\partial\bar{w}\partial z} = -\bar{z}(f(w))^{-2}\frac{\partial f}{\partial\bar{w}}.$$

Also, we compute the second derivative with respect to w and $\bar{w}$

$$\frac{\partial^2\rho}{\partial w\partial\bar{w}} = \frac{\partial}{\partial w}(t(-(f(w))^{-2}\frac{\partial f}{\partial\bar{w}})) = t(-(f(w))^{-2}\frac{\partial^2 f}{\partial w\partial\bar{w}} + 2(f(w))^{-3}\left|\frac{\partial f}{\partial w}\right|^2).$$

Let

$$L = \{\frac{\partial\rho}{\partial w}\frac{\partial}{\partial z} - \frac{\partial\rho}{\partial z}\frac{\partial}{\partial w}\}/\left|\partial\rho\right|.$$

Then the Levi form is given by

$$H_\rho(L,L)(p) = (\frac{\partial^2\rho}{\partial z\partial\bar{z}}\left|\frac{\partial\rho}{\partial w}\right|^2 - \frac{\partial^2\rho}{\partial\bar{z}\partial w}\frac{\partial\rho}{\partial z}\frac{\partial\rho}{\partial\bar{w}} - \frac{\partial^2\rho}{\partial z\partial\bar{w}}\frac{\partial\rho}{\partial w}\frac{\partial\rho}{\partial\bar{z}} + \frac{\partial^2\rho}{\partial w\partial\bar{w}}\left|\frac{\partial\rho}{\partial z}\right|^2)\frac{1}{\left|\partial\rho\right|^2}.$$

Using the above derivatives, we get

$$H_\rho(L,L)(p) = (f(w))^{-1}\left|\frac{\partial f}{\partial w}\right|^2 - \frac{\partial^2 f}{\partial w\partial\bar{w}}.$$

Turning to the full complex Hessian, we have

$$\begin{vmatrix} \frac{\partial^2\rho}{\partial z\partial\bar{z}} & \frac{\partial^2\rho}{\partial w\partial\bar{z}} \\ \frac{\partial^2\rho}{\partial z\partial\bar{w}} & \frac{\partial^2\rho}{\partial w\partial\bar{w}} \end{vmatrix} = \begin{vmatrix} (f(w))^{-1} & -z(f(w))^{-2}\frac{\partial f}{\partial w} \\ -\bar{z}(f(w))^{-2}\frac{\partial f}{\partial\bar{w}} & t(-(f(w))^{-2}\frac{\partial^2 f}{\partial w\partial\bar{w}} + 2(f(w))^{-3}\left|\frac{\partial f}{\partial w}\right|^2) \end{vmatrix}$$

$$= (f(w))^{-1}t(-(f(w))^{-2}\frac{\partial^2 f}{\partial w\partial\bar{w}} + 2(f(w))^{-3}\left|\frac{\partial f}{\partial w}\right|^2) - tf^{-4}$$

$$\overset{t=f|_{b\Omega}}{=} -(f(w))^{-2}\frac{\partial^2 f}{\partial w\partial\bar{w}} + (f(w))^{-3}\left|\frac{\partial f}{\partial w}\right|^2 = (f(w))^{-3}(\left|\frac{\partial^2 f}{\partial w}\right|^2 - f\frac{\partial^2 f}{\partial w\partial\bar{w}}) \geq 0.$$

From the assumption, our domain is pseudoconvex which implies the Levi form is positive semidefinite on the complex tangent space (the last inequality). Hence, ρ is a plurisubharmonic defining function. $\square$

Lemma 3.2.1. *Let Ω be a Hartogs domain in $\mathbb{C}^2$, and let K_1 and K_2 be two compact subsets of the boundary of Ω. Let U_1, and U_2 be neighborhoods of K_1, and K_2 respectively, such that, $K_1 \cap U_2 = \varnothing$, $K_2 \cap U_1 = \varnothing$, and the boundary of Ω is strictly pseudoconvex on $b\Omega \cap \bar{U}_1 \setminus K_1$ and $b\Omega \cap \bar{U}_2 \setminus K_2$. Let ρ_1 and ρ_2 be two defining functions for Ω defined on U_1 and U_2 respectively. Then, for every $0 < \tau_3 < 1$, there exists a neighborhood U_3 of $b\Omega$ and ρ_3 a defining function defined on $U_1 \cup U_2$ such that*

$$\rho_3 = \rho_1 \; on \, U_1 \setminus U_2$$

and

$$\rho_3 = \rho_2 \;\; on \, U_2 \setminus U_1.$$

Furthermore,

$$i\partial\bar{\partial}(-(-\rho_3)^{\tau_3}) \geq iM_3(-\rho_3)^{\tau_3}\partial\bar{\partial}\,|z|^2 \;\; on \, U_3 \cap \overline{U_1 \cap U_2}.$$

Proof. Given $\chi \in C^\infty(\mathbb{C}^2)$ such that $0 \leq \chi \leq 1, \chi = 1$ on a neighborhood of $\overline{U_1 \setminus U_2}$, and $\chi = 0$ on a neighborhood of $\overline{U_2 \setminus U_1}$, there exists a defining function $\rho_3 = \chi\rho_1 + (1-\chi)\rho_2$, and we know $e^{\lambda_3\rho_3} - 1$ is strictly plurisubharmonic on $\overline{U_1 \cap U_2}$ for λ_3 sufficiently large (see Theorem 3.4.4 in [7]). By Theorem 3.4.4 in [7], there exists some $N_3 > 0$ such that

$$i\partial\bar{\partial}(e^{\lambda_3\rho_3} - 1) > iN_3\partial\bar{\partial}\,|z|^2\,.$$

We compute

$$i\partial\bar{\partial}(e^{\lambda_3\rho_3} - 1) = i\lambda_3 e^{\lambda_3\rho_3}(\partial\bar{\partial}\rho_3 + \lambda_3\partial\rho_3 \wedge \bar{\partial}\rho_3) \geq iN_3\partial\bar{\partial}\,|z|^2\,.$$

Hence,

$$i\partial\bar{\partial}\rho_3 \geq ie^{-\lambda_3\rho_3}N_3\partial\bar{\partial}\,|z|^2\,\frac{1}{\lambda_3} - i\frac{\lambda_3}{\tau_3}\partial\rho_3 \wedge \bar{\partial}\rho_3 \tag{3.2.8}$$

On the other hand, we want to show

$$i\partial\bar{\partial}(-(-\rho_3)^{\tau_3}) \geq iM_3(-\rho_3)^{\tau_3}\partial\bar{\partial}\,|z|^2 .$$

Expanding the left hand side we need

$$-i\tau_3(\tau_3-1)(-\rho_3)^{\tau_3-2}\partial\rho_3\wedge\bar{\partial}\rho_3 + i\tau_3(-\rho_3)^{\tau_3-1}\partial\bar{\partial}\rho_3 \geq iM_3(-\rho_3)^{\tau_3}\partial\bar{\partial}\,|z|^2$$

which follows from

$$i\tau_3(-\rho_3)^{\tau_3-1}\partial\bar{\partial}\rho_3 \geq iM_3(-\rho_3)^{\tau_3}\partial\bar{\partial}\,|z|^2 + i\tau_3(\tau_3-1)(-\rho_3)^{\tau_3-2}\partial\rho_3\wedge\bar{\partial}\rho_3.$$

So, we need

$$i\partial\bar{\partial}\rho_3 \geq iM_3(-\rho_3)\frac{1}{\tau_3}\partial\bar{\partial}\,|z|^2 + i(\tau_3-1)(-\rho_3)^{-1}\partial\rho_3\wedge\bar{\partial}\rho_3. \tag{3.2.9}$$

Since ρ_3 is close to zero near the boundary, the first term $e^{-\lambda_3\rho_3}N_3\partial\bar{\partial}\,|z|^2\,\frac{1}{\lambda_3}$ in the right hand side of (3.2.8) is greater than the first term $iM_3(-\rho_3)\frac{1}{\tau_3}\partial\bar{\partial}\,|z|^2$ in the right hand side of (3.2.9). Similarly, since $0<\tau_3<1$, the second term in (3.2.8) bounds the second term (3.2.9). $\square$

Proof of Theorem 1.1.2. First we consider the case where $b\Omega$ contains an annulus on which $\frac{\partial\tilde{\delta}}{\partial z}=0$. From the assumption, we have $0<\tau<1$ and $\tau<\frac{\pi}{2\mathcal{C}\ln\frac{B}{A}+\pi}$. This implies

$$\frac{2\tau\mathcal{C}}{1-\tau}\ln B - \frac{2\tau\mathcal{C}}{1-\tau}\ln A < \pi,$$

Since $\frac{2\tau\mathcal{C}}{1-\tau}\ln B - \frac{2\tau\mathcal{C}}{1-\tau}\ln A < \pi$, there exists a constant ϕ, such that

$$\sin(\frac{2\tau\mathcal{C}}{1-\tau}\ln t+\phi) > 0$$

on $A\leq t\leq B$.

Then there exists $\varepsilon > 0$ such that a positive solution of

$$-\frac{1}{4}\tau(1-\tau)^2\left(\frac{\partial g}{\partial t}+t\frac{\partial^2 g}{\partial t^2}\right)-\frac{\mathcal{C}^2}{t^2}\tau^3 tg \geq \varepsilon\tau^3 t\frac{\mathcal{C}^2}{t^2} \tag{3.2.10}$$

on $A \leq t \leq B$ is given by

$$g = c_1\cos\left(\frac{2\tau\mathcal{C}}{1-\tau}\ln t\right)+c_2\sin\left(\frac{2\tau\mathcal{C}}{1-\tau}\ln t\right)-\varepsilon,$$

where $c_1 = \sin\phi$,and $c_2 = \cos\phi$. Then (3.2.10) becomes

$$-\frac{1}{4}\tau(1-\tau)^2\left(\frac{\partial g}{\partial t}+t\frac{\partial^2 g}{\partial t^2}\right) > \tau^3 tg\frac{\mathcal{C}^2}{t^2}. \tag{3.2.11}$$

Define $h = g^{1-\tau}$, and $\sigma = -h(-\tilde{\delta})^\tau$. Let a, b and c be as in the proof of Theorem 1.1.1. As in the proof of that theorem, we have

$$\begin{aligned} ab-|c|^2 = {}& -\tau^2(\tau-1)\,|\rho_{tw}|^2\,tg-\frac{1}{4}\tau(1-\tau)^2\left(\frac{\partial g}{\partial t}+t\left(\frac{\partial^2 g}{\partial t^2}-\tau g^{-1}\left(\frac{\partial g}{\partial t}\right)^2\right)\right) \\ & -\tau^2 t\left(g\,|\rho_{t\overline{w}}|^2+\frac{1}{4}\frac{1}{g}(1-\tau)^2\left(\frac{\partial g}{\partial t}\right)^2\right). \end{aligned}$$

We compute

$$\begin{aligned} ab-|c|^2 = {}& -\tau^2(\tau-1)tg\,|\rho_{tw}|^2-\frac{1}{4}\tau(1-\tau)^2\frac{\partial g}{\partial t}-\frac{1}{4}\tau(1-\tau)^2 t\frac{\partial^2 g}{\partial t^2}+\frac{1}{4}\tau^2(1-\tau)^2 tg^{-1}\left(\frac{\partial g}{\partial t}\right)^2 \\ & -\tau^2 tg\,|\rho_{t\overline{w}}|^2-\frac{1}{4}\tau^2 tg^{-1}(1-\tau)^2\left(\frac{\partial g}{\partial t}\right)^2. \end{aligned}$$

After simplifications, we get

$$ab - |c|^2 = -\tau^3 tg\, |\rho_{tw}|^2$$
$$+ \tau^2 tg\, |\rho_{tw}|^2 - \tau^2 tg\, |\rho_{t\overline{w}}|^2 - \frac{1}{4}\tau(1-\tau)^2 \left(\frac{\partial g}{\partial t} + t\frac{\partial^2 g}{\partial t^2}\right).$$

From equation (3.2.11), and the fact that our hypotheses imply $|\rho_{tw}| \leq \frac{\mathcal{C}}{t}$

$$ab - |c|^2 > 0 \Longrightarrow H_\sigma > 0$$

on M. Since H_σ is strictly positive on a compact set, it is greater than zero on a neighborhood of M.

Then $H_\sigma \geq 0$ on some neighborhood of the annulus (see also Harrington [19], and Liu [24]). Therefore, σ is plurisubharmonic in a neighborhood of M .

Next, Suppose M is a disc on which $\frac{\partial\tilde{\delta}}{\partial z} = 0$. Then we have shown in Lemma 3.2.1 that there exists h satisfying (3.2.1). Then on M

$$\frac{\partial h}{\partial z} = -\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}} / \frac{\partial\tilde{\delta}}{\partial\bar{w}} = -4\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}. \tag{3.2.12}$$

The second derivative is given by

$$\frac{\partial^2 h}{\partial\bar{z}\partial z} = -4\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^3\tilde{\delta}}{\partial\bar{z}\partial\bar{w}\partial z} - 4\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}.$$

Since h is real, we get

$$\frac{\partial^2 h}{\partial\bar{z}\partial z} = -4Re(\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^3\tilde{\delta}}{\partial\bar{z}\partial\bar{w}\partial z}) - 4\left|\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}\right|^2. \tag{3.2.13}$$

Using equation (3.2.6), the above (3.2.13) becomes

$$\frac{\partial^2 h}{\partial\bar{z}\partial z} = 0$$

on M. Let

$$\hat{h} = e^{\tau h - s|z|^2} \tag{3.2.14}$$

where $s > 0$ and

$$\sigma = \hat{h}(-\tilde{\delta})^{\tau}.$$

As in the proof of Theorem 1.1.1 we set

$$a = 4\hat{h}\tau \left|\rho_{tw}\pi(p)\right|^2 t - \frac{\partial^2 \hat{h}}{\partial z \partial \bar{z}}. \tag{3.2.15}$$

After computing the second derivative of $\hat{h}$, we get

$$a = 4\tau e^{\tau h - s|z|^2} \left|\rho_{tw}\right|^2 t - e^{\tau h - s|z|^2}(\frac{\tau\partial h}{\partial z} - s\bar{z})(\frac{\tau\partial h}{\partial \bar{z}} - sz) - e^{\tau h - s|z|^2}(\frac{\tau\partial^2 h}{\partial z\partial \bar{z}} - s).$$

After simplification, we get

$$a = 4\tau e^{\tau h - s|z|^2} \left|\rho_{tw}\right|^2 t - e^{\tau h - s|z|^2}(\tau^2 \left|\frac{\partial h}{\partial z}\right|^2 - \tau\frac{\partial h}{\partial z} sz - s\tau\bar{z}\frac{\partial h}{\partial \bar{z}} + s^2\left|z\right|^2) + s e^{\tau h - s|z|^2}.$$

So, (3.1.6) implies

$$a = 4\tau t e^{\tau h - s|z|^2} \left|\rho_{tw}\right|^2 - e^{\tau h - s|z|^2}(4t\left|\rho_{tw}\right|^2 \tau^2 + 2Re(\frac{\partial h}{\partial z} sz) + s^2 t + s).$$

Hence,

$$Re(\frac{\partial h}{\partial z} sz) = -4sRe(\frac{\partial\tilde{\delta}}{\partial w}\frac{\partial^2\tilde{\delta}}{\partial z\partial \bar{w}}.z) = Re(4\frac{\partial\tilde{\delta}}{\partial w}\rho_{t\bar{w}}\left|z\right|^2) = -4tRe(\rho_w\rho_{t\bar{w}}).$$

Rewriting the real part as the sum of a complex number and its conjugate, we get

$$Re(\frac{\partial h}{\partial z} sz) = -2t(\rho_w\rho_{t\bar{w}} + \rho_{\bar{w}}\rho_{tw}) = 0.$$

The last equality comes from (3.1.5). So

$$a = e^{\tau h - s|z|^2}(s + 4\tau(1-\tau)t\,|\rho_{tw}|^2 - s^2 t).$$

Let

$$b = -\frac{1}{4}\hat{h}\tau(\tau - 1). \tag{3.2.16}$$

Since $\tau < 1$, we get $b > 0$. Furthermore,

$$c = \tau\left(\hat{h}\rho_{t\bar{w}}\bar{z} + \frac{\partial \hat{h}}{\partial z}\frac{\partial \tilde{\delta}}{\partial \bar{w}}\right) = \tau e^{\tau h - s|z|^2}\rho_{t\bar{w}}\bar{z} + \tau e^{\tau h - s|z|^2}\frac{\partial \tilde{\delta}}{\partial \bar{w}}(\tau\frac{\partial h}{\partial z} - s\bar{z}). \tag{3.2.17}$$

Using (3.2.12), we get

$$c = \tau e^{\tau h - s|z|^2}\rho_{t\bar{w}}\bar{z} + \tau e^{\tau h - s|z|^2}\frac{\partial \tilde{\delta}}{\partial \bar{w}}(\tau\rho_{t\bar{w}}\bar{z} - s\bar{z}),$$

which is equal to

$$c = e^{\tau h - s|z|^2}(\tau(1-\tau)\rho_{t\bar{w}}\bar{z} - s\tau\bar{z}\frac{\partial \tilde{\delta}}{\partial \bar{w}}).$$

So, using (3.1.6)

$$ab - |c|^2 = \frac{1}{4}e^{2\tau h - 2s|z|^2}\{\tau(1-\tau)(s - s^2|z|^2) - \tau^2 s^2 |z|^2\}.$$

The above quantity $ab - |c|^2 > 0$ if $1 - \tau > s|z|^2$. From the assumption on the disc M, we have $|z|^2 \leq B$. So, if $s < \frac{1-\tau}{B}$, we get $ab - |c|^2 > 0$. Therefore, since $b > 0$ and the determinant of the matrix

$$L = \begin{bmatrix} a & c \\ \bar{c} & b \end{bmatrix}$$

is also greater than zero, the matrix L is positive definite. Then $H_\sigma > 0$ on the disc, so $H_\sigma \geq 0$ on a neighborhood of the disc.

Suppose $K \subset b\Omega$ is a set of weakly pseudoconvex points satisfying the hypotheses of

Lemma 3.2.2. Lemma 3.2.2 grants that there exists a plurisubharmonic defining function on K. Following the same procedure in [16], we can construct a Diederich-Fornæss Index of τ near K.

Finally, we decompose the weakly pseudoconvex points into $\{K_j\}$ where each K_j is either a disc, an annulus or $\{\text{weakly pseudoconvex} : \frac{\partial\tilde{\delta}}{\partial z} \neq 0\}$ in the boundary of Ω. Repeated use of Lemma 3.2.1, shows that there exists a defining function, say ρ_3, defined on $\cup_i U_i$, where each U_i is a neighborhood of K_i, such that $-(-\rho_3)^\tau$ is plurisubharmonic.

□

As shown in Liu [24], we can also show that our results are sharp on the worm domain.

Proof of Corollary 1.1.1. Let ρ_r be the defining function for the worm domain given by Definition 2.1.7. We choose r to be fixed, so remove the subscript from ρ. Let $w = u + iv$, and $z = x + iy$. Then

$$\begin{aligned} \frac{\partial}{\partial w} &= \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}), & \frac{\partial}{\partial \bar{w}} &= \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}), \\ \frac{\partial}{\partial u} &= (\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}), & \frac{\partial}{\partial v} &= i(\frac{\partial}{\partial w} - \frac{\partial}{\partial \bar{w}}). \end{aligned} \tag{3.2.18}$$

Substituting, we get

$$\frac{\partial^2\rho}{\partial u\partial\bar{z}} = \frac{\partial^2\rho}{\partial w\partial\bar{z}} + \frac{\partial^2\rho}{\partial\bar{w}\partial\bar{z}} \tag{3.2.19}$$

and

$$\frac{\partial^2\rho}{\partial v\partial\bar{z}} = i(\frac{\partial^2\rho}{\partial w\partial\bar{z}} - \frac{\partial^2\rho}{\partial\bar{w}\partial\bar{z}}). \tag{3.2.20}$$

For $(z, w) \in M_r$, we have

$$\begin{aligned} \frac{\partial\rho}{\partial w} &= e^{-i\ln|z|^2}, & \frac{\partial^2\rho}{\partial\bar{z}\partial w} &= \frac{-iz}{|z|^2}e^{-i\ln|z|^2}, \\ \frac{\partial^2\rho}{\partial z\partial w} &= \frac{-i\bar{z}}{|z|^2}e^{-i\ln|z|^2}, & \frac{\partial\rho}{\partial\bar{z}\partial\bar{w}} &= \frac{iz}{|z|^2}e^{i\ln|z|^2}. \end{aligned} \tag{3.2.21}$$

We also have $\frac{\partial\rho}{\partial z}=0$. Since

$$\frac{\partial\rho}{\partial w}=\frac{1}{2}\frac{\partial\rho}{\partial u}-i\frac{1}{2}\frac{\partial\rho}{\partial v}=\cos(\ln|z|^2)-i\sin(\ln|z|^2),$$

the real normal vector, denoted by ν, is given by

$$\nu=\bigtriangledown\rho=(0,0,2\cos(\ln|z|^2),2\sin(\ln|z|^2)),$$

and the real tangent space is spanned by the following components:

$$T_1=(1,0,0,0)\cdot\bigtriangledown=\frac{\partial}{\partial x}$$

$$T_2=(0,1,0,0)\cdot\bigtriangledown=\frac{\partial}{\partial y},$$

and

$$T_3=(0,0,-\sin(\ln|z|^2),\cos(\ln|z|^2))\cdot\bigtriangledown=-\sin(\ln|z|^2)\frac{\partial}{\partial u}+\cos(\ln|z|^2)\frac{\partial}{\partial v}.$$

We denote $\frac{\partial}{\partial\nu}=\frac{1}{2}\bigtriangledown\rho\cdot\bigtriangledown$.

Therefore,

$$\frac{\partial}{\partial w}=\frac{1}{2}(\frac{\partial}{\partial u}-i\frac{\partial}{\partial v})=\frac{1}{2}(\cos(\ln|z|^2)-i\sin(\ln|z|^2))\cdot(\cos(\ln|z|^2)\frac{\partial}{\partial u}+\sin(\ln|z|^2)\frac{\partial}{\partial v})$$

$$+\frac{1}{2}(-\sin(\ln|z|^2)-i\cos(\ln|z|^2)\{-\sin(\ln|z|^2)\frac{\partial}{\partial u}+\cos(\ln|z|^2)\frac{\partial}{\partial v}\}$$

$$=\frac{1}{2}e^{-i\ln|z|^2}\frac{\partial}{\partial\nu}-i\frac{1}{2}e^{-i\ln|z|^2}T_3.$$

Hence,

$$\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}=\frac{1}{2}e^{-i\ln|z|^2}\frac{\partial}{\partial\nu}\frac{\partial\tilde{\delta}}{\partial\bar{z}}-i\frac{1}{2}e^{-i\ln|z|^2}T_3\frac{\partial\tilde{\delta}}{\partial\bar{z}}.$$

Since T_3 and $\frac{\partial}{\partial\bar{z}}$ are tangential, $T_3\frac{\partial\tilde{\delta}}{\partial\bar{z}}=|\bigtriangledown\rho|^{-1}T_3\frac{\partial\rho}{\partial\bar{z}}$. For justification, see the argument preceding (2.9) in [17].

Using (2.1.4), we have $\frac{\partial}{\partial \nu}\frac{\partial \tilde{\delta}}{\partial \bar{z}} = 0$, where $\nu = (0, 0, 2\cos(\ln|z|^2), 2\sin(\ln|z|^2))$. We get

$$\frac{\partial^2 \tilde{\delta}}{\partial \bar{z}\partial w} = -i\frac{1}{2}e^{-i\ln|z|^2}\left|\bigtriangledown \rho\right|^{-1} T_3 \frac{\partial \rho}{\partial \bar{z}}.$$

We also have

$$T_3 = ie^{-i\ln|z|^2}\frac{\partial}{\partial w} - ie^{-i\ln|z|^2}\frac{\partial}{\partial \bar{w}}.$$

Using these formulas, we compute

$$T_3\frac{\partial \rho}{\partial \bar{z}} = -\sin(\ln|z|^2)\frac{\partial^2 \rho}{\partial u \partial \bar{z}} + \cos(\ln|z|^2)\frac{\partial^2 \rho}{\partial v \partial \bar{z}}.$$

Using (3.2.19) and (3.2.20), we get

$$T_3\frac{\partial \rho}{\partial \bar{z}} = -\sin(\ln|z|^2 + i\cos(\ln|z|^2))\frac{\partial^2 \rho}{\partial w \partial \bar{z}}(-\sin(\ln|z|^2)) - i\cos(\ln|z|^2)\frac{\partial^2 \rho}{\partial \bar{w} \partial \bar{z}}.$$

This can be simplified using Euler's Formula as follows:

$$T_3\frac{\partial \rho}{\partial \bar{z}} = ie^{i\ln|z|^2}\frac{\partial^2 \rho}{\partial w \partial \bar{z}} - ie^{-i\ln|z|^2}\frac{\partial^2 \rho}{\partial \bar{w} \partial \bar{z}}.$$

Subsititute (3.2.21), we compute

$$\frac{\partial^2 \tilde{\delta}}{\partial \bar{z}\partial w} = -\frac{i}{4}e^{-i\ln|z|^2}\left(ie^{i\ln|z|^2}\frac{-iz}{|z|^2}e^{-i\ln|z|^2} - ie^{-i\ln|z|^2}\frac{iz}{|z|^2}e^{i\ln|z|^2}\right)$$

$$= \frac{i}{4}e^{-i\ln|z|^2}\frac{2z}{|z|^2},$$

so,

$$\left|\frac{\partial^2 \tilde{\delta}}{\partial \bar{z}\partial w}\right| = \frac{1}{2}\frac{1}{|z|}. \tag{3.2.22}$$

From Theorem 1.1.1, we get

$$\tau < \frac{\pi}{2\mathcal{C}\ln r^2 + \pi},$$

for every $\mathcal{C} < \frac{1}{2}$, so

$$\tau \leq \frac{\pi}{\ln r^2 + \pi}.$$

Moreover, Theorem 1.1.2 grants the value of the Diederich-Fornæss Index is at least $\frac{\pi}{2\mathcal{C}\ln r^2+\pi}$. Hence, we get the equality. $\square$

Remark 3.2.1. *Boas and Straube defined Hartogs domains that were nowhere worm-like in* [5]*. They show that Condition R is satisfied on these domains. In our setting, Ω satisfies the condition of nowhere worm-like if and only if $\frac{\partial^2\tilde{\delta}}{\partial z \partial \bar{w}} = 0$ on any annulus in the boundary of Ω. Therefore, $\mathcal{C}$ can be chosen close to 0 in $\frac{\pi}{2\mathcal{C}\ln r+\pi}$. Hence, Theorem 1.1.2 grants that the Diederich-Fornæss Index τ for a nowhere worm-like is equal to one.*

Chapter 4

Good Vector Field Method

4.1 Existence of a Family of Good Vector Fields

Proof of Theorem 1.1.3. Assume we have a family of good vector fields. Since the hypotheses of Theorem 2.11 in [19] are satisfied, the Diederich-Fornæss Index of Ω equals one.

Conversely, assume that the Diederich-Fornæss Index of Ω is equal to one. In the following proof, we will make use of Boas and Straube's result [4], and others [26] which say the existence of a plurisubharmonic defining function on the boundary implies the existence of a good vector field.

We will consider each connected component of the weakly pseudoconvex points separately. First, suppose that $B \geq 0$, $\frac{\partial \tilde{\delta}}{\partial z} = 0$ and, $\frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{z}} = 0$ on M, where for some $w \in \mathbb{C}$

$$M = \{(z, w) : |z|^2 \leq B\} \subset b\Omega.$$

By Boas and Straube's result [6], the existence of a family of good vector fields in a neighborhood of a disc M in the boundary is granted. Let K satisfies the hypotheses of Lemma 3.2.1. Let α be defined by (3.2.2). We get that there exists a function h solving the differential equation (3.2.1), so $dh = \alpha$ on M. So, the plurisubharmonic defining function is given by $\rho = \tilde{\delta} e^h$. Therefore, the result of Boas and Straube [4] ensures the existence of a family of good vector fields in a neighborhood of M in the boundary.

Next, we will consider the case that we have an annulus in the boundary. From Theorem 1.1.1, if the supremum over all possible τ is equal to 1, we get $A = B$, or for every $\mathcal{C} > 0$, there exists $(z, w) \in M$ such that $\left|\frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{z}}\right| \leq \frac{\mathcal{C}}{|z|}$. The case $A = B$ will be considered later. Let $\mathcal{C}_j = \frac{1}{j}$. Then there exists a sequence $\{(z_j, w)\} \subset M$ such that $\left|\frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{z}}(z_j, w)\right| \leq \frac{1/j}{|z|}$, for every $j \in \mathbb{N}$. Since M is compact, then there exists a convergent subsequence $\{(z_{j_k}, w)\}$ converging to $(\tilde{z}, w) \in M$. By the continuity of $\frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{z}}$, we get

$$\left| \frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{z}}(\tilde{z}, w) \right| = 0,$$

so due to the circular symmetry of the Hartogs domain $\left| \frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{z}} \right| = 0$ on a circle. So, $\alpha = 0$ on the circle $|z| = C$, for some real constant C. Next, we will show that we also have a zero cohomology class $[\alpha|M] = 0$ on the annulus that contains $|z| = C$. Let γ be the path parametrized by $\gamma(t) = re^{it}$. With the notation

$$z = x + iy = re^{it}$$

$$\alpha(\gamma(t)) = \beta(\gamma(t))dz + \bar{\beta}(\gamma(t))d\bar{z}.$$

We compute

$$dz = ire^{it}dt$$

$$d\bar{z} = -ire^{-it}dt.$$

So,

$$\alpha(\gamma(t)) = \beta(\gamma(t))ire^{it}dt - \bar{\beta}(\gamma(t))ire^{-it}dt$$

and

$$\int_\gamma \alpha = ir \int_0^{2\pi} \beta(re^{it})e^{it} - \overline{\beta(re^{it})}e^{-it}dt.$$

Now,

$$\begin{aligned} d\alpha &= \frac{\partial \beta}{\partial \bar{z}} d\bar{z} \wedge dz + \frac{\partial \bar{\beta}}{\partial \bar{z}} dz \wedge d\bar{z} \\ &= (\frac{\partial \bar{\beta}}{\partial z} - \frac{\partial \beta}{\partial \bar{z}}) dz \wedge d\bar{z}. \end{aligned}$$

Since $d\alpha = 0$ (see Lemma 3.2.1), we get

$$\frac{\partial \bar{\beta}}{\partial z} = \frac{\partial \beta}{\partial \bar{z}} \implies \begin{aligned} \frac{\partial \beta}{\partial r} &= \frac{\partial \beta}{\partial z} e^{i\theta} + \frac{\partial \beta}{\partial \bar{z}} e^{-i\theta} \\ \frac{\partial \beta}{\partial \theta} &= \frac{\partial \beta}{\partial z} ire^{i\theta} - \frac{\partial B}{\partial \bar{z}} ire^{-i\theta}. \end{aligned}$$

Multiplying the first equation by ir, and then subtract it from the second equation we get

$$ir\frac{\partial \beta}{\partial r} - \frac{\partial \beta}{\partial \theta} = 2ire^{-i\theta}\frac{\partial \beta}{\partial \bar{z}}.$$

So,

$$e^{i\theta}(ir\frac{\partial \beta}{\partial r} - \frac{\partial \beta}{\partial \theta}) + e^{-i\theta}(-ir\frac{\partial \bar{\beta}}{\partial r} - \frac{\partial \bar{\beta}}{\partial \theta}) = 0.$$

We can write

$$ir(e^{i\theta}\frac{\partial \beta}{\partial r} - e^{-i\theta}\frac{\partial \bar{\beta}}{\partial \theta}) = e^{i\theta}\frac{\partial \bar{\beta}}{\partial \theta} + e^{-i\theta}\frac{\partial \bar{\beta}}{\partial \theta},$$

and hence,

$$\frac{\partial}{\partial r}\int_\gamma \alpha = \frac{1}{r}\int_\gamma \alpha + \int_0^{2\pi} e^{it}\frac{\partial \beta}{\partial t}(re^{it}) + e^{-it}\frac{\partial \bar{\beta}}{\partial t}(re^{-it})dt.$$

This can be written as

$$\frac{\partial}{\partial r}\int_\gamma \alpha = \frac{1}{r}\int_\gamma \alpha + \int_0^{2\pi} \frac{\partial}{\partial t}(e^{it}\beta + e^{-it}\bar{\beta})dt - \int_0^{2\pi} ie^{it}\beta(re^{it}) - ie^{-it}\bar{\beta}(re^{it})dt.$$

Compute

$$\frac{\partial}{\partial r}\int_\gamma \alpha = \frac{1}{r}\int_\gamma \alpha - \frac{1}{r}\int_\gamma \alpha + 0 = 0.$$

So,

$$\int_\gamma \alpha = 0, \qquad \forall \gamma,$$

where γ is a closed circle.

Therefore, we can define

$$h = \int_{\gamma} \alpha,$$

for an arbitrary path γ with fixed base point.

So,

$$dh = \alpha.$$

This shows that we have zero cohomology class on the annulus. By Boas and Straube's observation, Remark 5, section 4 of [6], there exists a family of good vector fields.

For the case $A = B$, we will have a circle on which the normal is constant by the rotational invariance of Hartogs domains. Using Theorem 1 in [5], we get a family of good vector fields, see Example 3 in the same reference.

$\square$

Chapter 5

Stein Neighborhood Basis

5.1 A Necessary and a Sufficient Condition for the Existence of a Strong Stein Neighborhood Basis

In this Chapter we provide a condition under which the Stein neighborhood basis exists. Bedford and Fornæss [2] have introduced a general criteria for the existence of a Stein neighborhood basis. This criteria is relatively easy to compute on a Hartogs domain. Suppose $M = \{(z, w) : \ A < |z|^2 < B\}$ is an annulus in the $b\Omega$ for some $w \in \mathbb{C}$. For $\theta \in \mathbb{R}$, we denote the boundary component of M by γ_0 and γ_1 which can be parametrized by $\gamma_0 = (\sqrt{B}e^{i\theta}, w)$ and $\gamma_1 = (\sqrt{A}e^{i\theta}, w)$, respectively. We define $c_1 = \int_{\gamma_1} \alpha$, where α is given by (3.2.2). Let $d^c\omega = i(\bar{\partial} - \partial)\omega$.

Theorem 5.1.1. *(Bedford and Fornæss) Let $\Omega \subset \mathbb{C}^2$ be a strongly pseudoconvex domain at all points of $\partial\Omega \setminus \bar{M}$ with C^4 boundary, where $M \subset \partial\Omega$. Also, assume γ_0 and γ_1(defined as above) are boundary components of an annulus in $\mathbb{C}$ which is conformally equivalent to M. Let ω be a solution of the following problem:*

1. *ω is harmonic on M, $\omega \in C^1(\bar{M})$*
2. *$\int_{\gamma_1} d^c\omega = c_1$*
3. *$\omega(z) = 0 \quad for \quad z \in \gamma_0$.*

Then there exists a constant a_1 satisfies $\omega(z) = a_1$ on γ_1.

If $|a_1| < \pi$ then $\bar{\Omega}$ has a Stein neighborhood basis, and if $|a_1| > \pi$ then $\bar{\Omega}$ does not have a Stein neighborhood basis.

In our setting,

$$c_1 = \int_{\gamma_1} \alpha = \int_{\gamma_1 = \sqrt{A}e^{i\theta}} -\frac{\partial^2 \tilde{\delta}}{\partial z \partial \bar{w}} / \frac{\partial \tilde{\delta}}{\partial \bar{w}} dz - \frac{\partial^2 \tilde{\delta}}{\partial \bar{z} \partial w} / \frac{\partial \tilde{\delta}}{\partial w} d\bar{z}$$
$$= \int_{\gamma_1 = \sqrt{A}e^{i\theta}} -\frac{\rho_{t\bar{w}}(|z|^2, w)\bar{z}}{\rho_{\bar{w}}(|z|^2, w)} dz - \frac{\rho_{tw}(|z|^2, w)z}{\rho_w(|z|^2, w)} d\bar{z} \tag{5.1.1}$$
$$= \int_{\gamma_1 = \sqrt{A}e^{i\theta}} -2Re(\frac{\rho_{t\bar{w}}(|z|^2, w)}{\rho_{\bar{w}}(|z|^2, w)} \bar{z} dz).$$

The integrand is constant with respect to θ, so we get

$$c_1 = \int_0^{2\pi} -2Re(\frac{\rho_{t\bar{w}}(A, w)}{\rho_{\bar{w}}(A, w)}(iA)d\theta) = -2Re(2\pi i A \frac{\rho_{t\bar{w}}(A, w)}{\rho_{\bar{w}}(A, w)}).$$

Using (3.1.6), we get

$$c_1 = -2Re\left(2\pi i A \frac{\frac{\partial^2 \tilde{\delta}}{\partial \bar{w} \partial z}}{\bar{z} \frac{\partial \tilde{\delta}}{\partial \bar{w}}}\right). \tag{5.1.2}$$

We can check

$$\omega = \frac{c_1}{4\pi} log \frac{|z|^2}{B}$$

satisfies the assumption of Theorem 5.1.1.

Using the above result we have

$$\int_{\gamma_1} d^c \omega = \int_{\gamma_1} i(\bar{\partial} - \partial)\omega = i \int_{\gamma_1} \frac{\partial \omega}{\partial \bar{z}} d\bar{z} - i \int_{\gamma_1} \frac{\partial \omega}{\partial z} dz$$
$$= i \int_{\gamma_1} \frac{c_1}{4\pi |z|^2} z d\bar{z} - i \int_{\gamma_1} \frac{c_1}{4\pi |z|^2} \bar{z} dz = c_1.$$

When $|z|^2 = A$, we have $\omega(z) = a_1$, where a_1 is a constant given by

$$a_1 = \frac{c_1}{4\pi} \log \frac{A}{B}.$$

Hence, if

$$\left| \frac{c_1}{4\pi} \log \frac{A}{B} \right| > \pi,$$

then there is no Stein neighborhood basis, and if

$$\left|\frac{c_1}{4\pi}\log\frac{A}{B}\right| < \pi,$$

then a Stein neighborhood basis exists when $|z|^2 = A$.

Proof of Theorem 1.1.4. Let us assume

$$\left|\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}\right| < \frac{\pi}{2\sqrt{A}\left|log\frac{A}{B}\right|}.$$

when $|z|^2 = A$. Using equation (3.1.7)

$$\left|\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}\right| = |\rho_{t\bar{w}}(A,w)\bar{z}| = |\rho_{t\bar{w}}(A,w)\bar{z}| = |\rho_{t\bar{w}}(A,w)|\sqrt{A}.$$

Using (5.1.2)

$$|c_1| < 8\pi A\frac{\left|\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}\right|}{\sqrt{A}},$$

so

$$|a_1| = \left|\frac{c_1}{4\pi}\log\frac{A}{B}\right| < 2\sqrt{A}\left|\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}\right|\left|\log\frac{A}{B}\right| < \pi.$$

This implies a Stein neighborhood basis exists.

Next assume $\left|\frac{\partial^2\tilde{\delta}}{\partial z\partial\bar{w}}\right| > \frac{\pi}{2\sqrt{A}}\left|\log\frac{A}{B}\right|$ when $|z| = \sqrt{A}$. The Taylor series in ω for $\tilde{\delta}$ near M is given by

$$\tilde{\delta}(z,w) = Re(we^{i\theta(|z|^2)}) + O(|w|^2),$$

for some smooth real-valued function θ. On M, we have

$$\frac{\partial\tilde{\delta}}{\partial w} = \frac{1}{2}e^{i\theta|z|^2}$$

and

$$\frac{\partial^2 \tilde{\delta}}{\partial \bar{z} \partial w} = \frac{1}{2} e^{i\theta(|z|^2)} i\theta'(|z|^2) z.$$

When $|z|^2 = A$, after plugging into (5.1.2), we get

$$c_1 = 2Re(2\pi i A \frac{(1/2) e^{i\theta(|z|^2)} i\theta'(|z|^2) z}{(1/2) z e^{i\theta(|z|^2)}}) = 4\pi A\theta'(A).$$

On M, we also have

$$|z| \left| \frac{\partial^2 \tilde{\delta}}{\partial \bar{z} \partial w} \right| = \frac{1}{2} \left| \theta'(|z|^2|) \right| |z|^2 .$$

Then

$$|c_1| = 8\pi\sqrt{A} \left| \frac{\partial^2 \tilde{\delta}}{\partial \bar{z} \partial w} \right| .$$

Using our hypothesis, we get

$$|a_1| = \left| \frac{c_1}{2\pi} \log \frac{A}{B} \right| > \pi.$$

Theorem 5.1.1 implies that no Stein neighborhood basis exists. $\square$

Proof of Corollary 1.1.2. Let $M = \{(z, w) : A \le |z|^2 \le B\}$, for some $w \in \mathbb{C}$, be an annulus in the boundary of Ω. For every $\varepsilon > 0$, define for some $w \in \mathbb{C}$,

$$M_\varepsilon = \{(z, w) : A \le |z|^2 \le A + \varepsilon\}.$$

Apply Theorem 1.1.1 to $M_\varepsilon, \forall \varepsilon > 0$, and conclude that $\frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{z}} = 0$ on a circle $(r_\varepsilon e^{i\theta}, w) \in M_\varepsilon$. On $|z|^2 = A$, we have $\frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{z}} = 0$, by the continuity of $\frac{\partial^2 \tilde{\delta}}{\partial w \partial \bar{z}}$. Hence, the existence of a Stein neighborhood basis is granted by Theorem 1.1.4. $\square$

Proof of Theorem 1.1.5 . Following the proof of Theorem 1.1.1, the complex Hessian of σ is positive semidefinite at p near M satisfying $\pi(p) \in M$ only if the matrix $L = \begin{bmatrix} a & c \\ \bar{c} & b \end{bmatrix}$ is positive semidefinite, where

$$a = \Big(-\hat{h}\tau \left(4\left|\rho_{tw}(\pi(p))\right|^2 t\right) + \frac{\partial^2 \hat{h}}{\partial z \partial \overline{z}} \Big),$$

$$b = \frac{1}{4}\hat{h}\tau(\tau - 1)$$

and

$$c = \tau \left(\hat{h}\rho_{t\overline{w}}\overline{z} + \frac{\partial \hat{h}}{\partial z}\frac{\partial \tilde{\delta}}{\partial \overline{w}} \right).$$

To show that L is positive semidefinite, since $\tau > 1$, $b > 0$. We are left with showing that $ab - |c|^2 \geq 0$.

As before we set $\hat{h} = g^{(1-\tau)}$ for some $g > 0$, so that

$ab - |c|^2 \geq 0$ which is equivalent to

$$-\frac{1}{4}\tau(1-\tau)^2 \left(\frac{\partial g}{\partial t} + t\frac{\partial^2 g}{\partial t^2} \right) - \tau^3 tg \left|\rho_{tw}\right|^2 \geq 0. \tag{5.1.3}$$

Our assumptions imply

$$|\rho_{tw}| > \frac{\mathcal{C}}{t}.$$

Then, (5.1.3) implies

$$-\frac{1}{4}\tau(1-\tau)^2 \left(\frac{\partial g}{\partial t} + t\frac{\partial^2 g}{\partial t^2} \right) - \tau^3 tg \left(\frac{\mathcal{C}^2}{t^2} \right) > 0.$$

We will show that this implies a contradiction.

Let $\tilde{g}(s) = g(e^s)$ i.e., we will use the following substitution $t = e^s, \ln A < s < \ln B$. So,

$$\frac{d\tilde{g}}{ds} = e^s \frac{dg}{dt}$$

and

$$\frac{d^2\tilde{g}}{ds^2} = e^s \frac{dg}{dt} + e^{2s}\frac{d^2 g}{dt^2}.$$

Therefore, (5.1.3) will be

$$\frac{1}{4}\tau(1-\tau)^2\left(\frac{d^2\tilde{g}}{ds^2}\right)+\tau^3\mathcal{C}^2\tilde{g}<0. \tag{5.1.4}$$

Assume we have a strictly positive function $\tilde{g}$ on some interval I satisfying (5.1.4). After making the substitution $u=\sqrt{\frac{\tau^3\mathcal{C}^2}{\frac{1}{4}\tau(1-\tau)^2}}s=\frac{2\tau\mathcal{C}}{\tau-1}s$, with

$$\frac{2\tau\mathcal{C}}{(\tau-1)}\ln A<u<\frac{2\tau\mathcal{C}}{(\tau-1)}\ln B.$$

We get

$$\frac{d^2\tilde{g}}{du^2}+\tilde{g}<0. \tag{5.1.5}$$

If $\frac{2\tau\mathcal{C}}{(\tau-1)}\ln\frac{B}{A}\geq\pi$, then we obtain a contradiction as in the proof of Theorem 1.1.1. Therefore, we must have

$$\frac{\tau}{\tau-1}<\frac{\pi}{2\mathcal{C}\ln\frac{B}{A}}.$$

From the assumption, we have $\tau>1$. So, we must have

$$1-\frac{2\mathcal{C}\ln\frac{B}{A}}{\pi}>0\quad\text{and}\quad\tau>\frac{1}{1-\frac{2\mathcal{C}\ln\frac{B}{A}}{\pi}}.$$

□

Proof of Corollary 1.1.3. From the proof of Corollary 1.1.1, (3.2.22), we know that

$$\left|\frac{\partial^2\tilde{\delta}}{\partial\bar{z}\partial w}\right|=\frac{1}{2}\frac{1}{|z|}$$

on M. Choosing $B=r^2$ and $A=1$, Theorem 1.1.4 grants the existence of a Stein neighborhood basis if $\frac{1}{2}<\frac{\pi}{2\ln r^2}$, and no Stein neighborhood basis can exist if $\frac{1}{2}>\frac{\pi}{2\ln r^2}$.

For any value of $\mathcal{C}<\frac{1}{2}$, the hypotheses of Theorem 1.1.5 are satisfied. So, if there exists a strictly positive function h such that $\sigma=h\tilde{\delta}^\tau$ is plurisubharmonic on some neighborhood of $\bar{\Omega}$, Theorem 1.1.5 implies that $\frac{\pi}{2\ln r^2}\geq\frac{1}{2}$ and $\tau\geq\frac{\pi}{\pi-\ln r^2}$.

Using (5.1.2)

$$c_1 = 2Re(2\pi(-\frac{\frac{2z}{4|z|^2}}{\frac{1}{2}z})) = 4\pi,$$

so,

$$\left| \frac{c_1}{4\pi} \log \frac{1}{r^2} \right| < \pi.$$

This implies a Stein neighborhood basis exists. From the above theorem $\tau > \frac{\pi}{\pi - 2\mathcal{C} \ln \frac{B}{A}}$, but for any value of $\mathcal{C} \leq \frac{1}{2}$. This implies $\tau \geq \frac{\pi}{\pi - \ln r^2}$.

$\square$

Bibliography

[1] D. E. Barrett, *Behavior of the Bergman projection on the Diederich-Fornæss worm*, Acta Math. **168** (1992), no. 1-2, 1–10.

[2] E. Bedford and J. E. Fornæss, *Domains with pseudoconvex neighborhood systems*, Invent. Math. **47** (1978), no. 1, 1–27.

[3] S. Bell and E. Ligocka, *A simplification and extension of Fefferman's theorem on biholomorphic mappings*, Invent. Math. **57** (1980), no. 3, 283–289.

[4] H. P. Boas and E. J. Straube, *Sobolev estimates for the $\overline{\partial}$-Neumann operator on domains in $\mathbf{C}^n$ admitting a defining function that is plurisubharmonic on the boundary*, Math. Z. **206** (1991), no. 1, 81–88.

[5] ———, *The Bergman projection on Hartogs domains in $\mathbf{C}^2$*, Trans. Amer. Math. Soc. **331** (1992), no. 2, 529–540.

[6] ———, *de Rham cohomology of manifolds containing the points of infinite type, and Sobolev estimates for the $\overline{\partial}$-Neumann problem*, J. Geom. Anal. **3** (1993), no. 3, 225–235.

[7] S-C Chen and M-C Shaw, *Partial differential equations in several complex variables*, AMS/IP Studies in Advanced Mathematics, vol. 19, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001.

[8] M. Christ, *Global C^∞ irregularity of the $\overline{\partial}$-Neumann problem for worm domains*, J. Amer. Math. Soc. **9** (1996), no. 4, 1171–1185.

[9] S. Şahutoğlu, *Strong Stein neighbourhood bases*, Complex Var. Elliptic Equ. **57** (2012), no. 10, 1073–1085.

[10] K. Diederich and J. E. Fornæss, *Pseudoconvex domains: an example with nontrivial Nebenhülle*, Math. Ann. **225** (1977), no. 3, 275–292.

[11] ———, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), no. 2, 129–141.

[12] ———, *Pseudoconvex domains with real-analytic boundary*, Ann. Math. (2) **107** (1978), no. 2, 371–384.

[13] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[14] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1–65.

[15] J. E. Fornæss and A.-K. Herbig, *A note on plurisubharmonic defining functions in* $\mathbb{C}^n$, Math. Ann. **342** (2008), no. 4, 749–772.

[16] ———, *A note on plurisubharmonic defining functions in* $\mathbb{C}^n$, Math. Ann. **342** (2008), no. 4, 749–772.

[17] P. Harrington and A. Raich, *Defining functions for unbounded* C^m *domains*, Rev. Mat. Iberoam. **29** (2013), no. 4, 1405–1420.

[18] P. S. Harrington, *Global regularity for the* $\overline{\partial}$*-Neumann operator and bounded plurisubharmonic exhaustion functions*, Adv. Math. **228** (2011), no. 4, 2522–2551.

[19] P. S. Harrington, *The Diederich-Fornæss Index and Good Vector Fields* (2017). preprint, arXiv:1705.05815.

[20] A.-K. Herbig and J. D. McNeal, *Convex defining functions for convex domains*, J. Geom. Anal. **22** (2012), no. 2, 433–454.

[21] L. Hörmander, L^2 *estimates and existence theorems for the* $\bar{\partial}$ *operator*, Acta Math. **113** (1965), 89–152.

[22] J. J. Kohn, *Quantitative estimates for global regularity*, Analysis and geometry in several complex variables (Katata, 1997), pp. 97–128.

[23] ———, *Regularity at the boundary of the $\bar{\partial}$-Neumann problem*, Proc. Nat. Acad. Sci. U.S.A. **49** (1963), 206–213.

[24] B. Liu, *The Diederich–Fornæss index I: for domains of non-trivial index*, preprint, arXiv:1701.00293 (2017).

[25] S. Pinton and G. Zampieri, *The Diederich-Fornæss index and the global regularity of the $\bar{\partial}$-Neumann problem*, Math. Z. **276** (2014), no. 1-2, 93–113.

[26] E. J. Straube and M. K. Sucheston, *Plurisubharmonic defining functions, good vector fields, and exactness of a certain one form*, Monatsh. Math. **136** (2002), no. 3, 249–258.

[27] B. M. Weinstock, *Some conditions for uniform H-convexity*, Illinois J. Math. **19** (1975), 400–404.

[28] J. Yum, *On the Steinness Index* (2018). preprint, arXiv:1804.04304.

[29] Y. E. Zeytuncu, *Regularity of canonical operators and Nebenhülle: Hartogs domains*, J. Math. Anal. Appl. **409** (2014), no. 1, 236–243.

[30] ———, *Regularity of canonical operators and Nebenhülle: Hartogs domains*, J. Math. Anal. Appl. **409** (2014), no. 1, 236–243.