Backstepping Control and Transformation of Multi-Input Multi-Output Affine Nonlinear Systems into a Strict Feedback Form

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Backstepping Control and Transformation of Multi-Input Multi-Output Affine Nonlinear Systems into a Strict Feedback Form

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering

by

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ABSTRACT

This dissertation presents an improved method for controlling multi-input multi-output affine nonlinear systems. A method based on Lie derivatives of the system’s outputs is proposed to transform the system into an equivalent strict feedback form. This enables using backstepping control approaches based on Lyapunov stability and integrator backstepping theory to be applied. The geometrical coordinate transformation of multi-input multi-output affine nonlinear systems into strict feedback form has not been detailed in previous publications. In this research, a new approach is presented that extends the transformation process of single-input single-output nonlinear. A general algorithm of the transformation process is formulated. The research will consider square feedback linearizable multi-input multi-output systems where the number of inputs equals to the number of outputs. The preliminary mathematical tools, necessary and sufficient feedback linearizability conditions, as well as a step-by-step transformation process is explained in this research. The approach is applied to the Western Electricity Coordinating Council (WECC) 3-machine nonlinear power system model. Detailed simulation results indicate that the proposed design method is effective in stabilizing the WECC power system when subjected to large disturbances.
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DEDICATION

To my parents, my wife and my children Leen, Salem, Heyam, Nawaf and Jurie.
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1 Introduction

1.1 Overview

It has always been a primary goal for engineers and scholars of control theory to find the most effective approaches and control methods to provide desired and best stability properties. However, in some cases, a good and reliable method may exist but for one reason or another is not suitable for the intended system and vise versa. Therefore, it is sometimes very helpful to modify the intended system or transform it into an equivalent form that meets the control method requirements [1]. Feedback linearization control, for instance, has been considered a successful approach in solving many control problems by transforming dynamical nonlinear models into at least equivalent canonical forms that are simpler than the original forms [2–8]. However, it still has some shortcomings and restrictions [9]. Exact feedback linearization known as input-state feedback linearization is based on the cancellation of system’s nonlinearities regardless of the importance of some of those nonlinearities to the system stability, that is other than the certain structural property required to perform such cancellation [10]. Input-output feedback linearization on the other hand accounts in many cases just to part of the closed-loop dynamics. The other part, which is the internal dynamics, is considered unobservable and the stability of this internal dynamics is essentially required for the input-output feedback linearization effectiveness and this is not the case in many real systems [9]. It is noteworthy to mention that input-state feedback linearization is simply an exceptional case of input-output feedback linearizable systems when successive differentiation of output function turns out equal relative degree and system’s order [11]. Furthermore, ordinary proportional-integral control has been widely adopted in controlling energy conversion systems and although it is applicable and easy to design, it ignores the transient states of the system and deals with the average steady-state model in the neighborhood of equilibrium points which makes dynamic response relatively slow. Moreover, it is difficult to tune PI control parameters [12]. Fortunately, the preceding imperfection of feedback linearization and PI controllers can be avoided and overcome using some other advanced nonlinear control approaches such as backstepping control method that is based on the Lyapunov theorem of stability. This is owing to that backstepping concentrates on construction of Lyapunov function whose derivative can be negative by a verity of control laws.
rather than one specific control law form. Thus, backstepping as a design approach is more flexible in avoiding the cancellation of useful nonlinearities [13]. Since it was developed in 1990, the backstepping control approach has been used in many control design applications. In [14] it was used to stabilize nonlinear spacecraft attitude considering the disturbances and delay due to the actuator based on the delay compensation method [15]. [16] based on [17] develops a controller for one-dimensional unstable heat equation through solving the kernel partial differential equation to transform the partial differential system into an exponential stable target system. A position controller was designed in [18] for an unmanned aerial vehicle with four rotors using adaptive backstepping and adding integral action as proposed in [19] in association with a linear PID controller for stabilizing attitude angle. Adaptive and non-adaptive indirect backstepping controller designs based on making an assumption of the virtual control functions was developed in [20] to track the output voltage of a dc-dc boost converter. Making the same assumptions and considering the effect of parasitic elements, an adaptive backstepping controller was designed in [21] to track the output voltage of a dc-dc boost converter. In [22], coping with the uncertainty of input voltage, inductance, capacitance, load resistance, and undesirable overestimation when choosing update laws was achieved by combining input-output linearization and backstepping methodology to design a dynamical adaptive controller for PWM power converters. Defining new state variables was the approach to transform a mismatched nonlinear dynamic system in [23] into strict feedback form such that the backstepping control method can be applied. In this research, the coordinate transformation of feedback linearizable systems based on the Lie derivative of the system outputs is proposed for multi-input multi-output systems. This method is an extension to the transformation process of single-input single-output systems proposed in [24].

1.2 Motivation and Contribution of The Dissertation

This research develops a technique and introduces preliminaries, required conditions and a step-by-step procedure to transform the mathematical model of MIMO affine nonlinear system into its equivalent strict feedback form such that the backstepping control method based on Lyapunov stability can be applied. The main motives to have MIMO affine nonlinear systems in the strict feedback form and using the backstepping control approach is its ability to accommodate useful nonlinearities and avoid wasteful cancellations,
unlike feedback linearization methods both input-output and input-state linearization that require precise structural property and often cancel useful nonlinearities. For example, to stabilize $z^* = 0$ for the system given in [25] as

$$\dot{z} = a \cos z - bz^3 + cu$$  \hspace{1cm} (1.1)

using feedback linearization, the control law:

$$u = \frac{1}{c} (-a \cos z + bz^3 - kz)$$  \hspace{1cm} (1.2)

will result in the linear feedback system:

$$\dot{z} = -kz$$  \hspace{1cm} (1.3)

which satisfies (5.2) with

$$\dot{V} \leq -W(z) = -kz^2$$  \hspace{1cm} (1.4)

as will be explained later in chapter five. This is actually illogical control law because in addition to $a \cos z$ it cancels $-bz^3$ which is helpful for stabilization at $z = 0$ especially for large values of $z$. Moreover, the existence of $bz^3$ in (1.2) will enlarge the value of $u$ which is harmful and may result in non-robustness. On the other hand, flexibility in choosing a control law to make a derivative of Lyapunov function negative helps to avoid such harmful cancellation. For instance, choosing the control law

$$u = \frac{1}{c} (-a \cos z - kz)$$  \hspace{1cm} (1.5)

satisfies (5.2) such that

$$\dot{V} \leq -W(z) = -(kz^2 + bz^4)$$  \hspace{1cm} (1.6)

and makes $u$ grows linearly with $|z|$. From this perspective, simplifying the process of transforming MIMO affine nonlinear systems into the required strict feedback form will add a very good tool to control engineers’ toolbox.

1.3 Dissertation Outline

This dissertation has seven chapters beyond the introduction (Chapter 1). Chapter 2 discusses briefly the mathematical tools that will be needed for understanding control methods covered in this dissertation and will also be useful when explaining the transformation
process into the strict feedback form. Topics that will be explained are the gradient of a scalar function, Jacobian matrix, coordinate transformation of linear and nonlinear systems, derived mapping, Lie derivative and Lie brackets of vector fields, affine nonlinear system, some function properties, Frobenius theorem, the notion of relative degree and lastly conditions required for a nonlinear system to be exactly feedback linearizable.

Chapter 3 explains the common nonlinear feedback control methods input-state and input-output feedback linearization where most of the math tools from chapter 2 as well as Frobenius theorem and feedback linearizability conditions will be applied. These methods are considered a good beginning to understand systems mapping and transformation. When a clear output function exists input-output feedback linearization for both single-input single-output and multi-input multi-output nonlinear systems is discussed. On the other hand, input-state feedback linearization is presented for both single-input single-output and multi-input multi-output nonlinear systems when a clear output function may or may not be given.

In chapter 4, the dissertation concentrates on discussing Sontag’s formula and Lyapunov theorem of stability which is considered one of the widely used methods to prove the stability of nonlinear systems. However, the interest will be in the part of this theorem where the behavior of mathematically designed function known as Lyapunov function candidate is studied to examine the stability of closed loop systems.

Chapter 5 converses about integrator backstepping theorem and presents a detailed systematic procedure to produce stabilizing controllers for nonlinear systems with a chain of integrators and also for systems in the strict feedback form.

Chapter 6 introduces a simple example to clarify the difference between static and dynamic control design methods and also explains both simple adaptive regulation and simple adaptive tracking controllers designs. Moreover, it introduces adaptive backstepping for second-order matched systems when the control input is in the same equation as the unknown parameter and then extended matching systems when the unknown parameter is one integrator before the control input. The disadvantage of increasing the number of parameter estimates due to overestimation in previous approaches is solved in the last section through mathematically overestimation reduction.

In chapter 7, the dissertation explains step by step procedure on how to transform single-input single-output and multi-input multi-output affine nonlinear systems into equivalent strict feedback form. The conclusion summarizes the research results and overall work of this dissertation and also addresses a few suggestions for future work.
2 Preliminary Mathematical Knowledge

The purpose of this chapter is to explain briefly the math tools and theorems that will frequently be used for understanding the common nonlinear methods covered in this dissertation and will also be very useful when using backstepping approach and transformation into the strict feedback form. The topics that will be explained are the gradient of a scalar function, Jacobian matrix, the coordinate transformation of linear and nonlinear systems, derived mapping, Lie derivative and Lie brackets of vector fields, affine nonlinear system, some function properties, Frobenius theorem, the notion of relative degree and in the last section of this chapter conditions required to feedback linearize nonlinear systems.

2.1 Gradient

Consider a smooth scalar function $h$ of the state $z$:

$$h(z)$$

(2.1)

The gradient of $h$ is denoted by:

$$\nabla h = \frac{\partial h}{\partial z}$$

(2.2)

and represented by a group of elements in a row vector [26].

$$\nabla h (z) = \left[ \frac{\partial h}{\partial z_1} \quad \frac{\partial h}{\partial z_2} \quad \frac{\partial h}{\partial z_3} \quad \ldots \quad \frac{\partial h}{\partial z_n} \right]$$

(2.3)

Example 2.1

The gradient of the function given by:

$$h(z) = z_1^2 + z_1 z_2 + z_3$$

(2.4)

is obtained as:

$$\nabla h (z) = \left[ \frac{\partial h}{\partial z_1} \quad \frac{\partial h}{\partial z_2} \quad \frac{\partial h}{\partial z_3} \right]$$

(2.5)

$$= \begin{bmatrix} 2z_1 + z_2 & z_1 & 1 \end{bmatrix} \triangle$$
2.2 Jacobian Matrix

Consider a vector field:

\[
f(z) = \begin{bmatrix}
f_1(z) \\
f_2(z) \\
\vdots \\
f_n(z)
\end{bmatrix}
\]

(2.6)

The Jacobian matrix of \( f(z) \) is designated by:

\[ \nabla h = \frac{\partial f}{\partial z} \]  

(2.7)

and represented by a matrix of \( n \times n \) dimension as follows [27]:

\[
\nabla f(z) = \begin{bmatrix}
\frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \cdots & \frac{\partial f_1}{\partial z_n} \\
\frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \cdots & \frac{\partial f_2}{\partial z_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial z_1} & \frac{\partial f_n}{\partial z_2} & \cdots & \frac{\partial f_n}{\partial z_n}
\end{bmatrix}
\]

(2.8)

Example 2.2

The Jacobian matrix of the vector field given by:

\[
f(z) = \begin{bmatrix}
f_1(z) \\
f_2(z) \\
f_3(z)
\end{bmatrix} = \begin{bmatrix}
-a z_1 + k_1 \\
-b z_2 + k_2 - c z_1 z_3 \\
\alpha z_2 z_3
\end{bmatrix}
\]

(2.9)

is obtained as:

\[
\nabla f(z) = \begin{bmatrix}
\frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \frac{\partial f_1}{\partial z_3} \\
\frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \frac{\partial f_2}{\partial z_3} \\
\frac{\partial f_3}{\partial z_1} & \frac{\partial f_3}{\partial z_2} & \frac{\partial f_3}{\partial z_3}
\end{bmatrix} = \begin{bmatrix}
-a & 0 & 0 \\
-c z_3 & -b & -c z_1 \\
\alpha z_2 & \alpha z_3 & 0
\end{bmatrix}
\]

(2.10)

2.3 Coordinate Transformation of Linear and Nonlinear Systems

2.3.1 Transformation of Linear System

Given a linear system of the following general form in \( z \) coordinates

\[
\dot{z} = Az + Bu \\
y = Cz + Du
\]

(2.11)
where $z \in R^n$, $u \in R^r$, and $y \in R^m$ are state vector, control input and output vector respectively. A, B, C, and D are matrices of corresponding dimensions. Introducing a new vector $x$ through the transformation
\[ Tz = x \] (2.12)
where $T$ is a non-singular matrix of $n \times n$ dimension such that
\[ T^{-1}x = z \] (2.13)
then the system (2.11) can be transformed into the following system of $x$ coordinates [28]
\[ \dot{x} = TAT^{-1}x + TBu \]
\[ y = CTA^{-1}x + Du \] (2.14)

### 2.3.2 Transformation of Nonlinear System

Given a SISO nonlinear system of the general form
\[ \dot{z} = f(z) + g(z)u \]
\[ y = h(z) \] (2.15)
where $z \in R^n$, $u \in R$, $y \in R$ are state variable, control input variable, and output variable respectively. The nonlinear state transformation
\[ x = T(z) = \begin{bmatrix} T_1(z) \\ T_2(z) \\ \vdots \\ T_{n-1}(z) \\ T_n(z) \end{bmatrix} \] (2.16)
can transform the system into an equivalent system of a new state $x$ according to the following definition and lemma [10].

**Definition 2.1** If a transformation $x = T(z)$ is smooth and its inverse $z = T^{-1}(x)$ exists and be smooth as well, then it is called a **diffeomorphism**.

**Lemma 2.1** Considering the coordinate transformation (2.16), one can write
\[ \dot{x} = \frac{dT(z)}{dz} = \frac{\partial T}{\partial z} \frac{dz}{dt} \] (2.17)
Thus, systems of the form (2.15) can be re-written as
\[ \dot{x} = \frac{\partial T(z)}{\partial z} f(z = T^{-1}(x)) + \frac{\partial T(z)}{\partial z} g(z = T^{-1}(x)) u \]
\[ y = h(z = T^{-1}(x)) \] (2.18)
2.4 Derived Mapping

Due to the importance of nonlinear system coordinate transformation in this research, it would be very helpful to acquaint the reader with the term derived mapping and the concept of vector field transformation as explained in [29] in following definition.

Definition 2.2 Given a diffeomorphism

\[
x = T(z) = \begin{bmatrix} T_1(z) \\ T_2(z) \\ \vdots \\ T_n(z) \end{bmatrix} \tag{2.19}
\]

from \( z \in \mathbb{R}^n \) to \( x \in \mathbb{R}^n \) and a vector field in \( z \) space as

\[
f(z) = \begin{bmatrix} f_1(z) \\ f_2(z) \\ \vdots \\ f_n(z) \end{bmatrix} \tag{2.20}
\]

then transformation of \( f(z) \) from \( z \) space to \( x \) space denoted by \( T.(f) \) is called derived mapping and defined as.

\[
T.(f(z)) = \frac{\partial T(z)}{\partial z}f\left(z = T^{-1}(x)\right) \tag{2.21}
\]

where \( \frac{\partial T(z)}{\partial z} \) is non-singular Jacobian matrix at \( z = z^* \) of \( T(z) \). In the same manner

\[
T^{-1}.(f(x)) = \frac{\partial T^{-1}(x)}{\partial x}f\left(x = T(z)\right) \tag{2.22}
\]

Figure 2.1: Derived mapping between z space and x space.
Example 2.3

Given the transformation matrix

\[ x = T(z) = \begin{bmatrix} z_1 + z_2 \\ z_2 \\ z_1 + z_3 \end{bmatrix} \]  

(2.23)

and the vector field

\[ f(z) = \begin{bmatrix} 1 \\ -1 \\ z_1 + z_2 \end{bmatrix} \]  

(2.24)

From (2.21), the derived mapping of \( f(z) \) is obtained as follows

\[ T. (f(z)) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ z_1 + z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 + z_1 + z_2 \end{bmatrix} \]  

(2.25)

and from (2.23), one can easily find

\[ z = T^{-1}(x) = \begin{bmatrix} x_1 - x_2 \\ x_2 \\ x_3 - x_1 + x_2 \end{bmatrix} \]  

(2.26)

From (2.26), substituting for \( z_1 \) and \( z_2 \) in (2.25) yields

\[ T. (f(z)) = \begin{bmatrix} 0 \\ -1 \\ 1 + x_1 \end{bmatrix} = f(x) \]  

(2.27)

In the same manner, from (2.22)

\[ T^{-1}. (f(x)) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 + x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ x_1 \end{bmatrix} \]  

(2.28)

From (2.23), substituting for \( x_1 \) in (2.28) yields

\[ T^{-1}. (f(x)) = \begin{bmatrix} 1 \\ -1 \\ z_1 + z_2 \end{bmatrix} = f(z) \]  

(2.29)
2.5 Lie Derivative

In differential geometry, Lie derivative is the directional derivative of a scalar function \( h(z) \) along a vector \( f(z) \) and according to [30] it is explained in the following definition:

**Definition 2.3** *Consider a smooth scalar function \( h(z) \) and a smooth vector field \( f(z) \). A new scalar function known as Lie derivative of \( h(z) \) with respect to \( f(z) \) designated by \( L_f h(z) \) is defined by:

\[
L_f h(z) = \nabla h(z) \cdot f(z) = \frac{\partial h(z)}{\partial z} f(z)
\]

(2.30)*

Accordingly, one can recursively define repeated Lie derivatives as follows:

\[
L_f^0 h(z) = h(z)
\]

\[
L_f^1 h(z) = \frac{\partial h(z)}{\partial z} f(z)
\]

\[
L_f^2 h(z) = L_f L_f h(z) = \frac{\partial (L_f h(z))}{\partial z} f(z)
\]

\[
\vdots
\]

\[
L_f^k h(z) = L_f (L_f^{k-1} h(z)) = \frac{\partial (L_f^{k-1} h(z))}{\partial z} f(z)
\]

(2.31)

The example on this part will be postponed to Section 3.1 of input-output feedback linearization for SISO affine nonlinear systems.

2.6 Lie Brackets

Lie bracket is another useful mathematical tool in this research and it was discussed in [31,32] as differentiating a vector field along another vector field according to the following definition.

**Definition 2.4** *Consider the vector fields \( f(z) \) and \( g(z) \) on \( \mathbb{R}^n \). The derivative of \( g(z) \) along \( f(z) \) results in another vector field known as Lie bracket of \( f(z) \) and \( g(z) \) and designated by \([f,g]\) or commonly as \( \text{ad}_f g \) and is defined as:

\[
[f,g] = \nabla g(z) f(z) - \nabla f(z) g(z)
\]

(2.32)*

where \( \nabla g(z) \) and \( \nabla f(z) \) are Jacobian matrices.
Accordingly, one can recursively define repeated Lie brackets as follows:

\[ ad^0_f g = g \]
\[ ad^k_f g = [f, ad^{k-1}_f g], \quad k \geq 1. \] (2.33)

**Example 2.4**

Consider the DC motor system given in [10] as:

\[ \dot{z} = f(z) + g(z) u \]

where:

\[ f(z) = \begin{bmatrix} -az_1 \\ -bz_2 + k - cz_1 z_3 \\ \theta z_1 z_2 \end{bmatrix}, \quad g(z) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \] (2.34)

The first and second Lie brackets can be calculated as follows:

\[ ad_f g = [f, g] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -az_1 \\ -bz_2 + k - cz_1 z_3 \\ \theta z_1 z_2 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -a \\ -cz_3 \\ \theta z_2 \end{bmatrix} = \begin{bmatrix} a \\ cz_3 \\ -\theta z_2 \end{bmatrix} \] (2.35)

\[ ad^2_f g = [f, ad_f g] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ -\theta & 0 \end{bmatrix} - \begin{bmatrix} -az_1 \\ -bz_2 + k - cz_1 z_3 \\ \theta z_1 z_2 \end{bmatrix} \]

\[ - \begin{bmatrix} -a & 0 & 0 \\ -cz_3 & -b & -cz_1 \\ \theta z_2 & \theta z_1 & 0 \end{bmatrix} \begin{bmatrix} a \\ cz_3 \\ -\theta z_2 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 \\ c\theta z_1 z_2 \\ b\theta z_2 - k\theta + c\theta z_1 z_3 \end{bmatrix} - \begin{bmatrix} -a^2 \\ -acz_3 - bcz_3 + c\theta z_1 z_2 \\ a\theta z_2 + c\theta z_1 z_3 \end{bmatrix} \]

\[ = \begin{bmatrix} a^2 \\ acz_3 + bcz_3 \\ b\theta z_2 - a\theta z_2 - k\theta \end{bmatrix} \triangleq \] (2.36)
2.7 Affine Nonlinear System

In nonlinear control theory, most of today’s systems have the general form (2.15) for SISO systems and the following general form for MIMO systems:

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\vdots \\
\dot{z}_n 
\end{bmatrix} = \begin{bmatrix} f_1(z) \\
f_2(z) \\
\vdots \\
f_n(z) \end{bmatrix} u_1 + \begin{bmatrix} g_{11}(z) \\
g_{12}(z) \\
\vdots \\
g_{1n}(z) \end{bmatrix} u_1 + \cdots + \begin{bmatrix} g_{m1}(z) \\
g_{m2}(z) \\
\vdots \\
g_{mn}(z) \end{bmatrix} u_m
\]

whose short form can be written as

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\vdots \\
\dot{y}_m 
\end{bmatrix} = \begin{bmatrix} h_1(z) \\
h_2(z) \\
\vdots \\
h_m(z) \end{bmatrix}
\]

where \(z \in \mathbb{R}^n\), \(u \in \mathbb{R}^m\), and \(y \in \mathbb{R}^m\) are state variables, control inputs, and system outputs respectively. Such nonlinear systems with nonlinear state vectors and linear control input/inputs are called affine nonlinear systems in nonlinear control theories [33,34].

2.8 Function Properties

One of the important things that readers will frequently encounter in this research is some functions’ properties explained in a well known nonlinear control textbook by Khalil [10]. As what so-called Lyapunov function denoted by \(V(z)\) will frequently be utilized, then if:

1. \(V(0) = 0\) and \(V(z) > 0\) with \(z \neq 0 \Rightarrow V(z)\) is Positive definite.
2. \(-V(z)\) is positive definite \(\Rightarrow V(z)\) is Negative definite.
3. \(V(0) = 0\) and \(V(z) \geq 0\) with \(z \neq 0 \Rightarrow V(z)\) is Positive semi-definite.
4. \(-V(z)\) is positive semi-definite \(\Rightarrow V(z)\) is Negative semi-definite.
5. \(V(z) \to \infty\) as \(|z| \to \infty \Rightarrow V(z)\) is Radially unbounded.
2.9 Frobenius Theorem

This theorem is needed when discussing the conditions required for feedback linearization of nonlinear systems. It was explained in detail in [35–37]. Consider a function \( h(z) \) and a function \( q_{ij}(z) \) where \( i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n \) and \( m < n \) such that:

\[
\begin{bmatrix}
\frac{\partial h(z)}{\partial z_1} & \frac{\partial h(z)}{\partial z_2} & \cdots & \frac{\partial h(z)}{\partial z_n} \\
q_{11}(z) & q_{21}(z) & \cdots & q_{m1}(z) \\
q_{12}(z) & q_{22}(z) & \cdots & q_{m2}(z) \\
\vdots & \vdots & \ddots & \vdots \\
q_{1n}(z) & q_{2n}(z) & \cdots & q_{mn}(z)
\end{bmatrix}
\]

(results in the following set of differential equations)

\[
\begin{align*}
\frac{\partial h(z)}{\partial z_1}q_{11}(z) + \frac{\partial h(z)}{\partial z_2}q_{12}(z) + \cdots + \frac{\partial h(z)}{\partial z_n}q_{1n}(z) &= 0 \\
\frac{\partial h(z)}{\partial z_1}q_{21}(z) + \frac{\partial h(z)}{\partial z_2}q_{22}(z) + \cdots + \frac{\partial h(z)}{\partial z_n}q_{2n}(z) &= 0 \\
& \quad \vdots \\
\frac{\partial h(z)}{\partial z_1}q_{m1}(z) + \frac{\partial h(z)}{\partial z_2}q_{m2}(z) + \cdots + \frac{\partial h(z)}{\partial z_n}q_{mn}(z) &= 0
\end{align*}
\]

If the matrix

\[
\begin{bmatrix}
q_1(z) & q_2(z) & \cdots & q_m(z)
\end{bmatrix}
\]

in (2.39) has rank \( m \) at point \( z = z^* \), then there are \( n - m \) scalar functions around \( z^* \) representing the solutions of (2.40) if and only if the rank of the matrix

\[
\begin{bmatrix}
q_1(z) & q_2(z) & \cdots & q_m(z) & [q_i, q_j]
\end{bmatrix}
\]

equals \( m \) as well for all \( z \) around \( z^* \) where \([q_i, q_j]\) is the Lie bracket of any two columns in (2.41) such that the Jacobian matrix

\[
\begin{bmatrix}
\frac{\partial h_1(z)}{\partial z_1} & \frac{\partial h_1(z)}{\partial z_2} & \cdots & \frac{\partial h_1(z)}{\partial z_n} \\
\frac{\partial h_2(z)}{\partial z_1} & \frac{\partial h_2(z)}{\partial z_2} & \cdots & \frac{\partial h_2(z)}{\partial z_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial h_{n-m}(z)}{\partial z_1} & \frac{\partial h_{n-m}(z)}{\partial z_2} & \cdots & \frac{\partial h_{n-m}(z)}{\partial z_n}
\end{bmatrix}
\]

has rank \( n - m \) at \( z = z^* \). These conditions of Frobenius theorem in terms of conditions of feedback linearization are known as involutivity of a set of vector fields. The next theorem summarizes the concept of Frobenius theorem.
Theorem 2.1 Given the partial differential equations set

\[ \frac{\partial h(z)}{\partial z} \begin{bmatrix} q_1(z) & q_2(z) & \ldots & q_m(z) \end{bmatrix}_{n \times m} = 0 \]

There exist \( h_1(z), h_2(z), \ldots, h_{n-m}(z) \) satisfying the given set of equations and the set of vectors in the Jacobian matrix

\[ \nabla h = \frac{\partial h(z)}{\partial z} \]

are linearly independent if and only if the set of vector fields

\[ \begin{bmatrix} q_1(z) & q_2(z) & \ldots & q_m(z) \end{bmatrix} \]

is involutive. ♦

2.10 Relative Degree

2.10.1 Relative Degree for SISO Affine Nonlinear Systems

Consider the SISO affine nonlinear system given in (2.15) as

\[ \dot{z} = f(z) + g(z) u \]
\[ y = h(z) \]

It is well known that this system is of a relative degree \( \rho \) if one needs to differentiate \( y = h(z) \) \( \rho \) times until \( u \) the control input appears for the first time and does not vanish for every \( z \subset D \subset \mathbb{R}^n \).

Example 2.5

Consider the system of order three given by:

\[ \begin{align*}
\dot{z}_1 &= z_2^2 + 5u \\
\dot{z}_2 &= z_1 + z_3 \\
\dot{z}_3 &= -z_1 z_2 + z_1 \\
y &= z_2
\end{align*} \tag{2.44} \]

Differentiating \( y = h(z) \) in time yields

\[ \dot{y} = \dot{z}_2 = z_1 + z_3 \tag{2.45} \]
Differentiating it one more time results in
\[ \ddot{y} = \dot{z}_1 + \dot{z}_3 \]
\[ = z_1 - z_1 z_2 + z_2^2 + 5u \quad (2.46) \]

Now, one can see that the control input \( u \) showed up in \( \ddot{y} \). Thus, one can say that this system is of a relative degree \( \rho = 2 \) in \( \mathbb{R}^3 \).

As explained in Khalil’s [10], if \( f(z), g(z) \) and \( h(z) \) are smooth enough in the domain \( D \subset \mathbb{R}^n \), then the first derivative of the output \( y = h(z) \) in terms of Lie derivative is:

\[ \dot{y} = \frac{\partial h}{\partial z} [f(z) + g(z) u] \]
\[ \overset{\text{def}}{=} L_f h(z) + L_g h(z) u \]

If \( L_g h(z) = 0 \), then the first derivative of \( y \) will be \( \dot{y} = L_f h(z) \) which is independent of input \( u \). In the same manner, the second derivative is given by:

\[ \ddot{y} = \frac{\partial (L_f h)}{\partial z} [f(z) + g(z) u] \]
\[ \overset{\text{def}}{=} L_f^2 h(z) + L_g L_f h(z) u \]

where \( L_f^2 h(z) \overset{\Delta}{=} \frac{\partial (L_f h)}{\partial z} f(z) \) and \( L_g L_f h(z) \overset{\Delta}{=} \frac{\partial (L_f h)}{\partial z} g(z) \) are Lie derivative of \( L_f h(z) \) along \( f(z) \) and Lie derivative of \( L_f h(z) \) along \( g(z) \) respectively. Again, if \( L_g L_f h(z) = 0 \), then the second derivative of \( y \) will be \( \ddot{y} = L_f^2 h(z) \) which is also independent of input \( u \). Continuing like this, if \( h(z) \) satisfies:

\[ L_g L_f^{i-1} h(z) = 0, \quad i = 1, \ldots, \rho - 1 \]
\[ L_g L_f^{\rho-1} h(z) \neq 0, \quad \forall z \subset D \subset \mathbb{R}^n \]

then one can say that the system is of a relative degree \( \rho \) and

\[ y^{(\rho)} = L_f^\rho h(z) + L_g L_f^{\rho-1} h(z) u. \]

That is the input appears for the first time in the \( \rho^{th} \) derivative of the output. In brief, the next definition summarizes the relative degree notion for SISO nonlinear systems

**Definition 2.5** A SISO affine nonlinear system of the general form (2.15)

\[ \dot{z} = f(z) + g(z) u \]
\[ y = h(z) \]
with smooth enough \( f(z), g(z), \) and \( h(z) \) in the domain \( D \subset \mathbb{R}^n \), has a relative degree \( \rho \) if \( h(z) \) satisfies:

\[
L_g L_f^{i-1} h(z) = 0, \quad i = 1, \ldots, \rho - 1
\]

\[
L_g L_f^{\rho-1} h(z) \neq 0, \quad \forall \, z \subset D \subset \mathbb{R}^n
\]

so that

\[
y^{(\rho)} = L_f^\rho h(z) + L_g L_f^{\rho-1} h(z) u.
\]

### 2.10.2 Relative Degree for MIMO Affine Nonlinear Systems

The analysis used so far for the relative degree of SISO affine nonlinear systems can be expanded to find the relative degree of MIMO affine nonlinear systems as explained in [29]. Consider MIMO affine nonlinear system given in (2.38) as

\[
\dot{z}_n = f(z) + g_1(z) u_1 + g_2(z) u_2 + \cdots + g_m(z) u_m
\]

\[
y_1 = h_1
\]

\[
\vdots
\]

\[
y_m = h_m
\]

If \( f(z), g_i(z), \) and \( h_i(z) \) are smooth enough in the domain \( D \subset \mathbb{R}^n \), then:

**Considering output function** \( y_1 = h_1(z) \)

**1st derivative**

\[
y_1^{(1)} = \frac{\partial h_1}{\partial z} [f(z) + g_1(z) u_1 + g_2(z) u_2 + \cdots + g_m(z) u_m]
\]

\[
def = L_f h_1(z) + L_{g_1} h_1(z) u_1 + L_{g_2} h_1(z) u_2 + \cdots + L_{g_m} h_1(z) u_m
\]

If

\[
L_{g_1} h_1(z) = L_{g_2} h_1(z) = \cdots = L_{g_m} h_1(z) = 0
\]

then

\[
y_1^{(1)} = L_f h_1(z)
\]

which is independent of \( u_i \). Similarly
2\textsuperscript{nd} derivative

\[ y_1^{(2)} = \frac{\partial (L_f h_1)}{\partial z} \left[ f(z) + g_1(z) u_1 + g_2(z) u_2 + \ldots + g_m(z) u_m \right] \]

\[ \overset{\text{def}}{=} L_f^2 h_1(z) + L_{g_1} L_f h_1(z) u_1 + L_{g_2} L_f h_1(z) u_2 + \ldots + L_{g_m} L_f h_1(z) u_m \]

If

\[ L_{g_1} L_f h_1(z) = L_{g_2} L_f h_1(z) = \ldots = L_{g_m} L_f h_1(z) = 0 \]

then

\[ y_1^{(2)} = L_f^2 h_1(z) \]

Continuing like this

\((\rho_1 - 1)\textsuperscript{th}\) derivative

\[ y_1^{(\rho_1 - 1)} = \frac{\partial (L_f^{\rho_1-2} h_1)}{\partial z} \left[ f(z) + g_1(z) u_1 + g_2(z) u_2 + \ldots + g_m(z) u_m \right] \]

\[ \overset{\text{def}}{=} L_f^{\rho_1-1} h_1(z) + L_{g_1} L_f^{\rho_1-2} h_1(z) u_1 + L_{g_2} L_f^{\rho_1-2} h_1(z) u_2 + \ldots + L_{g_m} L_f^{\rho_1-2} h_1(z) u_m \]

If

\[ L_{g_1} L_f^{\rho_1-2} h_1(z) = L_{g_2} L_f^{\rho_1-2} h_1(z) = \ldots = L_{g_m} L_f^{\rho_1-2} h_1(z) = 0 \]

then

\[ y_1^{(\rho_1 - 1)} = L_f^{\rho_1-1} h_1(z) \]

\((\rho_1)\textsuperscript{th}\) derivative

\[ y_1^{(\rho_1)} = \frac{\partial (L_f^{\rho_1-1} h_1)}{\partial z} \left[ f(z) + g_1(z) u_1 + g_2(z) u_2 + \ldots + g_m(z) u_m \right] \]

\[ \overset{\text{def}}{=} L_f^{\rho_1} h_1(z) + L_{g_1} L_f^{\rho_1-1} h_1(z) u_1 + L_{g_2} L_f^{\rho_1-1} h_1(z) u_2 + \ldots + L_{g_m} L_f^{\rho_1-1} h_1(z) u_m \]

If at least one

\[ L_{g_i} L_f^{\rho_1-1} h_1(z) \neq 0, \quad i = 1, 2, \ldots, m. \]

then MIMO affine nonlinear system (2.38) has a sub-relative degree \(\rho_1\) corresponding to output function \(y_1 = h_1(z)\).
Considering output function \( y_2 = h_2(z) \)

1\textsuperscript{st} derivative

\[
y^{(1)}_2 = \frac{\partial h_2}{\partial z} \left[ f(z) + g_1(z)u_1 + g_2(z)u_2 + \ldots + g_m(z)u_m \right]
\]

\[\overset{\text{def}}{=} L_fh_2(z) + L_{g_1}h_2(z)u_1 + L_{g_2}h_2(z)u_2 + \ldots + L_{g_m}h_2(z)u_m\]

If

\[
L_{g_1}h_2(z) = L_{g_2}h_2(z) = \ldots = L_{g_m}h_2(z) = 0
\]

then

\[
y^{(1)}_2 = L_fh_2(z)
\]

which is independent of \( u_i \). Similarly

2\textsuperscript{nd} derivative

\[
y^{(2)}_2 = \frac{\partial (L_fh_2)}{\partial z} \left[ f(z) + g_1(z)u_1 + g_2(z)u_2 + \ldots + g_m(z)u_m \right]
\]

\[\overset{\text{def}}{=} L^2_fh_2(z) + L_{g_1}L_fh_2(z)u_1 + L_{g_2}L_fh_2(z)u_2 + \ldots + L_{g_m}L_fh_2(z)u_m\]

If

\[
L_{g_1}L_fh_2(z) = L_{g_2}L_fh_2(z) = \ldots = L_{g_m}L_fh_2(z) = 0
\]

then

\[
y^{(2)}_2 = L^2_fh_2(z)
\]

Continuing like this

\((\rho_2 - 1)\textsuperscript{th}\) derivative

\[
y^{(\rho_2-1)}_2 = \frac{\partial (L^{\rho_2-2}_fh_2)}{\partial z} \left[ f(z) + g_1(z)u_1 + g_2(z)u_2 + \ldots + g_m(z)u_m \right]
\]

\[\overset{\text{def}}{=} L^{\rho_2-1}_fh_2(z) + L_{g_1}L^{\rho_2-2}_fh_2(z)u_1 + L_{g_2}L^{\rho_2-2}_fh_2(z)u_2 + \ldots + L_{g_m}L^{\rho_2-2}_fh_2(z)u_m\]
If
\[ L_{g_1}L_f^{\rho_2-2}h_2(z) = L_{g_2}L_f^{\rho_2-2}h_2(z) = \ldots = L_{g_m}L_f^{\rho_2-2}h_2(z) = 0 \]
then
\[ y_2^{(\rho_2-1)} = L_f^{\rho_2-1}h_2(z) \]

\((\rho_2)^{th}\) derivative

\[ y_2^{(\rho_2)} = \frac{\partial (L_f^{\rho_2-1}h_2)}{\partial z} \left[ f(z) + g_1(z)u_1 + g_2(z)u_2 + \ldots + g_m(z)u_m \right] \]
\[ \overset{\text{def}}{=} L_f^{\rho_2}h_2(z) + L_{g_1}L_f^{\rho_2-1}h_2(z)u_1 + L_{g_2}L_f^{\rho_2-1}h_2(z)u_2 + \ldots + L_{g_m}L_f^{\rho_2-1}h_2(z)u_m \]

If at least one
\[ L_{g_i}L_f^{\rho_2-1}h_2(z) \neq 0, \quad i = 1, 2, \ldots, m. \]
then MIMO affine nonlinear system (2.38) has a sub-relative degree \(\rho_2\) corresponding to output function \(y_2 = h_2(z)\). Continuing like this

Considering output function \(y_{m-1} = h_{m-1}(z)\)

1st derivative

\[ y_{m-1}^{(1)} = \frac{\partial h_{m-1}}{\partial z} \left[ f(z) + g_1(z)u_1 + g_2(z)u_2 + \ldots + g_m(z)u_m \right] \]
\[ \overset{\text{def}}{=} L_f h_{m-1}(z) + L_{g_1}h_{m-1}(z)u_1 + L_{g_2}h_{m-1}(z)u_2 + \ldots + L_{g_m}h_{m-1}(z)u_m \]

If
\[ L_{g_1}h_{m-1}(z) = L_{g_2}h_{m-1}(z) = \ldots = L_{g_m}h_{m-1}(z) = 0 \]
then
\[ y_{m-1}^{(1)} = L_f h_{m-1}(z) \]
which is independent of \(u_i\). Similarly
If then
Continuing like this
If
then $y_{m-1}^{(2)} = L_f^2 h_{m-1} (z)$
Continuing like this

$(\rho_{m-1} - 1)^{th}$ derivative

$y_{m-1}^{(\rho_{m-1} - 1)} = \frac{\partial \left( L_f^{\rho_{m-1} - 2} h_{m-1} \right)}{\partial z} \left[ f (z) + g_1 (z) u_1 + g_2 (z) u_2 + \ldots + g_m (z) u_m \right]$
\[= L_f^{\rho_{m-1} - 1} h_{m-1} (z) + L_g_1 L_f^{\rho_{m-1} - 2} h_{m-1} (z) u_1 + L_g_2 L_f^{\rho_{m-1} - 2} h_{m-1} (z) u_2 + \ldots + L_g_m L_f^{\rho_{m-1} - 2} h_{m-1} (z) u_m \]
If
$\quad L_g_1 L_f^{\rho_{m-1} - 2} h_{m-1} (z) = L_g_2 L_f^{\rho_{m-1} - 2} h_{m-1} (z) = \ldots = L_g_m L_f^{\rho_{m-1} - 2} h_{m-1} (z) = 0$
then
$\quad y_{m-1}^{(\rho_{m-1} - 1)} = L_f^{\rho_{m-1} - 1} h_{m-1} (z)$

$(\rho_{m-1})^{th}$ derivative

$y_{m-1}^{(\rho_{m-1})} = \frac{\partial \left( L_f^{\rho_{m-1} - 1} h_{m-1} \right)}{\partial z} \left[ f (z) + g_1 (z) u_1 + g_2 (z) u_2 + \ldots + g_m (z) u_m \right]$
\[= L_f^{\rho_{m-1}} h_{m-1} (z) + L_g_1 L_f^{\rho_{m-1} - 1} h_{m-1} (z) u_1 + L_g_2 L_f^{\rho_{m-1} - 1} h_{m-1} (z) u_2 + \ldots + L_g_m L_f^{\rho_{m-1} - 1} h_{m-1} (z) u_m \]
If at least one
\[ \quad L_g_i L_f^{\rho_{m-1} - 1} h_{m-1} (z) \neq 0, \quad i = 1, 2, \ldots, m. \]
then MIMO affine nonlinear system (2.38) has a sub-relative degree $\rho_{m-1}$ corresponding to output function $y_{m-1} = h_{m-1} (z)$.

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Considering output function \( y_m = h_m(z) \)

**1st derivative**

\[
y_m^{(1)} = \frac{\partial h_m}{\partial z} \left[ f(z) + g_1(z) u_1 + g_2(z) u_2 + \ldots + g_m(z) u_m \right]
\]

\[
def = L_f h_m(z) + L_{g_1} h_m(z) u_1 + L_{g_2} h_m(z) u_2 + \ldots + L_{g_m} h_m(z) u_m
\]

If \( L_{g_1} h_m(z) = L_{g_2} h_m(z) = \ldots = L_{g_m} h_m(z) = 0 \)

then

\[
y_m^{(1)} = L_f h_m(z)
\]

which is independent of \( u_i \). Similarly

**2nd derivative**

\[
y_m^{(2)} = \frac{\partial}{\partial z} \left( L_f h_m \right) \left[ f(z) + g_1(z) u_1 + g_2(z) u_2 + \ldots + g_m(z) u_m \right]
\]

\[
def = L_f^2 h_m(z) + L_{g_1} L_f h_m(z) u_1 + L_{g_2} L_f h_m(z) u_2 + \ldots + L_{g_m} L_f h_m(z) u_m
\]

If \( L_{g_1} L_f h_m(z) = L_{g_2} L_f h_m(z) = \ldots = L_{g_m} L_f h_m(z) = 0 \)

then

\[
y_m^{(2)} = L_f^2 h_m(z)
\]

Continuing like this

**(\( \rho_m - 1 \))th derivative**

\[
y_m^{(\rho_m - 1)} = \frac{\partial}{\partial z} \left( L_f^{\rho_m - 2} h_m \right) \left[ f(z) + g_1(z) u_1 + g_2(z) u_2 + \ldots + g_m(z) u_m \right]
\]

\[
def = L_f^{\rho_m - 1} h_m(z) + L_{g_1} L_f^{\rho_m - 2} h_m(z) u_1 + L_{g_2} L_f^{\rho_m - 2} h_m(z) u_2 + \ldots +
\]

\[
L_{g_m} L_f^{\rho_m - 2} h_m(z) u_m
\]
If 
\[ L_g L_f^{\rho_m - 2} h_m(z) = L_g L_f^{\rho_m - 2} h_m(z) = \ldots = L_g L_f^{\rho_m - 2} h_m(z) = 0 \]
then 
\[ y_m^{(\rho_m - 1)} = L_f^{\rho_m - 1} h_m(z) \]

(\(\rho_m\))th derivative 

\[ y_m^{(\rho_m)} = \frac{\partial (L_f^{\rho_m - 1} h_m)}{\partial z} \left[ f(z) + g_1(z) u_1 + g_2(z) u_2 + \ldots + g_m(z) u_m \right] \]

\[ \overset{\text{def}}{=} L_f^{\rho_m} h_m(z) + L_g L_f^{\rho_m - 1} h_m(z) u_1 + L_g L_f^{\rho_m - 1} h_m(z) u_2 + \ldots + L_g L_f^{\rho_m - 1} h_m(z) u_m \]

If at least one 
\[ L_g L_f^{\rho_m - 1} h_m(z) \neq 0, \quad i = 1, 2, \ldots, m. \]
then MIMO affine nonlinear system (2.38) has a sub-relative degree \(\rho_m\) corresponding to output function \(y_m = h_m(z)\) and hence if the matrix

\[
\begin{bmatrix}
L_{g_1} L_f^{\rho_1} h_1(z) & L_{g_2} L_f^{\rho_1} h_1(z) & \ldots & L_{g_m} L_f^{\rho_1} h_1(z) \\
L_{g_1} L_f^{\rho_2} h_2(z) & L_{g_2} L_f^{\rho_2} h_2(z) & \ldots & L_{g_m} L_f^{\rho_2} h_2(z) \\
\vdots & \vdots & \ddots & \vdots \\
L_{g_1} L_f^{\rho_m} h_m(z) & L_{g_2} L_f^{\rho_m} h_m(z) & \ldots & L_{g_m} L_f^{\rho_m} h_m(z)
\end{bmatrix}
\]

is non-singular at \(z = z^*\), then MIMO affine nonlinear system of the form (2.38) is of a vector relative degree

\[ \rho = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_m\} \]

The following definition summarizes the relative degree notion for MIMO affine nonlinear systems.

**Definition 2.6** A MIMO affine nonlinear system of the general form (2.38)

\[
\dot{z}_n = f(z) + g_1(z) u_1 + g_2(z) u_2 + \cdots + g_m(z) u_m \\
y_i = h_i(z), \quad i = 1, \ldots, m
\]

with smooth enough \(f(z), g_i(z),\) and \(h_i(z)\) in the domain \(D \subset \mathbb{R}^n\), has a vector relative degree

\[ \rho = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_m\} \quad (2.49) \]

if the following conditions are true for every output function \(y_i = h_i(z)\).
1. 

\[ L_{g_1} L_f^k h_i(z) = L_{g_2} L_f^k h_i(z) = \ldots = L_{g_m} L_f^k h_i(z) = 0, \quad k = 0, 1, \ldots, \rho_i - 2. \quad (2.50) \]

2. At least one element is not zero in the row vector

\[
\begin{bmatrix}
L_{g_1} L_f^{\rho_i-1} h_i(z) \\
L_{g_2} L_f^{\rho_i-1} h_i(z) \\
\vdots \\
L_{g_m} L_f^{\rho_i-1} h_i(z)
\end{bmatrix}
\]

so that

\[ y_{(\rho_i)} = L_f^{\rho_i} h_i(z) + \sum_{j=1}^{m} L_{g_j} L_f^{\rho_i-1} h_i(z) u_j \quad (2.52) \]

and for the given MIMO system

3. The following matrix is non-singular in the neighborhood of \( z = z^* \)

\[
\begin{bmatrix}
L_{g_1} L_f^{\rho_1-1} h_1(z) & L_{g_2} L_f^{\rho_1-1} h_1(z) & \ldots & L_{g_m} L_f^{\rho_1-1} h_1(z) \\
L_{g_1} L_f^{\rho_2-1} h_2(z) & L_{g_2} L_f^{\rho_2-1} h_2(z) & \ldots & L_{g_m} L_f^{\rho_2-1} h_2(z) \\
\vdots & \vdots & \ddots & \vdots \\
L_{g_1} L_f^{\rho_m-1} h_m(z) & L_{g_2} L_f^{\rho_m-1} h_m(z) & \ldots & L_{g_m} L_f^{\rho_m-1} h_m(z)
\end{bmatrix}
\]

Example 2.6

In this example, obtaining a MIMO system’s relative degree will be shown by considering the mathematical model for the proton exchange membrane fuel cell discussed in [38]. The multi-input single-output dynamic model was first derived and then using the extended system approach discussed in [9] and [39] to introduce extra states and outputs, the model was converted into what so-called square MIMO system where the number of control inputs equals the number of system outputs.

![Proton exchange membrane fuel cell](image-url)
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix}
= -\begin{bmatrix}
\frac{U_A V_a}{V_a} x_1 \\
\frac{U_A V_c}{V_c} x_2 \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
R_c T \\
0 \\
0 \\
0
\end{bmatrix}
u_1 + \begin{bmatrix}
0 \\
R_c T/V_c \\
1 \\
0
\end{bmatrix} u_2 - \begin{bmatrix}
\frac{R_c T}{2FV_a} \\
0 \\
0 \\
1
\end{bmatrix} u_3
\]  \hspace{1cm} (2.54a)

\[
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
= \begin{bmatrix}
h_1 (x) \\
h_2 (x) \\
h_3 (x)
\end{bmatrix}
= \begin{bmatrix}
V_{FC} \\
x_3 \\
x_4
\end{bmatrix}
\]  \hspace{1cm} (2.54b)

The output voltage for a single PEMFC is

\[
y_1 = h_1 (x) = V_{FC} = \frac{\Delta G}{2F} + \frac{\Delta S}{2F} (T - T_0) + \frac{R_c T}{2F} \left[ \ln x_1 + \frac{1}{2} \ln x_2 \right]
+ \left\{ -0.948 + \left[ (286 \times 10^{-5}) + (20 \times 10^{-5}) \ln A + (4.3 \times 10^{-5}) \ln \left( \frac{x_1}{1.09 \times 10^6 \times e^{(77/T)}} \right) \right] T
+ (7.6 \times 10^{-5}) T \ln \left( \frac{x_2}{5.08 \times 10^6 \times e^{(-408/T)}} \right) + (-1.93 \times 10^{-4}) T \ln u_3 \right\}
- (R_M + R_C) u_4 + b \ln \left[ 1 - \frac{J}{J_{max}} \right]
\]  \hspace{1cm} (2.55)

Tables 2.1 - 2.4 present the definition for each of the symbols and parameters used in (2.54) and (2.55)

### Table 2.1: PEMFC states legend.

<table>
<thead>
<tr>
<th>State</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$P_{H_2}$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$P_{O_2}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$y_2$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$y_3$</td>
</tr>
</tbody>
</table>

### Table 2.2: PEMFC inputs legend.

<table>
<thead>
<tr>
<th>Input</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>$v_{H_{2\text{in}}}$</td>
</tr>
<tr>
<td>$u_2$</td>
<td>$v_{O_{2\text{in}}}$</td>
</tr>
<tr>
<td>$u_3$</td>
<td>$i$</td>
</tr>
</tbody>
</table>
Table 2.3: PEMFC outputs legend.

<table>
<thead>
<tr>
<th>Output</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 )</td>
<td>( V_{FC} )</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>( x_3 )</td>
</tr>
<tr>
<td>( y_3 )</td>
<td>( x_4 )</td>
</tr>
</tbody>
</table>

Table 2.4: PEMFC parameters definitions.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_{H_2}, P_{O_2} )</td>
<td>Partial pressure of hydrogen and oxygen respectively</td>
</tr>
<tr>
<td>( v_{H_2}, v_{O_2} )</td>
<td>Inlet mole flow rate of hydrogen and oxygen respectively</td>
</tr>
<tr>
<td>( i )</td>
<td>Cell’s operating current (A)</td>
</tr>
<tr>
<td>( V_a, V_c )</td>
<td>Anode and cathode volumes respectively</td>
</tr>
<tr>
<td>( U )</td>
<td>Fuel rate</td>
</tr>
<tr>
<td>( A )</td>
<td>Flow area</td>
</tr>
<tr>
<td>( \Delta G )</td>
<td>Gibb’s free energy change (J/mol)</td>
</tr>
<tr>
<td>( F )</td>
<td>Faraday’s constant (96,487 C/mol)</td>
</tr>
<tr>
<td>( \Delta S )</td>
<td>Standard mole entropy change (J/mol)</td>
</tr>
<tr>
<td>( T )</td>
<td>Cell’s operating temperature (T)</td>
</tr>
<tr>
<td>( T^\circ )</td>
<td>Cell’s reference temperature (T)</td>
</tr>
<tr>
<td>( R_m )</td>
<td>Proton exchange membrane equivalent resistance</td>
</tr>
<tr>
<td>( R_C )</td>
<td>The equivalent resistance of external circuit and it is assumed to be constant</td>
</tr>
<tr>
<td>( R_o )</td>
<td>Gas constant (8.315 J/mol.k)</td>
</tr>
<tr>
<td>( b )</td>
<td>A variable coefficient subject to cell’s operating conditions (V)</td>
</tr>
<tr>
<td>( J )</td>
<td>Current density of the cell (A/cm²)</td>
</tr>
<tr>
<td>( J_{max} )</td>
<td>Maximum current density (500 – 1500 mA/cm²)</td>
</tr>
</tbody>
</table>

Considering output function \( y_1 = h_1(x) = V_{FC} \)

1st derivative

\[
y_1^{(1)} = \frac{\partial h_1}{\partial x} \left[ f(x) + g_1(x) u_1 + g_2(x) u_2 + g_3(x) u_3 \right]
\]

\[= L_f h_1(x) + L_{g_1} h_1(x) u_1 + L_{g_2} h_1(x) u_2 + L_{g_3} h_1(z) u_3\]
This yields

\[ L_f h_1(x) = \begin{bmatrix} \frac{R_c T}{2F x_1} + \left(4.3 \times 10^{-5}\right) T & \frac{R_c T}{4F x_2} + \left(7.6 \times 10^{-5}\right) T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{R_c T}{V_a} \\\n-\frac{U_A V_c}{V_a} \end{bmatrix} u_1 \]

\[ = -\left(\frac{R_c T}{2F} + (4.3 \times 10^{-5}) T\right) \frac{R_c T}{V_a x_1} u_1 \]

\[ L_{g_1} h_1(x) u_1 = \begin{bmatrix} \frac{R_c T}{2F x_1} + \left(4.3 \times 10^{-5}\right) T & \frac{R_c T}{4F x_2} + \left(7.6 \times 10^{-5}\right) T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{R_c T}{V_c} \end{bmatrix} u_2 \]

\[ = \left(\frac{R_c T}{4F} + (7.6 \times 10^{-5}) T\right) \frac{R_c T}{V_c x_2} u_2 \]

\[ L_{g_2} h_1(x) u_2 = \begin{bmatrix} \frac{R_c T}{2F x_1} + \left(4.3 \times 10^{-5}\right) T & \frac{R_c T}{4F x_2} + \left(7.6 \times 10^{-5}\right) T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{R_c T}{V_c} \\ 1 \end{bmatrix} u_3 \]

\[ = -\left(\frac{R_c T}{2F} + (4.3 \times 10^{-5}) T\right) \frac{R_c T}{2F V_a x_1} - \left(\frac{R_c T}{4F} + (7.6 \times 10^{-5}) T\right) \frac{R_c T}{4F V_c x_2} u_3 \]

Thus

\[ L_f^{\rho_1} h_1(x) = L_f h_1(x) \]

\[ L_{g_1} L_f^{\rho_1-1} h_1(x) u_1 = L_{g_1} L_f^{0} h_1(x) u_1 \]

\[ L_{g_2} L_f^{\rho_1-1} h_1(x) u_2 = L_{g_2} L_f^{0} h_1(x) u_2 \]

\[ L_{g_3} L_f^{\rho_1-1} h_1(x) u_3 = L_{g_3} L_f^{0} h_1(x) u_3 \]

and hence the system (2.54) has sub-relative degree \( \rho_1 = 1 \) corresponding to \( y_1 = h_1(x) = V_{FC} \). The same way,
Considering output function \( y_2 = h_2(x) = x_3 \)

1st derivative

\[
y_2^{(1)} = \frac{\partial h_2}{\partial x} \left[ f(x) + g_1(x) u_1 + g_2(x) u_2 + g_3(x) u_3 \right]
\]

This yields

\[
L_f h_2(x) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{UA}{V_a} x_1 \\ -\frac{UA}{V_c} x_2 \\ 0 \\ 0 \end{bmatrix} = 0
\]

\[
L_{g_1} h_2(x) u_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{R_v T}{V_a} \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 = 0
\]

\[
L_{g_2} h_2(x) u_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{R_v T}{V_c} \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 = u_2
\]

\[
L_{g_3} h_2(z) u_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{R_v T}{2FV_a} \\ -\frac{R_v T}{4FV_c} \\ 0 \\ -1 \end{bmatrix} u_3 = 0
\]

Thus

\[
L_f^{\rho_2} h_2(x) = L_f h_2(x)
\]

\[
L_{g_1} L_f^{\rho_2-1} h_2(x) u_1 = L_{g_1} L_f^0 h_2(x) u_1
\]

\[
L_{g_2} L_f^{\rho_2-1} h_2(x) u_2 = L_{g_2} L_f^0 h_2(x) u_2
\]

\[
L_{g_3} L_f^{\rho_2-1} h_2(x) u_3 = L_{g_3} L_f^0 h_2(x) u_3
\]

and hence the system (2.54) has sub-relative degree \( \rho_2 = 1 \) corresponding to \( y_2 = h_2(x) = x_3 \).

Similarly,
Considering output function $y_3 = h_3(x) = x_4$

1st derivative

$$y_3^{(1)} = \frac{\partial h_3}{\partial x} \left[ f(x) + g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3 \right]$$

$$= L_f h_3(x) + L_{g_1} h_3(x) u_1 + L_{g_2} h_3(x) u_2 + L_{g_3} h_3(z) u_3$$

This yields

$$L_f h_2(x) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-UA}{V_a} x_1 \\ \frac{-UA}{V_c} x_2 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$L_{g_1} h_2(x) u_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{R_o T}{V_a} \\ 0 \\ 0 \end{bmatrix} u_1 = 0$$

$$L_{g_2} h_2(x) u_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{R_o T}{V_c} \\ 1 \\ 0 \end{bmatrix} u_2 = u_2$$

$$L_{g_3} h_2(z) u_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-R_o T}{2 F V_a} \\ \frac{-R_o T}{4 F V_c} \\ 0 \\ -1 \end{bmatrix} u_3 = -u_3$$

Thus

$$L_f^{\rho_2} h_2(x) = L_f h_2(x)$$

$$L_{g_1} L_f^{\rho_3-1} h_3(x) u_1 = L_{g_1} L_f^0 h_3(x) u_1$$

$$L_{g_2} L_f^{\rho_3-1} h_3(x) u_2 = L_{g_2} L_f^0 h_3(x) u_2$$

$$L_{g_3} L_f^{\rho_3-1} h_2(x) u_3 = L_{g_3} L_f^0 h_3(x) u_3$$

and hence the system (2.54) has sub-relative degree $\rho_3 = 1$ corresponding to the output function $y_3 = h_3(x) = x_4$. Accordingly, the proton exchange membrane fuel cell system is of a vector relative degree $\rho = \{\rho_1, \rho_2, \rho_3\} = \{1, 1, 1\}$. △
2.11 Conditions for Feedback Linearization

Before discussing the feedback linearization as a common nonlinear control method, the dissertation reviews the required conditions for nonlinear systems to be exactly feedback linearizable and shows how they can be tested for those conditions. For SISO affine nonlinear systems, the necessary conditions were discussed thoroughly in [1], [29], [36] and [40–43].

Consider the SISO affine nonlinear system given in (2.15) without an output as

\[
\dot{z} = f(z) + g(z) u
\]

where \( z \in R^n \) and \( u \in R \) are respectively system’s states and control input. If there exists an output function say \( \phi(z) \) that satisfies

\[
L_g L_f^0 \phi(z) = L_g L_f^2 \phi(z) = \ldots = L_g L_f^{n-2} \phi(z) = 0 \quad (2.56a)
\]

\[
L_g L_f^{n-1} \phi(z) \neq 0 \quad (2.56b)
\]

such that the relative degree of the system equals to its order \( \rho = n \) and the Jacobian

\[
\frac{\partial T(z)}{\partial z} = \begin{bmatrix}
\frac{\partial \phi(z)}{\partial z_1} & \frac{\partial \phi(z)}{\partial z_2} & \ldots & \frac{\partial \phi(z)}{\partial z_n} \\
\frac{\partial (L_f \phi(z))}{\partial z_1} & \frac{\partial (L_f \phi(z))}{\partial z_2} & \ldots & \frac{\partial (L_f \phi(z))}{\partial z_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial (L_f^{n-1} \phi(z))}{\partial z_1} & \frac{\partial (L_f^{n-1} \phi(z))}{\partial z_2} & \ldots & \frac{\partial (L_f^{n-1} \phi(z))}{\partial z_n}
\end{bmatrix}
\]

(2.57)

is non-singular at \( z^* \), then the given SISO affine nonlinear system with smooth \( f(z) \) and \( g(z) \) is exactly feedback linearizable in the neighborhood of \( z^* \). This is summarized in [29] in the following lemma

**Lemma 2.2** A SISO affine nonlinear system

\[
\dot{z} = f(z) + g(z) u
\]

is exactly feedback linearizable in the neighborhood of \( z^* \) if there is an output function \( \phi(z) \) that results in the system’s relative degree \( \rho \) that equals to the system’s order \( n \). ◊

To obtain the output function \( \phi(z) \), one needs to solve (2.56) but first has to confirm its existence. Here, the use of Forbenius theorem comes to play. It was previously proved in [43] that the Lie derivative of \( \phi(z) \) along the Lie bracket of the two vectors \( f(z) \) and \( g(z) \) is defined as

\[
L_{[f,g]} \phi(z) = L_{ad_{f,g}} \phi(z) = L_f L_g \phi(z) - L_g L_f \phi(z)
\]

(2.58)
Proof 2.1

\[ L_{f(z)}L_{g(z)}\phi(z) = \frac{\partial (L_{g(z)}\phi(z))}{\partial z} f(z) \]

and

\[ \frac{\partial (L_{g(z)}\phi(z))}{\partial z} = \frac{\partial}{\partial z} \left( \frac{\partial \phi(z)}{\partial z} g(z) \right) = \left( g(z)^T \frac{\partial^2 \phi(z)}{\partial z^2} + \frac{\partial \phi(z)}{\partial z} \frac{\partial g(z)}{\partial z} \right) \]

Thus

\[ L_{f(z)}L_{g(z)}\phi(z) = g(z)^T \frac{\partial^2 \phi(z)}{\partial z^2} f(z) + \frac{\partial \phi(z)}{\partial z} \frac{\partial g(z)}{\partial z} f(z) \]

Similarly

\[ L_{g(z)}L_{f(z)}\phi(z) = f(z)^T \frac{\partial^2 \phi(z)}{\partial z^2} g(z) + \frac{\partial \phi(z)}{\partial z} \frac{\partial f(z)}{\partial z} g(z) \]

Hence

\[ L_{f(z)}L_{g(z)}\phi(z) - L_{g(z)}L_{f(z)}\phi(z) = \frac{\partial \phi(z)}{\partial z} \left( \frac{\partial g(z)}{\partial z} f(z) - \frac{\partial f(z)}{\partial z} g(z) \right) \]
\[ = \frac{\partial \phi(z)}{\partial z} [f, g] \]
\[ = L_{[f,g]}\phi(z) \quad \square \]

Consequently, according to detailed proof in [29] and [36] the partial differential equation (2.56) can be re-written as

\[ L_{g(z)}\phi(z) = L_{\text{ad}_f g} \phi(z) = L_{\text{ad}_f^2 g} \phi(z) = \ldots = L_{\text{ad}_f^{n-2} g} \phi(z) = 0 \quad (2.59a) \]
\[ L_{\text{ad}_f^{n-1} g} \phi(z) \neq 0 \quad (2.59b) \]

in which (2.59a) is the partial differential equations set that according to Lie derivative definition can be written as

\[ \frac{\partial \phi(z)}{\partial z} \left[ g(z) \text{ ad}_f g(z) \text{ ad}_f^2 g(z) \ldots \text{ ad}_f^{n-2} g(z) \right] = 0 \quad (2.60) \]

From [40], and in comparison with Frobenius theorem discussed in section 2.9, the following theorem can be deduced.

**Theorem 2.2** Given a SISO affine nonlinear system of the form

\[ \dot{z} = f(z) + g(z) u \]

with smooth enough \( f(z) \) and \( g(z) \), an output function \( \phi(z) \) that results in the system’s relative degree \( \rho = n \) must exist if and only if
1. The vector fields \([g(z), ad_fg(z), \ldots, ad_f^{n-1}g(z)]\) are linearly independent, that is the rank of the matrix \(G(z) = \rho([g(z), ad_fg(z), \ldots, ad_f^{n-1}g(z)])\) = \(n\).

2. The set \(\{g(z), ad_fg(z), \ldots, ad_f^{n-2}g(z)\}\) is involutive. ♦

where:

\[
\begin{align*}
ad_f^0g(z) &= g(z) \\
ad_f^1g(z) &= [f, g] \\
ad_f^2g(z) &= [f, [f, g]] \\
&\vdots \\
ad_f^i g(z) &= [f, ad_f^{i-1}g(z)], \quad i = 1, 2, \ldots
\end{align*}
\]

stand for the successive Lie brackets of the two vector fields \(f(z)\) and \(g(z)\). However, testing involutivity of a set of vector fields \(\{v_1, v_2, \ldots, v_n\}\) is accomplished by testing whether:

\[
\begin{align*}
rank(v_1(z), v_2(z), \ldots, v_n(z)) &= rank(v_1(z), v_2(z), \ldots, v_n(z), [v_i, v_j](z)) \quad \forall \quad z \ & \ & \forall \quad i, j.
\end{align*}
\]

This means that the new vector field \([v_i, v_j](z)\) is linearly dependent with the other \(n\) vector fields and still in the same span and does not create a new direction [29].

**Example 2.7**

Considering the nonlinear system given in nonlinear control notes by Prof. Raúl Ordóñez:

\[
\begin{align*}
\dot{z}_1 &= \sin(z_1 + z_3) + |z_2 - z_3^2| + 2u \\
\dot{z}_2 &= 2z_1z_3 + z_3 \\
\dot{z}_3 &= z_1
\end{align*}
\]

such that

\[
\begin{align*}
f(z) &= \begin{bmatrix} 
\sin(z_1 + z_3) + |z_2 - z_3^2| \\
2z_1z_3 + z_3 \\
z_1
\end{bmatrix}, \quad g(z) = \begin{bmatrix} 2 \\
0 \\
0
\end{bmatrix}
\end{align*}
\]

To check whether this system is exactly feedback linearizable or not, one needs to apply the conditions for exact feedback linearizability of nonlinear systems. Starting with finding the two Lie brackets \(ad_fg(z)\) and \(ad_f^2g(z)\)

\[
ad_fg(z) = \frac{\partial g(z)}{\partial z} f(z) - \frac{\partial f(z)}{\partial z} g(z)
\]
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
sin(z_1 + z_3) + |z_2 - z_3^2| \\
2z_1 z_3 + z_3 \\
z_1
\end{bmatrix}
- \begin{bmatrix}
cos z_1 \cos z_3 - \sin z_1 \sin z_3 \frac{z_2 - z_3^2}{|z_2 - z_3^2|} - \sin z_1 \sin z_3 + \cos z_1 \cos z_3 \frac{z_2 - z_3^2}{|z_2 - z_3^2|} 2z_3 \\
2z_3 & 0 & 2z_1 + 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
2 \\
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
2 \cos z_1 \cos z_3 - 2 \sin z_1 \sin z_3 \\
4z_3 \\
2
\end{bmatrix}
= \begin{bmatrix}
-2 \cos z_1 \cos z_3 + 2 \sin z_1 \sin z_3 \\
-4z_3 \\
-2
\end{bmatrix}
\]

\[
ad_{fg}^2 (z) = \frac{\partial (ad_{fg} (z))}{\partial z} f (z) - \frac{\partial f (z)}{\partial z} ad_{fg} (z)
\]

\[
\begin{bmatrix}
2 \sin z_1 \cos z_3 + 2 \cos z_1 \sin z_3 & 0 & 2 \cos z_1 \sin z_3 + 2 \sin z_1 \cos z_3 \\
0 & 0 & -4 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
sin(z_1 + z_3) + |z_2 - z_3^2| \\
2z_1 z_3 + z_3 \\
z_1
\end{bmatrix}
- \begin{bmatrix}
cos z_1 \cos z_3 - \sin z_1 \sin z_3 \frac{z_2 - z_3^2}{|z_2 - z_3^2|} - \sin z_1 \sin z_3 + \cos z_1 \cos z_3 \frac{z_2 - z_3^2}{|z_2 - z_3^2|} 2z_3 \\
2z_3 & 0 & 2z_1 + 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-2 \cos z_1 \cos z_3 + 2 \sin z_1 \sin z_3 \\
-4z_3 \\
-2
\end{bmatrix}
= \begin{bmatrix}
\alpha \\
-4z_1 \\
0
\end{bmatrix}
- \begin{bmatrix}
\beta \\
(-2 \cos z_1 \cos z_3 + 2 \sin z_1 \sin z_3) 2z_3 - 4z_1 - 2 \\
-2 \cos z_1 \cos z_3 + 2 \sin z_1 \sin z_3
\end{bmatrix}
\]

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One can easily find that the rank of the matrix (2.64) is 3. Therefore, the first condition for
where

\[
\begin{pmatrix}
\alpha - \beta \\
(2 \cos z_1 \cos z_3 - 2 \sin z_1 \sin z_3) \, 2z_3 + 2 \\
2 \cos z_1 \cos z_3 - 2 \sin z_1 \sin z_3
\end{pmatrix}
\]

Thus, the vector field \([g(z), \text{ad}_f g(z), \text{ad}^2_f g(z)]\) is given by

\[
\begin{bmatrix}
2 & -2 \cos z_1 \cos z_3 + 2 \sin z_1 \sin z_3 & \alpha - \beta \\
0 & -4z_3 & (2 \cos z_1 \cos z_3 - 2 \sin z_1 \sin z_3) \, 2z_3 + 2 \\
0 & -2 & 2 \cos z_1 \cos z_3 - 2 \sin z_1 \sin z_3
\end{bmatrix}
\]

where

\[
\alpha - \beta = 2 \sin^2 z_1 \cos^2 z_3 + 2 \sin z_1 \cos z_3 |z_2 - z_3^2| + 2 \cos^2 z_1 \sin^2 z_3 + \\
2 \cos z_1 \sin z_3 |z_2 - z_3^2| + 2z_1 \cos z_1 \sin z_3 + 2z_1 \sin z_1 \cos z_3 + \\
2 \cos^2 z_1 \cos^2 z_3 + 2 \sin^2 z_1 \sin^2 z_3 - 2 \sin z_1 \sin z_3 + 2 \cos z_1 \cos z_3
\]

One can easily find that the rank of the matrix (2.64) is 3. Therefore, the first condition for
exact feedback linearization was achieved. Next, testing the system for the second condition
where involutivity of the vector set \( \{ g(z), \, ad_f g(z) \} \) is investigated through finding the Lie bracket

\[
[g(z), \, ad_f g(z)] = \frac{\partial (ad_f g(z))}{\partial z} g(z) - \frac{\partial g(z)}{\partial z} ad_f g(z)
\]

\[
= \begin{bmatrix}
2 \sin z_1 \cos z_3 + 2 \cos z_1 \sin z_3 & 0 & 2 \cos z_1 \sin z_3 + 2 \sin z_1 \cos z_3 \\
0 & 0 & -4 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
2 \\
0 \\
0
\end{bmatrix}
\]

\[-
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-2 \cos z_1 \cos z_3 + 2 \sin z_1 \sin z_3 \\
-4z_3 \\
-2
\end{bmatrix}
\]

\[=
\begin{bmatrix}
4 \sin z_1 \cos z_3 + 4 \cos z_1 \sin z_3 \\
0 \\
0
\end{bmatrix}
\]

It is clear that the rank of the matrix

\[
\begin{bmatrix}
g(z) & ad_f g(z) & [g(z), \, ad_f g(z)]
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2 & -2 \cos z_1 \cos z_3 + 2 \sin z_1 \sin z_3 & 4 \sin z_1 \cos z_3 + 4 \cos z_1 \sin z_3 \\
0 & -4z_3 & 0 \\
0 & -2 & 0
\end{bmatrix}
\]

is \( 2 \ \forall \ z \). Therefore, the vector set \( \{ g(z), \, ad_f g(z) \} \) is involutive and hence the second condition was achieved as well. Thus, the system (2.63) is exactly feedback linearizable and there exists an output \( \phi(z) \) such that the relative degree of the system is \( \rho = n = 3 \). \( \triangle \)

Exact feedback linearizability for MIMO affine nonlinear systems was also studied previously in [36] and [44–48]. Consider the MIMO system of the form given in (2.38) without an output as

\[
\dot{z}_n = f(z) + \sum_{i=1}^{m} g_i(z) u_i
\]

where \( z \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) are respectively states of the system and control inputs. Similar to what has been done for SISO systems, if there exist \( m \) output functions \( \phi_1(z), \phi_2(z), \ldots, \phi_m(z) \) that satisfy

\[
L_{g_1} L_f^k \phi_i(z) = L_{g_2} L_f^k \phi_i(z) = \ldots = L_{g_m} L_f^k \phi_i(z) = 0 \quad (2.65)
\]

\[
k = 0, 1, \ldots, \rho_i - 2 \quad \forall \ i = 1, 2, \ldots, m.
\]
and at least one element is not zero in the row vector

\[
\begin{bmatrix}
L_{g_1} f_{\rho_1}^{-1} \phi_i(z) & L_{g_2} f_{\rho_1}^{-1} \phi_i(z) & \ldots & L_{g_m} f_{\rho_1}^{-1} \phi_i(z)
\end{bmatrix}, \quad i = 1, 2, \ldots, m
\] (2.66)

such that the system has a vector relative degree \( \rho = \{ \rho_1, \rho_2, \ldots, \rho_{m-1}, \rho_m \} \) in which \( \rho_1 + \rho_2 + \ldots + \rho_{m-1} + \rho_m = \nu \) where \( \nu \) is the system’s order, and the following two matrices are non-singular at \( z^* \)

\[
\begin{bmatrix}
L_{g_1} f_{\rho_1}^{-1} \phi_1(z) & L_{g_2} f_{\rho_1}^{-1} \phi_1(z) & \ldots & L_{g_m} f_{\rho_1}^{-1} \phi_1(z) \\
L_{g_1} f_{\rho_2}^{-1} \phi_2(z) & L_{g_2} f_{\rho_2}^{-1} \phi_2(z) & \ldots & L_{g_m} f_{\rho_2}^{-1} \phi_2(z) \\
\vdots & \vdots & \ddots & \vdots \\
L_{g_1} f_{\rho_m}^{-1} \phi_m(z) & L_{g_2} f_{\rho_m}^{-1} \phi_m(z) & \ldots & L_{g_m} f_{\rho_m}^{-1} \phi_m(z)
\end{bmatrix}
\] (2.67)

\[
\frac{\partial T^i(z)}{\partial z} = \begin{bmatrix}
\frac{\partial \phi_1(z)}{\partial z_1} & \frac{\partial \phi_1(z)}{\partial z_2} & \ldots & \frac{\partial \phi_1(z)}{\partial z_n} \\
\frac{\partial (L_f \phi_1(z))}{\partial z_1} & \frac{\partial (L_f \phi_1(z))}{\partial z_2} & \ldots & \frac{\partial (L_f \phi_1(z))}{\partial z_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial (L_f^{\rho_1-1} \phi_1(z))}{\partial z_1} & \frac{\partial (L_f^{\rho_1-1} \phi_1(z))}{\partial z_2} & \ldots & \frac{\partial (L_f^{\rho_1-1} \phi_1(z))}{\partial z_n} \\
\frac{\partial \phi_m(z)}{\partial z_1} & \frac{\partial \phi_m(z)}{\partial z_2} & \ldots & \frac{\partial \phi_m(z)}{\partial z_n} \\
\frac{\partial (L_f \phi_m(z))}{\partial z_1} & \frac{\partial (L_f \phi_m(z))}{\partial z_2} & \ldots & \frac{\partial (L_f \phi_m(z))}{\partial z_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial (L_f^{\rho_m-1} \phi_m(z))}{\partial z_1} & \frac{\partial (L_f^{\rho_m-1} \phi_m(z))}{\partial z_2} & \ldots & \frac{\partial (L_f^{\rho_m-1} \phi_m(z))}{\partial z_n}
\end{bmatrix}
\] (2.68)

then the given MIMO affine nonlinear system with smooth enough \( f(z) \) and \( g_i(z) \) is exactly feedback linearizable in the neighborhood of \( z^* \). This is summarized in [29] in the following lemma

**Lemma 2.3** Suppose that

\[
G(z) = \begin{bmatrix} g_1(z) & g_2(z) & \ldots & g_{m-1}(z) & g_m(z) \end{bmatrix}_{n \times m} \quad U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix}_{m \times 1}
\] (2.69)

where \( G(z) \) has rank \( m \) at \( z^* \). Then, the condensed expression of (2.38)

\[
\dot{z}_n = f(z) + G(z) U
\] (2.70)
is exactly feedback linearizable around $z^*$ if there exist $m$ output functions $\phi_1(z), \phi_2(z), \ldots, \phi_m(z)$ such that (2.70) has a vector relative degree $\rho = \{\rho_1, \rho_2, \ldots, \rho_{m-1}, \rho_m\}$ in which $\rho_1 + \rho_2 + \ldots + \rho_{m-1} + \rho_m = \rho = n$ where $n$ is the order of the system.

In the same way as SISO systems, if one can solve the set of partial differential equations (2.65), then it might be possible to find the set of output functions $\phi_1(z), \phi_2(z), \ldots, \phi_m(z)$ that guarantee the non-singularity of the matrices (2.67) and (2.68). However, what conditions guarantee the existence of $\phi_i$ in case of MIMO systems. Neglecting the proof, the dissertation presents those conditions in the next theorem as explained in Isidori’s [36].

**Theorem 2.3** If $G(z)$ has rank $m$ at $z^*$, then MIMO affine nonlinear system of the form (2.70)

$$\dot{z}_n = f(z) + G(z)U$$

is exactly feedback linearizable in the neighborhood of $z^*$ if and only if for the distribution:

$$D_i = \{ad^k g_j(z) : 0 \leq k \leq i, 1 \leq j \leq m\}, \quad i = 0, 1, \ldots, n-1. \quad (2.71)$$

1. For each $0 \leq i \leq n-1$, $D_i$ has a constant dimension in the neighborhood of $z^*$

2. $D_{n-1}$ has dimension $n$.

3. For each $0 \leq i \leq n-2$, $D_i$ is involutive.

Another useful study that provides an algorithm to explicitly compute the transformation mapping is presented in [49].
3 Common Nonlinear Control Methods

In the beginnings of controlling nonlinear systems, it was common to linearize the desired system around an equilibrium point such that linear control methods can be applied. This approach was valid as long as the system’s operation is in the neighborhood of that point. However, it often fails if the system operates in a wider range. An alternative approach known as gain scheduling was then introduced where the system is linearized at several operating points and at each one of them a linear controller is designed, then a controller that includes the group of designed linear controllers is implemented [46]. Several other approaches to control nonlinear systems were suggested afterward. This chapter deals with nonlinear feedback linearization methods both input-output and input-state feedback linearization. Most of the math tools including Lie derivative and Lie bracket that a reader acquainted with in the previous chapter as well as Frobenius theorem and feedback linearizability conditions will be brought into play in this chapter. These two nonlinear approaches deal with affine nonlinear system and transform it into an equivalent system of partially or completely linear dynamics basically through proper state transformation and feedback such that it is possible to apply linear control methods. Both methods are considered effective in many control problems although they have some shortcomings and restrictions which can all revolve around canceling all nonlinearities of a nonlinear system regardless of their positive or negative impact. The reason to discuss these two methods in this research is that they are considered very useful to understand systems mapping and transformation. Moreover, they are presented here to show their weaknesses that made backstepping be a better control approach. In section 3.1 input-output feedback linearization when output function is well known will be discussed for single-input single-output nonlinear systems, whereas section 3.2 explains how to extend the concepts to multi-input multi-output nonlinear systems. Section 3.3 assumes clear output function may or may not be given, therefore input-state feedback linearization for single-input single-output nonlinear systems is presented. Similarly, the extension to multi-input multi-output nonlinear systems is presented in section 3.4. Also, this chapter will show the steps to find the coordinate transformation matrices and feedback control functions at least for SISO systems in both approaches.
3.1 Input-Output Feedback Linearization for SISO Systems

When a controller is designed for a certain nonlinear system, states are all assumed to be measured and available for the design of control law. However, this is not always the case in practice, and instead, in most cases only the output of the system is available. Input-output feedback linearization can be used to create a relation between the system's input and output and design a feedback control law that cancels the system's nonlinearities. To explain this approach, the system given in (2.15) as

\[
\dot{z} = f(z) + g(z)u \\
y = h(z)
\]

will be considered with \( z \in R^n, u \in R \) and \( y \in R \). Referring to section 2.10 of relative degree, it is known that if the system (2.15) is of a relative degree \( \rho \) less than \( n \) the order of the system (\( \rho < n \)), then (2.47) and (2.48) are true and at this point, one can define a control law

\[
u = \frac{1}{L_fL_f^{\rho-1}h(z)}[-L_f^\rho h(z) + v], \quad v \in R^n
\]

that input-output linearize system (2.15) such that the input-output mapping turns to a chain of \( \rho \) integrators

\[
y^\rho = v
\]

If this is achievable, then it is possible to coordinate transform the system into its normal form through the diffeomorphism

\[
x = T(z)
\]

where

\[
x = \begin{bmatrix} T_1(z) \\ T_2(z) \\ \vdots \\ T_\rho(z) \\ T_{\rho+1}(z) \\ \vdots \\ T_{n-1}(z) \\ T_n(z) \end{bmatrix}, \quad y = \begin{bmatrix} h(z) \\ L_f h(z) \\ \vdots \\ L_f^{\rho-2} h(z) \\ L_f^{\rho-1} h(z) \\ \vdots \\ T_{\rho+1}(z) \\ T_{\rho+2}(z) \\ \vdots \\ T_{n-1}(z) \\ T_n(z) \end{bmatrix}
\]

\[
x \in R^{\rho} \quad \text{and} \quad q \in R^{n-\rho}
\]
such that, when differentiating the first element in (3.3), one will obtain

$$\dot{x}_1 = \frac{dT_1(z)}{dt} = \frac{dh(z)}{dt} = \frac{\partial h(z)}{\partial z} \cdot \frac{dz}{dt}$$

$$= \frac{\partial h(z)}{\partial z} \left[ f(z) + g(z)u \right]$$

$$= L_f h(z) + L_g h(z) u$$

(3.4)

If $\rho > 1$, then from the analysis of a relative degree for SISO systems and definition 2.5 it is known that

$$L_g h(z) = 0$$

(3.5)

Thus, (3.4) will become

$$\dot{x}_1 = L_f h(z) = x_2$$

(3.6)

Similarly, differentiating the second element in (3.3) yields

$$\dot{x}_2 = \frac{dT_2(z)}{dt} = \frac{d(L_f h(z))}{dt} = \frac{\partial (L_f h(z))}{\partial z} \cdot \frac{dz}{dt}$$

$$= \frac{\partial (L_f h(z))}{\partial z} \left[ f(z) + g(z)u \right]$$

$$= L_f^2 h(z) + L_g L_f h(z) u$$

(3.7)

Likewise, if $\rho > 2$, then

$$L_g L_f h(z) = 0$$

(3.8)

Thus, (3.7) will become

$$\dot{x}_2 = L_f^2 h(z) = x_3$$

(3.9)

Continuing doing the same for the next $\rho - 2$ elements in (3.3) results in

$$\dot{x}_3 = x_4$$

$$\vdots$$

$$\dot{x}_{\rho-1} = x_{\rho}$$

$$\dot{x}_{\rho} = L_f^\rho h(z) + L_g L_f^{\rho-1} h(z) u$$

(3.10)

where according to definition 2.5

$$L_g L_f^{\rho-1} h(z) \neq 0$$

Also, the elements $T_{\rho+1}, T_{\rho+2}, \ldots, T_{n-1}, T_n$ can be found such that

$$\frac{\partial T_{\rho+i}}{\partial z} g(z) = L_g T_{\rho+i}(z) = 0, \quad i = 1, 2, \ldots, n - \rho.$$
Thus, for the last \( n - \rho \) in (3.3)

\[
\dot{q}_{\rho+1} = \frac{dT_{\rho+1}(z)}{dt} = \frac{\partial T_{\rho+1}(z)}{\partial z} \cdot \frac{dz}{dt} = \frac{\partial T_{\rho+1}(z)}{\partial z} [f(z) + g(z) u] = L_f T_{\rho+1}(z) + L_g T_{\rho+1}(z) u
\]  

(3.12)

From (3.11), it is clear that

\[ L_g T_{\rho+1}(z) = 0 \]

Thus, (3.12) turns out to be

\[ \dot{q}_{\rho+1} = L_f T_{\rho+1}(z) \]  

(3.13)

Similarly,

\[
\dot{q}_{\rho+2} = \frac{dT_{\rho+2}(z)}{dt} = \frac{\partial T_{\rho+2}(z)}{\partial z} \cdot \frac{dz}{dt} = \frac{\partial T_{\rho+2}(z)}{\partial z} [f(z) + g(z) u] = L_f T_{\rho+2}(z) + L_g T_{\rho+2}(z) u
\]  

(3.14)

Again, from (3.11) it is known that

\[ L_g T_{\rho+2}(z) = 0 \]

Thus, (3.14) turns out to be

\[ \dot{q}_{\rho+2} = L_f T_{\rho+2}(z) \]  

(3.15)

Continuing doing the same for the rest of the elements in (3.3) yields

\[
\dot{q}_{\rho+3} = L_f T_{\rho+3}(z) \\
\vdots \\
\dot{q}_{n-1} = L_f T_{n-1}(z) \\
\dot{q}_n = L_f T_n(z)
\]  

(3.16)

From the previous analysis, it is clear that the system is transformed into two parts, known as external dynamics \( x \) and internal dynamics \( q \). It is also clear from (3.6), (3.9) and (3.10) that \( x_i \) where \( i = 1, 2, \ldots, \rho - 1 \) are all linear and only \( x_\rho \) is nonlinear. If the system’s internal dynamics are stable, the system is called a minimum-phase system and the feedback control law

\[
u = \frac{1}{L_g L_f^{\rho-1} h(z)} [-L_f h(z) + v]
\]

(3.17)
that linearizes $x_\rho$ can stabilize the external input-output dynamics hence the whole system can be stabilized such that

$$v = -k_1 x_1 - k_2 x_2 \ldots - k_\rho x_\rho$$

where **linear optimal control design with quadratic performance index** [50] can be used to design $k_1, k_2, \ldots, k_\rho$ such that $A - Bk$ is stable. However, if the system’s internal dynamics are unstable, the system is called a **non-minimum-phase** system and it would be useless to input-output feedback linearize it because the system’s internal dynamics are not observable. From the previous analysis, the transformation of the SISO affine nonlinear system into its normal form can be summarized in the following theorem as explained in [10] and [36].

**Theorem 3.1** If a SISO affine nonlinear system of the general form (2.15)

$$\dot{z} = f(z) + g(z) u$$
$$y = h(z)$$

is input-output linearizable and has a relative degree $\rho < n$, then one can choose the diffeomorphism

$$\begin{bmatrix} x \\ \vdots \\ h(z) \\ \vdots \\ L_\rho^0 f h(z) \\ \vdots \\ L_\rho^0 f h(z) \end{bmatrix} = (T(z))^{\rho+1}, \quad x \in \mathbb{R}^p \quad \text{and} \quad q \in \mathbb{R}^{n-\rho}$$

where

$$\frac{\partial T_{\rho+i}}{\partial z} g(z) = L_g T_{\rho+i}(z) = 0, \quad i = 1, 2, \ldots, n - \rho.$$
to put the system in the following normal form.

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_\rho &= L_f^\rho h(T^{-1}(x,q)) + L_g L_f^{\rho-1} h(T^{-1}(x,q)) u.
\end{align*}
\]

(3.21)

External dynamics

\[
\begin{align*}
\dot{q}_{\rho+i} &= L_f T_{\rho+i}(T^{-1}(x,q)), \quad i = 1, 2, \ldots, n - \rho.
\end{align*}
\]

Internal dynamics

\[
\begin{align*}
y &= h(T^{-1}(x,q))
\end{align*}
\]

Output

with the feedback control law

\[
u = \frac{1}{L_g L_f^{\rho-1} h(T^{-1}(x,q))} \left[-L_f^\rho h(T^{-1}(x,q)) + v\right]
\]

that can stabilize the external input-output dynamics such that

\[
v = -k_1 x_1 - k_2 x_2 \ldots - k_\rho x_\rho
\]

(3.23)

On the other hand, if the system (2.15) is of a relative degree \( \rho \) that equals the order \( n \) of the system \( (\rho = n) \), then there will be no internal dynamics and the system is by default minimum-phase and the transformation matrix according to the proof below will turn out to be

\[
\begin{bmatrix}
T_1(z) \\
T_2(z) \\
T_3(z) \\
\vdots \\
T_{n-1}(z) \\
T_n(z)
\end{bmatrix} = \begin{bmatrix}
h(z) \\
L_f h(z) \\
L_f^2 h(z) \\
\vdots \\
L_f^{n-2} h(z) \\
L_f^{n-1} h(z)
\end{bmatrix}, \quad x \in \mathbb{R}^n
\]

(3.24)

Proof 3.1

When the system’s relative degree \( \rho \) equals the system’s order \( n \), then the successive derivative of \( h(z) \) the output function of the system is obtained as follows

\[
y = h(z) \\
y^{(1)} = L_f h(z) + L_g L_f^0 h(z) u
\]
\[ \begin{align*}
y^{(2)} &= L_1^2 h(z) + L_g L_f h(z) u \\
y^{(3)} &= L_1^3 h(z) + L_g L_f^2 h(z) u \\
\vdots \\
y^{(n-1)} &= L_1^{n-1} h(z) + L_g L_f^{n-2} h(z) u \\
y^{(n)} &= L_1^n h(z) + L_g L_f^{n-1} h(z) u
\end{align*} \tag{3.25} \]

As \( \rho = n \), then in imitation of relative degree analysis for SISO affine nonlinear system and definition 2.5, one can say that

\[ \begin{align*}
L_g L_f^0 h(z) &= L_g L_f h(z) = L_g L_f^2 h(z) = \ldots = L_g L_f^{n-2} h(z) = 0 \tag{3.26a} \\
L_g L_f^{n-1} h(z) &\neq 0 \tag{3.26b}
\end{align*} \]

Therefore, (3.25) becomes

\[ \begin{align*}
y &= h(z) \\
y^{(1)} &= L_f h(z) \\
y^{(2)} &= L_f^2 h(z) \\
y^{(3)} &= L_f^3 h(z) \\
\vdots \\
y^{(n-1)} &= L_f^{n-1} h(z) \\
y^{(n)} &= L_f^n h(z) + L_g L_f^{n-1} h(z) u
\end{align*} \tag{3.27} \]

Let

\[ \begin{align*}
y &= x_1 \\
y^{(1)} &= x_2 \\
y^{(2)} &= x_3 \\
\vdots \\
y^{(n-2)} &= x_{n-1} \\
y^{(n-1)} &= x_n
\end{align*} \tag{3.28} \]

This yields

\[ \dot{x}_1 = y^{(1)} = x_2 \]
\[ \dot{x}_2 = y^{(2)} = x_3 \]
\[ \vdots \]
\[ \dot{x}_{n-1} = y^{(n-1)} = x_n \]
\[ \dot{x}_n = y^{(n)} \]  

Finally, (3.25)-(3.29) yields

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = x_3 \]
\[ \vdots \]
\[ \dot{x}_{n-1} = x_n \]
\[ \dot{x}_n = L^T_2 h \left( T^{-1}(x) \right) + L_g L_f^{-1} h \left( T^{-1}(x) \right) u \]
\[ y = h \left( T^{-1}(x) \right) = x_1 \]

It is clear now that equations \( \dot{x}_1 \) to \( \dot{x}_{n-1} \) are linear and the only nonlinear one is \( \dot{x}_n \) that can be linearized by

\[ u = \frac{1}{L_g L_f^{-1} h \left( T^{-1}(x) \right)} \left[ -L_f^n h \left( T^{-1}(x) \right) + v \right] \]  

where

\[ v = -k_1 x_1 - k_2 x_2 \ldots - k_n x_n \]  

from (3.27) and (3.28) can again be written as

\[ v = -k_1 h \left( (T^{-1}(x)) \right) - k_2 L_f h \left( (T^{-1}(x)) \right) \ldots - k_n L_f^{n-1} h \left( (T^{-1}(x)) \right) \]  

Again, linear optimal control design with quadratic performance index [50] can be used to design \( k_1, k_2, \ldots, k_n \) such that \( A - Bk \) is stable. □

**Example 3.1**

From the system given in example 2.5:

\[ f(z) = \begin{bmatrix} \frac{z^2}{z_2} \\ z_1 + z_3 \\ -z_1 z_2 + z_1 \end{bmatrix}, \quad g(z) = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \]  

and  

\[ y = z_2 \]
It was explained earlier that this system is of a relative degree \( \rho = 2 \) in \( \mathbb{R}^3 \) and hence to find a diffeomorphism that coordinate transform the system into its normal form, one would have external dynamics with \( x \in \mathbb{R}^2 \) and internal dynamics with \( q \in \mathbb{R}^1 \) such that

\[
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
q_1
\end{bmatrix} = T(z) =
\begin{bmatrix}
T_1(z) \\
T_\rho(z) \\
\vdots \\
T_{\rho+1}(z)
\end{bmatrix}
\]

(3.34)

\[
\begin{bmatrix}
T_1(z) \\
T_2(z) \\
\vdots \\
T_3(z)
\end{bmatrix} = \begin{bmatrix}
h(z) \\
h_f(z) \\
\vdots \\
T_3(z)
\end{bmatrix} = \begin{bmatrix}
z_2 \\
z_1 + z_3
\end{bmatrix}
\]

From (3.11), \( T_3(z) \) can be obtained as

\[
\frac{\partial T_3}{\partial z}g(z) = 0
\]

\[
\Rightarrow \begin{bmatrix}
\frac{\partial T_3}{\partial z_1} & \frac{\partial T_3}{\partial z_2} & \frac{\partial T_3}{\partial z_3}
\end{bmatrix} \begin{bmatrix}
5 \\
0 \\
0
\end{bmatrix} \Rightarrow 5 \times \frac{\partial T_3}{\partial z_1} = 0
\]

This means that \( T_3 \) is independent of \( z_1 \), that is to say that \( T_3 = T_3(z_2, z_3) \). One can try \( T_3 = z_3 \). This yields

\[
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
q_1
\end{bmatrix} = \begin{bmatrix}
T_1(z) \\
T_2(z) \\
\vdots \\
T_3(z)
\end{bmatrix} = \begin{bmatrix}
z_2 \\
z_1 + z_3 \\
\vdots \\
z_3
\end{bmatrix}
\]

(3.35)

whose Jacobian matrix \( \frac{\partial T(z)}{\partial z} \) is easily found non-singular and thus its inverse can by some algebra be obtained as

\[
T^{-1}(x, q) = \begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_3
\end{bmatrix} = \begin{bmatrix}
x_2 - q_1 \\
x_1 \\
\vdots \\
q_1
\end{bmatrix}
\]

(3.36)
Therefore, (3.35) according to definition 2.1 is a diffeomorphism. Taking its derivative with respect to time

\[
\begin{align*}
\dot{x}_1 &= \dot{z}_2 \\
&= z_1 + z_3 \\
&= x_2 - q_1 + q_1 \\
&= x_2 \\
\dot{x}_2 &= \dot{z}_1 + \dot{z}_3 \\
&= z_2^2 + 5u - z_1 z_2 + z_1 \\
&= x_1^2 + 5u - (x_2 - q_1) x_1 + x_2 - q_1 \\
\dot{q}_1 &= \dot{z}_3 \\
&= -z_1 z_2 + z_1 \\
&= (-x_2 + q_1) x_1 + x_2 - q_1
\end{align*}
\]

That is the system (2.44) in its normal form is

\[
\begin{align*}
\text{External dynamics} & \quad \left\{ \begin{array}{l}
\dot{x}_1 = x_2 \\
\dot{x}_2 = - (x_2 - q_1) x_1 + x_2 - q_1 + x_1^2 + \frac{5}{L_f} u \\
\end{array} \right. \\
\text{Internal dynamics} & \quad \left\{ \begin{array}{l}
\dot{q}_1 = (-x_2 + q_1) x_1 + x_2 - q_1 \\
\end{array} \right. \\
\text{Output} & \quad \left\{ \begin{array}{l}
y = x_1 \\
\end{array} \right.
\end{align*}
\]

(3.37)

Looking at the system’s internal dynamics and setting \( x = 0 \) to examine the zero dynamics stability yields

\[
\dot{q}_1 = (0 + q_1) 0 + 0 - q_1
\]

\[
= -q_1
\]

Thus, \( q_1^* \) is exponentially stable and in this case, the following controller can be implemented

\[
u = \frac{1}{5} \left[ (x_2 - q_1) x_1 - x_2 + q_1 - x_1^2 - k_1 x_1 - k_2 x_2 \right], \quad k_1, k_2 > 0
\]

(3.38)

This results in \( x_1 \to 0, \ x_2 \to 0 \) and \( q_1 \to 0 \) and also \( u \) is bounded. \( \Delta \)
3.2 Input-Output Feedback Linearization for MIMO Systems

The idea of input-output feedback linearization for single-input single-output non-linear systems is extended here to multi-input multi-output systems. The affine nonlinear system of the general form (2.38) will be considered

$$\dot{z}_n = f(z) + g_1(z)u_1 + g_2(z)u_2 + \cdots + g_m(z)u_m$$

$$y_1 = h_1$$

$$\vdots$$

$$y_m = h_m$$

with \( z \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), and \( y \in \mathbb{R}^m \). Referring to section 2.10 of relative degree it is known that if the system (2.38) is of a vector relative degree

$$\rho = \{\rho_1, \rho_2, \ldots, \rho_{m-1}, \rho_m\}$$

such that

$$\rho_1 + \rho_2 + \cdots + \rho_{m-1} + \rho_m = \rho$$

then (2.50), (2.51) and (2.52) are true and at this point, one can write the input-output equation

$$\begin{bmatrix}
y_1^{(\rho)} \\
y_2^{(\rho)} \\
\vdots \\
y_{m-1}^{(\rho)} \\
y_m^{(\rho)}
\end{bmatrix} = \begin{bmatrix}
L_{f}^{\rho_1}h_1(z) \\
L_{f}^{\rho_2}h_2(z) \\
\vdots \\
L_{f}^{\rho_{m-1}}h_{m-1}(z) \\
L_{f}^{\rho_m}h_m(z)
\end{bmatrix} + \begin{bmatrix}
L_{g_1}L_{f}^{\rho_1-1}h_1(z) \\
L_{g_2}L_{f}^{\rho_2-1}h_2(z) \\
\vdots \\
L_{g_m}L_{f}^{\rho_{m-1}-1}h_{m-1}(z) \\
L_{g_m}L_{f}^{\rho_m-1}h_m(z)
\end{bmatrix}\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{m-1} \\
u_m
\end{bmatrix}$$

(3.39)

which in a short form is re-written as

$$y^{(\rho)} = A(z) + B(z)u$$

(3.40)

Accordingly, if \( B(z) \) is non-singular, then the input-output control law is obtained as

$$u = B^{-1}(z)(-[A(z) + v]), \quad v \in \mathbb{R}^m$$

(3.41)

This is known as a static case. However, if \( B^{-1}(z) \) doesn’t exist, then it might be possible to make it work by using the dynamic feedback linearization \([51]\). Furthermore, if a system
of the form (2.38) is input-output linearizable and has a vector relative degree \( \rho < n \), then it is possible to transform it into its normal form using the diffeomorphism

\[
\begin{bmatrix}
  T_{1,1}(z) & \cdots & T_{1,n}(z) \\
  T_{2,1}(z) & \cdots & T_{2,n}(z) \\
  \vdots & \ddots & \vdots \\
  T_{m,1}(z) & \cdots & T_{m,n}(z)
\end{bmatrix}
= \begin{bmatrix}
  h_1(z) \\
  L_f h_1(z) \\
  L^2_f h_1(z) \\
  \vdots \\
  L_{f^{m-2}} h_1(z) \\
  L_{f^{m-1}} h_1(z)
\end{bmatrix}, \quad x \in \mathbb{R}^\rho \quad \text{and} \quad q \in \mathbb{R}^{n-\rho} \quad (3.42)
\]

such that, when differentiating the first element in (3.42), one would obtain

\[
\dot{x}_{1,1} = \frac{dT_{1,1}(z)}{dt} = \frac{dh_1(z)}{dt} = \frac{\partial h_1(z)}{\partial z} \cdot \frac{dz}{dt} = \frac{\partial h_1(z)}{\partial z} \left[ f(z) + g_1(z) u_1 + g_2(z) u_2 + \ldots + g_m(z) u_m \right] \quad (3.43)
\]

\[
= L_f h_1(z) + L_{g_1} h_1(z) u_1 + L_{g_2} h_1(z) u_2 + \ldots + L_{g_m} h_1(z) u_m
\]
If $\rho_1 > 1$, then from the analysis of a relative degree for MIMO systems and definition 2.6, it is known that

$$L_{g_1} h_1(z) = L_{g_2} h_1(z) = \ldots = L_{g_m} h_1(z) = 0 \quad (3.44)$$

Thus, (3.43) will become

$$\dot{x}_{1,1} = L_f h_1(z) = x_{1,2} \quad (3.45)$$

Similarly, differentiating the second element in (3.42) yields

$$\dot{x}_{1,2} = \frac{dT_{1,2}(z)}{dt} = \frac{d(L_f h_1(z))}{dt} = \frac{\partial (L_f h_1(z))}{\partial z} \cdot \frac{dz}{dt}$$

$$= \frac{\partial (L_f h_1(z))}{\partial z} \left[ f(z) + g_1(z) u_1 + g_2(z) u_2 + \ldots + g_m(z) u_m \right]$$

$$= L_f^2 h_1(z) + L_{g_1} L_f h_1(z) u_1 + L_{g_2} L_f h_1(z) u_2 + \ldots + L_{g_m} L_f h_1(z) u_m \quad (3.46)$$

Likewise, if $\rho_1 > 2$, then

$$L_{g_1} L_f h_1(z) = L_{g_2} L_f h_1(z) = \ldots = L_{g_m} L_f h_1(z) = 0 \quad (3.47)$$

Thus, (3.46) will become

$$\dot{x}_{1,2} = L_f^2 h_1(z) = x_{1,3} \quad (3.48)$$

Continuing doing the same for the next $\rho_1 - 2$ elements in (3.42) results in

$$\dot{x}_{1,3} = x_{1,4}$$

$$\vdots$$

$$\dot{x}_{1,\rho_1-1} = x_{1,\rho_1}$$

$$\dot{x}_{1,\rho_1} = L_f^{\rho_1-1} h_1(z) + L_{g_1} L_f^{\rho_1-1} h_1(z) u_1 + L_{g_2} L_f^{\rho_1-1} h_1(z) u_2 + \ldots + L_{g_m} L_f^{\rho_1-1} h_1(z) u_m \quad (3.49)$$

where according to definition 2.6 at least one element is not zero in the row vector

$$\left[ g_1 L_f^{\rho_1-1} h_1(z) \quad g_2 L_f^{\rho_1-1} h_1(z) \quad \ldots \quad g_m L_f^{\rho_1-1} h_1(z) \right]$$

In the same way, for the next $\rho - \rho_1$ elements in (3.42)

$$\dot{x}_{2,1} = x_{2,2}$$

$$\dot{x}_{2,2} = x_{2,3}$$

$$\dot{x}_{2,3} = x_{2,4}$$
\[
\begin{align*}
\dot{x}_{2,\rho_2-1} &= x_{2,\rho_2} \\
\dot{x}_{2,\rho_2} &= L_f^2 h_2(z) + L_{g_1} L_f^2 h_2(z) u_1 + L_{g_2} L_f^2 h_2(z) u_2 + \ldots + L_{g_m} L_f^2 h_2(z) u_m \\
&\quad \ldots \\
&\quad \ldots \\
\dot{x}_{m,1} &= x_{m,2} \\
\dot{x}_{m,2} &= x_{m,3} \\
\dot{x}_{m,3} &= x_{m,4} \\
&\quad \ldots \\
\dot{x}_{m,\rho_m} &= x_{m,\rho_m} \\
\dot{x}_{m,\rho_m} &= L_f^{\rho_m} h_m(z) + L_{g_1} L_f^{\rho_m-1} h_m(z) u_1 + L_{g_2} L_f^{\rho_m-1} h_m(z) u_2 + \ldots + L_{g_m} L_f^{\rho_m-1} h_m(z) u_m 
\end{align*}
\] (3.50)

Furthermore, it’s proved in [36] that if the vector fields set \{g_1(z), g_2(z), \ldots, g_m(z)\} is involutive, then \(T_{\rho+1}, T_{\rho+2}, \ldots, T_{n-1}, T_n\) can be found such that

\[
\frac{\partial T_{\rho+i} g_j(z)}{\partial z} = L_{g_j} T_{\rho+i}(z) = 0, \quad i = 1, 2, \ldots, n - \rho \quad \text{for every} \quad j = 1, 2, \ldots, m. \quad (3.51)
\]

Thus, for the last \(n - \rho\) in (3.42)

\[
\dot{q}_{\rho+1} = \frac{dT_{\rho+1}(z)}{dt} = \frac{\partial T_{\rho+1}(z)}{\partial z} \cdot \frac{dz}{dt} = \frac{\partial T_{\rho+1}(z)}{\partial z} [f(z) + g_1(z) u_1 + g_2(z) u_2 + \ldots + g_m(z) u_m] = L_f T_{\rho+1}(z) + L_{g_1} T_{\rho+1}(z) u_1 + L_{g_2} T_{\rho+1}(z) u_2 + \ldots + L_{g_m} T_{\rho+1}(z) u_m 
\] (3.52)

From (3.51), it is clear that

\[
L_{g_1} T_{\rho+1}(z) = L_{g_2} T_{\rho+1}(z) = \ldots = L_{g_m} T_{\rho+1}(z) = 0
\]

Thus, (3.52) turns out to be

\[
\dot{q}_{\rho+1} = L_f T_{\rho+1}(z) \quad (3.53)
\]
Similarly,
\[
\dot{q}_{\rho+2} = \frac{dT_{\rho+2}(z)}{dt} = \frac{\partial T_{\rho+2}(z)}{\partial z} \cdot \frac{dz}{dt} = \frac{\partial T_{\rho+2}(z)}{\partial z} \left[ f(z) + g_1(z)u_1 + g_2(z)u_2 + \ldots + g_m(z)u_m \right] = L_f T_{\rho+2}(z) + L_{g_1} T_{\rho+2}(z) u_1 + L_{g_2} T_{\rho+2}(z) u_2 + \ldots + L_{g_m} T_{\rho+2}(z) u_m
\]

Again, from (3.51), it is known that
\[
L_{g_1} T_{\rho+2}(z) = L_{g_2} T_{\rho+2}(z) = \ldots = L_{g_m} T_{\rho+2}(z) = 0
\]

Thus, (3.54) turns out to be
\[
\dot{q}_{\rho+2} = L_f T_{\rho+2}(z)
\]

Continuing doing the same for the rest of the elements in (3.42) yields
\[
\begin{align*}
\dot{q}_{\rho+3} &= L_f T_{\rho+3}(z) \\
\vdots & \ \\
\dot{q}_{n-1} &= L_f T_{n-1}(z) \\
\dot{q}_n &= L_f T_n(z)
\end{align*}
\]

It is worth noting here that the condition of involutivity to find the last \( n - \rho \) elements in the transformation matrix was not mentioned in the case of the SISO system because it has just one vector \( g(z) \) and thus the condition is always satisfied. From the previous analysis, it’s clear that the system is transformed into two parts known as external dynamics \( x \) and internal dynamics \( q \) exactly like what has been done in the SISO system transformation. It is also clear from (3.45), (3.48), (3.49) and (3.50) that \( x_{i,j} \) where \( j = 1, 2, \ldots, \rho_i - 1 \) for every \( i = 1, 2, \ldots, m \) are all linear and only \( x_{i,\rho_i} \) is nonlinear. Thus, one can write the equation

\[
\begin{bmatrix}
\dot{x}_{1,\rho_1} \\
\dot{x}_{2,\rho_2} \\
\vdots \\
\dot{x}_{m,\rho_m}
\end{bmatrix} = \begin{bmatrix}
L_f^{\rho_1} h_1(z) \\
L_f^{\rho_2} h_2(z) \\
\vdots \\
L_f^{\rho_m} h_m(z)
\end{bmatrix} + \begin{bmatrix}
L_{g_1} L_f^{\rho_1-1} h_1(z) & L_{g_2} L_f^{\rho_1-1} h_1(z) & \ldots & L_{g_m} L_f^{\rho_1-1} h_1(z) \\
L_{g_1} L_f^{\rho_2-1} h_2(z) & L_{g_2} L_f^{\rho_2-1} h_2(z) & \ldots & L_{g_m} L_f^{\rho_2-1} h_2(z) \\
\vdots & \vdots & \vdots & \vdots \\
L_{g_1} L_f^{\rho_m-1} h_m(z) & L_{g_2} L_f^{\rho_m-1} h_m(z) & \ldots & L_{g_m} L_f^{\rho_m-1} h_m(z)
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_m
\end{bmatrix}
\]

which in a short form is re-written as
\[
\dot{x} = A(z) + B(z)u
\]
Accordingly, if $B(z)$ is non-singular, then (3.58) can be linearized by

$$u = B^{-1}(z) [-A(z) + v]$$

(3.59)

where

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} -k_{1,1}x_{1,1} - k_{1,2}x_{1,2} - \ldots - k_{1,\rho_1}x_{1,\rho_1} \\ -k_{2,1}x_{2,1} - k_{2,2}x_{2,2} - \ldots - k_{2,\rho_2}x_{2,\rho_2} \\ \vdots \\ -k_{m,1}x_{m,1} - k_{m,2}x_{m,2} - \ldots - k_{m,\rho_m}x_{m,\rho_m} \end{bmatrix}$$

(3.60)

and $k_{1,1}, \ldots, k_{1,\rho_1}, k_{2,1}, \ldots, k_{2,\rho_2}, \ldots, k_{m,1}, \ldots, k_{m,\rho_m}$ is designed such that $A - Bk$ is stable. From the previous analysis the coordinate transformation of MIMO affine nonlinear system into its normal form can be summarized in the following theorem as explained in [29] and [36].

**Theorem 3.2** If a MIMO affine nonlinear system of the form (2.38)

$$\begin{align*}
\dot{z}_n &= f(z) + g_1(z)u_1 + g_2(z)u_2 + \cdots + g_m(z)u_m \\
y_1 &= h_1 \\
&\vdots \\
y_m &= h_m
\end{align*}$$

is input-output linearizable and has a vector relative degree $\rho = \{\rho_1, \rho_2, \ldots, \rho_{m-1}, \rho_m\}$ such that $\rho_1 + \rho_2 + \cdots + \rho_{m-1} + \rho_m = \rho < n$, then one can choose the diffeomorphism

$$T(z) = \begin{bmatrix} x_{1,1} \\ x_{1,2} \\ \vdots \\ x_{1,\rho_1} \\ \vdots \\ x_{m,1} \\ x_{m,2} \\ \vdots \\ x_{m,\rho_m} \\ \vdots \\ q_{\rho+1} \\ q_{\rho+2} \\ q_{\rho+3} \\ \vdots \\ q_{n-1} \\ q_n \end{bmatrix} \begin{bmatrix} T_{1,1}(z) \\ T_{1,2}(z) \\ \vdots \\ T_{1,\rho_1}(z) \\ \vdots \\ T_{m,1}(z) \\ T_{m,2}(z) \\ \vdots \\ T_{m,\rho_m}(z) \\ \vdots \\ T_{\rho+1}(z) \\ T_{\rho+2}(z) \\ T_{\rho+3}(z) \\ \vdots \\ T_{n-1}(z) \\ T_n(z) \end{bmatrix} = \begin{bmatrix} h_1(z) \\ L_fh_1(z) \\ \vdots \\ L_f^{\rho-1}h_1(z) \\ \vdots \\ h_m(z) \\ L_fh_m(z) \\ \vdots \\ L_f^{\rho-1}h_m(z) \end{bmatrix}, \quad x \in \mathbb{R}^p \quad \text{and} \quad q \in \mathbb{R}^{n-\rho} \quad (3.61)
where if \{g_1(z), g_2(z), \ldots, g_m(z)\} is involutive, then \(T_{\rho+1}, T_{\rho+2}, \ldots, T_{n-1}, T_n\) can be found such that
\[
\frac{\partial T_{\rho+i}}{\partial z} g_j(z) = L_{g_j} T_{\rho+i} (z) = 0, \quad i = 1, 2, \ldots, n - \rho \quad \text{for every} \quad j = 1, 2, \ldots, m. \tag{3.62}
\]
to put the system in the normal form
\[
\begin{align*}
\dot{x}_{1,1} &= x_{1,2} \\
\dot{x}_{1,2} &= x_{1,3} \\
& \vdots \\
\dot{x}_{1,\rho_1} &= L_f^{\rho_1} h_1 (T^{-1} (x, q)) + L_g L_f^{\rho_1-1} h_1 (T^{-1} (x, q)) u_1 + \ldots + L_g L_f^{\rho_1-1} h_1 (T^{-1} (x, q)) u_m \\
& \quad \vdots \\
\dot{x}_{m,1} &= x_{m,2} \\
\dot{x}_{m,2} &= x_{m,3} \\
& \vdots \\
\dot{x}_{m,\rho_m} &= L_f^{\rho_m} h_m (T^{-1} (x, q)) + L_g L_f^{\rho_m-1} h_m (T^{-1} (x, q)) u_1 + \ldots + L_g L_f^{\rho_m-1} h_m (T^{-1} (x, q)) u_m \\
\dot{q}_{\rho+1} &= L_f T_{\rho+1} (T^{-1} (x, q)) \\
& \quad \vdots \\
\dot{q}_n &= L_f T_n (T^{-1} (x, q)) \\
y_1 &= h_1 (T^{-1} (x, q)) \\
& \quad \vdots \\
y_m &= h_m (T^{-1} (x, q))
\end{align*}
\tag{3.63}
\]
with the feedback control law
\[
u = B^{-1} (z) [-A (z) + v] \tag{3.64}
\]
such that
\[
A = \begin{bmatrix}
L_f^{\rho_1} h_1 (T^{-1} (x, q)) \\
L_f^{\rho_2} h_2 (T^{-1} (x, q)) \\
\vdots \\
L_f^{\rho_m} h_m (T^{-1} (x, q))
\end{bmatrix} \\
B = \begin{bmatrix}
L_g L_f^{\rho_1-1} h_1 (T^{-1} (x, q)) \\
L_g L_f^{\rho_2-1} h_1 (T^{-1} (x, q)) \\
\vdots \\
L_g L_f^{\rho_m-1} h_1 (T^{-1} (x, q)) \\
L_g L_f^{\rho_1-1} h_2 (T^{-1} (x, q)) \\
L_g L_f^{\rho_2-1} h_2 (T^{-1} (x, q)) \\
\vdots \\
L_g L_f^{\rho_m-1} h_2 (T^{-1} (x, q)) \\
\vdots \\
L_g L_f^{\rho_1-1} h_m (T^{-1} (x, q)) \\
L_g L_f^{\rho_2-1} h_m (T^{-1} (x, q)) \\
\vdots \\
L_g L_f^{\rho_m-1} h_m (T^{-1} (x, q))
\end{bmatrix}
\]
On the other hand, if the system (2.38) is of a vector relative degree, the formation matrix (3.42) turns out to be

\[
\begin{bmatrix}
-k_{1,1}x_{1,1} - k_{1,2}x_{1,2} - \ldots - k_{1,\rho_1}x_{1,\rho_1} \\
-k_{2,1}x_{2,1} - k_{2,2}x_{2,2} - \ldots - k_{2,\rho_2}x_{2,\rho_2} \\
\vdots \\
-k_{m,1}x_{m,1} - k_{m,2}x_{m,2} - \ldots - k_{m,\rho_m}x_{m,\rho_m}
\end{bmatrix}
\]

such that

\[
\rho = \{\rho_1, \rho_2, \ldots, \rho_{m-1}, \rho_m\}
\]

such that

\[
\rho_1 + \rho_2 + \ldots + \rho_{m-1} + \rho_m = \rho = n
\]

where \(n\) is the order of the system, then there will be no internal dynamics and the transformation matrix (3.42) turns out to be

\[
\begin{bmatrix}
x_{1,1} \\
x_{1,2} \\
x_{1,3} \\
\vdots \\
x_{1,\rho_1-1} \\
x_{1,\rho_1} \\
\vdots \\
x_{2,1} \\
x_{2,2} \\
x_{2,3} \\
\vdots \\
x_{2,\rho_2-1} \\
x_{2,\rho_2} \\
\vdots \\
\vdots \\
x_{m,1} \\
x_{m,2} \\
x_{m,3} \\
\vdots \\
x_{m,\rho_m-1} \\
x_{m,\rho_m}
\end{bmatrix}
= T(z)
= \begin{bmatrix}
T_{1,1}(z) \\
T_{1,2}(z) \\
T_{1,3}(z) \\
\vdots \\
T_{1,\rho_1-1}(z) \\
T_{1,\rho_1}(z) \\
\vdots \\
T_{2,1}(z) \\
T_{2,2}(z) \\
T_{2,3}(z) \\
\vdots \\
T_{2,\rho_2-1}(z) \\
T_{2,\rho_2}(z) \\
\vdots \\
\vdots \\
T_{m,1}(z) \\
T_{m,2}(z) \\
T_{m,3}(z) \\
\vdots \\
T_{m,\rho_m-1}(z) \\
T_{m,\rho_m}(z)
\end{bmatrix}
= \begin{bmatrix}
h_1(z) \\
L_f h_1(z) \\
L_f^2 h_1(z) \\
\vdots \\
L_f^{\rho_1-2} h_1(z) \\
L_f^{\rho_1-1} h_1(z) \\
\vdots \\
h_2(z) \\
L_f h_2(z) \\
L_f^2 h_2(z) \\
\vdots \\
L_f^{\rho_2-2} h_2(z) \\
L_f^{\rho_2-1} h_2(z) \\
\vdots \\
h_m(z) \\
L_f h_m(z) \\
L_f^2 h_m(z) \\
\vdots \\
L_f^{\rho_m-2} h_m(z) \\
L_f^{\rho_m-1} h_m(z)
\end{bmatrix}, \quad x \in \mathbb{R}^n
\]
Thus, MIMO affine nonlinear system of the general form (3.38) and a vector relative degree \( \rho = n \) in its normal form will be

\[
\begin{align*}
\dot{x}_{1,1} &= x_{1,2} \\
\dot{x}_{1,2} &= x_{1,3} \\
\dot{x}_{1,3} &= x_{1,4} \\
&\vdots \\
\dot{x}_{1,\rho_1-1} &= x_{1,\rho_1} \\
\dot{x}_{1,\rho_1} &= L_{f_1}^{\rho_1} h_1(z) + L_{g_1} L_{f_1}^{\rho_1-1} h_1(z) u_1 + L_{g_2} L_{f_1}^{\rho_1-1} h_1(z) u_2 + \ldots + L_{g_n} L_{f_1}^{\rho_1-1} h_1(z) u_m \\
&\vdots \\
\dot{x}_{2,1} &= x_{2,2} \\
\dot{x}_{2,2} &= x_{2,3} \\
\dot{x}_{2,3} &= x_{2,4} \\
&\vdots \\
\dot{x}_{2,\rho_2-1} &= x_{2,\rho_2} \\
\dot{x}_{2,\rho_2} &= L_{f_2}^{\rho_2} h_2(z) + L_{g_1} L_{f_2}^{\rho_2-1} h_2(z) u_1 + L_{g_2} L_{f_2}^{\rho_2-1} h_2(z) u_2 + \ldots + L_{g_n} L_{f_2}^{\rho_2-1} h_2(z) u_m \\
&\vdots \\
&\vdots \\
\dot{x}_{m,1} &= x_{m,2} \\
\dot{x}_{m,2} &= x_{m,3} \\
\dot{x}_{m,3} &= x_{m,4} \\
&\vdots \\
\dot{x}_{m,\rho_m-1} &= x_{m,\rho_m} \\
\dot{x}_{m,\rho_m} &= L_{f_m}^{\rho_m} h_m(z) + L_{g_1} L_{f_m}^{\rho_m-1} h_m(z) u_1 + L_{g_2} L_{f_m}^{\rho_m-1} h_m(z) u_2 + \ldots + L_{g_n} L_{f_m}^{\rho_m-1} h_m(z) u_m \\
y_1 &= h_1(z) \\
y_2 &= h_2(z) \\
&\vdots \\
y_m &= h_m(z)
\end{align*}
\]
3.3 Input-State Feedback Linearization for SISO Systems

It was previously explained that if the two conditions of feedback linearizability in section 2.11 have been fulfilled, then the system is feedback linearizable system. However, one may wonder how it is possible to design a controller $u$ and a state transformation such that an equivalent completely linear system is obtained. The answer to this question will be in two approaches that will lead to the same results. In the first approach, the affine nonlinear system given in (2.15) is considered without an output as

$$\dot{z} = f(z) + g(z) u$$

Assuming the transformation matrix given in (2.16) as

$$x = T(z) = \begin{bmatrix}
T_1(z) \\
T_2(z) \\
\vdots \\
T_{n-1}(z) \\
T_n(z)
\end{bmatrix}$$

is a diffeomorphism, then (2.15) can be transformed into the following controllable canonical form:

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\quad \vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= v
\end{align*}$$

(3.68)

This is equivalent to (3.30) that results from input-output feedback linearization when $\rho = n$ where

$$v = L^n f \left( T^{-1} (x) \right) + L^g L^n f \left( T^{-1} (x) \right) u$$

(3.69)

Equation (3.69) is written in a short form as

$$v = \alpha(x) + \beta(x) u$$

(3.70)

Thus, (3.68) can be stabilized by choosing $u$ in (3.70) as

$$u = \frac{1}{\beta(x)} \left[ -\alpha(x) + v \right]$$

(3.71)
The question is how to find the transformation matrix (2.16). Obviously, (3.68) in a matrix form is re-written as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_{n-1} \\
\dot{x}_n
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n
\end{bmatrix} +
\begin{bmatrix}
\alpha(x) + \beta(x) u
\end{bmatrix}
\]  

(3.72)

which in a short form is written as

\[
\dot{x} = Ax + Bu
\]  

(3.73)

by considering (2.16) and equating (2.17) and (3.73), one can get:

\[
\frac{\partial T}{\partial z} f(z) = Ax + B\alpha(x)
\]

\[
= \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
T_1(z) \\
T_2(z) \\
\vdots \\
T_{n-1}(z) \\
T_n(z)
\end{bmatrix} +
\begin{bmatrix}
\alpha(x)
\end{bmatrix}
\]  

(3.74)

\[
= \begin{bmatrix}
T_2(z) \\
T_3(z) \\
\vdots \\
T_{n-1}(z) \\
0
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\alpha(x)
\end{bmatrix} =
\begin{bmatrix}
T_2(z) \\
T_3(z) \\
\vdots \\
T_{n-1}(z) \\
\alpha(x)
\end{bmatrix}
\]

and can also get:

\[
\frac{\partial T}{\partial z} g(z) = B\beta(x)
\]

\[
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix} \beta(x) =
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\beta(x)
\end{bmatrix}
\]

(3.75)
From the previous analysis, one can conclude that $T(z)$ should satisfy:

$$\frac{\partial T_i}{\partial z} g(z) = 0, \quad i = 1, \ldots, n - 1. \quad (3.76)$$

$$\frac{\partial T_n}{\partial z} g(z) = \beta(x) \neq 0. \quad (3.77)$$

such that:

$$T_{i+1} = \frac{\partial T_i}{\partial z} f(z) \quad i = 1, \ldots, n - 1. \quad (3.78)$$

and then:

$$\alpha(x) = \frac{\partial T_n}{\partial z} f(z) = \frac{\partial T_n}{\partial z} f(T^{-1}(x)). \quad (3.79)$$

$$\beta(x) = \frac{\partial T_n}{\partial z} g(z) = \frac{\partial T_n}{\partial z} g(T^{-1}(x)). \quad (3.80)$$

**Example 3.2**

Consider the nonlinear system given in example 2.7 as:

$$\dot{z}_1 = \sin(z_1 + z_3) + |z_2 - z_3^2| + 2u$$

$$\dot{z}_2 = 2z_1z_3 + z_3$$

$$\dot{z}_3 = z_1$$

where

$$f(z) = \begin{bmatrix} \sin(z_1 + z_3) + |z_2 - z_3^2| \\ 2z_1z_3 + z_3 \\ z_1 \end{bmatrix}, \quad g(z) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

It was proved in example 2.7 that this system is feedback linearizable. In this example stabilizing its states through the input-state feedback linearization will be explained. However, this system is not in its controllable canonical form. Therefore, to transform it into the controllable canonical form, one has first to use the conditions (3.76) through (3.78) and then apply feedback linearization. To put the system in the controllable canonical form, one needs to find $x = T(z)$ that satisfies:

1. $\frac{\partial T_1}{\partial z} g(z) = 0$
2. $\frac{\partial T_2}{\partial z} g(z) = 0$
3. $\frac{\partial T_3}{\partial z} g(z) \neq 0$
4. $T_2 = \frac{\partial T_1}{\partial z} f(z)$
5. $T_3 = \frac{\partial T_2}{\partial z} f(z)$

Starting with writing out all conditions in more details:
1. \[ \frac{\partial T_1}{\partial z} g(z) = \begin{bmatrix} \frac{\partial T_1}{\partial z_1} & \frac{\partial T_1}{\partial z_2} & \frac{\partial T_1}{\partial z_3} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \frac{\partial T_1}{\partial z_1} = 0 \quad \Rightarrow \quad T_1 = T_1(z_2, z_3) \]

2. \[ \frac{\partial T_2}{\partial z} g(z) = \begin{bmatrix} \frac{\partial T_2}{\partial z_1} & \frac{\partial T_2}{\partial z_2} & \frac{\partial T_2}{\partial z_3} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \frac{\partial T_2}{\partial z_1} = 0 \quad \Rightarrow \quad T_2 = T_2(z_2, z_3) \]

3. \[ \frac{\partial T_3}{\partial z} g(z) = \begin{bmatrix} \frac{\partial T_3}{\partial z_1} & \frac{\partial T_3}{\partial z_2} & \frac{\partial T_3}{\partial z_3} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \frac{\partial T_3}{\partial z_1} \neq 0 \]

4. \[ T_2 = \frac{\partial T_1}{\partial z} f(z) = \begin{bmatrix} \frac{\partial T_1}{\partial z_1} & \frac{\partial T_1}{\partial z_2} & \frac{\partial T_1}{\partial z_3} \end{bmatrix} \begin{bmatrix} \sin(z_1 + z_3) + |z_2 - z_3|^2 \\ 2z_1z_3 + z_3 \\ z_1 \end{bmatrix} \]

From condition 1, it is known that \( \frac{\partial T_1}{\partial z_1} = 0 \). Hence:

\[ T_2 = \frac{\partial T_1}{\partial z_2} (2z_1z_3 + z_3) + \frac{\partial T_1}{\partial z_3} z_1 \]

5. \[ T_3 = \frac{\partial T_2}{\partial z} f(z) = \begin{bmatrix} \frac{\partial T_2}{\partial z_1} & \frac{\partial T_2}{\partial z_2} & \frac{\partial T_2}{\partial z_3} \end{bmatrix} \begin{bmatrix} \sin(z_1 + z_3) + |z_2 - z_3|^2 \\ 2z_1z_3 + z_3 \\ z_1 \end{bmatrix} \]

From condition 2, it is known that \( \frac{\partial T_2}{\partial z_1} = 0 \). Hence:

\[ T_3 = \frac{\partial T_2}{\partial z_2} (2z_1z_3 + z_3) + \frac{\partial T_2}{\partial z_3} z_1 \]

From condition 3, one can try \( T_3 = z_1 \). Hence:

\[ \frac{\partial T_3}{\partial z_1} g(z) = 2 \neq 0. \]

So, condition 3 is satisfied. Substituting \( T_3 \) in condition 5 results in:

\[ z_1 = \frac{\partial T_2}{\partial z_2} (2z_1z_3 + z_3) + \frac{\partial T_2}{\partial z_3} z_1 \]

If one lets \( T_2 = z_3 \), then \( \frac{\partial T_2}{\partial z_2} = 0 \) and \( \frac{\partial T_2}{\partial z_3} = 1 \) and hence:

\[ z_1 = (0) (2z_1z_3 + z_3) + (1) z_1 = z_1 \]
Then, condition 5 as well as condition 2 are satisfied. Substituting $T_2$ in condition 4 yields:

$$z_3 = \frac{\partial T_1}{\partial z_2} (2z_1z_3 + z_3) + \frac{\partial T_1}{\partial z_3} z_1$$

The only way to make both sides of this equation equal is to make $\frac{\partial T_1}{\partial z_2} = 1$ and $\frac{\partial T_1}{\partial z_3} = -2z_3$. This will result in:

$$T_1 = z_2 - z_3^2$$

and hence condition 4 will turn out to be:

$$z_3 = (1) (2z_1z_3 + z_3) + (-2z_3) z_1$$

$$= 2z_1z_3 + z_3 - 2z_1z_3 = z_3$$

Hence, condition 4 as well as condition 1 are satisfied. From the previous analysis, one can deduce that:

$$x = T(z) \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} T_1(z) \\ T_2(z) \\ T_3(z) \end{bmatrix} = \begin{bmatrix} z_2 - z_3^2 \\ z_3 \\ z_1 \end{bmatrix}$$

(3.81)

whose Jacobian matrix $\frac{\partial T(z)}{\partial z}$ is non-singular and hence its inverse can easily be obtained as:

$$z = T^{-1}(x) \Rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} T_1^{-1}(x) \\ T_2^{-1}(x) \\ T_3^{-1}(x) \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 + x_2^2 \\ x_2 \end{bmatrix}$$

(3.82)

According to definition 2.1, $T(z)$ is a diffeomorphism. To verify this work, one can consider the transformation matrix (3.81) and find the derivative in time for each row element.

$$\dot{x}_1 = \dot{z}_2 - 2z_3\dot{z}_3$$

$$= 2z_1z_3 + z_3 - 2z_1z_3 = z_3 = x_2$$

$$\dot{x}_2 = \dot{z}_3$$

$$= z_1 = x_3$$

$$\dot{x}_3 = \dot{z}_1 = \sin(z_1 + z_3) + |z_2 - z_3^2| + 2u$$

$$= \sin(x_2 + x_3) + |x_1| + 2u$$

As a result, the system (2.63) in controllable canonical form becomes:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = \underbrace{\sin(x_2 + x_3)}_{\alpha(x)} + \underbrace{|x_1|}_{\beta(x)} + \frac{2}{\beta(x)} u$$

(3.83)
Thus, it can be linearized via the state feedback

\[ u = \frac{1}{\beta(x)} \left[ -\alpha(x) + v \right] \]

\[ = \frac{1}{2} [ -\sin(x_2 + x_3) - |x_1| + v ] \]  

(3.84)

to obtain the linear state equation

\[
\dot{x} = Ax + Bv  \\
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} v
\]

(3.85)

Linear optimal control design with quadratic performance index [50] can be used to design

\[ v = - \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \] such that \( A - Bk \) is stable. So, the overall state feedback is:

\[ u(x, v) = \frac{1}{2} \left[ -\sin(x_2 + x_3) - |x_1| - k_1x_1 - k_2x_2 - k_3x_3 \right] \]

(3.86)

In terms of the original states \( z_1, z_2, \) and \( z_3, \) this controller is corresponding to the original input:

\[ u(z, v) = \frac{1}{2} \left[ -\sin(z_3 + z_1) - |z_2 - z_3^2| - k_1(z_2 - z_3^2) - k_2z_3 - k_3z_1 \right] \Delta \]

(3.87)

Fig. 3.1 shows the block diagram representing the closed-loop system with input-state linearization in the inner-loop and dynamic stabilization in the outer-loop.
Another easier approach to find the transformation matrix and get the system in its controllable canonical form known as Brunovsky normal form is to find the output function $\phi(z)$ that is the solution to the partial differential equations set

$$
\left[ \frac{\partial \phi(z)}{\partial z_1} \quad \frac{\partial \phi(z)}{\partial z_2} \quad \ldots \quad \frac{\partial \phi(z)}{\partial z_n} \right] \left[ \begin{array}{c} g(z) \\ ad_f g(z) \\ \ldots \\ ad_f^{n-2} g(z) \end{array} \right] = 0 \quad (3.88a)
$$

$$
\left[ \frac{\partial \phi(z)}{\partial z_1} \quad \frac{\partial \phi(z)}{\partial z_2} \quad \ldots \quad \frac{\partial \phi(z)}{\partial z_n} \right] ad_f^{n-1} g \neq 0 \quad (3.88b)
$$

which for the system of example 3.2 can be written as

$$
\left[ \frac{\partial \phi(z)}{\partial z_1} \quad \frac{\partial \phi(z)}{\partial z_2} \quad \frac{\partial \phi(z)}{\partial z_3} \right] \left[ \begin{array}{c} g(z) \\ ad_f g(z) \end{array} \right] = 0 \quad (3.89a)
$$

$$
\left[ \frac{\partial \phi(z)}{\partial z_1} \quad \frac{\partial \phi(z)}{\partial z_2} \quad \frac{\partial \phi(z)}{\partial z_3} \right] ad_f^{n-1} g \neq 0 \quad (3.89b)
$$

Substituting for $g(z)$, $ad_f g(z)$ and $ad_f^{n-1} g$ from (2.64) yields

$$
\left[ \frac{\partial \phi(z)}{\partial z_1} \quad \frac{\partial \phi(z)}{\partial z_2} \quad \frac{\partial \phi(z)}{\partial z_3} \right] \left[ \begin{array}{c} 2 \\ -2 \cos z_1 \cos z_3 + 2 \sin z_1 \sin z_3 \\ 0 \\ -4z_3 \\ 0 \\ -2 \end{array} \right] = 0 \quad (3.90a)
$$

$$
\left[ \frac{\partial \phi(z)}{\partial z_1} \quad \frac{\partial \phi(z)}{\partial z_2} \quad \frac{\partial \phi(z)}{\partial z_3} \right] \left[ \begin{array}{c} \alpha - \beta \\ (2 \cos z_1 \cos z_3 - 2 \sin z_1 \sin z_3) 2z_3 + 2 \\ 2 \cos z_1 \cos z_3 - 2 \sin z_1 \sin z_3 \end{array} \right] \neq 0 \quad (3.90b)
$$

Hence, from (3.90a), the following set of partial differential equations is obtained

$$
2 \frac{\partial \phi(z)}{\partial z_1} = 0 \quad (3.91)
$$

$$
\frac{\partial \phi(z)}{\partial z_1} (-2 \cos z_1 \cos z_3 + 2 \sin z_1 \sin z_3) - 4 \frac{\partial \phi(z)}{\partial z_2} z_3 - 2 \frac{\partial \phi(z)}{\partial z_3} = 0 \quad (3.92)
$$

from which it is easy to find the output function candidate as

$$
\phi(z_1, z_2, z_3) = z_2 - z_3^2 = h(z) \quad (3.93)
$$

The resulting output function (3.93) satisfies (3.90b) such that

$$
\left[ \begin{array}{ccc} 0 & 1 & -2z_3 \end{array} \right] \left[ \begin{array}{c} \alpha - \beta \\ (2 \cos z_1 \cos z_3 - 2 \sin z_1 \sin z_3) 2z_3 + 2 \\ 2 \cos z_1 \cos z_3 - 2 \sin z_1 \sin z_3 \end{array} \right] = 2 \neq 0 \quad (3.94)
$$
Hence, it is a valid output function and the transformation matrix can be obtained as

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} =
\begin{bmatrix}
  h(z) \\
  L_f h(z) \\
  L_f^2 h(z)
\end{bmatrix}
\tag{3.95}
\]

where

\[
L_f h(z) = \frac{\partial h(z) f(z)}{\partial z} = \begin{bmatrix} \frac{\partial h(z)}{\partial z_1} & \frac{\partial h(z)}{\partial z_2} & \frac{\partial h(z)}{\partial z_3} \end{bmatrix}
\begin{bmatrix}
  \sin(z_1 + z_3) + |z_2 - z_3^2| \\
  2z_1 z_3 + z_3 \\
  z_1
\end{bmatrix}
\tag{3.96}
\]

\[
= \begin{bmatrix} 0 & 1 & -2z_3 \end{bmatrix} \begin{bmatrix}
  \sin(z_1 + z_3) + |z_2 - z_3^2| \\
  2z_1 z_3 + z_3 \\
  z_1
\end{bmatrix} = z_3
\]

\[
L_f^2 h(z) = \frac{\partial(L_f h(z)) f(z)}{\partial z} = \begin{bmatrix} \frac{\partial(L_f h(z))}{\partial z_1} & \frac{\partial(L_f h(z))}{\partial z_2} & \frac{\partial(L_f h(z))}{\partial z_3} \end{bmatrix}
\begin{bmatrix}
  \sin(z_1 + z_3) + |z_2 - z_3^2| \\
  2z_1 z_3 + z_3 \\
  z_1
\end{bmatrix}
\tag{3.97}
\]

\[
= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix}
  \sin(z_1 + z_3) + |z_2 - z_3^2| \\
  2z_1 z_3 + z_3 \\
  z_1
\end{bmatrix} = z_1
\]

This results in the following transformation matrix

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} =
\begin{bmatrix}
  h(z) \\
  L_f h(z) \\
  L_f^2 h(z)
\end{bmatrix} =
\begin{bmatrix}
  z_2 - 2z_3^2 \\
  z_3 \\
  z_1
\end{bmatrix}
\tag{3.98}
\]

which is equivalent to the diffeomorphism (3.81) found in the first approach. Similar to what has been done in the first approach, one can get \( z = T^{-1}(x) \) and then easily put the system in its Brunovsky normal form and design the control function. \( \triangle \)

A more detailed algorithm to find the output function \( \phi(z) \) to transform the SISO nonlinear system of the form (2.15) into its Brunovsky normal form without solving partial differential equation (3.88) is explained in [29].
3.4 Input-State Feedback Linearization for MIMO Systems

If the feedback linearizability conditions for MIMO systems previously explained in section 2.11 are fulfilled, then it is possible to extend the input-state feedback linearization process for SISO systems to be compatible with MIMO systems. The departure will be from the general form of affine nonlinear systems given in (2.38) without an output as

\[
\dot{z}_n = f(z) + \sum_{i=1}^{m} g_i(z) u_i, \quad i = 1, \ldots, m
\]

Let

\[
G(z) = \begin{bmatrix} g_1(z) & g_2(z) & \cdots & g_{m-1}(z) & g_m(z) \end{bmatrix}_{n \times m}
\]

\[
U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix}_{m \times 1}
\]

Then, the MIMO system given above is re-written as

\[
\dot{z}_n = f(z) + G(z) U
\]

Assume a state transformation matrix given as

\[
=T(z) = \begin{bmatrix} T_{1,1}(z) \\ T_{1,2}(z) \\ \vdots \\ T_{1,\rho_1 - 1}(z) \\ T_{1,\rho_1}(z) \\ \vdots \\ T_{2,1}(z) \\ T_{2,2}(z) \\ \vdots \\ T_{2,\rho_2 - 1}(z) \\ T_{2,\rho_2}(z) \\ \vdots \\ \vdots \\ \vdots \\ T_{m,1}(z) \\ T_{m,2}(z) \\ \vdots \\ T_{m,\rho_m - 1}(z) \\ T_{m,\rho_m}(z) \end{bmatrix}, \quad x \in \mathbb{R}^n
\]
is a diffeomorphism, such that (3.99) can be coordinate transformed into the following controllable canonical form:

\[
\begin{align*}
\dot{x}_{1,1} &= x_{1,2} \\
\dot{x}_{1,2} &= x_{1,3} \\
\dot{x}_{1,3} &= x_{1,4} \\
& \vdots \\
\dot{x}_{1,\rho_1-1} &= x_{1,\rho_1} \\
\dot{x}_{1,\rho_1} &= v_1 \\
& \vdots \\
\dot{x}_{2,1} &= x_{2,2} \\
\dot{x}_{2,2} &= x_{2,3} \\
\dot{x}_{2,3} &= x_{2,4} \\
& \vdots \\
\dot{x}_{2,\rho_2-1} &= x_{2,\rho_2} \\
\dot{x}_{2,\rho_2} &= v_2 \\
& \vdots \\
& \vdots \\
\dot{x}_{m,1} &= x_{m,2} \\
\dot{x}_{m,2} &= x_{m,3} \\
\dot{x}_{m,3} &= x_{m,4} \\
& \vdots \\
\dot{x}_{m,\rho_m-1} &= x_{m,\rho_m} \\
\dot{x}_{m,\rho_m} &= v_m
\end{align*}
\]

(3.101)

The form (3.101) is equivalent to that in (3.67) which results from input-output feedback linearization when \( \rho = n \) where

\[
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_m
\end{bmatrix}
= 
\begin{bmatrix}
L_{f_1}^{\rho_1} h_1(z) \\
L_{f_2}^{\rho_2} h_2(z) \\
\vdots \\
L_{f_m}^{\rho_m} h_m(z)
\end{bmatrix}
+ 
\begin{bmatrix}
L_{g_1} L_{f_1}^{\rho_1-1} h_1(z) & L_{g_2} L_{f_1}^{\rho_1-1} h_1(z) & \ldots & L_{g_m} L_{f_1}^{\rho_1-1} h_1(z) \\
L_{g_1} L_{f_2}^{\rho_2-1} h_2(z) & L_{g_2} L_{f_2}^{\rho_2-1} h_2(z) & \ldots & L_{g_m} L_{f_2}^{\rho_2-1} h_2(z) \\
\vdots & \vdots & \ddots & \vdots \\
L_{g_1} L_{f_m}^{\rho_m-1} h_m(z) & L_{g_2} L_{f_m}^{\rho_m-1} h_m(z) & \ldots & L_{g_m} L_{f_m}^{\rho_m-1} h_m(z)
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_m
\end{bmatrix}
\]

(3.102)
Equation (3.102) is written in a short form as
\[ \mathbf{v} = \alpha(z) + \beta(z) \mathbf{u} \] (3.103)
in which
\[
\alpha(x) = \begin{bmatrix}
\alpha_1(x) \\
\alpha_2(x) \\
\vdots \\
\alpha_m(x)
\end{bmatrix}_{m \times 1}, \quad
\beta(x) = \begin{bmatrix}
\beta_{11}(x) & \beta_{12}(x) & \cdots & \beta_{1m}(x) \\
\beta_{21}(x) & \beta_{22}(x) & \cdots & \beta_{2m}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{m1}(x) & \beta_{m2}(x) & \cdots & \beta_{mm}(x)
\end{bmatrix}_{m \times m}
\]

If \( \beta(x) \) is non-singular, then (3.101) can be stabilized by the control law
\[ \mathbf{u} = \frac{1}{\beta(x)} [-\alpha(x) + \mathbf{v}] \] (3.104)

The question is how to find the state transformation matrix (3.100). Obviously, (3.101) in a matrix form can be re-written as
\[
\begin{bmatrix}
\dot{x}_{11} \\
\dot{x}_{12} \\
\vdots \\
\dot{x}_{1m-1} \\
\dot{x}_{21} \\
\vdots \\
\dot{x}_{m,1} \\
\dot{x}_{m,2} \\
\vdots \\
\dot{x}_{m,m-1} \\
\dot{x}_{m,\alpha_m}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}_{n \times 1}
\begin{bmatrix}
x_{11} \\
x_{12} \\
\vdots \\
x_{1m-1} \end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}_{n \times m} \begin{bmatrix}
\alpha(x) + \beta(x) \mathbf{u}
\end{bmatrix}_{m \times 1}
\] (3.105)
whose short form is
\[ \dot{x} = \mathbf{A} \mathbf{x} + \mathbf{B} [\alpha(x) + \beta(x) \mathbf{u}] \] (3.106)

From (3.100),
\[ \dot{x} = \frac{\partial T(z)}{\partial z} [f(z) + G(z) U] \] (3.107)
By equating (3.106) with (3.107), one can get:

\[
\frac{\partial T}{\partial z} f(z) = Ax + B \alpha(x)
\]

\[
\begin{bmatrix}
  0 & 1 & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  0 & 0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1 \\
  0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  \vdots \\
  0 \\
  0
\end{bmatrix}
= 
\begin{bmatrix}
  T_{1,1} \\
  T_{1,2} \\
  T_{1,3} \\
  \vdots \\
  T_{1,\rho_1-1} \\
  T_{1,\rho_1}
\end{bmatrix}
+ 
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  \vdots \\
  0 \\
  1
\end{bmatrix}
\alpha(x)_{m \times 1}
\]

\[
\begin{bmatrix}
  T_{2,1} \\
  T_{2,2} \\
  T_{2,3} \\
  \vdots \\
  T_{2,\rho_2-1} \\
  T_{2,\rho_2}
\end{bmatrix}
+ 
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  \vdots \\
  0 \\
  1
\end{bmatrix}
\alpha(x)_{m \times 1}
\]

\[
\begin{bmatrix}
  0 & 1 & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  0 & 0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1 \\
  0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
  T_{2,1} \\
  T_{2,2} \\
  T_{2,3} \\
  \vdots \\
  T_{2,\rho_2-1} \\
  T_{2,\rho_2}
\end{bmatrix}
+ 
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  \vdots \\
  0 \\
  1
\end{bmatrix}
\alpha(x)_{m \times 1}
\]

(3.108)
and can also get:

\[ \frac{\partial T(z)}{\partial z} G(z) = B \beta(x) \]

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
1 & & & \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
1 & & & \\
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\beta_{m1} & \beta_{m2} & \cdots & \beta_{mm} \\
\end{bmatrix}
\]

(3.109)

From the previous analysis, one can conclude that \( T(z) \) should satisfy:

\[
\frac{\partial T_i,j}{\partial z} f(z) = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}, \quad i = 1, 2, \ldots, m \quad \forall \quad j = 1, \ldots, \rho_i - 1.
\]

(3.110)

\[
\frac{\partial T_i,\rho_i}{\partial z} f(z) = \begin{bmatrix} \beta_{i1} & \beta_{i2} & \cdots & \beta_{im} \end{bmatrix}, \quad i = 1, 2, \ldots, m.
\]

(3.111)

where at least one element in \( \begin{bmatrix} \beta_{i1} & \beta_{i2} & \cdots & \beta_{im} \end{bmatrix} \) is not zero. Such that:

\[
T_{i,j+1} = \frac{\partial T_{i,j}}{\partial z} f(z), \quad i = 1, 2, \ldots, m \quad \forall \quad j = 1, \ldots, \rho_i - 1.
\]

(3.112)

and then:

\[
\alpha_i(x) = \frac{\partial T_i,\rho_i}{\partial z} f(z) = \frac{\partial T_i,\rho_i}{\partial z} f(T^{-1}(x)).
\]

(3.113)

\[
\begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{im} \end{bmatrix} = \frac{\partial T_i,\rho_i}{\partial z} G(z) = \frac{\partial T_i,\rho_i}{\partial z} G(T^{-1}(x)).
\]

(3.114)

The analysis that has been done so far for the MIMO state feedback linearization is enough to understand the concept. However, it may not be easy to apply the conditions (3.110)-(3.114) to find the transformation matrix (3.100). This is going to be beyond our research scope. The reader is referred to [29] and [36] for more step-by-step details on how to find the transformation matrix (3.100).
4 Lyapunov Based Control Design

One of the most widely used methods in control theory to prove the stability of nonlinear systems besides input-output stability is the Lyapunov theorem of stability [52]. The interest in this research is in the part of this theorem where a closed-loop system’s stability is examined by studying the behavior of a function known as the Lyapunov function. This function is designed mathematically to provide an easy way to measure control objectives such that nonlinear systems stability can be tested through a scalar differential equation [13].

4.1 Lyapunove Stability Theorem

To explain the notion of Lyapunov theorem of stability, the time-invariant system with a state \( z \) and a dynamic:

\[
\dot{z} = f(z) + g(z) u
\]

(4.1)

is considered where \( z \in \mathbb{R}^n \), and the control input \( u \in \mathbb{R} \). The goal is to derive a feedback control function \( u = \alpha(z) \) such that

\[
\dot{z} = f(z) + g(z) \alpha(z)
\]

(4.2)

will have equilibrium points that are globally asymptotically stable at the origin. According to Lyapunov, to achieve this goal, one needs to find a smooth, positive definite and radially unbounded function \( V(z) \) known as Lyapunov function such that its derivative along the solution of (4.1) satisfies:

\[
\dot{V}(z) = \frac{\partial V(z)}{\partial z} \left[ f(z) + g(z) \alpha(z) \right] \leq -W(z)
\]

(4.3)

where \( W(z) \) is positive definite. If a good choice of \( V(z) \) is made and (4.3) is satisfied, then the control law \( \alpha(z) \) results in globally asymptotically stable equilibrium points of (4.1) [53–56].

Example 4.1

Consider the scalar nonlinear system given in (1.1) by

\[
\dot{z} = -a \cos z - b z^3 + cu
\]
Choosing the smooth, positive definite and radially unbounded Lyapunov function

\[ V(z) = \frac{1}{2}z^2 \]  

(4.4)

and differentiating it in time yields

\[ \dot{V} = z\dot{z} = -az \cos z - bz^4 + czu \]  

(4.5)

Now, one can choose the control function

\[ u = \frac{1}{c} (a \cos z + bz^3 - kz) \]  

(4.6)

that results in

\[ \dot{V} = -kz^2 \]  

(4.7)

which is negative definite. Thus, (4.6) will stabilize the equilibrium at the origin. △

4.2 Sontag’s Formula

If a nonlinear system has the form (4.1) with \( f(0) = 0 \) and a smooth, positive definite and radially unbounded Lyapunov function exists, then another way to choose a stabilizing control function is through using what so-called Sontag’s formula [57,58].

\[
u = \alpha(z) = \begin{cases} 
-\frac{\partial V}{\partial z} f(z) + \sqrt{\left(\frac{\partial V}{\partial z} f(z)\right)^2 + \left(\frac{\partial V}{\partial z} g(z)\right)^2}, & \text{for } \frac{\partial V}{\partial z} g(z) \neq 0 \\
0, & \text{for } \frac{\partial V}{\partial z} g(z) = 0
\end{cases}
\]  

(4.8)

Example 4.2

Reconsider the scalar nonlinear system (1.1), to use Sontag’s formula (4.8) to find the stabilizing function \( u \), one needs first to fulfill that \( f(0) = 0 \). This can easily be done by assuming that

\[ u = \frac{1}{c} (a \cos z + \alpha) \]  

(4.9)

Thus, the new system will be

\[ \dot{z} = -bz^3 + \alpha \]  

(4.10)
with \( f(z) = -bz^3 \) and \( g(z) = 1 \). Using the same Lyapunov function (4.4) and differentiating it in time results in
\[
\dot{V} = -bz^4 + z\alpha \quad (4.11)
\]
Based on (4.4) and (4.10), \( \frac{\partial V}{\partial z} g = az \neq 0 \). Thus, the stabilization function can be obtained using Sontag’s formula as follows
\[
\alpha = -\frac{\frac{\partial V}{\partial z} f + \sqrt{\left(\frac{\partial V}{\partial z} f\right)^2 + \left(\frac{\partial V}{\partial z} g\right)^4}}{\frac{\partial V}{\partial z} g}
\]
\[
= -bz^4 + \sqrt{(-bz^4)^2 + z^4}
\]
\[
= b^3 z - z\sqrt{b^2 z^4 + 1} \quad (4.12)
\]
This results in the following overall stabilizing function
\[
u = u = \frac{1}{c} \left( a \cos z + bz^3 - z\sqrt{b^2 z^4 + 1} \right) \triangle
\]
Chapters five and six present the integrator backstepping theorem, backstepping design for systems with a chain of integrators, backstepping design for systems in the strict feedback form and adaptive backstepping design. Both chapters are based significantly on the great book of nonlinear and adaptive control design by M. Krstić, I. Kanellakopoulos and P.V. Kokotović [13].
5 Backstepping Control

5.1 Integrator Backstepping

The well-known integrator backstepping theorem is a useful tool to analyze systems in a particular form by applying a systematic procedure to produce a stabilizing controller \cite{59}. Consider the system shown in Fig. 5.1 and given by:

\begin{equation}
\begin{aligned}
\dot{z} &= f(z) + g(z) q \\
\dot{q} &= u_f
\end{aligned}
\end{equation}

where \( z \in \mathbb{R}^n \), \( q \in \mathbb{R} \) and the control input \( u_f \in \mathbb{R} \).

Assuming that a state feedback control function \( q = u_1(z) \) that stabilizes \( \dot{z} \) is known where \( u_1(0) = 0 \). That is to say, there is a positive definite, radially unbounded Lyapunov function \( V_1(z) \) so that when using \( u_1(z) \):

\begin{equation}
\dot{V}_1(z) = \frac{\partial V_1(z)}{\partial z} \left[ f(z) + g(z) u_1(z) \right] \leq -W(z)
\end{equation}

where \( W(z) \) is positive definite. Accordingly, if one can make \( q \) equals \( u_1(z) \), then the desired behavior can be achieved. Defining the error variable

\begin{equation}
x = q - u_1(z)
\end{equation}

Figure 5.1: Block diagram of the affine nonlinear system.
results in introducing the virtual control function \( u_1(z) \) as shown in Fig. 5.2. That yields:

\[
\dot{z} = f(z) + g(z) u_1(z) + g(z) [q - u_1(z)]
\]  

(5.4)

\[\text{Figure 5.2: Introducing virtual control function } u_1(z).\]

Taking the derivative in time of (5.3) results in:

\[
\dot{x} = \dot{q} - \dot{u}_1(z) = u_f - \dot{u}_1(z)
\]  

(5.5)

The preceding steps will transform system (5.1) into the following system, which includes backstepping of the virtual control function \( u_1(z) \) as shown in Fig. 5.3.

\[
\begin{aligned}
\dot{z} &= f(z) + g(z) u_1(z) + g(z) x \\
\dot{x} &= u_f - \dot{u}_1(z)
\end{aligned}
\]  

(5.6)

\[\text{Figure 5.3: Backstepping of the virtual control function } u_1(z).\]
The augmented Lyapunov function for the system (5.6) is given by:

\[ V(z, x) = V_1(z) + \frac{1}{2}x^2 \]  

(5.7)

and its derivative with respect to time along the solution of system (5.6) is computed as:

\[
\dot{V}(z, x) = \frac{\partial V(z)}{\partial z} f(z) + \frac{\partial V(z)}{\partial z} g(z) u_1(z) + \frac{\partial V(z)}{\partial z} g(z) x + xu_f - xu_1(z)
\]  

(5.8)

Choosing:

\[ u_f = -\frac{\partial V(z)}{\partial z} g(z) + \dot{u}_1(z) - kx \quad k > 0 \]  

(5.9)

yields:

\[
\dot{V}(z, x) = \frac{\partial V(z)}{\partial z} [f(z) + g(z) u_1(z)] - kx^2 \leq -W(z) - kx^2
\]  

(5.10)

This proves according to Lyapunov theorem of stability that \((z, x) = (0, 0)\) is globally asymptotically stable and as \(x = q - u_1(z)\) and \(u_1(0) = 0\), \((z, q) = (0, 0)\) is globally asymptotically stable as well.

5.2 Backstepping for Systems with a Chain of Integrators

![Figure 5.4: Nonlinear system with a chain of integrators.](image-url)
Consider the system shown in Fig. 5.4 and given by the general form

\[
\begin{align*}
\dot{z} &= f(z) + g(z) q_1 \\
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= q_3 \\
&\quad \vdots \\
\dot{q}_{n-1} &= q_n \\
\dot{q}_n &= u_f 
\end{align*}
\] (5.11)

Backstepping can be used to stabilize systems with a chain of integrators by the successive repetition of the design procedure used for the second-order system (5.1). For more clarification, deriving a control function for the following third-order system will be considered.

\[
\begin{align*}
\dot{z} &= f(z) + g(z) q_1 \\
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= u_f 
\end{align*}
\] (5.12a)

5.2.1 Step:1

One can start the design by considering the first subsystem consisting of equation (5.12a):

\[
\begin{align*}
\begin{cases}
\dot{z} &= f(z) + g(z) q_1 \\
\end{cases} 
\end{align*}
\] (5.13)

Let \( q_1 \) be a virtual control function and assume that there is a feedback control function \( u_1(z) = q_1 \) with \( u_1(0) = 0 \) that stabilizes (5.13). There also exist \( V_1(z) \), a corresponding positive definite and radially unbounded Lyapunov function such that when using \( u_1(z) \):

\[
\dot{V}_1(z) = \frac{\partial V_1(z)}{\partial z} [f(z) + g(z) u_1(z)] \leq -W_1(z) 
\] (5.14)

where \( W_1(z) \) is positive definite. With this assumption, one can proceed to derive the stabilizing function for the whole system using backstepping as follows.

5.2.2 Step:2

Consider the second subsystem consisting of the two equations (5.12a) and (5.12b):

\[
\begin{align*}
\begin{cases}
\dot{z} &= f(z) + g(z) q_1 \\
\dot{q}_1 &= q_2 \\
\end{cases} 
\end{align*}
\] (5.15)
Let $q_2$ be the virtual control function and assume that there is a control function $u_2(z, q_1) = q_2$ that stabilizes (5.15). Define the error parameter representing the difference between the first virtual function $q_1$ and the proposed control function $u_1(z)$ as:

$$x_1 = q_1 - u_1(z) \quad (5.16)$$

Accordingly, the first subsystem (5.13) will equivalently change to:

$$\dot{z} = f(z) + g(z)[u_1(z) + x_1] = f(z) + g(z)u_1(z) + g(z)x_1 \quad (5.17)$$

Differentiating (5.16) in time results in:

$$\dot{x}_1 = \dot{q}_1 - \dot{u}_1(z) = g(z_2) - \dot{u}_1(z) \quad (5.18)$$

where $\dot{u}_1(z) = \frac{\partial u_1(z)}{\partial z}[f(z) + g(z)q_1]$. Thus, subsystem (5.15) in the new coordinates will be:

$$\dot{z} = f(z) + g(z)u_1(z) + g(z)x_1 \quad (5.19)$$

To stabilize (5.19), consider the augmented Lyapunov function:

$$V_2(z, x_1) = V_1(z) + \frac{1}{2}x_1^2 \quad (5.20)$$

and differentiate it in time:

$$\dot{V}_2(z, x_1) = \dot{V}_1(z) + x_1\dot{x}_1$$

$$= \frac{\partial V_1(z)}{\partial z}[f(z) + g(z)u_1(z) + g(z)x_1] + x_1[q_2 - \dot{u}_1(z)]$$

$$= \frac{\partial V_1(z)}{\partial z}f(z) + \frac{\partial V_1(z)}{\partial z}g(z)u_1(z) + \frac{\partial V_1(z)}{\partial z}g(z)x_1 + x_1q_2 - x_1\dot{u}_1(z) \quad (5.21)$$

Choosing $q_2 = u_2(z, q_1)$ as:

$$q_2 = u_2(z, q_1) = -\frac{\partial V_1(z)}{\partial z}g(z) + \dot{u}_1 - k_1x_1 \quad (5.22)$$

yields that (5.21) turns out to be:

$$\dot{V}_2(z, x_1) = \frac{\partial V_1(z)}{\partial z}[f(z) + g(z)u_1(z)] - k_1x_1^2$$

$$\leq -W(z) - k_1x_1^2 \quad (5.23)$$

In turn, this results in a global asymptotic stabilization of the the origin $(z, x_1) = (0, 0)$ according to Lyapunov theorem of stability. Since $x_1 = q_1 - u_1(z)$ and $u_1(0) = 0$, then $(z, q_1) = (0, 0)$ is globally asymptotically stable as well.
5.2.3 Step:3

It is time now to derive the control function that stabilizes the whole system. Considering the third subsystem consisting of equations (5.12a), (5.12b) and (5.12c)

\[
\begin{align*}
\dot{z} &= f(z) + g(z) q_1 \\
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= u_f 
\end{align*}
\]  
(5.24)

where \(u_f\) is the control input. In step:2, \(q_2\) was the virtual control function, and its proposed value with Lyapunov function \(V_2(z, x_1)\) was \(u_2(z, q_1)\). Defining the error parameter representing the difference between the second virtual control function \(q_2\) and the proposed function \(u_2(z, q_1)\) as:

\[x_2 = q_2 - u_2(z, q_1)\]  
(5.25)

This, in turn, will change the second equation in (5.24) equivalently to:

\[\dot{q}_1 = u_2(z, q_1) + x_2\]  
(5.26)

Taking the time derivative of (5.25) results in:

\[\dot{x}_2 = \dot{q}_2 - \dot{u}_2(z, q_1) = u_f - \dot{u}_2(z, q_1)\]  
(5.27)

Thus, the subsystem (5.24) will turn out to be:

\[
\begin{align*}
\dot{z} &= f(z) + g(z) q_1 \\
\dot{q}_1 &= u_2(z, q_1) + x_2 \\
\dot{x}_2 &= u_f - \dot{u}_2 
\end{align*}
\]  
(5.28)

The augmented Lyapunov function for this system can be given by:

\[V(z, x_1, x_2) = V_2(z, x_1) + \frac{1}{2} x_2^2\]  
(5.29)

Differentiating (5.29) in time results in:

\[\dot{V} = \dot{V}_1(z) + x_1 \dot{x}_1 + x_2 \dot{x}_2\]

\[= \frac{\partial V_1(z)}{\partial z} [f(z) + g(z) q_1] + x_1 (\dot{q}_1 - \dot{u}_1(z)) + x_2 (u_f - \dot{u}_2(z, q_1))\]

\[= \frac{\partial V_1(z)}{\partial z} [f(z) + g(z) u_1(z) + g(z) x_1] + x_1 (u_2(z, q_1) + x_2) - x_1 \dot{u}_1(z) + x_2 u_f - x_2 \dot{u}_2(z, q_1)\]  
(5.30)
Substituting (5.22) in (5.30) yields:

\[
\dot{V}(z, x_1, x_2) = \frac{\partial V_1(z)}{\partial z} [f(z) + g(z) u_1(z)] - k_1 x_1^2 + x_1 x_2 + x_2 u_f - x_2 u_2(z, q_1)
\] (5.31)

To guarantee that (5.31) is negative definite, \( u_f \) can be chosen to be:

\[
u_f = \dot{u}_2(z, q_1) - k_2 x_2 - x_1
\] (5.32)

Thus, (5.31) will become:

\[
\dot{V}(z, x_1, x_2) = \frac{\partial V_1(z)}{\partial z} [f(z) + g(z) u_1(z)] - k_1 x_1^2 + k_2 x_2^2
\]

\[
\leq -W(z) - k_1 x_1^2 + k_2 x_2^2
\] (5.33)

which means according to Lyapunov that the origin of the third-order system (5.12) is globally asymptotically stable with control law (5.32). For more clarification in this regard, the following example of a third-order system is considered:

**Example 5.1**

\[
\dot{z} = z^2 + q_1
\] (5.34a)

\[
\dot{q}_1 = q_2
\] (5.34b)

\[
\dot{q}_2 = u_f
\] (5.34c)

\[
y = q_1
\] (5.34d)

This system has a relative degree of 2 because \( \ddot{y} = u_f \). It also has an internal dynamic with state \( z \). However, this internal dynamic is not stable because:

\[
\dot{z} = \phi(z, q_1) = \phi(z, 0) = z^2
\] (5.35)

The instability of the internal system dynamic means that the equilibrium point \( z^* = 0 \) at the origin is not stable and hence the system (5.34) can not be stabilized using feedback linearization method. Fortunately, the backstepping design approach can solve this problem in three steps.

**Step:1**

Consider (5.34a) representing the first subsystem:

\[
\left\{ \begin{array}{l}
\dot{z} = z^2 + q_1
\end{array} \right.
\] (5.36)
Let $q_1$ be a virtual control function and assume that there is a feedback control function $u_1(z) = q_1$ that stabilizes (5.36). Also, assume a positive definite and radially unbounded Lyapunov function is given by:

$$V_1(z) = \frac{1}{2}z^2$$  \hspace{1cm} (5.37)

such that when using $u_1(z)$:

$$\dot{V}_1(z) \leq -W(z)$$  \hspace{1cm} (5.38)

where $W(z)$ is positive definite. Taking the derivative in time of (5.37) results in:

$$\dot{V}_1 = z\dot{z}
= z [z^2 + u_1(z)]
= z^3 + zu_1(z)$$  \hspace{1cm} (5.39)

Choosing $u_1(z)$ as:

$$u_1(z) = -z - z^2$$  \hspace{1cm} (5.40)

and substituting (5.40) in (5.39). Thus, (5.39) will become:

$$\dot{z} = z^3 + z [-z - z^2]
= z^3 - z^2 - z^3
= -z^2
\leq -W(z)$$  \hspace{1cm} (5.41)

At this point, one can proceed to the second step towards deriving the stabilizing state feedback law for (5.34) using backstepping approach.

**Step:2**

Consider the second subsystem represented by the set of equations (5.34a) and (5.34b):

$$\begin{cases}
\dot{z} = z^2 + q_1 \\
\dot{q}_1 = q_2
\end{cases}$$  \hspace{1cm} (5.42)

Now, $q_2$ is considered as a control input. Define the error parameter:

$$x_1 = q_1 - u_1(z)$$  \hspace{1cm} (5.43)

and re-write subsystem (5.36) using the error system (5.43)

$$\begin{cases}
\dot{\hat{z}} = z^2 + u_1(z) + x_1
\end{cases}$$  \hspace{1cm} (5.44)
Differentiating (5.43) in time yields:

\[
\dot{x}_1 = q_1 - \dot{u}_1(z) \\
= q_2 - \dot{u}_1(z)
\]  

(5.45)

Then, the subsystem (5.42) will accordingly be updated to:

\[
\begin{cases}
\dot{z} = z^2 + u_1(z) + x_1 \\
\dot{x}_1 = q_2 - \dot{u}_1(z)
\end{cases}
\]  

(5.46)

where \(\dot{u}_1(z)\) is calculated as:

\[
\dot{u}_1(z) = -\dot{z} - 2z\dot{z} \\
= -z^2 - u_1(z) - 2z(z^2 + u_1(z)) \\
= -z^2 + z + z^2 - 2z(z^2 - z - z^2) \\
= z + 2z^2
\]  

(5.47)

Choosing the augmented Lyapunov function to stabilize (5.46):

\[
V_2(z, x_1) = V_1(z) + \frac{1}{2}x_1^2
\]  

(5.48)

The time derivative of (5.48) is obtained as:

\[
\dot{V}_2(z, x_1) = \dot{V}_1(z) + x_1\dot{x}_1 \\
= -z^2 + x_1[q_2 - \dot{u}_1(z)] \\
= -z^2 + x_1q_2 - x_1\dot{u}_1(z)
\]  

(5.49)

Choosing \(q_2 = u_2(z, q_1)\) as:

\[
q_2 = u_2(z, q_1) = \dot{u}_1(z) - k_1x_1
\]  

(5.50)

and substituting (5.50) in (5.49). Thus, (5.49) turns out to be:

\[
\dot{V}_2(z, x_1) = -z^2 - k_1x_1^2 \\
\leq -W(z) - k_1x_1^2
\]  

(5.51)

which means that the origin \((z, x_1) = (0,0)\) is globally asymptotically stable and hence \((z, q_1) = (0,0)\) is also globally asymptotically stable.
Step: 3

In the third step, the subsystem consisting of the set of equations (5.34a), (5.34b) and (5.34c) is considered:

\[
\begin{align*}
\dot{z} &= z^2 + q_1 \\
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= u_f
\end{align*}
\]  
(5.52)

where now the control input is \( u_f \). Define the error parameter:

\[ x_2 = q_2 - u_2(z, q_1) \]  
(5.53)

such that from (5.40), (5.43), (5.47) and (5.50), \( u_2(z, q_1) \) can further be calculated as:

\[ u_2(z, q_1) = \dot{u}_1(z) - k_1 x_1 \\
= z + 2z^2 - k_1 [q_1 - u_1(z)] \]  
(5.54)

\[ = z + 2z^2 - k_1 [q_1 + z + z^2] \]

As a result of (5.53), the second equation in the subsystem (5.52) turns out to be:

\[ \dot{q}_1 = u_2(z, q_1) + x_2 \]  
(5.55)

The time derivative of the error parameter (5.53) is:

\[ \dot{x}_2 = \dot{q}_2 - \dot{u}_2(z, q_1) \]
\[ = u_f - \dot{u}_2(z, q_1) \]  
(5.56)

From (5.54), \( \dot{u}_2(z, q_1) \) can be derived as:

\[ \dot{u}_2(z, q_1) = \dot{z} + 4z\dot{z} - k_1 [\dot{q}_1 + \dot{z} + 2z\ddot{z}] \]
\[ = \dot{z} + 4z\dot{z} - k_1 \dot{q}_1 - k_1 \dot{z} - 2k_1 z \ddot{z} \]
\[ = z^2 + q_1 + 4z (z^2 + q_1) - k_1 q_2 - k_1 (z^2 + q_1) - 2k_1 z (z^2 + q_1) \]  
(5.57)

Consequently, the subsystem (5.52) will equivalently turn out to be:

\[
\begin{align*}
\dot{z} &= z^2 + q_1 \\
\dot{q}_1 &= u_2(z, q_1) + x_2 \\
\dot{x}_2 &= u_f - \dot{u}_2(z, q_1)
\end{align*}
\]  
(5.58)
The augmented Lyapunov function is chosen as:

$$V(z, x_1, x_2) = V_1(z) + \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$$  \hspace{1cm} (5.59)

The derivative in time of (5.59) is calculated as follows:

$$\dot{V}(z, x_1, x_2) = \dot{V}_1(z) + x_1 \dot{x}_1 + x_2 \dot{x}_2$$

$$= -z^2 + x_1 [q_2 - \dot{u}_1(z)] + x_2 [u_f - \dot{u}_2(z, q_1)]$$  \hspace{1cm} (5.60)

Substitute for $q_2$ from (5.53)

$$\dot{V}(z, x_1, x_2) = -z^2 + x_1 [u_2(z, q_1) + x_2] - x_1 \dot{u}_1 + x_2 [u_f - \dot{u}_2(z, q_1)]$$  \hspace{1cm} (5.61)

Substituting for $u_2(z, q_1)$ from (5.54)

$$\dot{V}(z, x_1, x_2) = -z^2 - k_1 x_1^2 + x_1 x_2 + x_2 u_f - x_2 \dot{u}_2(z, q_1)$$  \hspace{1cm} (5.62)

To make (5.62) negative definite according to Lyapunov theorem, one can choose $u_f$ as:

$$u_f = \dot{u}_2(z, q_1) - k_2 x_2 - x_1$$  \hspace{1cm} (5.63)

Substituting (5.63) in (5.62) yields:

$$\dot{V}(z, x_1, x_2) = -z^2 - k_1 x_1^2 - k_2 x_2^2$$

$$\leq -W(z) - k_1 x_1^2 - k_2 x_2^2$$  \hspace{1cm} (5.64)

**Figure 5.5:** Simulation results of stabilizing the system (5.34).
It can be concluded that according to the Lyapunov theorem of stability, the origin \((z, q_1, q_2) = (0, 0, 0)\) as simulation results show in Fig. 5.5 is globally asymptotically stable. △

5.3 Backstepping for Systems in Strict Feedback Form

In nonlinear systems control, the system is said to be in strict feedback form if it is of the following form [52] and [24]:

\[
\begin{align*}
\dot{z}_1 &= f_1(z_1) + g_1(z_1) z_2 \\
\dot{z}_2 &= f_2(z_1, z_2) + g_2(z_1, z_2) z_3 \\
&\vdots \\
\dot{z}_{n-1} &= f_{n-1}(z_1, z_2, \ldots, z_{n-1}) + g_{n-1}(z_1, z_2, \ldots, z_{n-1}) z_n \\
\dot{z}_n &= f_n(z_1, z_2, \ldots, z_{n-1}, z_n) + g_n(z_1, z_2, \ldots, z_{n-1}, z_n) u_f
\end{align*}
\]

where \(z \in \mathbb{R}^n\) is the states of the system and \(u_f \in \mathbb{R}\) is the control input. The notion of integrator backstepping can be expanded to be suitable for such systems. To clarify how control function is designed through backstepping for systems of strict feedback form, the following third-order example is considered.

\[
\begin{cases}
\dot{z}_1 = f_1(z_1) + g_1(z_1) z_2 \\
\dot{z}_2 = f_2(z_1, z_2) + g_2(z_1, z_2) z_3 \\
\dot{z}_3 = f_3(z_1, z_2, z_3) + g_3(z_1, z_2, z_3) u_f
\end{cases}
\]

5.3.1 Step:1

The design starts by considering the first subsystem \(\dot{z}_1\):

\[
\dot{z}_1 = f_1(z_1) + g_1(z_1) z_2
\]

Let \(z_2\) be a virtual control function and assume that a control function \(u_1(z_1)\) that stabilizes \(\dot{z}_1\) is known. Besides, assume that there is a positive definite and radially unbounded Lyapunov function \(V_1(z_1)\) such that when using \(u_1(z_1)\):

\[
\dot{V}_1(z_1) = \frac{\partial V}{\partial z_1} [f_1(z_1) + g_1(z_1) u_1] \leq -W_1
\]

where \(W_1\) is positive definite.
5.3.2 **Step:2**

Consider the set of subsystems $\dot{z}_1$ and $\dot{z}_2$

$$\begin{cases} 
\dot{z}_1 = f_1(z_1) + g_1(z_1) z_2 \\
\dot{z}_2 = f_2(z_1, z_2) + g_2(z_1, z_2) z_3 
\end{cases} \quad (5.69)$$

Now, let $z_3$ be the virtual control function and the desired stabilizing function be $u_2$. From **step:1**, one can introduce the error parameter that represents the difference between the virtual control function $z_2$ and the desired control function $u_1(z_1)$ as follows:

$$x_1 = z_2 - u_1(z_1) \quad (5.70)$$

Differentiating it in time and substituting for $z_2 = u_1 + x_1$ and the virtual control input $z_3$ with the desired stabilizing function $u_2$, the set of subsystems $\dot{z}_1$ and $\dot{z}_2$ in new coordinates will be:

$$\dot{z}_1 = f_1(z_1) + g_1(z_1)[u_1 + x_1] = f_1(z_1) + g_1(z_1) u_1 + g_1(z_1) x_1 \quad (5.71)$$

$$\dot{x}_1 = \dot{z}_2 - \dot{u}_1 = f_2(z_1, z_2) + g_2(z_1, z_2) u_2 - \dot{u}_1$$

Choosing the Lyapunov function:

$$V_2(z_1, x_1) = V_1 + \frac{1}{2}x_1^2 = \frac{1}{2}z_1^2 + \frac{1}{2}x_1^2 \quad (5.72)$$

Differentiating $V_2$ in time yields:

$$\dot{V}_2 = z_1 \dot{z}_1 + x_1 \dot{x}_1$$

$$= z_1 [f_1 + g_1 u_1 + g_1 x_1] + x_1 [f_2 + g_2 u_2 - \dot{u}_1]$$

$$= z_1 [f_1 + g_1 u_1] + x_1 [f_2 + g_2 u_2 - \dot{u}_1 + z_1 g_1]$$

$$\leq - W_1 + x_1 [f_2 + g_2 u_2 - \dot{u}_1 + z_1 g_1] \quad (5.73)$$

Choosing $u_2(z_1, x_1)$:

$$u_2(z_1, x_1) = \frac{1}{g_2} [- f_2 + \dot{u}_1 - z_1 g_1 - k_1 x_1] \quad (5.74)$$

Substituting for $u_2(z_1, x_1)$ in $\dot{V}_2$ results in:

$$\dot{V}_2 \leq - W_1 - k_1 x_1^2 \leq - W_2 \quad (5.75)$$
5.3.3 Step:3

In the last step, the set of subsystems \( \dot{z}_1, \dot{z}_2 \) and \( \dot{z}_3 \) will be considered.

\[
\begin{align*}
\dot{z}_1 &= f_1(z_1) + g_1(z_1) z_2 \\
\dot{z}_2 &= f_2(z_1, z_2) + g_2(z_1, z_2) z_3 \\
\dot{z}_3 &= f_3(z_1, z_2, z_3) + g_3(z_1, z_2, z_3) u_f
\end{align*}
\]

(5.76)

Now, \( u_f \) is the control input for the whole system. From step:2, one can introduce the error parameter that represents the difference between the virtual control function \( z_3 \) and the desired control function \( u_2(z_1, x_1) \) as follows:

\[
x_2 = z_3 - u_2(z_1, x_1)
\]

(5.77)

Differentiating it in time and substituting for \( z_3 = u_2 + x_2 \), the set of subsystems \( \dot{z}_1, \dot{z}_2 \) and \( \dot{z}_3 \) in new coordinates will be:

\[
\begin{align*}
\dot{z}_1 &= f_1(z_1) + g_1(z_1) z_2 \\
\dot{z}_2 &= f_2(z_1, z_2) + g_2(z_1, z_2)[u_2 + x_2] \\
&= f_2(z_1, z_2) + g_2(z_1, z_2) u_2 + g_2(z_1, z_2) x_2 \\
\dot{x}_2 &= f_3(z_1, z_2, z_3) + g_3(z_1, z_2, z_3) u_f - \dot{u}_2
\end{align*}
\]

(5.78)

Choosing the Lyapunov function:

\[
V_3(z_1, x_1, x_2) = V_2(z_1, x_1) + \frac{1}{2} x_2^2
\]

\[
= V_1(z_1) + \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2
\]

(5.79)

\[
= \frac{1}{2} x_1^2 + \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2
\]

Differentiating \( V_3 \) in time results in:

\[
\dot{V}_3 = z_1 \dot{z}_1 + x_1 \dot{x}_1 + x_2 \dot{x}_2
\]

\[
\begin{align*}
&= z_1 [f_1 + g_1 z_2] + x_1 [\dot{z}_2 - \dot{u}_1] + x_2 [\dot{z}_3 - \dot{u}_2] \\
&= z_1 [f_1 + g_1 u_1 + g_1 x_1] + x_1 [f_2 + g_2 u_2 + g_2 x_2 - \dot{u}_1] + x_2 [f_3 + g_3 u_f - \dot{u}_2] \\
&= z_1 [f_1 + g_1 u_1] + x_1 [f_2 + g_2 u_2 - \dot{u}_1 + g_1 z_1] + x_2 [f_3 + g_3 u_f - \dot{u}_2 + g_2 x_1]
\end{align*}
\]

(5.80)

Substituting for \( u_2 \) from (5.74) yields:

\[
\dot{V}_3 = z_1 [f_1 + g_1 u_1] - k_1 x_1^2 + x_2 [f_3 + g_3 u_f - \dot{u}_2 + g_2 x_1]
\]

\[
\leq -W_2 + x_2 [f_3 + g_3 u_f - \dot{u}_2 + g_2 x_1]
\]

(5.81)
Choosing $u_f$

$$u_f = \frac{1}{g_3} [-f_3 + \dot{u}_1 - g_2x_1 - k_2x_2] \quad (5.82)$$

Substituting for $u_f$ in $\dot{V}_3$ results in:

$$\dot{V}_3 \leq -W_2 - k_2x_2^2 \leq -W_3 \quad (5.83)$$

According to Lyapunov, the designed $u_f$ guarantees the stability of the system equilibria. Following this iterative procedure explains how backstepping is used to design stabilizing functions for systems in the strict feedback form. However, it may get more complicated when the system is of a higher order. Therefore, the algorithm in the next section can be applied to design stabilizing function in a much easier and more organized manner.

### 5.4 Backstepping Control Algorithm for Systems in Strict Feedback Form

#### 5.4.1 Step:1

Let $x_1 = z_1$. Define the error variable:

$$x_2 = z_2 - u_1 \quad (5.84)$$

where $z_2$ is the virtual control signal, and $u_1$ is the proposed stabilizing function. Choose:

$$u_1 = \frac{1}{g_1 (z_1)} (-f_1 (z_1) - k_1 x_1), \quad k_1 > 0 \quad (5.85)$$

Taking the derivative in time of $x_1$ yields:

$$\dot{x}_1 = f_1 (z_1) + g_1 (z_1) [x_2 + u_1]$$

$$= f_1 (z_1) + g_1 (z_1) x_2 + g_1 (z_1) u_1 \quad (5.86)$$

Substituting for the value of $u_1$ results in:

$$\dot{x}_1 = -k_1 x_1 + g_1 (z_1) x_2 \quad (5.87)$$

Consider the Lyapunov function:

$$V_1 (x_1) = \frac{1}{2} x_1^2 \quad (5.88)$$

Differentiating it in time:

$$\dot{V}_1 (x_1) = x_1 \dot{x}_1$$

$$= x_1 (-k_1 x_1 + g_1 (z_1) x_2) \quad (5.89)$$

$$= -k_1 x_1^2 + g_1 (z_1) x_1 x_2$$
5.4.2 Step:2

From step:1, \( x_2 = z_2 - u_1 \). Define the error variable:

\[
x_3 = z_3 - u_2
\]

where \( z_3 \) is the virtual control signal, and \( u_2 \) is the proposed stabilizing function. Choose:

\[
u_2 = \frac{1}{g_2(z_1,z_2)} \left[ -f_2(z_1,z_2) + \dot{u}_1 - k_2 x_2 - g_1(z_1) x_1 \right], \quad k_2 > 0
\]

Taking the derivative in time of the error variable \( x_2 \) results in:

\[
\dot{x}_2 = \dot{z}_2 - \dot{u}_1
\]

\[
= f_2(z_1,z_2) + g_2(z_1,z_2) [x_3 + u_2] - \dot{u}_1
\]

\[
= f_2(z_1,z_2) + g_2(z_1,z_2) x_3 + g_2(z_1,z_2) u_2 - \dot{u}_1
\]

Substituting for the value of \( u_2 \) yields:

\[
\dot{x}_2 = -k_2 x_2 + g_2(z_1,z_2) x_3 - g_1(z_1) x_1
\]

Consider the augmented Lyapunov function:

\[
V_2(x_1,x_2) = V_1(x_1) + \frac{1}{2} x_2^2
\]

Differentiating it in time results in:

\[
\dot{V}_2(x_1,x_2) = \dot{V}_1(x_1) + x_2 \dot{x}_2
\]

\[
= -k_1 x_1^2 + g_1(z_1) x_1 x_2 + x_2 \dot{x}_2
\]

\[
= -k_1 x_1^2 + g_1(z_1) x_1 x_2 - k_2 x_2 + g_2(z_1,z_2) x_3 - g_1(z_1) x_1
\]

\[
= -k_1 x_1^2 + g_1(z_1) x_1 x_2 - k_2 x_2^2 + g_2(z_1,z_2) x_2 x_3 - g_1(z_1) x_1 x_2
\]

\[
= -k_1 x_1^2 - k_2 x_2^2 + g_2(z_1,z_2) x_2 x_3
\]

5.4.3 Step:n-1

From step:n-2, \( x_{n-1} = z_{n-1} - u_{n-2} \). Define the error variable:

\[
x_n = z_n - u_{n-1}
\]

where \( z_n \) is the virtual control signal, and \( u_{n-1} \) is the proposed stabilizing function. Choose:

\[
u_{n-1} = \frac{1}{g_{n-1}(z_1,\ldots,z_{n-1})} \left[ -f_{n-1}(z_1,\ldots,z_{n-1}) + \dot{u}_{n-2} - k_{n-1} x_{n-1} - g_{n-2}(z_1,\ldots,z_{n-2}) x_{n-2} \right], \quad k_{n-1} > 0
\]
Taking the derivative in time of the error variable $x_{n-1}$ results in:

$$\dot{x}_{n-1} = \dot{z}_{n-1} - \dot{u}_{n-2}$$

$$= f_{n-1}(z_1, \ldots, z_{n-1}) + g_{n-1}(z_1, \ldots, z_{n-1})[x_n + u_{n-1}] - \dot{u}_{n-2}$$

$$= f_{n-1}(z_1, \ldots, z_{n-1}) + g_{n-1}(z_1, \ldots, z_{n-1})x_n + g_{n-1}(z_1, \ldots, z_{n-1})u_{n-1} - \dot{u}_{n-2}$$

Substituting for the value of $u_{n-1}$ yields:

$$\dot{x}_{n-1} = -k_{n-1}x_{n-1} - g_{n-2}(z_1, \ldots, z_{n-2})x_{n-2} + g_{n-1}(z_1, \ldots, z_{n-1})x_n$$

(5.99)

Consider the augmented Lyapunov function:

$$V_{n-1}(x_1, \ldots, x_{n-1}) = V_{n-2}(x_1, \ldots, x_{n-2}) + \frac{1}{2}x_n^2$$

(5.100)

Differentiating it in time:

$$\dot{V}_{n-1}(x_1, \ldots, x_{n-1}) = -\sum_{i=1}^{n-2} k_i x_i^2 + g_{n-2}(z_1, \ldots, z_{n-2})x_{n-2}x_{n-1} - k_{n-1}x_{n-1}^2$$

$$+ g_{n-1}(z_1, \ldots, z_{n-1})x_{n-1}x_n - g_{n-2}(z_1, \ldots, z_{n-2})x_{n-2}x_{n-1}$$

(5.101)

$$- \sum_{i=1}^{n-1} c_i x_i^2 + g_{n-1}(z_1, \ldots, z_{n-1})x_{n-1}x_n$$

5.4.4 Step:n

From step:n-1, $x_n = z_n - u_{n-1}$. Differentiating $x_n$ in time results in:

$$\dot{x}_n = \dot{z}_n - \dot{u}_{n-1}$$

$$= f_n(z_1, \ldots, z_n) + g_n(z_1, \ldots, z_n)u_f - \dot{u}_{n-1}$$

(5.102)

Choose:

$$u_f = \frac{1}{g_n(z_1, \ldots, z_n)}[-f_n(z_1, \ldots, z_n) + \dot{u}_{n-1} - k_nx_n - g_{n-1}(z_1, \ldots, z_{n-1})x_{n-1}], \quad k_n > 0$$

(5.103)

Consider the augmented Lyapunov function:

$$V_n(x_1, \ldots, x_n) = V_{n-1}(x_1, \ldots, x_{n-1}) + \frac{1}{2}x_n^2$$

$$= \frac{1}{2} \sum_{i=1}^{n} x_i^2$$

(5.104)
Differentiating it in time:

\[
\dot{V}_n (x_1, \ldots, x_n) = - \sum_{i=1}^{n} k_i x_i^2
\]  

(5.105)

which is negative definite and hence one can deduce that the origin \([x_1, \ldots, x_n]\) is globally asymptotically stable. Furthermore, as it has been assumed that \(x_i = z_i - u_{i-1}\) and \(u_{i-1} (0) = 0\), then \([z_1, \ldots, z_n]\), the origin of the system is globally asymptotically stable as well.
6 Adaptive Backstepping Control

6.1 Simple Adaptive Regulation Control

This section introduces an illustrative example to clarify the difference between static, which is non-adaptive and dynamic, which is adaptive control design methods. Consider the scalar nonlinear system given by:

\[ \dot{z} = \theta f(z) + g(z)u \]  

(6.1)

where the parameter \( \theta \) is an unknown constant. The goal is to regulate \( z(t) \rightarrow 0 \) or in other words to make \( z = 0 \) into a stable equilibrium point. Assume that there is a positive definite and radially unbounded Lyapunov function:

\[ V(z) = \frac{1}{2}z^2 \]  

(6.2)

If \( \theta \) were known, then the controller:

\[ u = \frac{1}{g(z)}[-\theta f(z) - kz] , \quad k > 0 \]  

(6.3)

would make the derivative of Lyapunov function (6.2) negative definite.

\[ \dot{V} = -kz^2 \]  

(6.4)

However, \( \theta \) is unknown, and hence, the controller (6.3) is irrational. This problem of uncertainty of \( \theta \) can be solved by replacing \( \theta \) by an estimate \( \hat{\theta} \) so that (6.3) becomes:

\[ u = \frac{1}{g(z)}[-\hat{\theta} f(z) - kz] , \quad k > 0 \]  

(6.5)

In turn, (6.1) will accordingly change as follows:

\[ \dot{z} = \theta f(z) - \hat{\theta} f(z) - kz \]

\[ = (\theta - \hat{\theta}) f(z) - kz \]  

(6.6)

where \( \hat{\theta} \) represents the error parameter for the difference between the actual unknown value \( \theta \) and its estimate \( \hat{\theta} \). Using the same Lyapunov function given in (6.2) would not help to design
a suitable controller because its derivative would contain an indefinite term that would not be easy to get rid of, as shown below:

\[
\dot{V} = zz' \\
= z\left(\tilde{\theta}f(z) - kz\right) \\
= \tilde{\theta}zf(z) - kz^2 \tag{6.7}
\]

A solution to this problem was proposed in [13] where (6.2) is augmented by a quadratic term of \(\tilde{\theta}\) as follows

\[
V(z, \tilde{\theta}) = \frac{1}{2}z^2 + \frac{1}{2\gamma}\tilde{\theta}^2 \tag{6.8}
\]

where \(\gamma\) is an adaptation gain. Thus, the controller dynamic will contain an update law for the estimate \(\hat{\theta}\) and then with a proper choice of \(\dot{\hat{\theta}}\), the indefinite term can be canceled. The derivative of (6.8) is obtained as:

\[
\dot{V} = zz' + \frac{1}{\gamma}\tilde{\theta}z' \\
= z\left(\tilde{\theta}f(z) - kz\right) + \frac{1}{\gamma}\tilde{\theta}z' \\
= -kz^2 + \tilde{\theta}\left[zf(z) + \frac{1}{\gamma}\tilde{\theta}\right] \tag{6.9}
\]

The term above the brace is still indefinite. However as \(\dot{\tilde{\theta}} = -\dot{\hat{\theta}}\), then (6.9) turns out to be:

\[
\dot{V} = -kz^2 + \tilde{\theta}\left[zf(z) - \frac{1}{\gamma}\tilde{\theta}\right] \tag{6.10}
\]

Choosing the update law:

\[
\dot{\hat{\theta}} = \gamma zf(z) \tag{6.11}
\]

yields:

\[
\dot{V} = -kz^2 \leq 0 \tag{6.12}
\]

Hence, \([z \quad \hat{\theta}]^T = [0 \quad 0]^T\) is stable, and the resulting adaptive system is:

**nonlinear system** \(\Rightarrow\) \[
\begin{align*}
\dot{z} &= \theta f(z) + g(z) u \\
u &= \frac{1}{g(z)} \left[-\dot{\tilde{\theta}}f(z) - kz\right]
\end{align*}
\]

**adaptive controller** \(\Rightarrow\) \[
\begin{align*}
\dot{\hat{\theta}} &= \gamma zf(z) \\
\hat{\theta} &= \gamma zf(z)
\end{align*}
\]
The system (6.13) can be re-written as:

\[
\begin{align*}
\dot{z} &= \hat{\theta} f(z) - kz \\
\dot{\hat{\theta}} &= -\gamma z f(z)
\end{align*}
\]  

(6.14)

In the previous example, the unknown parameter \( \theta \) is matched by \( u \), this means it is in the same equation with the control function \( u \), and hence it was so easy to cancel the term containing \( \theta \) by the control law. Later in this chapter, a more general case when the unknown parameter is unmatched by \( u \) will be discussed, that is when the control function is separated from it by a number of integrators.
6.2 Simple Adaptive Tracking Control

Reconsider the scalar nonlinear system given in (6.1). The goal now is to track a reference \( r(t) \). Therefore, one can start by defining the tracking error parameter:

\[
e = z - r
\]  
(6.15)

Taking the derivative in time of (6.15)

\[
\dot{e} = \dot{z} - \dot{r}
\]  
(6.16)

Substituting (6.1) in (6.16)

\[
\dot{e} = \theta f(z) + g(z) u - \dot{r}
\]  
(6.17)

Considering the estimate \( \hat{\theta} \) of the unknown constant parameter \( \theta \), the control law

\[
u = \frac{1}{g(z)} \left[ -\hat{\theta} f(z) + \dot{r} - ke \right]
\]  
(6.18)

will try to regulate \( e \to 0 \). To see if that will work, one can use the Lyapunov function

\[
V(e, \hat{\theta}) = \frac{1}{2} e^2 + \frac{1}{2\gamma} \hat{\theta}^2
\]  
(6.19)

whose derivative in time is obtained as

\[
\dot{V} = e \dot{e} + \frac{1}{\gamma} \hat{\theta} \dot{\hat{\theta}}
\]  
(6.20)

Substituting (6.17) and (6.18) into (6.20) yields

\[
\dot{V} = e \left[ \theta f(z) - \hat{\theta} f(z) + \dot{r} - ke - \dot{r} \right] - \frac{1}{\gamma} \hat{\theta} \dot{\hat{\theta}}
\]  
(6.21)

Choosing the adaptation law

\[
\dot{\hat{\theta}} = \gamma f(z) e
\]  
(6.22)

will change (6.21) to

\[
\dot{V} = -ke^2
\]  
(6.23)

Thus, according to the Lyapunov theorem of stability \([e \ \hat{\theta}]^T = [0 \ 0]^T\) is stable, and one can say that \(|\hat{\theta}|\) and \(|e|\) are bounded.
6.3 Adaptive Backstepping for Second Order Matched System

In sections 6.1 and 6.2, the controller design was straightforward. This simplicity of the design is because the unknown parameters appeared in the same equations with the control function. This section will explain how this is considered an advantage by considering the following second-order system.

\[
\begin{align*}
\dot{z}_1 &= f_1 (z_1) + g_1 (z_1) z_2 \\
\dot{z}_2 &= \theta f_2 (z) + g_2 (z) u_f
\end{align*}
\] (6.24a, 6.24b)

Assuming that \(\theta\) is known and following procedure of section 5.3 or 5.4, the non-adaptive controller can be designed by first considering subsystem (6.24a) and assuming that \(z_2\) is a virtual control input whose desired value will be denoted by \(u_1\). Let \(x_1 = z_1\) so that \(\dot{x}_1\) equals (6.24a). Choosing the Lyapunov function

\[
V_1 (x_1) = \frac{1}{2} x_1^2
\] (6.25)

whose derivative in time is obtained as

\[
\dot{V}_1 = x_1 \dot{x}_1 \\
= x_1 [f_1 (z_1) + g_1 (z_1) z_2]
\] (6.26)

and choosing

\[
z_2 = u_1 = \frac{1}{g_1 (z_1)} [-f_1 (z_1) - k_1 x_1]
\] (6.27)

would make (6.26) become

\[
\dot{V}_1 = x_1 \dot{x}_1 \\
= x_1 \left[ f_1 (z_1) + g_1 (z_1) \left( \frac{1}{g_1 (z_1)} [-f_1 (z_1) - k_1 x_1] \right) \right] \\
= x_1 [f_1 (z_1) - f_1 (z_1) - k_1 x_1] \\
= -k_1 x_1^2 \leq -W_1 \leq 0
\] (6.28)

The control function \(u_1\) is not the valid one but instead is the desired control function. Therefore, an error parameter that represents the difference between the virtual control function \(z_2\) and the desired control function \(u_1\) can be defined as

\[
x_2 = z_2 - u_1
\] (6.29)
Taking its derivative in time and using (6.24b)

\[
\dot{x}_2 = \dot{z}_2 - \dot{u}_1 = \theta f_2(z) + g_2(z) u_f - \dot{u}_1
\]  

(6.30)

From (6.27), (6.29) and (6.30), the system (6.24) in new coordinates will be

\[
\dot{x}_1 = -k_1 x_1 + g_1(z_1) x_2
\]

(6.31a)

\[
\dot{x}_2 = \theta f_2(z) + g_2(z) u_f - \dot{u}_1
\]

(6.31b)

Choosing the augmented Lyapunov function

\[
V_2(x_1, x_2) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2
\]

(6.32)

and differentiating it in time yields

\[
\dot{V}_2 = x_1 \dot{x}_1 + x_2 \dot{x}_2
\]

\[
= x_1 [-k_1 x_1 + g_1(z_1) x_2] + x_2 [\theta f_2(z) + g_2(z) u_f - \dot{u}_1]
\]

(6.33)

\[
= -k_1 x_1^2 + x_2 [g_1(z_1) x_1 + \theta f_2(z) + g_2(z) u_f - \dot{u}_1]
\]

Choosing \( u_f \) as

\[
u_f = \frac{1}{g_2(z)} [-k_2 x_2 - g_1(z_1) x_1 - \theta f_2(z) + \dot{u}_1]
\]

(6.34)

yields

\[
\dot{V}_2 = -k_1 x_1^2 - k_2 x_2^2
\]

\[
\leq -W_1 - k_2 x_2^2 \leq -W_2 \leq 0
\]

(6.35)

This implies that \([x_1 \ x_2]^T = [0 \ 0]^T\) is stable, and hence, \([z_1 \ z_2]^T = [0 \ 0]^T\) is also stable. However, this design was based on the assumption that \(\theta\) is known, but it is not. Therefore, it will again be replaced in (6.34) with an estimate \(\hat{\theta}\) such that the following adaptive controller is obtained

\[
u_f = \frac{1}{g_2(z)} [-k_2 x_2 - g_1(z_1) x_1 - \hat{\theta} f_2(z) + \dot{u}_1]
\]

(6.36)

Updating system (6.31) accordingly, one will get

\[
\dot{x}_1 = -k_1 x_1 + g_1(z_1) x_2
\]

(6.37a)

\[
\dot{x}_2 = \hat{\theta} f_2(z) - k_2 x_2 - g_1(z_1) x_1
\]

(6.37b)
where $\tilde{\theta}$ is the difference between the unknown parameter $\theta$ and its estimate $\hat{\theta}$. Augmenting (6.32) with a quadratic term of $\tilde{\theta}$ results in the following Lyapunov function

$$V_3(x_1, x_2, \tilde{\theta}) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 + \frac{1}{2\gamma} \tilde{\theta}^2$$

(6.38)

whose derivative in time is

$$\dot{V}_3 = x_1 \dot{x}_1 + x_2 \dot{x}_2 + \frac{1}{\gamma} \dot{\tilde{\theta}}$$

$$= x_1 [-k_1 x_1 + g_1(z_1) x_2] + x_2 [\tilde{\theta} f_2(z) - k_2 x_2 - g_1(z_1) x_1] + \frac{1}{\gamma} \dot{\tilde{\theta}}$$

(6.39)

To eliminate the indefinite term, one can choose the update law

$$\dot{\hat{\theta}} = \gamma f_2(z) x_2$$

(6.40)

Consequently, the derivative of Lyapunov function (6.39) will become

$$\dot{V}_3 = -k_1 x_1^2 - k_2 x_2^2$$

$$\leq 0$$

(6.41)

which implies that $[x_1 \ x_2 \ \tilde{\theta}]^T = [0 \ 0 \ 0]^T$ is stable and the resulting adaptive system is:

**nonlinear system** ⇒

$$\begin{cases}
\dot{x}_1 = -k_1 x_1 + g_1(z_1) x_2 \\
\dot{x}_2 = \theta f_2(z) + g_2(z) u_f - \dot{u}_1
\end{cases}$$

**adaptive controller** ⇒

$$\begin{cases}
u_f = \frac{1}{g_2(z)} \left[-k_2 x_2 - g_1(z_1) x_1 - \dot{\theta} f_2(z) + \dot{u}_1\right] \\
\dot{\hat{\theta}} = \gamma f_2(z) x_2
\end{cases}$$

(6.42)

which can be re-written as

$$\begin{cases}
\dot{x}_1 = -k_1 x_1 + g_1(z_1) x_2 \\
\dot{x}_2 = \tilde{\theta} f_2(z) - k_2 x_2 - g_1(z_1) x_1 \\
\dot{\tilde{\theta}} = -\gamma f_2(z) x_2
\end{cases}$$

(6.43)

Fig. 6.3 and Fig. 6.4 are the block diagrams for systems (6.42) and (6.43) respectively.
Figure 6.3: Block diagram for system (6.42)

Figure 6.4: Block diagram for system (6.43)
6.4 Adaptive Backstepping for Second Order Extended Matching System

Extended matching is a case where the unknown parameter is one integrator separated from the control input. This section considers the following second-order system with the unknown constant parameter \( \theta \) being one integrator before the control input \( u_f \).

\[
\dot{z}_1 = \theta f_1 (z_1) + g_1 (z_1) z_2 \quad (6.44a)
\]
\[
\dot{z}_2 = f_2 (z) + g_2 (z) u_f \quad (6.44b)
\]

Again, assuming that \( \theta \) is known, then using the concept of integrator backstepping and procedure of section 5.3 or 5.4 would stabilize the system as follows. Let \( z_2 \) be a virtual control input to the subsystem (6.44a) and denote its desired value as \( u_1 \). Let \( x_1 = z_1 \) so that \( \dot{x}_1 \) equals (6.44a). Choosing the following Lyapunov function for subsystem (6.44a)

\[
V_1 (x_1) = \frac{1}{2} x_1^2 \quad (6.45)
\]

Differentiating it in time as

\[
\dot{V}_1 = x_1 \dot{x}_1 = x_1 [\theta f_1 (z_1) + g_1 (z_1) z_2] \quad (6.46)
\]

and choosing the virtual control input

\[
z_2 = u_1 (x_1, \theta) = \frac{1}{g_1 (z_1)} [-\theta f_1 (z_1) - k_1 x_1] \quad (6.47)
\]

would make (6.46) become

\[
\dot{V}_1 = x_1 \dot{x}_1 = x_1 \left[ \theta f_1 (z_1) + g_1 (z_1) \left( \frac{1}{g_1 (z_1)} [-\theta f_1 (z_1) - k_1 x_1] \right) \right]
\]
\[
= x_1 [\theta f_1 (z_1) - \theta f_1 (z_1) - k_1 x_1]
\]
\[
= -k_1 x_1^2
\]
\[
\leq -W_1
\]
\[
\leq 0
\]

Now, as \( u_1 (x_1, \theta) \) is not a valid control function but the desired one, the following error parameter can be defined

\[
x_2 = z_2 - u_1 (x_1, \theta) \quad (6.49)
\]
The derivative of this error parameter is obtained as

\[ \dot{x}_2 = \dot{z}_2 - \dot{u}_1 = f_2(z) + g_2(z) u_f - \dot{u}_1 \]  

(6.50)

Considering (6.47), (6.49) and (6.50) will modify the system (6.44) to be

\[ \begin{align*}
\dot{x}_1 &= -k_1 x_1 + g_1(z_1) x_2 \\
\dot{x}_2 &= f_2(z) + g_2(z) u_f - \dot{u}_1
\end{align*} \]  

(6.51a, 6.51b)

Choosing the Lyapunov function

\[ V_2(x_1, x_2) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \]  

(6.52)

and then taking its derivative in time yields

\[ \begin{align*}
\dot{V}_2 &= x_1 \dot{x}_1 + x_2 \dot{x}_2 \\
&= x_1 [-k_1 x_1 + g_1(z_1) x_2] + x_2 [f_2(z) + g_2(z) u_f - \dot{u}_1] \\
&= -k_1 x_1^2 + x_2 [g_1(z_1) x_1 + f_2(z) + g_2(z) u_f - \dot{u}_1]
\end{align*} \]  

(6.53)

One can now choose the system control input as

\[ u_f = \frac{1}{g_2(z)} [-k_2 x_2 - g_1(z_1) x_1 - f_2(z) + \dot{u}_1] \]  

(6.54)

This control function \( u_f \) will make the derivative of Lyapunov function (6.53) negative definite.

\[ \begin{align*}
\dot{V}_2 &= -k_1 x_1^2 - k_2 x_2^2 \\
&= -W_1 - k_2 x_2^2 \\
&\leq -W_2 \\
&\leq 0
\end{align*} \]  

(6.55)

This is the objective of the Lyapunov theorem for the control function to stabilize the system. However, as \( \theta \) is unknown, the aforementioned method cannot be implemented. Fortunately, one can still use the notion of integrator backstepping to design the adaptive controller for extended matching systems problems. Back to the same system given in (6.44), as it is of order two, two steps are needed to design the control function.

\[ \begin{align*}
\dot{z}_1 &= \theta f_1(z_1) + g_1(z_1) z_2 \\
\dot{z}_2 &= f_2(z) + g_2(z) u_f
\end{align*} \]
Step: 1

Starting with the subsystem (6.44a), assume that $z_2$ is the virtual control input whose desired value is denoted by $u_1$. Let $x_1 = z_1$ and $x_2 = z_2 - u_1$. Also, assume that $\hat{\theta}_1$ is an estimate for the unknown parameter $\theta$. Based on these assumptions, the adaptive version of the controller (6.47) will be

$$z_2 = u_1 \left( x_1, \hat{\theta}_1 \right) = \frac{1}{g_1 (z_1)} \left[ -\hat{\theta}_1 f_1 (z_1) - k_1 x_1 \right]$$  \hspace{1cm} (6.56)

Taking the derivative of $x_1$ in time and substituting for $z_2 = u_1 + x_2$

$$\dot{x}_1 = \theta f_1 (z_1) + g_1 (z_1) u_1 + g_1 (z_1) x_2$$  \hspace{1cm} (6.57)

Substituting for $u_1$ from (6.56)

$$\dot{x}_1 = \theta f_1 (z_1) - \hat{\theta}_1 f_1 (z_1) - k_1 x_1 + g_1 (z_1) x_2$$

$$= \left( \theta - \hat{\theta}_1 \right) f_1 (z_1) - k_1 x_1 + g_1 (z_1) x_2$$  \hspace{1cm} (6.58)

Choosing the following Lyapunov function

$$V_1 \left( x_1, \hat{\theta}_1 \right) = \frac{1}{2} x_1^2 + \frac{1}{2\gamma} \hat{\theta}_1^2$$  \hspace{1cm} (6.59)

where $\gamma$ is the adaptation gain and differentiating it in time results in

$$\dot{V}_1 = x_1 \dot{x}_1 + \frac{1}{\gamma} \hat{\theta}_1 \dot{\theta}_1$$

$$= x_1 \left[ \hat{\theta}_1 f_1 (z_1) - k_1 x_1 + g_1 (z_1) x_2 \right] - \frac{1}{\gamma} \hat{\theta}_1 \dot{\theta}_1$$

$$= \hat{\theta}_1 f_1 (z_1) x_1 - k_1 x_1^2 + g_1 (z_1) x_1 x_2 - \frac{1}{\gamma} \hat{\theta}_1 \dot{\theta}_1$$  \hspace{1cm} (6.60)

Choosing the adaptation law

$$\dot{\theta}_1 = \gamma f_1 (z_1) x_1$$  \hspace{1cm} (6.61)

will make (6.60) become

$$\dot{V}_1 = -k_1 x_1^2 + g_1 (z_1) x_1 x_2$$  \hspace{1cm} (6.62)
Step: 2

Differentiating the error parameter $x_2$, for the difference between the virtual control function $z_2$ and the desired control function $u_1$

$$\dot{x}_2 = \dot{z}_2 - \dot{u}_1$$  \hspace{1cm} (6.63)

where

$$\dot{u}_1 (x_1, \dot{\theta}_1) = \frac{\partial u_1}{\partial x_1} \dot{x}_1 + \frac{\partial u_1}{\partial \dot{\theta}_1} \dot{\theta}_1$$  \hspace{1cm} (6.64)

Substituting (6.44b) and (6.64) into (6.63) yields

$$\dot{x}_2 = f_2 (z) + g_2 (z) u_f - \frac{\partial u_1}{\partial x_1} \dot{x}_1 - \frac{\partial u_1}{\partial \dot{\theta}_1} \dot{\theta}_1$$  \hspace{1cm} (6.65)

Substituting (6.44a) and (6.61) into (6.65) yields

$$\dot{x}_2 = f_2 (z) + g_2 (z) u_f - \frac{\partial u_1}{\partial x_1} \theta f_1 (z_1) + g_1 (z_1) z_2 - \frac{\partial u_1}{\partial \dot{\theta}_1} \gamma f_1 (z_1) x_1$$

$$= f_2 (z) + g_2 (z) u_f - \frac{\partial u_1}{\partial x_1} \theta f_1 (z_1) - \frac{\partial u_1}{\partial x_1} g_1 (z_1) z_2 - \frac{\partial u_1}{\partial \dot{\theta}_1} \gamma f_1 (z_1) x_1$$  \hspace{1cm} (6.66)

Now, one can choose the Lyapunov function and design $u_f$ that makes its derivative negative.

$$V_2 (x_1, x_2, \hat{\theta}_1) = V_1 + \frac{1}{2} x_2^2$$  \hspace{1cm} (6.67)

Differentiating it in time yields

$$\dot{V}_2 = \dot{V}_1 + x_2 \dot{x}_2$$

$$= -k_1 x_1^2 + g_1 (z_1) x_1 x_2 + x_2 \left[ f_2 (z) + g_2 (z) u_f - \frac{\partial u_1}{\partial x_1} \theta f_1 (z_1) - \frac{\partial u_1}{\partial x_1} g_1 (z_1) z_2 - \frac{\partial u_1}{\partial \dot{\theta}_1} \gamma f_1 (z_1) x_1 \right]$$

$$= -k_1 x_1^2 + x_2 \left[ g_1 (z_1) x_1 + f_2 (z) + g_2 (z) u_f - \frac{\partial u_1}{\partial x_1} \theta f_1 (z_1) - \frac{\partial u_1}{\partial x_1} g_1 (z_1) z_2 - \frac{\partial u_1}{\partial \dot{\theta}_1} \gamma f_1 (z_1) x_1 \right]$$  \hspace{1cm} (6.68)

Choosing $u_f$ that may cancel indefinite terms in $\dot{V}_2$, and for the unknown parameter $\theta$ we will see if using the first estimate $\hat{\theta}_1$ instead of $\theta$ can help.

$$u_f = \frac{1}{g_2 (z)} \left( -g_1 (z_1) x_1 - f_2 (z) + \frac{\partial u_1}{\partial x_1} \hat{\theta}_1 f_1 (z_1) + \frac{\partial u_1}{\partial x_1} g_1 (z_1) z_2 + \frac{\partial u_1}{\partial \dot{\theta}_1} \gamma f_1 (z_1) x_1 - k_2 x_2 \right)$$  \hspace{1cm} (6.69)

This $u_f$ will renders (6.68) become

$$\dot{V}_2 = -k_1 x_1^2 - k_2 x_2^2 + \frac{\partial u_1}{\partial x_1} \hat{\theta}_1 f_1 (z_1) x_2 - \frac{\partial u_1}{\partial x_1} \theta f_1 (z_1) x_2$$

$$= -k_1 x_1^2 - k_2 x_2^2 - (\theta - \hat{\theta}_1) \frac{\partial u_1}{\partial x_1} f_1 (z_1) x_2$$

$$= -k_1 x_1^2 - k_2 x_2^2 - \hat{\theta}_1 \frac{\partial u_1}{\partial x_1} f_1 (z_1) x_2$$  \hspace{1cm} (6.70)
It is very clear that using the first estimate $\hat{\theta}_1$ does not help in canceling the indefinite term $\tilde{\theta}_1$ in (6.70). As a solution to this problem, one can replace $\hat{\theta}_1$ in (6.69) with a new estimate $\hat{\theta}_2$ so that

$$
u_f = \frac{1}{g_2(z)} \left[ -g_1(z_1)x_1 - f_2(z) + \frac{\partial u_1}{\partial x_1} \hat{\theta}_2 f_1(z_1) + \frac{\partial u_1}{\partial x_1} g_1(z_1) z_2 + \frac{\partial u_1}{\partial \theta_1} \gamma f_1(z_1) x_1 - k_2 x_2 \right]$$

(6.71)

Based on the preceding, (6.66) will change to

$$\dot{x}_2 = -g_1(z_1)x_1 - k_2 x_2 - (\theta - \hat{\theta}_2) \frac{\partial u_1}{\partial x_1} f_1(z_1)$$

$$\dot{x}_2 = -g_1(z_1)x_1 - k_2 x_2 - \hat{\theta}_2 \frac{\partial u_1}{\partial x_1} f_1(z_1)$$

(6.72)

where $\hat{\theta}_2$ represents the difference between the unknown parameter $\theta$ and the second estimate $\hat{\theta}_2$. This change in $\dot{x}_2$ with the existence of the error parameter $\tilde{\theta}_2$ results in need to use the following Lyapunov function

$$V_2(x_1, x_2, \tilde{\theta}_1, \tilde{\theta}_2) = V_1 + \frac{1}{2} x_2^2 + \frac{1}{2\gamma} \tilde{\theta}_2^2$$

(6.73)

whose derivative is obtained as

$$\dot{V}_2 = \dot{V}_1 + x_2 \dot{x}_2 + \frac{1}{\gamma} \dot{\tilde{\theta}}_2$$

$$\dot{V}_2 = -k_1 x_1^2 + g_1(z_1)x_1 x_2 + x_2 \left( -g_1(z_1)x_1 - k_2 x_2 - \hat{\theta}_2 \frac{\partial u_1}{\partial x_1} f_1(z_1) \right) - \frac{1}{\gamma} \dot{\tilde{\theta}}_2$$

$$\dot{V}_2 = -k_1 x_1^2 + g_1(z_1)x_1 x_2 - g_1(z_1)x_1 x_2 - k_2 x_2^2 - \hat{\theta}_2 \frac{\partial u_1}{\partial x_1} f_1(z_1) x_2 - \frac{1}{\gamma} \dot{\tilde{\theta}}_2$$

$$\dot{V}_2 = -k_1 x_1^2 - k_2 x_2^2 - \tilde{\theta}_2 \left( \frac{\partial u_1}{\partial x_1} f_1(z_1) x_2 + \frac{1}{\gamma} \tilde{\theta}_2 \right)$$

(6.74)

Choosing the adaptation law

$$\dot{\tilde{\theta}}_2 = -\gamma \frac{\partial u_1}{\partial x_1} f_1(z_1) x_2$$

(6.75)

will make the derivative of Lyapunov function (6.74)

$$\dot{V}_2 = -k_1 x_1^2 - k_2 x_2^2$$

(6.76)

This result fulfills Lyapunov theorem of stability, and from (6.58) and (6.61) along with (6.72) and (6.75) the resulting closed-loop system will be
\[
\begin{align*}
\dot{x}_1 &= \tilde{\theta}_1 f_1 (z_1) - k_1 x_1 + g_1 (z_1) x_2 \\
\dot{x}_2 &= -g_1 (z_1) x_1 - k_2 x_2 - \tilde{\theta}_2 \frac{\partial u_1}{\partial x_1} f_1 (z_1) \\
\tilde{\theta}_1 &= -\gamma f_1 (z_1) x_1 \\
\tilde{\theta}_2 &= \gamma \frac{\partial u_1}{\partial x_1} f_1 (z_1) x_2
\end{align*}
\] (6.77)

6.5 Overestimation Reduction

Although the method discussed in section 6.4 is useful to some extent, it still has a disadvantage of increasing the number of parameter estimates due to overestimation which in turn increases the order of resulting adaptive controller. This section illustrates how overestimation can be reduced by slightly modifying the previous procedure [60]. Reconsidering system (6.44), two steps are needed for controller design as follows.

\[
\begin{align*}
\dot{z}_1 &= \theta f_1 (z_1) + g_1 (z_1) z_2 \\
\dot{z}_2 &= f_2 (z) + g_2 (z) u_f
\end{align*}
\]

Step:1

Starting again with the subsystem (6.44a) where \( z_2 \) is the virtual control input whose desired value is \( u_1 \). Similarly, let \( x_1 = z_1 \) and \( x_2 = z_2 - u_1 \). Assume that \( \hat{\theta} \) instead of \( \hat{\theta}_1 \) is the estimate for the unknown parameter \( \theta \) to indicate that only one estimate of the unknown parameter is used in this design procedure. Based on these assumptions

\[
\begin{align*}
z_2 &= u_1 (x_1, \hat{\theta}) = \frac{1}{g_1 (z_1)} \left[ -\tilde{\theta}_1 f_1 (z_1) - k_1 x_1 \right]
\end{align*}
\] (6.78)

Taking the derivative of \( x_1 \) in time and substituting for \( z_2 = u_1 + x_2 \)

\[
\begin{align*}
\dot{x}_1 &= \theta f_1 (z_1) + g_1 (z_1) u_1 + g_1 (z_1) x_2
\end{align*}
\] (6.79)

Substituting for \( u_1 \) from (6.78)

\[
\begin{align*}
\dot{x}_1 &= \theta f_1 (z_1) - \tilde{\theta}_1 f_1 (z_1) - k_1 x_1 + g_1 (z_1) x_2 \\
&= \left( \theta - \tilde{\theta}_1 \right) f_1 (z_1) - k_1 x_1 + g_1 (z_1) x_2 \\
&= \tilde{\theta}_1 f_1 (z_1) - k_1 x_1 + g_1 (z_1) x_2
\end{align*}
\] (6.80)
Choosing the following Lyapunov function

\[ V_1(x_1, \tilde{\theta}) = \frac{1}{2} x_1^2 + \frac{1}{2\gamma} \tilde{\theta}^2 \]  

(6.81)

where \( \gamma \) is the adaptation gain and differentiating it in time yields

\[ \dot{V}_1 = x_1 \dot{x}_1 + \frac{1}{\gamma} \dot{\tilde{\theta}} \dot{\tilde{\theta}} \]

\[ = x_1 \left[ \tilde{\theta} f_1(z_1) - k_1 x_1 + g_1(z_1) x_2 \right] - \frac{1}{\gamma} \tilde{\theta} \dot{\tilde{\theta}} \]  

(6.82)

\[ = \tilde{\theta} f_1(z_1) x_1 - k_1 x_1^2 + g_1(z_1) x_1 x_2 - \frac{1}{\gamma} \tilde{\theta} \dot{\tilde{\theta}} \]

Unlike step 1 in section 6.4, choosing the adaptation law will be postponed to the next step.

**Step: 2**

Differentiating the error parameter \( x_2 \).

\[ \dot{x}_2 = \dot{z}_2 - \dot{u}_1 \]  

(6.83)

where

\[ \dot{u}_1(x_1, \tilde{\theta}) = \frac{\partial u_1}{\partial x_1} \dot{x}_1 + \frac{\partial u_1}{\partial \tilde{\theta}} \dot{\tilde{\theta}} \]  

(6.84)

Substituting (6.44b) and (6.84) into (6.83) yields

\[ \dot{x}_2 = f_2(z) + g_2(z) u_f - \frac{\partial u_1}{\partial x_1} \dot{x}_1 - \frac{\partial u_1}{\partial \tilde{\theta}} \dot{\tilde{\theta}} \]  

(6.85)

Substituting (6.44a) into (6.85) yields

\[ \dot{x}_2 = f_2(z) + g_2(z) u_f - \frac{\partial u_1}{\partial x_1} \theta f_1(z_1) - \frac{\partial u_1}{\partial x_1} g_1(z_1) z_2 - \frac{\partial u_1}{\partial \tilde{\theta}} \dot{\tilde{\theta}} \]

(6.86)

As the error parameter \( \tilde{\theta} = \theta - \dot{\theta} \), then one can say that the unknown parameter \( \theta = \tilde{\theta} + \dot{\theta} \). Thus,

\[ \dot{x}_2 = f_2(z) + g_2(z) u_f - \left( \tilde{\theta} + \dot{\theta} \right) \frac{\partial u_1}{\partial x_1} f_1(z_1) - \frac{\partial u_1}{\partial x_1} g_1(z_1) z_2 - \frac{\partial u_1}{\partial \tilde{\theta}} \dot{\tilde{\theta}} \]

(6.87)
Now, choosing the Lyapunov function

\[ V_2(x_1, x_2, \theta) = V_1 + \frac{1}{2} x_2^2 = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 + \frac{1}{2\gamma} \hat{\theta}^2 \]  

(6.88)

It is clear that this Lyapunov function still has the parameter estimate \( \hat{\theta} \) that design procedure started with, unlike overestimation design procedure where Lyapunov function (6.73) had \( \hat{\theta} \) instead. Differentiating (6.88) in time yields

\[
\dot{V}_2 = \dot{V}_1 + x_2 \dot{x}_2 \\
= \tilde{\theta} f_1(z_1) x_1 - k_1 x_1^2 + g_1(z_1) x_1 x_2 - \frac{1}{\gamma} \dot{\hat{\theta}} + \\
x_2 \left( f_2(z) + g_2(z) u_f - \tilde{\theta} \frac{\partial u_1}{\partial x_1} f_1(z_1) - \hat{\theta} \frac{\partial u_1}{\partial x_1} f_1(z_1) - \frac{\partial u_1}{\partial x_1} g_1(z_1) z_2 - \frac{\partial u_1}{\partial \theta} \dot{\hat{\theta}} \right)
\]  

(6.89)

Now, one can group all terms having \( \tilde{\theta} \) so that

\[
\dot{V}_2 = -k_1 x_1^2 + \tilde{\theta} \left[ f_1(z_1) x_1 - \frac{\partial u_1}{\partial x_1} f_1(z_1) x_2 - \frac{1}{\gamma} \dot{\hat{\theta}} \right] + \\
x_2 \left[ g_1(z_1) x_1 + f_2(z) + g_2(z) u_f - \tilde{\theta} \frac{\partial u_1}{\partial x_1} f_1(z_1) - \hat{\theta} \frac{\partial u_1}{\partial x_1} f_1(z_1) - \frac{\partial u_1}{\partial x_1} g_1(z_1) z_2 - \frac{\partial u_1}{\partial \theta} \dot{\hat{\theta}} \right]
\]  

(6.90)

and choose the update law \( \dot{\hat{\theta}} \) as

\[ \dot{\hat{\theta}} = \gamma \left[ f_1(z_1) x_1 - \frac{\partial u_1}{\partial x_1} f_1(z_1) x_2 \right] \]  

(6.91)

and the stabilizing function \( u_f \) as

\[ u_f = \frac{1}{g_2(z)} \left[ -k_2 x_2 - g_1(z_1) x_1 - f_2(z) + \tilde{\theta} \frac{\partial u_1}{\partial x_1} f_1(z_1) + \hat{\theta} \frac{\partial u_1}{\partial x_1} f_1(z_1) + \frac{\partial u_1}{\partial x_1} g_1(z_1) z_2 + \frac{\partial u_1}{\partial \theta} \dot{\hat{\theta}} \right] \]  

(6.92)

Substituting for \( \dot{\hat{\theta}} \) from (6.91) and for \( u_f \) from (6.92) in (6.90) results in

\[
\dot{V}_2 = -k_1 x_1^2 + k_2 x_2^2 \\
\leq 0
\]  

(6.93)

This result also fulfills the Lyapunov theorem of stability. However, overestimation was avoided here. From (6.80) and (6.87) along with (6.91) and (6.92) the resulting closed-loop
system will be

\[
\begin{align*}
\dot{x}_1 &= \tilde{\theta} f_1(z_1) - k_1 x_1 + g_1(z_1) x_2 \\
\dot{x}_2 &= f_2(z) + g_2(z) u_f - \tilde{\theta} \frac{\partial u_1}{\partial x_1} f_1(z_1) - \dot{\tilde{\theta}} \frac{\partial u_1}{\partial x_1} f_1(z_1) - \frac{\partial u_1}{\partial \theta} g_1(z_1) z_2 - \frac{\partial u_1}{\partial \tilde{\theta}} \dot{\tilde{\theta}} \\
\dot{u}_f &= \frac{1}{g_2(z)} \left[ -k_2 x_2 - g_1(z_1) x_1 - f_2(z) + \tilde{\theta} \frac{\partial u_1}{\partial x_1} f_1(z_1) + \frac{\partial u_1}{\partial x_1} g_1(z_1) z_2 + \frac{\partial u_1}{\partial \theta} \dot{\tilde{\theta}} \right] \\
\dot{\tilde{\theta}} &= \gamma \left[ f_1(z_1) x_1 - \frac{\partial u_1}{\partial x_1} f_1(z_1) x_2 \right]
\end{align*}
\] (6.94)

which can be re-written as

\[
\begin{align*}
\dot{x}_1 &= \hat{\theta} f_1(z_1) - k_1 x_1 + g_1(z_1) x_2 \\
\dot{x}_2 &= -k_2 x_2 + g_1(z_1) x_1 - \hat{\theta} \frac{\partial u_1}{\partial x_1} f_1(z_1) \\
\dot{\hat{\theta}} &= -\gamma \left[ f_1(z_1) x_1 - \frac{\partial u_1}{\partial x_1} f_1(z_1) x_2 \right]
\end{align*}
\] (6.95)
7 Transformation of Affine Nonlinear Systems into Strict Feedback Form

7.1 Transformation of SISO Nonlinear Systems into Strict Feedback Form

The transformation of the single-input single-output affine nonlinear system into an equivalent strict feedback form was studied previously in [24]. This part of the dissertation will explain a step-by-step procedure of the transformation process. In nonlinear control theory, SISO affine nonlinear system is said to be in a strict feedback form if it is of the following form [52]

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1) + g_1(x_1) x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2) x_3 \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3) x_4 \\
&\vdots \\
\dot{x}_{n-2} &= f_{n-2}(x_1, x_2, \ldots, x_{n-2}) + g_{n-2}(x_1, x_2, \ldots, x_{n-2}) x_{n-1} \\
\dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \ldots, x_{n-1}) + g_{n-1}(x_1, x_2, \ldots, x_{n-1}) x_n \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_{n-1}, x_n) + g_n(x_1, x_2, \ldots, x_{n-1}, x_n) u
\end{align*}
\]

(7.1)

where \( x \in \mathbb{R}^n \) is states of the system and \( u \in \mathbb{R} \) is a control input. Considering the SISO affine nonlinear system given in (2.15) as

\[
\begin{align*}
\dot{z} &= f(z) + g(z) u \\
y &= h(z)
\end{align*}
\]

with \( z \in \mathbb{R}^n, u \in \mathbb{R} \) and \( y \in \mathbb{R} \) are state variables, control input, and system output respectively. It is clear from section 2.10 and definition 2.5 that if this system has a relative degree \( \rho \) that is less than \( n \) the order of the system \( (\rho < n) \), then

\[
L_g L_f^{i-1} h(z) = 0, \quad i = 1, \ldots, \rho - 1
\]

\[
L_g L_f^{\rho-1} h(z) \neq 0, \quad \forall z \subset D \subset \mathbb{R}^n
\]

However, according to proof 3.1 when the system’s relative degree \( \rho \) equals to \( n \) the order of the system \( (\rho = n) \), then

\[
L_g L_f^{i-1} h(z) = 0, \quad i = 1, \ldots, n - 1
\]

\[
L_g L_f^{n-1} h(z) \neq 0, \quad \forall z \subset D \subset \mathbb{R}^n
\]
and if \( f(z), g(z) \) and \( h(z) \) are smooth enough in the domain \( D \subset \mathbb{R}^n \), then a diffeomorphism

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_{n-1} \\
x_n
\end{bmatrix} = T(z) = \begin{bmatrix}
T_1(z) \\
T_2(z) \\
T_3(z) \\
\vdots \\
T_{n-1}(z) \\
T_n(z)
\end{bmatrix} = \begin{bmatrix}
h(z) \\
-h(z) + L_fh(z) \\
L_fh(z) + L_f^2h(z) \\
\vdots \\
(-1)^{n-2}L_f^{n-3}h(z) + L_f^{n-2}h(z) \\
(-1)^{n-1}L_f^{n-2}h(z) + L_f^{n-1}h(z)
\end{bmatrix}, \quad x \in \mathbb{R}^n
\] (7.2)

where:

\[
L_f^{-1}h(z) = 0 \quad (7.3)
\]
\[
L_f^0h(z) = h(z) \quad (7.4)
\]

and the state feedback:

\[
u = \frac{1}{L_gL_f^{n-1}h(z)}[-L_f^n h(z) + v] \quad (7.5)
\]

can transform the SISO affine nonlinear system of the form (2.15) into its equivalent strict feedback form (7.1) such that:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_{n-1} \\
\dot{x}_n
\end{bmatrix} = \begin{bmatrix}
\otimes & 1 & 0 & 0 & \cdots & 0 & 0 \\
\otimes & \otimes & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\otimes & \otimes & \otimes & \otimes & \cdots & \otimes & 1 \\
\otimes & \otimes & \otimes & \otimes & \cdots & \otimes & \otimes
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix} + \begin{bmatrix}
0
\end{bmatrix}
\] (7.6)

\[
y = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n
\end{bmatrix}
\]

where \( \otimes \) stands for values other than zero. For more clarification in this regard and without loss of generality the transformation process will be proved for a system of fourth-order for
which a local transformation matrix can easily be obtained from (7.2) as

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
\end{bmatrix}
= \begin{bmatrix}
  T_1(z) \\
  T_2(z) \\
  T_3(z) \\
  T_4(z) \\
\end{bmatrix}
= \begin{bmatrix}
  h(z) \\
  -h(z) + L_fh(z) \\
  L_fh(z) + L_f^2h(z) \\
  -L_f^2h(z) + L_f^3h(z) \\
\end{bmatrix}
\]

(7.7)

with a state feedback control law:

\[
u = \frac{1}{L_gL_f^3h(z)}[-L_f^4h(z) + v]
\]

(7.8)

easily obtained from (7.5).

**Proof 7.1** *(Transforming a 4th order SISO nonlinear system into strict feedback form)*

Considering first row element in (7.7): \( x_1 = T_1(z) = h(z) \)

Taking its derivative in time yields

\[
\dot{x}_1 = \frac{dT_1(z)}{dt} = \frac{dh(z)}{dt} = \frac{\partial h(z)}{\partial z} \cdot \frac{dz}{dt}
= \frac{\partial h(z)}{\partial z} [f(z) + g(z)u]
= L_fh(z) + L_gh(z)u
\]

(7.9)

If system’s relative degree \( \rho = n = 4 \). Then, it is clear form (3.26) that \( L_gL_f^k h(z) = 0 \) \( \forall k < 3 \) and \( L_gL_f^3 h(z) \neq 0 \). Furthermore, considering first and second elements in the transformation matrix (7.7), (7.9) turns out to be

\[
\dot{x}_1 = L_fh(z)
= h(z) + x_2
= x_1 + x_2
\]

(7.10)

Considering second row element in (7.7): \( x_2 = T_2(z) = -h(z) + L_fh(z) \)

Taking its derivative in time yields

\[
\dot{x}_2 = \frac{dT_2(z)}{dt} = -(x_1 + x_2) + \frac{d}{dt}L_fh(z)
= -(x_1 + x_2) + \frac{\partial}{\partial z}L_fh(z) \cdot \frac{dz}{dt}
= -(x_1 + x_2) + \frac{\partial}{\partial z}L_fh(z) [f(z) + g(z)u]
= -(x_1 + x_2) + L_f^2h(z) + L_gL_fh(z)u
\]

(7.11)
It is well known that $L_g L_f h(z) = 0$. Moreover, considering first, second and third elements in the transformation matrix (7.7), (7.11) turns out to be
\[
\dot{x}_2 = -(x_1 + x_2) + L_f^2 h(z) \\
= -(x_1 + x_2) + x_3 - L_f h(z) \\
= -(x_1 + x_2) + x_3 - (x_1 + x_2) \\
= -2(x_1 + x_2) + x_3
\]
\[(7.12)\]

Considering third row element in (7.7): $x_3 = T_3(z) = L_f h(z) + L_f^2 h(z)$

Taking its derivative in time yields
\[
\dot{x}_3 = \frac{dT_3(z)}{dt} = x_3 - (x_1 + x_2) + \frac{d}{dt} L_f^2 h(z) \\
= x_3 - (x_1 + x_2) + \frac{\partial}{\partial z} L_f^2 h(z) \cdot \frac{dz}{dt} \\
= x_3 - (x_1 + x_2) + \frac{\partial}{\partial z} L_f^2 h(z) [f(z) + g(z) u] \\
= x_3 - (x_1 + x_2) + L_f^3 h(z) + L_g L_f^2 h(z) u
\]
\[(7.13)\]

It is also known that $L_g L_f^2 h(z) = 0$. Moreover, considering first, second, third and fourth elements in the transformation matrix (7.7), (7.13) turns out to be
\[
\dot{x}_3 = x_3 - (x_1 + x_2) + L_f^3 h(z) \\
= x_3 - (x_1 + x_2) + x_4 + L_f^3 h(z) \\
= x_3 - (x_1 + x_2) + x_4 + (x_3 - L_f h(z)) \\
= x_3 - (x_1 + x_2) + x_4 + x_3 - (h(z) + x_2) \\
= x_3 - (x_1 + x_2) + x_4 + x_3 - (x_1 + x_2) \\
= 2(x_3 - (x_1 + x_2)) + x_4
\]
\[(7.14)\]

Considering fourth row element in (7.7): $x_4 = T_4(z) = -L_f^2 h(z) + L_f^3 h(z)$

Taking its derivative in time yields
\[
\dot{x}_4 = \frac{dT_4(z)}{dt} = -x_4 - x_3 + x_2 + x_1 + \frac{d}{dt} L_f^3 h(z) \\
= -x_4 - x_3 + x_2 + x_1 + \frac{\partial}{\partial z} L_f^3 h(z) \cdot \frac{dz}{dt} \\
= -x_4 - x_3 + x_2 + x_1 + \frac{\partial}{\partial z} L_f^3 h(z) [f(z) + g(z) u] \\
= -x_4 - x_3 + x_2 + x_1 + L_f^4 h(z) + L_g L_f^3 h(z) u
\]
\[(7.15)\]
It is already known that $L_g L_f^3 h(z) \neq 0$. Thus, substituting the state feedback (7.8) in (7.15) yields

$$\dot{x}_4 = -x_4 - x_3 + x_2 + x_1 + v$$

(7.16)

From (7.10), (7.12), (7.14) and (7.16) the SISO affine nonlinear system of fourth-order in its strict feedback form will be

$$\dot{x}_1 = x_1 + x_2$$
$$\dot{x}_2 = -2(x_1 + x_2) + x_3$$
$$\dot{x}_3 = 2(x_3 - (x_1 + x_2)) + x_4$$
$$\dot{x}_4 = -x_4 - x_3 + x_2 + x_1 + v$$

(7.17)

### 7.2 Transformation into Strict Feedback Form and Backstepping Control of a Surface Permanent-Magnet Wind Generator with Boost Converter

![Diagram of wind energy conversion system](image)

**Figure 7.1**: Direct-drive SPMSG based wind energy conversion system with boost converter.

This example considers the mathematical model of a surface permanent-magnet synchronous wind generator with a boost converter shown in Fig. 7.1 and given in [12] by:

$$\begin{cases}
\dot{x}_1 = k_1 x_1 + k_2 x_2 \sin x_3 + k_3 + k_7 u \\
\dot{x}_2 = k_4 x_1 \sin x_3 + k_5 x_2 + k_6 \\
\dot{x}_3 = x_2 \\
y = x_3
\end{cases}$$

(7.18)
such that

\[
\begin{bmatrix}
  k_1 x_1 + k_2 x_2 \sin x_3 + k_3 \\
  k_4 x_1 \sin x_3 + k_5 x_2 + k_6 \\
  x_2
\end{bmatrix}, \quad
\begin{bmatrix}
  k_7 \\
  0 \\
  0
\end{bmatrix}, \quad
h(x) = x_3
\]

The variables \(x_1, x_2, x_3\) and \(u\) are the current of the dc-link inductor, electrical speed of generator rotor, electrical angle of generator rotor, the duty ratio of the switching signal respectively. The reader is referred to [12] for the values of \(k_1 - k_7\). The two conditions for feedback linearizability discussed in section 2.11, and the relative degree of this system model have already been tested and fulfilled in [12] where the vector fields \([g(x), adfg(x), ad^2fg(x)]\) are linearly independent and the set \(\{g(x), adfg(x)\}\) is involutive and the relative degree of the system is 3. Consequently, this system is feedback linearizable, and according to (7.2), the diffeomorphism:

\[
\begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3
\end{bmatrix} = \begin{bmatrix}
  h(x) \\
  -h(x) + L_fh(x) \\
  L_fh(x) + L_f^2h(x)
\end{bmatrix}
\]

\[(7.19)\]

can transform the system into its strict feedback form. Considering that the system’s relative degree equals to its dimension (\(\rho = n = 3\)), then \(L_gL_f^k h(z) = 0 \ \forall \ k < 2\) and hence taking the derivative in time of \(z_1, z_2\) and \(z_3\) yields:

\[
\dot{z}_1 = \frac{\partial h}{\partial x} [f(x) + g(x)u] = L_fh(x) + L_g h(x)u = L_fh(x) = h(x) + z_2 = z_1 + z_2
\]

\[
\dot{z}_2 = -L_fh(x) + \frac{\partial L_fh(x)}{\partial x} [f(x) + g(x)u] = -L_fh(x) + L_f^2h(x) + L_g L_fh(x)u
\]

\[(7.20)\]
\[ \dot{L}f(x) + L_j^2h(x) = -(z_1 + z_2) + z_3 - z_2 - z_1 = -2z_1 - 2z_2 + z_3 \]  
\quad (7.21)

\[ \dot{z}_3 = L_j^2h(x) + \frac{\partial L_j^2h(x)}{\partial x} [f(x) + g(x)u] 
= L_j^2h(x) + L_j^3h(x) + Lg L_j^2h(x)u \]  
\quad (7.22)

Choosing:
\[ u = \frac{1}{L_g L_j^2h(x)} [-L_j^3h(x) + v] \]  
\quad (7.23)
yields
\[ \dot{z}_3 = -z_1 - z_2 + z_3 + v \]  
\quad (7.24)

Thus, the system in strict feedback form is:
\[ \begin{cases} 
\dot{z}_1 = z_1 + z_2 \\
\dot{z}_2 = -2z_1 - 2z_2 + z_3 \\
\dot{z}_3 = -z_1 - z_2 + z_3 + v 
\end{cases} \]  
\quad (7.25)

To design a stabilizing control function, the procedure in section 5.4 is applied as follows:

**Step:1**

Let \( x_1 = z_1 \). Define the error variable
\[ x_2 = z_2 - u_1(z_1) \]  
\quad (7.26)

Choose:
\[ u_1(z_1) = -z_1 - k_1 x_1, \quad k_1 > 0 \]  
\quad (7.27)

Taking the derivative in time of \( x_1 \) yields:
\[ \dot{x}_1 = -k_1 x_1 + x_2 \]  
\quad (7.28)

Consider the Lyapunov function:
\[ V_1(x_1) = \frac{1}{2} x_1^2 \]  
\quad (7.29)
Differentiating it in time results in:

\[
\dot{V}_1(x_1) = x_1 \dot{x}_1 \\
= x_1(-k_1x_1 + x_2) \\
= -k_1x_1^2 + x_1x_2
\]  

(7.30)

**Step:2**

From step 1, \(x_2 = z_2 - u_1(z_1)\). Define the error variable

\[x_3 = z_3 - u_2(z_1, z_2)\]  

(7.31)

Choose:

\[u_2(z_1, z_2) = 2z_1 + 2z_2 + \dot{u}_1(z_1) - k_2x_2 - x_1, \ k_2 > 0\]  

(7.32)

Taking the derivative in time of the error variable \(x_2\) yields:

\[
\dot{x}_2 = \dot{z}_2 - \dot{u}_1(z_1) \\
= -2z_1 - 2z_2 + x_3 + u_2(z_1, z_2) - \dot{u}_1(z_1)
\]  

(7.33)

Substituting for \(u_2(z_1, z_2)\) results in:

\[\dot{x}_2 = -k_2x_2 + x_3 - x_1\]  

(7.34)

Consider the augmented Lyapunov function:

\[V_2(x_1, x_2) = V_1(x_1) + \frac{1}{2}x_2^2\]  

(7.35)

Differentiating it in time:

\[
\dot{V}_2(x_1, x_2) = \dot{V}_1(x_1) + x_2 \dot{x}_2 \\
= -k_1x_1^2 + x_1x_2 + x_2(-k_2x_2 + x_3 - x_1) \\
= -k_1x_1^2 + x_1x_2 - k_2x_2^2 + k_2x_3 - x_1x_2 \\
= -k_1x_1^2 - k_2x_2^2 + x_2x_3
\]  

(7.36)

**Step:3**

From step 2, \(x_3 = z_3 - u_2(z_1, z_2)\). Differentiating \(x_3\) in time yields:

\[\dot{x}_3 = \dot{z}_3 - \dot{u}_2(z_1, z_2)\]
\[ = -z_1 - z_2 + z_3 + v - \dot{u}_2(z_1, z_2) \]  

(7.37)

Choose:

\[ v = z_1 + z_2 - z_3 + \dot{u}_2(z_1, z_2) - k_3 x_3 - x_2, \quad k_3 > 0 \]  

(7.38)

Substituting for \( v \):

\[ \dot{x}_3 = -k_3 x_3 + x_2 \]  

(7.39)

Consider the augmented Lyapunov function candidate:

\[ V_3(x_1, x_2, x_3) = V_2(x_1, x_2) + \frac{1}{2} x_3^2 \]  

(7.40)

Differentiating it in time and substituting for \( v \) results in:

\[
\dot{V}_3(x_1, x_2, x_3) = \dot{V}_2(x_1, x_2) + x_3 \dot{x}_3 \\
= -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2
\]

(7.41)

which is negative definite and thus the origin \([x_1, x_2, x_3]\) is globally asymptotically stable

and based on the assumption that \(x_i = z_i - u_{i-1}\) and \(u_{i-1}(0) = 0\), \([z_1, z_2, z_3]\) the origin of

the system is globally asymptotically stable as well. \(k_1 = 3.70, k_2 = 5.12\) and \(k_3 = 2.35\) were

calculated using optimal control method and quadratic performance index [50]. Simulation

results in Fig.7.2 show the stability of the system using the proposed stabilizing controller. △

![Time Response](attachment:image.png)

(a) \(z_1(0) = -2, z_2(0) = 4, z_3(0) = 6\)
Figure 7.2: States stabilization using backstepping control after transformation into SFBF.

Despite the fact that nonlinear systems can indirectly be transformed into a strict feedback form using some assumptions that may simplify the transformation process, the preceding approach is considered the direct approach to get the system in its equivalent strict feedback form through feedback and transformation.
7.3 Transformation of MIMO Nonlinear Systems into Strict Feedback Form

The main contribution of this research is to extend the concept and procedure of transforming SISO affine nonlinear system into strict feedback form to transform MIMO affine nonlinear system into its equivalent strict feedback form given by the general form [52]

\[
\begin{align*}
\dot{x}_{1,1} &= f_{1,1}(x_{1,1}) + g_{1,1}(x_{1,1}) x_{1,2} \\
\dot{x}_{1,2} &= f_{1,2}(x_{1,1}, x_{1,2}) + g_{1,2}(x_{1,1}, x_{1,2}) x_{1,3} \\
&\vdots \\
\dot{x}_{1,p_1-1} &= f_{1,p_1-1}(x_{1,1}, \ldots, x_{1,p_1-1}) + g_{1,p_1-1}(x_{1,1}, \ldots, x_{1,p_1-1}) x_{1,p_1} \\
\dot{x}_{1,p_1} &= f_{1,p_1}(x_{1,1}, \ldots, x_{1,p_1}) + \sum_{i=1}^{m} g_{1,p_1}^i(x_{1,1}, \ldots, x_{1,p_1}) u_i \\
&\vdots \\
\dot{x}_{2,1} &= f_{2,1}(x_{2,1}) + g_{2,1}(x_{2,1}) x_{2,2} \\
\dot{x}_{2,2} &= f_{2,2}(x_{2,1}, x_{2,2}) + g_{2,2}(x_{2,1}, x_{2,2}) x_{2,3} \\
&\vdots \\
\dot{x}_{2,p_2-1} &= f_{2,p_2-1}(x_{2,1}, \ldots, x_{2,p_2-1}) + g_{2,p_2-1}(x_{2,1}, \ldots, x_{2,p_2-1}) x_{2,p_2} \\
\dot{x}_{2,p_2} &= f_{2,p_2}(x_{2,1}, \ldots, x_{2,p_2}) + \sum_{i=1}^{m} g_{2,p_2}^i(x_{2,1}, \ldots, x_{2,p_2}) u_i \\
&\vdots \\
\dot{x}_{m-1,1} &= f_{m-1,1}(x_{m-1,1}) + g_{m-1,1}(x_{m-1,1}) x_{m-2} \\
\dot{x}_{m-1,2} &= f_{m-1,2}(x_{m-1,1}, x_{m-1,2}) + g_{m-1,2}(x_{m-1,1}, x_{m-1,2}) x_{m-1,3} \\
&\vdots \\
\dot{x}_{m-1,p_{m-1}-1} &= f_{m-1,p_{m-1}-1}(x_{m-1,1}, \ldots, x_{m-1,p_{m-1}-1}) + g_{m-1,p_{m-1}-1}(x_{m-1,1}, \ldots, x_{m-1,p_{m-1}-1}) x_{m-1,p_{m-1}} \\
\dot{x}_{m-1,p_{m-1}} &= f_{m-1,p_{m-1}}(x_{m-1,1}, \ldots, x_{m-1,p_{m-1}}) + \sum_{i=1}^{m} g_{m-1,p_{m-1}}^i(x_{m-1,1}, \ldots, x_{m-1,p_{m-1}}) u_i \\
&\vdots \\
\dot{x}_{m,1} &= f_{m,1}(x_{m,1}) + g_{m,1}(x_{m,1}) x_{m,2} \\
\dot{x}_{m,2} &= f_{m,2}(x_{m,1}, x_{m,2}) + g_{m,2}(x_{m,1}, x_{m,2}) x_{m,3} \\
&\vdots \\
\dot{x}_{m,p_m-1} &= f_{m,p_m-1}(x_{m,1}, \ldots, x_{m,p_m-1}) + g_{m,p_m-1}(x_{m,1}, \ldots, x_{m,p_m-1}) x_{m,p_m} \\
\dot{x}_{m,p_m} &= f_{m,p_m}(x_{m,1}, \ldots, x_{m,p_m}) + \sum_{i=1}^{m} g_{m,p_m}^i(x_{m,1}, \ldots, x_{m,p_m}) u_i
\end{align*}
\]

\[(7.42)\]
where \( x \in \mathbb{R}^n \) is the states of the system and \( u \in \mathbb{R}^m \) is the control inputs, such that if

\[
G(x) = \begin{bmatrix}
g_{1,\rho_1} & g_{2,\rho_1} & \cdots & g_{m-1,\rho_1} & g_{m,\rho_1} \\
g_{1,\rho_2} & g_{2,\rho_2} & \cdots & g_{m-1,\rho_2} & g_{m,\rho_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
g_{1,\rho_{m-1}} & g_{2,\rho_{m-1}} & \cdots & g_{m-1,\rho_{m-1}} & g_{m,\rho_{m-1}} \\
g_{1,\rho_m} & g_{2,\rho_m} & \cdots & g_{m-1,\rho_m} & g_{m,\rho_m}
\end{bmatrix}
\] (7.43)

is non-singular \( \forall x \in \mathbb{R}^n \), then integrator backstepping can be used to design a controller for the system (7.42) as follows:

Assume that \( x_{1,i} \) has to track a certain reference \( r_i \), then for each \( x_{1,i} \) where \( i = 1, 2, \ldots, m \) let

\[
e_{i,j} = x_{i,j} - \alpha_{i,j-1} \quad \text{for} \quad i = 1, 2, \ldots, m \quad \& \quad j = 1, 2, \ldots, \rho_i,
\] (7.44)

where \( \alpha_{i,j-1} \) is the virtual control function and is obtained by

\[
\alpha_{i,j} = \frac{1}{g_{i,j}} \left[-f_{i,j} + \dot{\alpha}_{i,j-1} - e_{i,j-1} - ke_{i,j}\right].
\] (7.45)

If the Lyapunov function candidate is given by

\[
V = \sum_{i=1}^{m} \sum_{j=1}^{\rho_i} e_{i,j}^2,
\] (7.46)

then it is derivative in time will be

\[
\dot{V} = \frac{1}{2} \sum_{i=1}^{m} \left[ g_{i,\rho_i-1} e_{i,\rho_i-1} e_{i,\rho_i} + e_{i,\rho_i} \left(-\dot{\alpha}_{i,\rho_i-1} + f_{i,\rho_i} + \left[g_{i,\rho_i} g_{i,\rho_i}^2 \cdots g_{i,\rho_i}^{m-1} g_{i,\rho_i}^m\right] u\right) - \sum_{j=1}^{\rho_i-1} ke_{i,j}^2 \right].
\] (7.47)

Choosing \( u \) as

\[
u = G(x)^{-1}
\begin{bmatrix}
-f_{1,\rho_1} + \dot{\alpha}_{1,\rho_1-1} - e_{1,\rho_1-1} - ke_{1,\rho_1} \\
-f_{2,\rho_2} + \dot{\alpha}_{2,\rho_2-1} - e_{2,\rho_2-1} - ke_{2,\rho_2} \\
\vdots \\
-f_{m-1,\rho_{m-1}} + \dot{\alpha}_{m-1,\rho_{m-2}} - e_{m-1,\rho_{m-2}} - ke_{m-1,\rho_{m-1}} \\
-f_{m,\rho_m} + \dot{\alpha}_{m,\rho_m-1} - e_{m,\rho_m-1} - ke_{m,\rho_m}
\end{bmatrix}
\] (7.48)

will render

\[
\dot{V} = -2kV
\] (7.49)
Thus, the equilibrium point $e = 0$ is exponentially stable. Considering again the affine nonlinear system given in (2.38) as

$$\dot{z}_n = f(z) + \sum_{i=1}^{m} g_i(z) u_i$$

$$y_i = h_i(z), \quad i = 1, \ldots, m$$

with $z \in R^n$, $u \in R^m$ and $y \in R^m$ are state variables, control inputs, and system outputs respectively. It is clear from relative degree analysis for MIMO systems and definition 2.6 in section 2.10 that if this system has a vector relative degree $\rho = \{\rho_1, \rho_2, \ldots, \rho_m\}$, then

$$L_{g_1} L_{f}^{k} h_i (z) = L_{g_2} L_{f}^{k} h_i (z) = \ldots = L_{g_m} L_{f}^{k} h_i (z) = 0, \quad k = 0, 1, \ldots, \rho_i - 2.$$  \hspace{1cm} (7.50)

and at least one element is not zero in the row vector

$$\left[ L_{g_1} L_{f}^{\rho_1 - 1} h_i (z) \quad L_{g_2} L_{f}^{\rho_2 - 1} h_i (z) \quad \ldots \quad L_{g_m} L_{f}^{\rho_m - 1} h_i (z) \right]$$ \hspace{1cm} (7.51)

Furthermore, if the vector relative degree $\rho = \rho_1 + \rho_2 + \ldots + \rho_m = n$ and $f(z)$, $g(z)$ and $h(z)$ are smooth enough in the domain $D \subset R^n$, then the diffeomorphism

$$x = T(z) = \begin{bmatrix}
    x_{1,1} \\
    x_{1,2} \\
    x_{1,3} \\
    \vdots \\
    x_{1,\rho_1 - 1} \\
    x_{1,\rho_1} \\
    \vdots \\
    x_{2,1} \\
    x_{2,2} \\
    x_{2,3} \\
    \vdots \\
    x_{2,\rho_2 - 1} \\
    x_{2,\rho_2} \\
    \vdots \\
    \vdots \\
    x_{m,1} \\
    x_{m,2} \\
    x_{m,3} \\
    \vdots \\
    x_{m,\rho_m - 1} \\
    x_{m,\rho_m}
\end{bmatrix} = \begin{bmatrix}
    T_{1,1} (z) \\
    T_{1,2} (z) \\
    T_{1,3} (z) \\
    \vdots \\
    T_{1,\rho_1 - 1} (z) \\
    T_{1,\rho_1} (z) \\
    \vdots \\
    T_{2,1} (z) \\
    T_{2,2} (z) \\
    T_{2,3} (z) \\
    \vdots \\
    T_{2,\rho_2 - 1} (z) \\
    T_{2,\rho_2} (z) \\
    \vdots \\
    \vdots \\
    T_{m,1} (z) \\
    T_{m,2} (z) \\
    T_{m,3} (z) \\
    \vdots \\
    T_{m,\rho_m - 1} (z) \\
    T_{m,\rho_m} (z)
\end{bmatrix} = \begin{bmatrix}
    h_1 (z) \\
    -h_1 (z) + L_i h_1 (z) \\
    L_i h_1 (z) + L_i^2 h_1 (z) \\
    \vdots \\
    (-1)^{\rho_1 - 2} L_i^{\rho_1 - 3} h_1 (z) + L_i^{\rho_1 - 2} h_1 (z) \\
    (-1)^{\rho_1 - 1} L_i^{\rho_1 - 2} h_1 (z) + L_i^{\rho_1 - 1} h_1 (z) \\
    \vdots \\
    (-1)^{\rho_2 - 2} L_i^{\rho_2 - 3} h_2 (z) + L_i^{\rho_2 - 2} h_2 (z) \\
    (-1)^{\rho_1 - 1} L_i^{\rho_2 - 2} h_2 (z) + L_i^{\rho_2 - 1} h_2 (z) \\
    \vdots \\
    (-1)^{\rho_m - 2} L_i^{\rho_m - 3} h_m (z) + L_i^{\rho_m - 2} h_m (z) \\
    (-1)^{\rho_m - 1} L_i^{\rho_m - 2} h_m (z) + L_i^{\rho_m - 1} h_m (z)
\end{bmatrix}, \quad x \in R^n \hspace{1cm} (7.52)
can transform MIMO affine nonlinear system of the form (2.38) into its equivalent strict feedback form (7.42). For more clarification in this regard and without loss of generality the transformation process will be proved for a system of fifth-order with two inputs and two outputs. That is, the system to be discussed is of the form

\[
\dot{z} = f(z) + g_1(z) u_1 + g_2(z) u_2 \\
y_1 = h_1(z) \\
y_2 = h_2(z)
\]  

(7.53)

Assuming that this system has sub-relative degrees \( \rho_1 = 2 \) and \( \rho_2 = 3 \) such that its relative degree is \( \rho = \rho_1 + \rho_2 = 5 = n \) where \( n \), as was mentioned earlier, is the order of the system. Then, the transformation matrix that transforms the system into its equivalent strict feedback form can easily be obtained from (7.52) as

\[
\begin{bmatrix}
x_{1,1} \\
x_{1,2} \\
\vdots \\
x_{2,1} \\
x_{2,2} \\
x_{2,3}
\end{bmatrix} = T(z) =
\begin{bmatrix}
T_{1,1}(z) \\
T_{1,2}(z) \\
\vdots \\
T_{2,1}(z) \\
T_{2,2}(z) \\
T_{2,3}(z)
\end{bmatrix}
\begin{bmatrix}
h_1(z) \\
-h_1(z) + Lf h_1(z) \\
\vdots \\
h_2(z) \\
-h_2(z) + Lf h_2(z) \\
Lf h_2(z) + L^2 f h_2(z)
\end{bmatrix}
, \quad x \in \mathbb{R}^5
\]  

(7.54)

**Proof 7.2 (Transforming a 5th order MIMO nonlinear system into strict feedback form)**

Considering first row element in (7.54): \( x_{1,1} = T_{1,1}(z) = h_1(z) \)

Taking its derivative in time yields

\[
\dot{x}_{1,1} = \frac{dT_{1,1}(z)}{dt} = \frac{dh_1(z)}{dt} = \frac{\partial h_1(z)}{\partial z} \cdot \frac{dz}{dt} = \frac{\partial h_1(z)}{\partial z} \left[ f(z) + g_1(z) u_1 + g_2(z) u_2 \right] \\
= Lf h_1(z) + Lg_1 h_1(z) u_1 + Lg_2 h_1(z) u_2
\]  

(7.55)

It is clear from (7.50) and (7.51) that as \( \rho_1 = 2 \), then \( Lg_1 L^0 f h_1(z) = Lg_2 L^0 f h_1(z) = 0 \) and at least one element is not zero in the row vector \( \left[ Lg_1 Lf h_1(z) \ Lg_2 Lf h_1(z) \right] \). Thus, considering first and second elements in (7.54), (7.55) turns out to be

\[
\dot{x}_{1,1} = Lf h_1(z) \\
= h_1(z) + x_{1,2} \\
= x_{1,1} + x_{1,2}
\]  

(7.56)
Considering second row element in (7.54): \(x_{1,2} = T_{1,2}(z) = -h_1(z) + L_f h_1(z)\)

Taking its derivative in time yields
\[
\dot{x}_{1,2} = \frac{dT_{1,2}(z)}{dt} = - (x_{1,1} + x_{1,2}) + \frac{d}{dt} L_f h_1(z)
\]
\[
= - (x_{1,1} + x_{1,2}) + \frac{\partial}{\partial z} L_f h_1(z) \cdot \frac{dz}{dt}
\]
\[
= - (x_{1,1} + x_{1,2}) + \frac{\partial}{\partial z} L_f h_1(z) [f(z) + g_1(z) u_1 + g_2(z) u_2]
\]
\[
= - (x_{1,1} + x_{1,2}) + L_f^2 h_1(z) + L_{g_1} L_f h_1(z) u_1 + L_{g_2} L_f h_1(z) u_2
\]  
\[\text{(7.57)}\]

Considering third row element in (7.54):
\[\dot{x}_{2,1} = \frac{dT_{2,1}(z)}{dt} = \frac{dh_{2}(z)}{dt} = \frac{\partial h_{2}(z)}{\partial z} \cdot \frac{dz}{dt}
\]
\[
= \frac{\partial h_{2}(z)}{\partial z} [f(z) + g_1(z) u_1 + g_2(z) u_2]
\]
\[
= L_f h_2(z) + L_{g_1} h_2(z) u_1 + L_{g_2} h_2(z) u_2
\]  
\[\text{(7.58)}\]

It is also clear from (7.50) and (7.51) that as \(\rho_2 = 3\), then \(L_{g_1} L_f^k h_2(z) = L_{g_2} L_f^k h_2(z) = 0\) \(\forall k < 2\) and at least one element is not zero in the row vector \(\begin{bmatrix} L_{g_1} L_f^2 h_2(z) & L_{g_2} L_f^2 h_2(z) \end{bmatrix}\).

Thus, considering third and fourth elements in (7.54), (7.58) turns out to be
\[
\dot{x}_{2,1} = L_f h_2(z)
\]
\[
= h_2(z) + x_{2,2}
\]
\[
= x_{2,1} + x_{2,2}
\]  
\[\text{(7.59)}\]

Considering fourth row element in (7.54):
\[x_{2,2} = T_{2,2}(z) = -h_2(z) + L_f h_2(z)\]

Taking its derivative in time yields
\[
\dot{x}_{2,2} = \frac{dT_{2,2}(z)}{dt} = - (x_{2,1} + x_{2,2}) + \frac{d}{dt} L_f h_2(z)
\]
\[
= - (x_{2,1} + x_{2,2}) + \frac{\partial}{\partial z} L_f h_2(z) \cdot \frac{dz}{dt}
\]
\[
= - (x_{2,1} + x_{2,2}) + \frac{\partial}{\partial z} L_f h_2(z) [f(z) + g_1(z) u_1 + g_2(z) u_2]
\]
\[
= - (x_{2,1} + x_{2,2}) + L_f^2 h_2(z) + L_{g_1} L_f h_2(z) u_1 + L_{g_2} L_f h_2(z) u_2
\]
\[
= - (x_{2,1} + x_{2,2}) + L_f^2 h_2(z)
\]  
\[\text{(7.60)}\]
Thus, considering third, fourth and fifth elements in (7.54), (7.60) turns out to be
\[ \dot{x}_{2,2} = -(x_{2,1} + x_{2,2}) + L_f^2 h_2 (z) \]
\[ = -(x_{2,1} + x_{2,2}) + x_{2,3} - L_f h_2 (z) \]
\[ = -(x_{2,1} + x_{2,2}) + x_{2,3} - (h_2 (z) + x_{2,2}) \]
\[ = -(x_{2,1} + x_{2,2}) + x_{2,3} - (x_{2,1} + x_{2,2}) \]
\[ = -2(x_{2,1} + x_{2,2}) + x_{2,3} \] (7.61)

Considering fifth row element in (7.54): \( x_{2,3} = T_{2,3} (z) = L_f h_2 (z) + L_f^2 h_2 (z) \)

Taking its derivative in time yields
\[ \dot{x}_{2,3} = \frac{dT_{2,3} (z)}{dt} = x_{2,3} - (x_{2,1} + x_{2,2}) + \frac{d}{dt} L_f^2 h_2 (z) \]
\[ = x_{2,3} - (x_{2,1} + x_{2,2}) + \frac{\partial}{\partial z} L_f^2 h_2 (z) \cdot \frac{dz}{dt} \]
\[ = x_{2,3} - (x_{2,1} + x_{2,2}) + \frac{\partial}{\partial z} L_f^2 h_2 (z) \left[f (z) + g_1 (z) u_1 + g_2 (z) u_2 \right] \]
\[ = x_{2,3} - (x_{2,1} + x_{2,2}) + L_f^3 h_2 (z) + L_g L_f^2 h_2 (z) u_1 + L_g L_f^2 h_2 (z) u_2 \] (7.62)

So far, from the previous analysis, particularly (7.56), (7.57), (7.59), (7.61) and (7.62), the system (7.53) has been transformed into the following form
\[ \begin{align*}
\dot{x}_{1,1} &= x_{1,1} + x_{1,2} \\
\dot{x}_{1,2} &= -(x_{1,1} + x_{1,2}) + L_f^2 h_1 (z) + L_g L_f h_1 (z) u_1 + L_g L_f h_1 (z) u_2 \\
\dot{x}_{2,1} &= x_{2,1} + x_{2,2} \\
\dot{x}_{2,2} &= -2(x_{2,1} + x_{2,2}) + x_{2,3} \\
\dot{x}_{2,3} &= x_{2,3} - (x_{2,1} + x_{2,2}) + L_f^3 h_2 (z) + L_g L_f^2 h_2 (z) u_1 + L_g L_f^2 h_2 (z) u_2
\end{align*} \] (7.63)

To illustrate and clarify what has been theoretically explained so far on the transformation of the multi-input multi-output system into equivalent strict feedback form, the next example considers the mathematical model for permanent magnet synchronous generator-based wind energy system discussed in [61] where a single MIMO controller was designed such that wind energy battery storage system can operate when the grid is connected or not without the need to switch between two different controllers. The feedback linearizability conditions discussed in section 2.11 and theorem 2.3 are assumed to be fulfilled for this system because the exact feedback linearization controller has been successfully designed in [61].
Example 7.1

\[ \frac{di_{ds}}{dt} = -\frac{R_s}{L_d}i_{ds} + \frac{L_q}{L_d}\omega_r i_{qs} + \frac{1}{L_d}u_{ds} \]
\[ \frac{di_{qs}}{dt} = -\frac{L_d}{L_q}\omega_r i_{ds} - \frac{R_s}{L_q}i_{qs} - \frac{\lambda_r}{L_q}\omega_r + \frac{1}{L_q}u_{qs} \]
\[ \frac{d\omega_r}{dt} = \frac{P}{J}T_m - \frac{3P^2}{2J}\lambda_r i_{qs} + \frac{3P^2}{2J}(L_d - L_q) i_{ds} i_{qs} \]
\[ \frac{di_{di}}{dt} = \omega_g i_{qi} - \frac{1}{L_f}u_{dl} + \frac{1}{L_f}u_{di} \]
\[ \frac{di_{qi}}{dt} = -\omega_g i_{di} - \frac{1}{L_f}u_{qf} + \frac{1}{L_f}u_{qi} \]
\[ \frac{di_{lb}}{dt} = \frac{1}{L_b}u_{bat} + \frac{1}{L_b}D u_{dc} \]
\[ dE_{dc} = u_{dc}i_{sdc} - \frac{3}{2}u_{di}i_{di} + P_{bat} \]
\[ \frac{di_{ds}}{dt} = i_{ds} - \tilde{i}_{ds} \]
\[ d\tilde{\omega}_r = \omega_r - \omega_r^* \]
\[ \frac{d\tilde{u}_{dl}}{dt} = u_{dl} - u_{dl}^* \]
\[ \frac{d\tilde{u}_{ql}}{dt} = u_{ql} - u_{ql}^* \]
\[ \frac{dE_{dc}}{dt} = E_{dc} - E_{dc}^* \]
\[ \frac{di_{lb}}{dt} = i_{lb} - i_{lb}^* \]

\[ y_1 = \tilde{i}_{ds} = \int(i_{ds} - i_{ds}^*) \, d\delta \]
\[ y_2 = \tilde{\omega}_r = \int(\omega_r - \omega_r^*) \, d\delta \]
\[ y_3 = \tilde{u}_{dl} = \int(u_{dl} - u_{dl}^*) \, d\delta \]
\[ y_4 = \tilde{u}_{ql} = \int(u_{ql} - u_{ql}^*) \, d\delta \]
\[ y_5 = \tilde{E}_{dc} = \int(E_{dc} - E_{dc}^*) \, d\delta \]
\[ y_6 = \tilde{i}_{lb} = \int(i_{lb} - i_{lb}^*) \, d\delta \]
### Table 7.1: Wind energy system parameters definitions.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_{ds}, i_{qs}$</td>
<td>Generator stator dq-axis currents ($A$)</td>
</tr>
<tr>
<td>$u_{ds}, u_{qs}$</td>
<td>Generator stator dq-axis voltages ($V$)</td>
</tr>
<tr>
<td>$R_s$</td>
<td>Resistance of stator winding ($\Omega$)</td>
</tr>
<tr>
<td>$L_d, L_q$</td>
<td>Stator dq-axis self inductances ($H$)</td>
</tr>
<tr>
<td>$\omega_r$</td>
<td>Rotating speed of the blades</td>
</tr>
<tr>
<td>$\lambda_r$</td>
<td>Rotor flux ($Wb$)</td>
</tr>
<tr>
<td>$P$</td>
<td>Pole pairs number</td>
</tr>
<tr>
<td>$J$</td>
<td>Generator moment of inertia</td>
</tr>
<tr>
<td>$T_m$</td>
<td>Turbine developed torque</td>
</tr>
<tr>
<td>$i_{di}, i_{qi}$</td>
<td>Output dq-axis currents of grid side converter ($A$)</td>
</tr>
<tr>
<td>$u_{di}, u_{qi}$</td>
<td>Output dq-axis voltages of grid side converter ($V$)</td>
</tr>
<tr>
<td>$R_{th}, L_{th}$</td>
<td>Real and imaginary parts of Thévenin impedance $Z_{th}$</td>
</tr>
<tr>
<td>$E_{thd}, E_{thq}$</td>
<td>Thévenin dq-axis voltage ($V$)</td>
</tr>
<tr>
<td>$u_{dl}, u_{ql}$</td>
<td>Load dq-axis voltage ($V$)</td>
</tr>
<tr>
<td></td>
<td>$u_{dl} = R_{th}i_{di} - \omega_g L_{th}i_{qi} + E_{thd}$</td>
</tr>
<tr>
<td></td>
<td>$u_{ql} = R_{th}i_{qi} - \omega_g L_{th}i_{di} + E_{thq}$</td>
</tr>
<tr>
<td>$L_f$</td>
<td>Filter inductance of grid side ($H$)</td>
</tr>
<tr>
<td>$\omega_g$</td>
<td>Grid electrical angular speed ($rad/s$)</td>
</tr>
<tr>
<td>$i_{lb}$</td>
<td>Current through the inductor $L_b$</td>
</tr>
<tr>
<td>$L_b$</td>
<td>Inductance of bi-directional buck-boost converter ($H$)</td>
</tr>
<tr>
<td>$C_{dc}$</td>
<td>Capacitance of bi-directional buck-boost converter ($F$)</td>
</tr>
<tr>
<td>$E_{dc}$</td>
<td>Energy storage of the dc-link capacitor $C_{dc}$</td>
</tr>
<tr>
<td>$u_{bat}$</td>
<td>Battery voltage ($V$)</td>
</tr>
<tr>
<td>$D$</td>
<td>Duty cycle</td>
</tr>
<tr>
<td>$u_{dc}$</td>
<td>The dc-link capacitor voltage ($V$)</td>
</tr>
<tr>
<td>$i_{sdc}$</td>
<td>dc current of generator side converter ($A$)</td>
</tr>
<tr>
<td>$P_{bat}$</td>
<td>Battery power</td>
</tr>
<tr>
<td>$\tilde{i}_{ds}$</td>
<td>Integral of the difference between $i_{ds}$ and its reference of $i_{ds}^*$</td>
</tr>
<tr>
<td>$\tilde{\omega}_r$</td>
<td>Integral of the difference between $\omega_r$ and its reference of $\omega_r^*$</td>
</tr>
<tr>
<td>$\tilde{u}_{dl}$</td>
<td>Integral of the difference between $u_{dl}$ and its reference of $u_{dl}^*$</td>
</tr>
<tr>
<td>$\tilde{u}_{ql}$</td>
<td>Integral of the difference between $u_{ql}$ and its reference of $u_{ql}^*$</td>
</tr>
<tr>
<td>$\tilde{E}_{dc}$</td>
<td>Integral of the difference between $E_{dc}$ and its reference of $E_{dc}^*$</td>
</tr>
<tr>
<td>$\tilde{i}_{lb}$</td>
<td>Integral of the difference between $i_{lb}$ and its reference of $i_{lb}^*$</td>
</tr>
</tbody>
</table>
Table 7.2: Wind energy system states legend.

<table>
<thead>
<tr>
<th>State</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$i_{ds}$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$i_{qs}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\omega_r$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$i_{di}$</td>
</tr>
<tr>
<td>$x_5$</td>
<td>$i_{qi}$</td>
</tr>
<tr>
<td>$x_6$</td>
<td>$i_{lb}$</td>
</tr>
<tr>
<td>$x_7$</td>
<td>$E_{dc}$</td>
</tr>
<tr>
<td>$x_8$</td>
<td>$\tilde{i}_{ds}$</td>
</tr>
<tr>
<td>$x_9$</td>
<td>$\tilde{\omega}_r$</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>$\tilde{u}_{dl}$</td>
</tr>
<tr>
<td>$x_{11}$</td>
<td>$\tilde{u}_{ql}$</td>
</tr>
<tr>
<td>$x_{12}$</td>
<td>$\tilde{E}_{dc}$</td>
</tr>
<tr>
<td>$x_{13}$</td>
<td>$\tilde{i}_{lb}$</td>
</tr>
</tbody>
</table>

Table 7.3: Wind energy system inputs legend.

<table>
<thead>
<tr>
<th>Input</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>$u_{ds}$</td>
</tr>
<tr>
<td>$u_2$</td>
<td>$u_{qs}$</td>
</tr>
<tr>
<td>$u_3$</td>
<td>$u_{di}$</td>
</tr>
<tr>
<td>$u_4$</td>
<td>$u_{qi}$</td>
</tr>
<tr>
<td>$u_5$</td>
<td>$P_{bat}$</td>
</tr>
<tr>
<td>$u_6$</td>
<td>$D$</td>
</tr>
</tbody>
</table>

Table 7.4: Wind energy system outputs legend.

<table>
<thead>
<tr>
<th>Output</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$\tilde{i}_{ds}$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$\tilde{\omega}_r$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$\tilde{u}_{dl}$</td>
</tr>
<tr>
<td>$y_4$</td>
<td>$\tilde{u}_{ql}$</td>
</tr>
<tr>
<td>$y_5$</td>
<td>$\tilde{E}_{dc}$</td>
</tr>
<tr>
<td>$y_6$</td>
<td>$\tilde{i}_{lb}$</td>
</tr>
</tbody>
</table>
Denoting system’s states, inputs, and outputs according to Tables 7.2, 7.3 and 7.4, and considering the values of \( u_{dl} \) and \( u_{ql} \) from Table 7.1, as well as assuming that \( L_d = L_q \), will make the system (7.64) become

\[
\begin{align*}
\dot{x}_1 &= -\frac{R_s}{L_d} x_1 + \frac{L_q}{L_d} x_2 x_3 + \frac{1}{L_d} u_1 \\
\dot{x}_2 &= -\frac{L_d}{L_q} x_1 x_3 - \frac{R_s}{L_q} x_2 - \frac{\lambda_r}{L_q} x_3 + \frac{1}{L_q} u_2 \\
\dot{x}_3 &= \frac{P}{J} T_m - \frac{3P^2}{2J} \lambda_r x_2 \\
\dot{x}_4 &= -\frac{R_{th}}{L_f} x_4 + \left( \frac{L_f \omega_g + L_{th} \omega_g}{L_f} \right) x_5 - \frac{1}{L_f} E_{thd} + \frac{1}{L_f} u_3 \\
\dot{x}_5 &= \left( \frac{L_{th} \omega_g - L_f \omega_g}{L_f} \right) x_4 - \frac{R_{th}}{L_f} x_5 - \frac{1}{L_f} E_{thq} + \frac{1}{L_f} u_4 \\
\dot{x}_6 &= \frac{u_{bat}}{L_b} - \frac{u_{dc}}{L_b} u_6 \\
\dot{x}_7 &= u_{dc} i_{sdc} - \frac{3}{2} x_4 u_3 + u_5 \\
\dot{x}_8 &= x_1 - x_1^* \\
\dot{x}_9 &= x_3 - x_3^* \\
\dot{x}_{10} &= R_{th} x_4 - \omega_g L_{th} x_5 + E_{thd} - u_{dl}^* \\
\dot{x}_{11} &= -\omega_g L_{th} x_4 + R_{th} x_5 + E_{thq} - u_{ql}^* \\
\dot{x}_{12} &= x_7 - x_7^* \\
\dot{x}_{13} &= x_6 - x_6^* \\
\ldots
\end{align*}
\tag{7.65}
\]

\[
\begin{align*}
y_1 &= x_8 = \int (x_1 - x_1^*) \, dx \\
y_2 &= x_9 = \int (x_3 - x_3^*) \, dx \\
y_3 &= x_{10} = \int (R_{th} x_4 - \omega_g L_{th} x_5 + E_{thd} - u_{dl}^*) \, dx \\
y_4 &= x_{11} = \int (-\omega_g L_{th} x_4 + R_{th} x_5 + E_{thq} - u_{ql}^*) \, dx \\
y_5 &= x_{12} = \int (x_7 - x_7^*) \, dx \\
y_6 &= x_{13} = \int (x_6 - x_6^*) \, dx
\end{align*}
\]
It was previously found in [61] that the aforementioned mathematical model for the permanent magnet synchronous generator-based wind energy system is of a vector relative degree

$$\rho = \{ \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6 \} = \{ 2, 3, 2, 2, 2, 2 \}$$  \hspace{1cm} (7.66)$$

and also

$$\rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 = 13 = n$$  \hspace{1cm} (7.67)$$

Thus, a transformation matrix that can be used to transform (7.65) into its equivalent strict feedback form can be obtained from (7.52) as

$$\begin{bmatrix} z_{1,1} \\ z_{1,2} \\ \vdots \\ z_{2,1} \\ z_{2,2} \\ z_{2,3} \\ \vdots \\ z_{3,1} \\ z_{3,2} \\ \vdots \\ z_{4,1} \\ z_{4,2} \\ \vdots \\ z_{5,1} \\ z_{5,2} \\ \vdots \\ z_{6,1} \\ z_{6,2} \end{bmatrix} = \begin{bmatrix} T_{1,1} (x) \\ T_{1,2} (x) \\ \vdots \\ T_{2,1} (x) \\ T_{2,2} (x) \\ T_{2,3} (x) \\ \vdots \\ T_{3,1} (x) \\ T_{3,2} (x) \\ \vdots \\ T_{4,1} (x) \\ T_{4,2} (x) \\ \vdots \\ T_{5,1} (x) \\ T_{5,2} (x) \\ \vdots \\ T_{6,1} (x) \\ T_{6,2} (x) \end{bmatrix} = \begin{bmatrix} y_1 \\ -y_1 + L_f y_1 \\ \vdots \\ y_2 \\ -y_2 + L_f y_2 \\ L_f y_2 + L_i^2 y_2 \\ \vdots \\ y_3 \\ -y_3 + L_f y_3 \\ \vdots \\ y_4 \\ -y_4 + L_f y_4 \\ \vdots \\ y_5 \\ -y_5 + L_f y_5 \\ \vdots \\ y_6 \\ -y_6 + L_f y_6 \end{bmatrix} = \begin{bmatrix} x_8 \\ -x_8 + x_1 - x_1^* \\ \vdots \\ x_9 \\ -x_9 + x_3 - x_3^* \\ x_3 - x_3^* + \frac{P T_m}{2 J} \lambda_r x_2 \\ \vdots \\ x_{10} \\ -x_{10} + R_{th} x_4 - \omega_y L_{th} x_5 + E_{thd} - u_{d}^* \\ \vdots \\ x_{11} \\ -x_{11} - \omega_y L_{th} x_4 + R_{th} x_5 + E_{thq} - u_{d}^* \\ \vdots \\ x_{12} \\ -x_{12} + x_7 - x_7^* \\ \vdots \\ x_{13} \\ -x_{13} + x_6 - x_6^* \end{bmatrix}, \quad z \in R^{13}$$  \hspace{1cm} (7.68)$$

The determinant of the Jacobian matrix of (7.68) is computed as

$$d (\nabla T (x)) = \frac{-3 P^2 (-L_{th}^2 \omega_y^2 + R_{th}^2)}{2 J}$$  \hspace{1cm} (7.69)$$
which is nonzero, and hence, the Jacobian $\nabla T(x)$ is non-singular, and the inverse $x = T^{-1}(z)$ exists and is computed as

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7 \\
  x_8 \\
  x_9 \\
  x_{10} \\
  x_{11} \\
  x_{12} \\
  x_{13}
\end{bmatrix} = T^{-1}(z) = \begin{bmatrix}
  x_1^1 + z_{1,1} \\
  x_2^1 + z_{1,2} \\
  x_3^1 + z_{2,1} + z_{2,2} \\
  x_4^1 + z_{3,1} + z_{3,2} \\
  x_5^1 + z_{4,1} + z_{4,2} \\
  x_6^1 + z_{5,1} + z_{5,2} \\
  x_7^1 + z_{6,1} + z_{6,2} \\
  x_8^1 + z_{7,1} + z_{7,2} \\
  x_9^1 + z_{8,1} + z_{8,2} \\
  x_{10}^1 + z_{9,1} + z_{9,2} \\
  x_{11}^1 + z_{10,1} + z_{10,2} \\
  x_{12}^1 + z_{11,1} + z_{11,2} \\
  x_{13}^1 + z_{12,1} + z_{12,2} + z_{13,1} + z_{13,2}
\end{bmatrix}
$$

(7.70)

According to definition 2.1, the transformation matrix (7.68) is a diffeomorphism. From (7.65), $f(x)$ and $g_t(x)$ where $i = 1, 2, \ldots, 6$ are obtained as

$$
f(x) = \begin{bmatrix}
  -\frac{R_x}{L_d} x_1 + \frac{L_q}{L_d} x_2 x_3 \\
  -\frac{L_d}{L_q} x_1 x_3 - \frac{R_x}{L_q} x_2 - \frac{\lambda}{L_q} x_3 \\
  PT_m - \frac{3P^2}{2J} \lambda r x_2 \\
  -\frac{R_{th}}{L_f} x_4 + \left( \frac{L_f \omega_y + L_{th} \omega_y}{L_f} \right) x_5 - \frac{1}{L_f} E_{thd} \\
  \left( \frac{L_{th} \omega_y - L_f \omega_y}{L_f} \right) x_4 - \frac{R_{th}}{L_f} x_5 - \frac{1}{L_f} E_{thq} \\
  \frac{u_{dL}}{L_b} \\
  \frac{u_{dcL}}{L_{sd}} \\
  x_1 - x_1^* \\
  x_3 - x_3^* \\
  R_{th} x_4 - \omega_y L_{th} x_5 + E_{thd} - u_{dL}^* \\
  -\omega_y L_{th} x_4 + R_{th} x_5 + E_{thq} - u_{qL}^* \\
  x_7 - x_7^* \\
  x_6 - x_6^*
\end{bmatrix}
$$

(7.71)
Considering (7.71) and (7.72), each element in the transformation matrix (7.68) is differentiated in time as follows

1) \[
\dot{z}_{1,1} = \frac{dT_{1,1}(x)}{dt} = \frac{dy_1}{dt} = \frac{\partial y_1}{\partial x} [f(x) + g_1(x) u_1 + \cdots + g_6(x) u_6] \\
= L_f y_1 \Rightarrow x_1 - x_1^* \\
= y_1 + z_{1,2} \\
= z_{1,1} + z_{1,2}
\]

2) \[
\dot{z}_{1,2} = \frac{dT_{1,2}(x)}{dt} = -L_f y_1 + \frac{\partial L_f y_1}{\partial x} [f(x) + g_1(x) u_1 + \cdots + g_6(x) u_6] \\
= - (z_{1,1} + z_{1,2}) + L_f^2 y_1 + L_g L_f y_1 u_1 + \cdots + L_{g6} L_f y_1 u_6 \\
= -z_{1,1} - z_{1,2} + L_f^2 y_1 + \frac{1}{L_d} u_1 \\
= -z_{1,1} - z_{1,2} - \frac{R_s}{L_d} x_1 + \frac{L_q}{L_d} x_2 x_3 + \frac{1}{L_d} u_1
\]

3) \[
\dot{z}_{2,1} = \frac{dT_{2,1}(x)}{dt} = \frac{dy_2}{dt} = \frac{\partial y_2}{\partial x} [f(x) + g_1(x) u_1 + \cdots + g_6(x) u_6] \\
= L_f y_2 \Rightarrow x_3 - x_3^* \\
= y_2 + z_{2,2} \\
= z_{2,1} + z_{2,2}
\]
4) \[
\dot{z}_{2,2} = \frac{dT_{2,2}(x)}{dt} = -L_f y_2 + \frac{\partial L_f y_2}{\partial x} [f(x) + g_1(x) u_1 + \cdots + g_6(x) u_6] \\
= -(z_{2,1} + z_{2,2}) + L_f^2 y_2 = -(z_{2,1} + z_{2,2}) + \frac{P}{J} T_m - \frac{3P^2}{2J} \lambda_r x_2 \quad (7.76)
\]

5) \[
\dot{z}_{2,3} = \frac{dT_{2,3}(x)}{dt} = L_f^2 y_2 + \frac{\partial L_f y_2}{\partial x} [f(x) + g_1(x) u_1 + \cdots + g_6(x) u_6] \\
= z_{2,3} - (z_{2,1} + z_{2,2}) + L_f^2 y_2 + L_g L_f^2 y_2 u_1 + \cdots + L_g L_f^2 y_2 u_6 \\
= z_{2,3} - (z_{2,1} + z_{2,2}) + L_f^2 y_2 - \frac{3P^2}{2J L_q} \lambda_r u_2 \\
= z_{2,3} - (z_{2,1} + z_{2,2}) + \frac{3P^2 \lambda_r}{2J} \left( -\frac{L_d}{L_q} x_1 x_3 - \frac{R_z}{L_q} x_2 - \frac{\lambda_r}{L_q} x_3 \right) - \frac{3P^2}{2J L_q} \lambda_r u_2 \\
(7.77)
\]

6) \[
\dot{z}_{3,1} = \frac{dT_{3,1}(x)}{dt} = \frac{dy_3}{dt} = \frac{\partial y_3}{\partial x} [f(x) + g_1(x) u_1 + \cdots + g_6(x) u_6] \\
= L_f y_3 \Rightarrow R_{th} x_4 - \omega_g L_{th} x_5 + E_{thd} - u^*_d \\
\quad = y_3 + z_{3,2} \\
\quad = z_{3,1} + z_{3,2} \\
\quad (7.78)
\]

7) \[
\dot{z}_{3,2} = \frac{dT_{3,2}(x)}{dt} = -L_f y_3 + \frac{\partial L_f y_3}{\partial x} [f(x) + g_1(x) u_1 + \cdots + g_6(x) u_6] \\
= -(z_{3,1} + z_{3,2}) + L_f^2 y_3 + L_g L_f y_3 u_1 + \cdots + L_g L_f y_3 u_6 \\
= -z_{3,1} - z_{3,2} + L_f^2 y_3 + \frac{R_{th}}{L_f} u_3 - \frac{\omega_g L_{th}}{L_f} u_4 \\
= -z_{3,1} - z_{3,2} - \left( \frac{R_{th}^2 + L_{th} \omega_g^2 - L_{th} L_f \omega_g^2}{L_f} \right) x_4 - \frac{R_{th}}{L_f} E_{thd} \\
\quad + \left( \frac{R_{th} L_f \omega_g + 2R_{th} L_{th} \omega_g}{L_f} \right) x_5 + \frac{\omega_g L_{th}}{L_f} E_{thd} + \frac{R_{th}}{L_f} u_3 - \frac{\omega_g L_{th}}{L_f} u_4 \\
\quad (7.79)
\]
\[ \dot{z}_{4,1} = \frac{dT_{4,1}(x)}{dt} = \frac{dy_1}{dt} = \frac{\partial y_1}{\partial x} [f(x) + g_1(x)u_1 + \cdots + g_6(x)u_6] \\
= -L_f y_4 \Rightarrow -\omega_y L_{th} x_4 + R_{th} x_5 + E_{thq} - u_{q*} \tag{7.80} \]

\[ z_{4,1} + z_{4,2} \]

\[ \dot{z}_{4,2} = \frac{dT_{4,2}(x)}{dt} = -L_f y_4 + \frac{\partial L_f y_4}{\partial x} [f(x) + g_1(x)u_1 + \cdots + g_6(x)u_6] \\
= - (z_{4,1} + z_{4,2}) + \frac{L_f^2 y_4}{L_f} y_4 + L_{g_1} L_f y_4 u_1 + \cdots + L_{g_6} L_f y_4 u_6 \\
= -z_{4,1} - z_{4,2} + \frac{\omega_y L_{th}}{L_f} u_3 + \frac{R_{th}}{L_f} u_4 \\
= -z_{4,1} - z_{4,2} + \left( \frac{2 R_{th} L_{th} \omega_y}{L_f} - \frac{L_f R_{th} \omega_y}{L_f} \right) x_4 - \frac{R_{th}}{L_f} E_{thd} \\
- \left( \frac{R_{th}^2 + L_f L_{th} \omega_y^2 + L_{th} \omega_y^2}{L_f} \right) x_5 + \frac{L_{th} \omega_y}{L_f} E_{thd} - \frac{\omega_y L_{th}}{L_f} u_3 + \frac{R_{th}}{L_f} u_4 \tag{7.81} \]

\[ \dot{z}_{5,1} = \frac{dT_{5,1}(x)}{dt} = \frac{dy_5}{dt} = \frac{\partial y_5}{\partial x} [f(x) + g_1(x)u_1 + \cdots + g_6(x)u_6] \\
= -L_f y_5 \Rightarrow x_7 - x_7^* \tag{7.82} \]

\[ z_{5,1} + z_{5,2} \]

\[ \dot{z}_{5,2} = \frac{dT_{5,2}(x)}{dt} = -L_f y_5 + \frac{\partial L_f y_5}{\partial x} [f(x) + g_1(x)u_1 + \cdots + g_6(x)u_6] \\
= - (z_{5,1} + z_{5,2}) + u_{dc} \dot{i}_{sd} + L_{g_1} L_f y_5 u_1 + \cdots + L_{g_6} L_f y_5 u_6 \tag{7.83} \]

\[ -z_{5,1} - z_{5,2} + u_{dc} \dot{i}_{sd} - \frac{3}{2} x_4 u_3 + u_5 \]

\[ \dot{z}_{6,1} = \frac{dT_{6,1}(x)}{dt} = \frac{dy_6}{dt} = \frac{\partial y_6}{\partial x} [f(x) + g_1(x)u_1 + \cdots + g_6(x)u_6] \\
= -L_f y_6 \Rightarrow x_6 - x_6^* \tag{7.84} \]

\[ z_{6,1} + z_{6,2} \]
\[ \dot{z}_{6,2} = \frac{dT_{6,2}(x)}{dt} = -L_f y_6 + \frac{\partial L_f y_6}{\partial x} [f(x) + g_1(x) u_1 + \cdots + g_6(x) u_6] \\
= -(z_{6,1} + z_{6,2}) + \frac{u_{\text{bat}}}{L_b} + L_{g_1} L_f y_6 u_1 + \cdots + L_{g_6} L_f y_6 u_6 \tag{7.85} \\
= -z_{6,1} - z_{6,2} + \frac{u_{\text{bat}}}{L_b} + u_{dc} u_6 \]

From (7.73) - (7.85), the permanent magnet synchronous generator-based wind energy system (7.65) in its strict feedback form will be

\[
\begin{align*}
\dot{z}_{1,1} &= z_{1,1} + z_{1,2} \\
\dot{z}_{1,2} &= -z_{1,1} - z_{1,2} - \frac{R_s}{L_d} x_1 + \frac{L_q}{L_d} x_2 x_3 + \frac{1}{L_d} u_1 \\
\vdots \\
\dot{z}_{2,1} &= z_{2,1} + z_{2,2} \\
\dot{z}_{2,2} &= -2 (z_{2,1} + z_{2,2}) + z_{2,3} \\
\dot{z}_{2,3} &= z_{2,3} - (z_{2,1} + z_{2,2}) + \frac{3 P^2 \lambda_r}{2 J} \left( -\frac{L_d}{L_q} x_1 x_3 - \frac{R_s}{L_q} x_2 - \frac{\lambda_r}{L_q} x_3 \right) - \frac{3 P^2}{2 J L_q} \lambda_r u_2 \\
\vdots \\
\dot{z}_{3,1} &= z_{3,1} + z_{3,2} \\
\dot{z}_{3,2} &= -z_{3,1} - z_{3,2} - \left( \frac{R_{th}^2 + L_{th}^2 \omega_g^2 - L_{th} L_f \omega_g^2}{L_f} \right) x_4 - \frac{R_{th}}{L_f} E_{thd} + \left\{ \frac{R_{th} L_f \omega_g + 2 R_{th} L_f \omega_g}{L_f} \right\} x_5 \\
&\quad + \frac{\omega_g L_{th}}{L_f} E_{thd} + \frac{R_{th}}{L_f} u_3 - \frac{\omega_g L_{th}}{L_f} u_4 \\
\vdots \\
\dot{z}_{4,1} &= z_{4,1} + z_{4,2} \\
\dot{z}_{4,2} &= -z_{4,1} - z_{4,2} + \left( \frac{2 R_{th} L_{th} \omega_g - L_f R_{th} \omega_g}{L_f} \right) x_4 - \frac{R_{th}}{L_f} E_{thd} - \left\{ \frac{R_{th}^2 + L_f L_{th} \omega_g^2 + L_{th}^2 \omega_g^2}{L_f} \right\} x_5 \\
&\quad + \frac{L_{th} \omega_g}{L_f} E_{thd} - \frac{\omega_g L_{th}}{L_f} u_3 + \frac{R_{th}}{L_f} u_4 \\
\vdots \\
\dot{z}_{5,1} &= z_{5,1} + z_{5,2} \\
\dot{z}_{5,2} &= -z_{5,1} - z_{5,2} + u_{dc} i_{dc} - \frac{3}{2} x_4 u_3 + u_5 \\
\vdots \\
\dot{z}_{6,1} &= z_{6,1} + z_{6,2} \\
\dot{z}_{6,2} &= -z_{6,1} - z_{6,2} + \frac{u_{\text{bat}}}{L_b} + \frac{u_{dc}}{L_b} u_6 \tag{7.86}
\end{align*}
\]
It is easy to substitute for \( x \) values from (7.70). If the matrix

\[
\begin{bmatrix}
\frac{1}{L_d} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3P^2}{2JL_q} \lambda_r & 0 & 0 & 0 & 0 \\
0 & 0 & R_{th} & 0 & -\frac{\omega_L L_{th}}{L_j} & 0 \\
0 & 0 & 0 & R_{th} & 0 & 0 \\
0 & 0 & -\frac{3}{2} \dot{x}_4 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{u_{dc}}{L_b}
\end{bmatrix}
\]

(7.87)

is non-singular \( \forall \ z \in \mathbb{R}^{13} \), then integrator backstepping can be used to design a controller for the system (7.86) following (7.44)-(7.49). 

7.4 Transformation into Strict Feedback Form and Backstepping Control of Western System Coordinating Council (WECC) 3-Machine system

One of the features that distinguishes the electric utility industry is the way that participants are interconnected. The purpose of substations and transmission lines is to deliver energy to customers upon demand. Currently, electricity of a large scale cannot be stored. That is, the amount of electricity provided to customers must always be balanced by generators and power sources [62]. Therefore, if the design and operation of a large scale power system is not coordinated, then there might be outages and other disruptions [63]. Avoiding this situation becomes more important as communities become more dependent on electrical infrastructure. As examples, this includes electrical power for telecommunications and computer networks; hospitals and healthcare equipment; wastewater treatment facilities and municipal water pumping stations; fuel distribution pipelines such as natural gas; transportation systems such as electric rail, vehicle traffic signaling and air traffic control centers. Throughout history, there have been severe power outages across North America. In the summer of 2003, a severe power outage occurred in the Northeast US impacting 50 million customers for almost 30 hours [64]. Consequently, the Federal Power Act was passed in 2005 to develop standards that would increase the reliability of the electric grid. Moreover, this act resulted in the establishment of the Electric Reliability Organization (ERO). This mission was delegated to the North American Electric Reliability Corporation (NERC). NERC subsequently assigned some of this authority to regional entities. Western Electricity Coordinating Council (WECC) is one of those entities [65,66]. In this research, a novel stabilization approach of the WECC 3-machine system using backstepping control methodology
based on the Lyapunov theorem of stability is proposed as a means to improve system stability. This is achieved by injecting power into the system from large-scale battery sources. The coordinates of the feedback linearizable WECC 3-machine system is transformed based on the Lie derivative of the system outputs. In the literature, the WECC 3-machine system was considered in several previous studies. In [67], a case study of the WECC 3-machine system was considered to illustrate an approach for vulnerability analysis proposed for coordinated variable structure switching. This is a class of cyber-physical reconfiguration attacks in which the system is destabilized when the opponent applies a switching sequence to the target circuit components. However, this approach was based on the linearized model of the system. In [68], a reduced-order model for synchronous generator dynamics was developed and then used with a balancing authority area dynamic model in which to design a dynamic automatic generation control. The idea was demonstrated on the WECC 3-Machine system. A comparison study in [69] showed a satisfactory response from applying power oscillation damping (POD) function versus that of a power system stabilizer (PSS) on the WECC 3-machine system with PV. In [62], local feedback and global control were used to investigate the stability of the power grid network. The results in this case were not feasible for implementation. The WECC 3-machine system was also a case study in [70] where screening stability and remedial action tool were proposed. The approach depends on the Lyapunov functions to select a suitable remedial action that stabilizes the power system. Consider the classical model for Western System Coordinating Council (WECC) 3-Machine system appearing in [71]

\[ \begin{align*}
\dot{x}_1 &= C_{12} \sin(x_2) - D_{12} \cos(x_2) + C_{13} \sin(x_4) - D_{13} \cos(x_4) - P_1 + u_1 \\
\dot{x}_2 &= x_3 - x_1 \\
\dot{x}_3 &= -C_{21} \sin(x_2) - D_{21} \cos(x_2) - C_{23} \sin(x_2 - x_4) - D_{23} \cos(x_2 - x_4) - P_2 + u_2 \\
\dot{x}_4 &= x_5 - x_1 \\
\dot{x}_5 &= -C_{31} \sin(x_4) - D_{31} \cos(x_4) - C_{32} \sin(x_4 - x_2) - D_{32} \cos(x_4 - x_2) - P_3 + u_3 \\
\vdots \\
y_1 &= x_1 \\
y_2 &= x_2 \\
y_3 &= x_4
\end{align*} \] (7.88)
where

\[
    x = \begin{bmatrix} \omega_1 & \delta_2 & \omega_2 & \hat{\delta}_3 & \omega_3 \end{bmatrix}^T
\]

\[
    u = \begin{bmatrix} T_{BAT_1} & T_{BAT_2} & T_{BAT_3} \end{bmatrix}^T
\]

\[
    y = \begin{bmatrix} \omega_1 & \hat{\delta}_2 & \hat{\delta}_3 \end{bmatrix}^T
\]

**Table 7.5**: WECC 3-machine system parameters definition.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_i )</td>
<td>Rotor i angular velocity</td>
</tr>
<tr>
<td>( \delta_i )</td>
<td>Rotor i angle</td>
</tr>
<tr>
<td>( \hat{\delta}_i )</td>
<td>Rotor i angular difference</td>
</tr>
<tr>
<td>( T_{BAT_i} )</td>
<td>Battery i power</td>
</tr>
<tr>
<td>( C_{ij} )</td>
<td>Constant</td>
</tr>
<tr>
<td>( D_{ij} )</td>
<td>Constant</td>
</tr>
<tr>
<td>( P_i )</td>
<td>Load power</td>
</tr>
</tbody>
</table>

From (7.88), \( f(x) \) and \( g_i(x) \) where \( i = 1, 2, 3 \) are obtained as

\[
    f(x) = \begin{bmatrix}
    C_{12} \sin(x_2) - D_{12} \cos(x_2) + C_{13} \sin(x_4) - D_{13} \cos(x_4) - P_1 \\
    x_3 - x_1 \\
    -C_{21} \sin(x_2) - D_{21} \cos(x_2) - C_{23} \sin(x_2 - x_4) - D_{23} \cos(x_2 - x_4) - P_2 \\
    x_5 - x_1 \\
    -C_{31} \sin(x_4) - D_{31} \cos(x_4) - C_{32} \sin(x_4 - x_2) - D_{32} \cos(x_4 - x_2) - P_3
    \end{bmatrix} \quad (7.89)
\]

\[
    g_1(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad g_3(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (7.90)
\]
Starting by finding the systems relative degree through differentiating each output until at least one input appears.

\[ y_1 = x_1 \]
\[ y_1^{(1)} = \dot{x}_1 \]
\[ = C_{12} \sin(x_2) - D_{12} \cos(x_2) + C_{13} \sin(x_4) - D_{13} \cos(x_4) - P_1 + u_1 \]  
(7.91)

Thus, system (7.88) has sub-relative degree \( \rho_1 = 1 \) corresponding to output \( y_1 \).

\[ y_2 = x_2 \]
\[ y_2^{(1)} = \dot{x}_2 \]
\[ = x_3 - x_1 \]
\[ y_2^{(2)} = \dot{x}_3 - \dot{x}_1 \]
\[ = -C_{21} \sin(x_2) - D_{21} \cos(x_2) - C_{23} \sin(x_2 - x_4) - D_{23} \cos(x_2 - x_4) - P_2 + u_2 \]
\[ - C_{12} \sin(x_2) + D_{12} \cos(x_2) - C_{13} \sin(x_4) + D_{13} \cos(x_4) + P_1 - u_1 \]  
(7.92)

Thus, system (7.88) has sub-relative degree \( \rho_2 = 2 \) corresponding to output \( y_2 \).

\[ y_3 = x_4 \]
\[ y_3^{(1)} = \dot{x}_4 \]
\[ = x_5 - x_1 \]
\[ y_3^{(2)} = \dot{x}_5 - \dot{x}_1 \]
\[ = -C_{31} \sin(x_4) - D_{31} \cos(x_4) - C_{32} \sin(x_4 - x_2) - D_{32} \cos(x_4 - x_2) - P_3 + u_3 \]
\[ - C_{12} \sin(x_2) + D_{12} \cos(x_2) - C_{13} \sin(x_4) + D_{13} \cos(x_4) + P_1 - u_1 \]  
(7.93)

Thus, the system (7.88) has sub-relative degree \( \rho_3 = 2 \) corresponding to output \( y_3 \). From (7.91)-(7.93), system (7.88) has a vector relative degree \( \rho = \{\rho_1, \rho_2, \rho_3\} \) also \( \rho_1 + \rho_2 + \rho_3 = 1 + 2 + 2 = 5 = n \) which is the system’s dimension. Hence, the transformation matrix that may transform the system (7.88) into its strict feedback form can easily be obtained from (7.52) as

\[
\begin{bmatrix}
  z_{1,1} \\
  z_{2,1} \\
  z_{2,2} \\
  z_{3,1} \\
  z_{3,2}
\end{bmatrix}
= T(x) =
\begin{bmatrix}
  y_1 \\
  y_2 \\
  -y_2 + L_f y_2 \\
  y_3 \\
  -y_3 + L_f y_3
\end{bmatrix}
= \begin{bmatrix}
  x_1 \\
  x_2 \\
  -x_2 + x_3 - x_1 \\
  x_4 \\
  -x_4 + x_5 - x_1
\end{bmatrix}
\]  
(7.94)
The determinant of the Jacobian matrix of (7.94) is easily computed as

\[ d(\nabla T(x)) = 1 \neq 0 \]  

(7.95)

Thus, (7.94) is non-singular and its inverse exists and is obtained as

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{bmatrix} = T^{-1}(z) =
\begin{bmatrix}
  z_{1,1} \\
  z_{2,1} \\
  z_{2,1} + z_{2,2} + z_{1,1} \\
  z_{3,1} \\
  z_{3,1} + z_{3,2} + z_{1,1}
\end{bmatrix}
\]  

(7.96)

Differentiating each row element in (7.94)

1) \[ \dot{z}_{1,1} = \frac{\partial y_1}{\partial x} [f(x) + g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3] \]
\[ = L_f y_1 + L_{g1} y_1 u_1 \]
\[ = C_{12} \sin(z_{2,1}) - D_{12} \cos(z_{2,1}) + C_{13} \sin(z_{3,1}) - D_{13} \cos(z_{3,1}) - P_1 + u_1 \]

(7.97)

2) \[ \dot{z}_{2,1} = \frac{\partial y_2}{\partial x} [f(x) + g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3] \]
\[ = L_f y_2 \]
\[ = x_3 - x_1 \]
\[ = z_{2,1} + z_{2,2} \]

(7.98)

3) \[ \dot{z}_{2,2} = -(z_{2,1} + z_{2,2}) + \frac{\partial (L_f y_2)}{\partial x} [f(x) + g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3] \]
\[ = -z_{2,1} - z_{2,2} + L_f^2 y_2 + L_{g1} L_f y_2 u_1 + L_{g2} L_f y_2 u_2 + L_{g3} L_f y_2 u_3 \]
\[ = -z_{2,1} - z_{2,2} - C_{12} \sin(z_{2,1}) + D_{12} \cos(z_{2,1}) - C_{13} \sin(z_{3,1}) + D_{13} \cos(z_{3,1}) + P_1 \]
\[ - C_{21} \sin(z_{2,1}) - D_{21} \cos(z_{2,1}) - C_{23} \sin(z_{2,1} - z_{3,1}) - D_{23} \cos(z_{2,1} - z_{3,1}) - P_2 \]
\[ - u_1 + u_2 \]
\[ = -z_{2,1} - z_{2,2} - (C_{12} + C_{21}) \sin(z_{2,1}) + (D_{12} - D_{21}) \cos(z_{2,1}) - C_{13} \sin(z_{3,1}) \]
\[ + D_{13} \cos(z_{3,1}) + P_1 - C_{23} \sin(z_{2,1} - z_{3,1}) - D_{23} \cos(z_{2,1} - z_{3,1}) - P_2 - u_1 + u_2 \]

(7.99)
From (7.97)-(7.101) the system (7.88) in its strict feedback form will be

\[
\begin{align*}
\dot{z}_{3,1} &= \frac{\partial y_3}{\partial x} [f(x) + g_1(x) u_1 + g_2(x) u_2 + g_3(x) u_3] \\
&= L_f y_3 \\
&= x_5 - x_1 \\
&= z_{3,1} + z_{3,2}
\end{align*}
\]  
(7.100)

5)

\[
\begin{align*}
\dot{z}_{3,2} &= -(z_{3,1} + z_{3,2}) + \frac{\partial (L_f y_3)}{\partial x} [f(x) + g_1(x) u_1 + g_2(x) u_2 + g_3(x) u_3] \\
&= -z_{3,1} - z_{3,2} + L_f y_3 + L_{g_1} L_f y_3 u_1 + L_{g_2} L_f y_3 u_2 + L_{g_3} L_f y_3 u_3 \\
&= -z_{3,1} - z_{3,2} - C_{12} \sin(z_{2,1}) + D_{12} \cos(z_{2,1}) - C_{13} \sin(z_{3,1}) + D_{13} \cos(z_{3,1}) + P_1 \\
&- C_{31} \sin(z_{3,1}) - D_{31} \cos(z_{3,1}) - C_{32} \sin(z_{3,1} - z_{2,1}) - D_{32} \cos(z_{3,1} - z_{2,1}) - P_3 \\
&- u_1 + u_3 \\
&= -(z_{3,1} - z_{3,2} - C_{12} \sin(z_{2,1}) + D_{12} \cos(z_{2,1}) - (C_{13} + C_{31}) \sin(z_{3,1}) \\
&+ (D_{13} - D_{31}) \cos(z_{3,1}) + P_1 - C_{32} \sin(z_{3,1} - z_{2,1}) - D_{32} \cos(z_{3,1} - z_{2,1}) \\
&- P_3 - u_1 + u_3
\end{align*}
\]  
(7.101)

From (7.97)-(7.101) the system (7.88) in its strict feedback form will be

\[
\begin{align*}
\dot{z}_{1,1} &= C_{12} \sin(z_{2,1}) - D_{12} \cos(z_{2,1}) + C_{13} \sin(z_{3,1}) - D_{13} \cos(z_{3,1}) - P_1 + u_1 \\
&\quad \ldots \\
\dot{z}_{2,1} &= z_{2,1} + z_{2,2} \\
\dot{z}_{2,2} &= -z_{2,1} - z_{2,2} - (C_{12} + C_{21}) \sin(z_{2,1}) + (D_{12} - D_{21}) \cos(z_{2,1}) - C_{13} \sin(z_{3,1}) \\
&+ D_{13} \cos(z_{3,1}) + P_1 - C_{32} \sin(z_{2,1} - z_{3,1}) - D_{32} \cos(z_{2,1} - z_{3,1}) - P_2 - u_1 + u_2 \\
&\quad \ldots \\
\dot{z}_{3,1} &= z_{3,1} + z_{3,2} \\
\dot{z}_{3,2} &= -z_{3,1} - z_{3,2} - C_{12} \sin(z_{2,1}) + D_{12} \cos(z_{2,1}) - (C_{13} + C_{31}) \sin(z_{3,1}) + (D_{13} - D_{31}) \cos(z_{3,1}) \\
&+ P_1 - C_{32} \sin(z_{3,1} - z_{2,1}) - D_{32} \cos(z_{3,1} - z_{2,1}) - P_3 - u_1 + u_3
\end{align*}
\]  
(7.102)
It is clear from (7.102) that

\[
G(x) = \begin{bmatrix}
g^1_{1,\rho_1} & g^2_{1,\rho_1} & g^3_{1,\rho_1} \\
g^1_{2,\rho_2} & g^2_{2,\rho_2} & g^3_{2,\rho_2} \\
g^1_{3,\rho_3} & g^2_{3,\rho_3} & g^3_{3,\rho_3}
\end{bmatrix}
\] (7.103)

is non-singular. Thus, to design a stabilizing control function, the procedure in section 5.4 is applied to each subsystem in (7.102) as follows:

**Sub-system** \[ z_{1,1} \]

**Step:1**

Let \[ e_{1,1} = z_{1,1} \]. Take its derivative in time

\[
\dot{e}_{1,1} = \dot{z}_{1,1} = C_{12} \sin(z_{2,1}) - D_{12} \cos(z_{2,1}) + C_{13} \sin(z_{3,1}) - D_{13} \cos(z_{3,1}) - P_1 + u_1
\] (7.104)

Choose the Lyapunov function:

\[
V_{1,1}(e_{1,1}) = \frac{1}{2} e_{1,1}^2
\] (7.105)

Differentiating it in time results in:

\[
\dot{V}_{1,1}(e_{1,1}) = e_{1,1} \dot{e}_{1,1} = e_{1,1} \left[ C_{12} \sin(z_{2,1}) - D_{12} \cos(z_{2,1}) + C_{13} \sin(z_{3,1}) - D_{13} \cos(z_{3,1}) - P_1 + u_1 \right]
\] (7.106)

Choosing:

\[
u_1 = -C_{12} \sin(z_{2,1}) + D_{12} \cos(z_{2,1}) - C_{13} \sin(z_{3,1}) + D_{13} \cos(z_{3,1}) + P_1 - k_{1,1} e_{1,1}, \quad k_{1,1} > 0
\] (7.107)

yields:

\[
\dot{V}_{1,1}(e_{1,1}) = -k_{1,1} e_{1,1}^2
\] (7.108)
Sub-system $z_{2,i}$

Step: 1

Let $z_{2,2}$ be a virtual control input whose desired value is denoted by $\alpha_1$. Let $e_{2,1} = z_{2,1}$. Take its derivative in time

$$\dot{e}_{2,1} = \dot{z}_{2,1} = z_{2,1} + z_{2,2}$$

(7.109)

Choose the Lyapunov function:

$$V_{2,1}(e_{2,1}) = \frac{1}{2} e_{2,1}^2$$

(7.110)

Differentiating it in time results in:

$$\dot{V}_{2,1}(e_{2,1}) = e_{2,1} \dot{e}_{2,1} = e_{2,1} [z_{2,1} + z_{2,2}]$$

(7.111)

Choosing:

$$z_{2,2} = \alpha_1 = -z_{2,1} - k_{2,1} e_{2,1}, \quad k_{2,1} > 0$$

(7.112)

yields:

$$\dot{V}_{2,1}(e_{2,1}) = -k_{2,1} e_{2,1}^2$$

(7.113)

Step: 2

The stabilizing function $z_{2,2}$ in step 1 is not the valid function but instead is the desired one. Therefore, one can define the error parameter

$$e_{2,2} = z_{2,2} - \alpha_1$$

(7.114)

Take its derivative in time

$$\dot{e}_{2,2} = \dot{z}_{2,2} - \dot{\alpha}_1 = -z_{2,1} - z_{2,2} - (C_{12} + C_{21}) \sin(z_{2,1}) + (D_{12} - D_{21}) \cos(z_{2,1}) - C_{13} \sin(z_{3,1})$$

$$+ D_{13} \cos(z_{3,1}) + P_1 - C_{23} \sin(z_{2,1} - z_{3,1}) - D_{23} \cos(z_{2,1} - z_{3,1}) - P_2 - u_1 + u_2 - \dot{\alpha}_1$$

(7.115)

From (7.109), (7.114) and (7.115), sub-system $z_{2,i}$ in new coordinates is

$$\dot{e}_{2,1} = z_{2,1} + \alpha_1 + e_{2,2}$$

$$\dot{e}_{2,2} = -z_{2,1} - z_{2,2} - (C_{12} + C_{21}) \sin(z_{2,1}) + (D_{12} - D_{21}) \cos(z_{2,1}) - C_{13} \sin(z_{3,1})$$

$$+ D_{13} \cos(z_{3,1}) + P_1 - C_{23} \sin(z_{2,1} - z_{3,1}) - D_{23} \cos(z_{2,1} - z_{3,1}) - P_2 - u_1 + u_2 - \dot{\alpha}_1$$

(7.116)
Choose the augmented Lyapunov function:

\[ V_{2,2} (e_{2,1}, e_{2,2}) = \frac{1}{2} e_{2,1}^2 + \frac{1}{2} e_{2,2}^2 \]  

(7.117)

Differentiating it in time results in:

\[
\dot{V}_{2,2} (e_{2,1}, e_{2,2}) = e_{2,1} \dot{e}_{2,1} + e_{2,2} \dot{e}_{2,2} \\
= e_{2,1} [ \dot{z}_{2,1} + \alpha_1 + e_{2,2} ] + e_{2,2} [ - z_{2,1} - z_{2,2} - (C_{12} + C_{21}) \sin(z_{2,1}) \\
+ (D_{12} - D_{21}) \cos(z_{2,1}) - C_{13} \sin(z_{3,1}) + D_{13} \cos(z_{3,1}) + P_1 \\
- C_{23} \sin(z_{2,1} - z_{3,1}) - D_{23} \cos(z_{2,1} - z_{3,1}) - P_2 - u_1 + u_2 - \dot{\alpha}_1 ]
\]

(7.118)

Substituting for \( \alpha_1 \) from (7.112) yields

\[
\dot{V}_{2,2} (e_{2,1}, e_{2,2}) = -k_{2,1} e_{2,1}^2 + e_{2,2} [ e_{2,1} - \dot{z}_{2,1} - z_{2,2} - (C_{12} + C_{21}) \sin(z_{2,1}) \\
+ (D_{12} - D_{21}) \cos(z_{2,1}) - C_{13} \sin(z_{3,1}) + D_{13} \cos(z_{3,1}) + P_1 \\
- C_{23} \sin(z_{2,1} - z_{3,1}) - D_{23} \cos(z_{2,1} - z_{3,1}) - P_2 - u_1 + u_2 - \dot{\alpha}_1 ]
\]

(7.119)

Choosing:

\[
u_2 = -e_{2,1} + \dot{z}_{2,1} + z_{2,2} + (C_{12} + C_{21}) \sin(z_{2,1}) \\
- (D_{12} - D_{21}) \cos(z_{2,1}) + C_{13} \sin(z_{3,1}) - D_{13} \cos(z_{3,1}) - P_1 \\
+ C_{23} \sin(z_{2,1} - z_{3,1}) + D_{23} \cos(z_{2,1} - z_{3,1}) + P_2 + u_1 - k_{2,2} e_{2,2} + \dot{\alpha}_1, \quad k_{2,2} > 0
\]

yields:

\[
\dot{V}_{2,2} (e_{2,1}, e_{2,2}) = -k_{2,1} e_{2,1}^2 - k_{2,2} e_{2,2}^2
\]

(7.120)

**Sub-system** \( z_{3,i} \)

**Step:1**

Let \( z_{3,2} \) be a virtual control input whose desired value is denoted by \( \alpha_2 \). Let \( e_{3,1} = z_{3,1} \). Take its derivative in time

\[
\dot{e}_{3,1} = \dot{z}_{3,1} \\
= z_{3,1} + z_{3,2}
\]

(7.122)

Choose the Lyapunov function as:

\[
V_{3,1} (e_{3,1}) = \frac{1}{2} e_{3,1}^2
\]

(7.123)
Differentiating it in time results in:

\[ \dot{V}_{3,1} (e_{3,1}) = e_{3,1} \dot{e}_{3,1} = e_{3,1} [z_{3,1} + z_{3,2}] \]  

(7.124)

Choosing:

\[ z_{3,2} = \alpha_2 = -z_{3,1} - k_{3,1} e_{3,1}, \quad k_{3,1} > 0 \]  

(7.125)

yields:

\[ \dot{V}_{3,1} (e_{3,1}) = -k_{3,1} \dot{e}_{3,1}^2 \]  

(7.126)

**Step:2**

The stabilizing function \( z_{3,2} \) in step 1 is not the valid function but instead is the desired one. Therefore, one can define the error parameter

\[ e_{3,2} = z_{3,2} - \alpha_2 \]  

(7.127)

Taking its derivative in time

\[ \dot{e}_{3,2} = \dot{z}_{3,2} - \dot{\alpha}_2 = -z_{3,1} - z_{3,2} - C_{12} \sin(z_{2,1}) + D_{12} \cos(z_{2,1}) - (C_{13} + C_{31}) \sin(z_{3,1}) + (D_{13} - D_{31}) \cos(z_{3,1}) + P_1 - C_{32} \sin(z_{3,1} - z_{2,1}) \]

(7.128)

From (7.122), (7.127) and (7.128), sub-system \( z_{3,i} \) in new coordinates is

\[ \dot{\hat{e}}_{3,1} = z_{3,1} + \alpha_2 + e_{3,2} \]

\[ \dot{\hat{e}}_{3,2} = -z_{3,1} - z_{3,2} - C_{12} \sin(z_{2,1}) + D_{12} \cos(z_{2,1}) - (C_{13} + C_{31}) \sin(z_{3,1}) + (D_{13} - D_{31}) \cos(z_{3,1}) + P_1 - C_{32} \sin(z_{3,1} - z_{2,1}) \]

(7.129)

Choose the augmented Lyapunov function:

\[ V_{3,2} (e_{3,1}, e_{3,2}) = \frac{1}{2} e_{3,1}^2 + \frac{1}{2} e_{3,2}^2 \]  

(7.130)

Differentiating it in time results in:

\[ \dot{V}_{3,2} (e_{3,1}, e_{3,2}) = e_{3,1} \dot{e}_{3,1} + e_{3,2} \dot{e}_{3,2} \]

\[ = e_{3,1} [z_{3,1} + \alpha_2 + e_{3,2}] + e_{3,2} [-z_{3,1} - z_{3,2} - C_{12} \sin(z_{2,1}) + D_{12} \cos(z_{2,1}) - (C_{13} + C_{31}) \sin(z_{3,1}) + (D_{13} - D_{31}) \cos(z_{3,1}) + P_1 - C_{32} \sin(z_{3,1} - z_{2,1}) - D_{32} \cos(z_{3,1} - z_{2,1}) - P_3 - u_1 + u_3 - \dot{\alpha}_2] \]  

(7.131)
Substituting for $\alpha_2$ from (7.125) yields

$$
\dot{V}_{3,2}(e_{3,1}, e_{3,2}) = -k_{3,1}e_{3,1}^2 + e_{3,2}\left[e_{3,1} - z_{3,1} - z_{3,2} - C_{12}\sin(z_{2,1}) + D_{12}\cos(z_{2,1})
- (C_{13} + C_{31})\sin(z_{3,1}) + (D_{13} - D_{31})\cos(z_{3,1}) + P_1 - C_{32}\sin(z_{3,1} - z_{2,1})
- D_{32}\cos(z_{3,1} - z_{2,1}) - P_3 - u_1 + u_3 - \dot{\alpha}_2\right]
$$

(7.132)

Choosing:

$$
u_3 = -e_{3,1} + z_{3,1} + z_{3,2} + C_{12}\sin(z_{2,1}) - D_{12}\cos(z_{2,1})
+ (C_{13} + C_{31})\sin(z_{3,1}) - (D_{13} - D_{31})\cos(z_{3,1}) - P_1 + C_{32}\sin(z_{3,1} - z_{2,1})
+ D_{32}\cos(z_{3,1} - z_{2,1}) + P_3 + u_1 - k_{3,2}e_{3,2} + \dot{\alpha}_2, \quad k_{3,2} > 0
$$

(7.133)

yields:

$$
\dot{V}_{3,2}(e_{3,1}, e_{3,2}) = -k_{3,1}e_{3,1}^2 - k_{3,2}e_{3,2}^2
$$

(7.134)

From (7.103), (7.107), (7.120) and (7.133), the overall stabilizing function is given by:

$$
u_s = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -C_{12}\sin(z_{2,1}) + D_{12}\cos(z_{2,1}) - C_{13}\sin(z_{3,1}) + D_{13}\cos(z_{3,1}) + P_1 - k_{1,1}e_{1,1} \\ -e_{2,1} + z_{2,1} + z_{2,2} + (C_{12} + C_{31})\sin(z_{2,1}) - (D_{12} - D_{31})\cos(z_{2,1}) + C_{13}\sin(z_{3,1}) - D_{13}\cos(z_{3,1}) - P_1 + C_{21}\sin(z_{2,1} - z_{3,1})
+ D_{23}\cos(z_{2,1} - z_{3,1}) + P_2 - k_{2,2}e_{2,2} + \dot{\alpha}_1 \\ -e_{3,1} + z_{3,1} + z_{3,2} + C_{12}\sin(z_{2,1}) - D_{12}\cos(z_{2,1}) + (C_{13} + C_{31})\sin(z_{3,1}) - (D_{13} - D_{31})\cos(z_{3,1}) - P_1 + C_{32}\sin(z_{3,1} - z_{2,1})
+ D_{32}\cos(z_{3,1} - z_{2,1}) + P_3 - k_{3,2}e_{3,2} + \dot{\alpha}_2 \end{bmatrix}
$$

(7.135)

Using the transformation matrix (7.94), the stabilizing function (7.135) can be transformed back to the following stabilizing function in the original system’s coordinates

$$
u_x = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -C_{12}\sin(x_2) + D_{12}\cos(x_2) - C_{13}\sin(x_4) + D_{13}\cos(x_4) + P_1 - k_{1,1}x_1 \\ -x_2 + x_3 - x_1 + (C_{12} + C_{31})\sin(x_2) - (D_{12} - D_{21})\cos(x_2) + C_{13}\sin(x_4) - D_{13}\cos(x_4) - P_1 + C_{23}\sin(x_2 - x_4)
+ D_{23}\cos(x_2 - x_4) + P_2 - k_{2,2}(-x_2 + k_3 - x_1 - \alpha_1) + \dot{\alpha}_1 \\ -x_4 + x_5 - x_1 + C_{12}\sin(x_2) - D_{12}\cos(x_2) + (C_{13} + C_{31})\sin(x_4) - (D_{13} - D_{31})\cos(x_4) - P_1 + C_{32}\sin(x_4 - x_2)
+ D_{32}\cos(x_4 - x_2) + P_3 - k_{3,2}(-x_4 + x_5 - x_1 - \alpha_2) + \dot{\alpha}_2 \end{bmatrix}
$$

(7.136)
where

\[
\begin{align*}
\alpha_1 &= -x_2 - k_{2,1}x_2 \\
\dot{\alpha}_1 &= -x_3 + x_1 - k_{2,1}(x_3 - x_1) \\
\alpha_2 &= -x_4 - k_{3,1}x_4 \\
\dot{\alpha}_2 &= -x_5 + x_1 - k_{3,1}(x_5 - x_1)
\end{align*}
\]  
(7.137)

A comparison between the system’s response to random initial conditions in both open-loop and closed-loop is given using Matlab/Simulink. The WECC 3-machine system (7.88) of parameters obtained from [71] and listed in Table 7.6 was put in a closed-loop configuration with the stabilizing function (7.136) as illustrated in the block diagram in Fig. 7.3. The linear optimal control design with quadratic performance index [50] was used to design gain values as \( k_{1,1} = 0.5774, k_{2,1} = 2.2188, k_{2,2} = 1.4023, k_{3,1} = 2.2188 \) and \( k_{3,2} = 1.4023 \).

**Figure 7.3:** Closed-loop block diagram for transformation into strict feedback form and backstepping control of WECC 3-machine system.
Table 7.6: WECC 3-machine system Parameters values.

<table>
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<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
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<td>$P_1$</td>
<td>$D_{11}$</td>
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</tbody>
</table>

Fig. 7.4 - Fig. 7.7 show system states, rotor angular differences, rotor angles and velocities respectively in both open-loop and closed-loop situations. Simulation results show the effectiveness of backstepping approach in stabilizing the system. Fig. 7.4a shows the stabilization of the system’s states $\left(\omega_1, \delta_2, \omega_2, \delta_3, \omega_3\right)$ in closed-loop configuration whereas Fig. 7.4b shows the system’s response for the same initial conditions in open-loop configuration. Rotor angular differences $\left(\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3\right)$ in closed-loop configuration are shown in Fig. 7.5a and in open-loop in Fig. 7.5b. Fig. 7.6a shows the stabilization of the rotors angles $\left(\delta_1, \delta_2, \delta_3\right)$ in closed-loop and Fig. 7.6b shows the instability in open-loop. Finally, Fig. 7.7a shows the stability of the rotor angular velocities $\left(\omega_1, \omega_2, \omega_3\right)$ in closed-loop configuration and Fig. 7.7b shows their instability when no controller is applied.
(a) System states in closed-loop.

(b) System states in open-loop.

Figure 7.4: WECC 3-machine system states.

(a) Rotor angular differences in closed-loop.

(b) Rotor angular differences in open-loop.

Figure 7.5: WECC 3-machine system rotor angular differences.
Figure 7.6: WECC 3-machine system rotor angles.

Figure 7.7: WECC 3-machine system rotor angular velocities.
8 Conclusion

This dissertation focuses basically on how to transform multi-input multi-output nonlinear system into an equivalent strict feedback form. This enables applying backstepping control approach based on Lyapunov stability and integrator backstepping theorem. After the introduction, the necessary and required mathematical tools were discussed in the second chapter. In the third chapter, two of the most common nonlinear control methods were explained in detail for both single-input single-output and multi-input multi-output systems. The Lyapunov theorem of stability, as well as Sontag’s formula, were both explained in chapter four. The notion of integrator backstepping, stabilization of systems with a chain of integrators and systems in the strict feedback form using backstepping control methodology were respectively explained in chapter five. In chapter six, simple adaptive regulation and tracking backstepping controllers were both covered as well as backstepping for second-order matched systems and extended matching systems. Avoiding overestimation when designing controllers for extended matching systems was also covered in chapter six. Finally, in chapter seven, the transformation process into a strict feedback form for both single-input single-output and multi-input multi-output systems was explained. The mathematical model of direct-drive surface permanent-magnet synchronous wind generator with boost converter as a SISO system and the mathematical model for wind energy battery storage system as a MIMO system were both transformed into their equivalent strict feedback forms in chapter seven as well. The backstepping stabilizing controller was designed to stabilize permanent magnet synchronous motor after transforming its mathematical model into its equivalent strict feedback form. Similarly, a stabilizing MIMO controller was designed to stabilize the Western Electricity Coordinating Council (WECC) 3-machines system. Both controllers were tested using Matlab/Simulink to show their effectiveness. As a suggestion for future work, the transformation of non-square multi-input multi-output systems were the number of inputs is not equal to the number of outputs of the system might be of interest.
Bibliography


