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A Structure Theorem for Bad 3-Orbifolds

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Mathematics

by

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Abstract

We explicitly construct 10 families of bad 3-orbifolds, $\mathcal{X}$, having the following property: given any bad 3-orbifold, $\mathcal{O}$, it admits an embedded suborbifold $X \in \mathcal{X}$ such that after removing this member from $\mathcal{O}$, and capping the resulting boundary, and then iterating this process finitely many times, you obtain a good 3-orbifold. Reversing this process gives us a procedure to obtain any possible bad 3-orbifold starting with a good 3-orbifold. Each member of $\mathcal{X}$ has 1 or 2 spherical boundary components and has underlying topological space $S^2 \times I$ or $(S^2 \times S^1)\backslash B^3$.

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1 Introduction

Throughout this dissertation we will denote $\mathcal{O}$ as a closed, orientable bad 3-orbifold. In this paper a spherical 2-orbifold is a 2-orbifold whose universal cover is a 2-sphere, and in particular the two types of bad 2-orbifolds, teardrops and bad-footballs are not considered a spherical 2-orbifold. Thurston's Orbifold Theorem (see [1]) which states that given a closed, orientable 3-orbifold $\mathcal{O}$ with non-empty singular set, we can first decompose $\mathcal{O}$ along spherical 2-orbifolds and then toroidal 2-orbifolds; then the remaining pieces satisfy exactly one of the following:

1. geometric, or

2. contains an embedded bad 2-orbifold.

Consequently a closed, orientable 3-orbifold is bad if and only if it contains an embedded bad 2-orbifold, such a 2-orbifold is either a teardrop $S^2(a)$ where $a > 1$ or a bad-football $S^2(a, b)$ where $b > a > 1$ (see Proposition 2.1). Our goal is to understand these 3-orbifolds. We show that if $\mathcal{O}$ is bad, then it contains a compact 3-suborbifold, X, satisfying the following properties:

1. X is a member of 1 of 10 possible families,

2. X contains a bad 2-suborbifold,

3. ∂X contains 1 or 2 spherical 2-orbifolds, and

4. X has underlying topological space $S^2 \times I$ or $(S^2 \times S^1) \backslash B^3$.

Let the process of removing X from $\mathcal{O}$ and capping the boundary be called *cutting and capping* X.

The main result of this paper is:

Theorem 1.1. *Let $\mathcal{O}$ be a bad 3-orbifold. Then there exists $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{n+m}$ (for some $n, m \geq 0$) so that $\mathcal{O}_i$ is obtained from $\mathcal{O}_{i-1}$ by cutting and cappping X_i, where:*

1. $\mathcal{O} = \mathcal{O}_0$,
2. $(1 \leq i \leq n)$ X_i *is listed in Theorem 6.2,*
3. $(n+1 \leq i \leq n+m)$ X_i *is listed in Theorem 5.2,*
4. $\mathcal{O}_{n+m}$ *is good.*

In particular, the boundary of each X_i consists of 1 or 2 spherical 2-orbifolds, and its underlying topological space is $S^2 \times I$ or $(S^2 \times S^1) \backslash B^3$.

Reversing the process of Theorem 1.1 gives us the following corollary.

Corollary 1.2. *Any closed, orientable, bad 3-orbifold is obtained from some good 3-orbifold by a repeated iteration of removing 1 or 2 orbifold 3-balls and attaching X_i along the boundary.*

Remark 1. As some $X \in \mathcal{X}$ have 2 boundary components, when we attach an X of this form it is possible to connect two components of a 3-orbifold together.

Outline. In Section 2 we cover notation, definitions, and basic information about 3-orbifolds. In Section 3 we we show that there exists a bound on the number of disjoint essential teardrops contained in $\mathcal{O}$. In Section 4 we show that in the absence of teardrops there exists a bound on the number of pairwise non-parallel, incompressible bad-footballs contained in $\mathcal{O}$. Sections 5 and 6 contain the main content of this paper, where we classify all possible families of compact 3-suborbifolds, X, as in Theorem 1.1. Together there are 10 possible families, where each member of X contains 1 or 2 spherical boundary components, and has underlying topological space $S^2 \times I$ or $(S^2 \times S^1) \backslash B^3$. In Section 7 we use everything together to prove the main theorem.

2 Preliminaries

Background. In order to properly have a discussion about 3-orbifolds we must first begin with 3-manifolds. A *3-manifold* is a topological Hausdorff, second countable space that is locally Euclidean in dimension 3. When studying these objects we may study them from a purely topological standpoint, but also we may study them from a geometric perspective by putting a metric on them.

When studying 3-manifolds we may break them up into smaller pieces that we understand. One way to do this is by prime decomposition. A connected 3-manifold M is *prime* if $M = P \# Q$ implies either P or Q equals S^3, the 3-sphere. Equivalently, every separating 2-sphere bounds a 3-ball. *Prime Decomposition Theorem* states that given a compact, connected, orientable 3-manifold, there exists a decomposition $M = P_1 \# \cdots \# P_n$ where each P_i is prime, and this decomposition is unique up to homeomorphism. Existence of prime decomposition was proven by Kneser in 1928, while Uniqueness was proven by Milnor in 1962.

Another type of decomposition of prime 3-manifolds is the JSJ Decomposition. A 2-torus, T, is a *JSJ torus* if it is incompressible and not boundary parallel (essential), and any incompressible torus in the manifold can be isotoped off of T. The *JSJ Decomposition Theorem* states that for a compact, irreducible, orientable 3-manifold M there exists a finite collection $\mathcal{T} \subset M$ of disjoint JSJ tori such that each component of $M - \mathcal{T}$ is either atoroidal (hyperbolic due to Thurston-Perelman) or a Seifert manifold (or Sol), and a maximal such collection $\mathcal{T}$ is unique up to isotopy. This was proven by Jaco, Shalen, and Johannson in 1978 (Jaco and Shalen together, while Johannson independently). An irreducible manifold M is *atoroidal* if every incompressible torus in M is boundary parallel.

In the early 1970s Thurston conjectured that every closed 3-manifold can be first decomposed along spheres and then JSJ-tori so that each piece admits one of eight possible geometric structures. Those geometries are: Euclidean, Spherical, Hyperbolic, $S^2 \times \mathbb{R}$,

$H^2 \times \mathbb{R}$, The universal cover of $\mathrm{SL}(2,\mathbb{R})$, Nil, and Sol. Where all pieces are prime, with one exception. Thurston proved that there are exactly eight geometries, where each geometry falls into three categories:

1. Seifert Fibered,
2. Sol, which have no boundary and do not appear after non-trivial JSJ decomposition,
3. Hyperbolic.

Hyperbolic geometry is the main structure of interest for many, as it is considered to be the geometry we know least about. Mostow's Rigidity Theorem, Margulis' Lemma, and Thurston's Hyperbolic Dehn Filling Theorem are all tools that have been developed to aid our understanding of this geometry.

Mostow's Rigidity Theorem states that for $n \geq 3$ if two hyperbolic n-manifolds M and N are π_1-isomorphic then they are isometric. This is extremely powerful because that implies that the following string of implications can be reversed for hyperbolic manifolds of dimension 3 or higher.

$$\text{Isometric} \Rightarrow \text{Diffeomorphic} \Rightarrow \text{Homeomorphic} \Rightarrow \text{Homotopy equivalent} \Rightarrow \pi_1\text{-Isomorphic}$$

Hyperbolic n-manifolds have hyperbolic space as their universal cover, which is contractible. That is, hyperbolic space is a $K(\pi,1)$ space. $K(\pi,1)$ spaces are Homotopy equivalent $\Leftrightarrow$ π_1-Isomorphic, which is why we may reduce the theorem to a question about π_1.

Margulis' Lemma states for an orientable hyperbolic n-manifold $M = \mathbb{H}^n/\Gamma$ there exists $\varepsilon_n \geq 0$ such that for Γ and for any $x \in \mathbb{H}^n$ the group $\Gamma_{\varepsilon_n} = \langle \{\gamma \in \Gamma | d(x, \gamma(x)) \leq \varepsilon_n\} \rangle$ is virtually nilpotent. In particular, for dimensions 2 or 3 the group is abelian. For dimension 3, this says that there exists a Margulis constant μ such that all points of radius of injectivity less than μ gives you an embedded neighborhood of short geodesics which form solid tori or cusps.

Thurston's Hyperbolic Dehn Filling Theorem states that for M a finite volume hyperbolic manifold where ∂M consists of n tori $T_1, \ldots, T_n$, for each i there exists a finite set E_i of slopes of T_i such that the Dehn filling, $M(s_1, \ldots, s_n)$, is hyperbolic provided each $s_i \notin E_i$. Moreover, the core of each attached solid torus is a geodesic, and if the size of E_i is big enough then these are the shortest geodesics in $M(s_1, \ldots, s_n)$. Another result of Gromov's Hyperbolic Dehn filling is that volume decreases under this process.

Orbifolds. Orbifolds have an atlas of charts $\{(u_i, \varphi_i)\}$ similar to that of manifolds, but rather than sending open sets of $\mathbb{R}^n$ to open sets of the topological space, orbifolds have maps that take open sets of the orbifold to open sets in $\mathbb{R}^n$. To each open set u_i of the orbifold is associated a finite group Γ_i which acts on the open set $\widetilde{u_i} \subset \mathbb{R}^n$. The map $\varphi_i : u_i \to \widetilde{u_i}/\Gamma_i$ is a homeomorphism.

When the finite group Γ_i is trivial for all i we have no fixed points and the orbifold is a manifold. In dimension 2 since we have The Classification of Surfaces, we may classify all closed, orientable, 2-orbifolds as well. In Table 1 below there are three rows, the first row contains the closed bad 2-orbifolds, the second row contains the spherical 2-orbifolds, and the last row contains the toroidal 2-orbifolds.

There is a notion of *Orbifold Euler Characteristic*, similar to that of manifold Euler Characteristic, that is calculated in dimension 2 by #vertices $- \#edges + \#faces$, where each vertex and edge is given weight $1/n$ with n being the multiplicity. Now, any finite simplicial complex X has an Euler characteristic $\chi(X) = \sum_{i \geq 0}(-1)^i \alpha_i$, where α_i is the number of i-simplexes. It follows from this definition that if $\widetilde{X}$ is the n-sheeted cover of X, then $\chi(\widetilde{X}) = n \cdot \chi(X)$.

Consider the orbifold with the singular set containing three points, p, q, and r, and the underlying space being the 2-sphere. If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, then we have positive Euler characteristic. So, if it is covered by a compact manifold, it has to be covered by the

Underlying Topological Space	Weights of the Singular Points	Manifold Universal Cover
S^2	$(n), n \geq 2$	None
S^2	(n, m), where $n \neq m$, $n, m \geq 2$	None
S^2	$(n, n), n \geq 2$	S^2
S^2	$(2, 2, n), n \geq 2$	S^2
S^2	$(2, 3, 3)$	S^2
S^2	$(2, 3, 4)$	S^2
S^2	$(2, 3, 5)$	S^2
S^2	None	S^2
S^2	$(2, 4, 4)$	$\mathbb{R}^2$
S^2	$(2, 3, 6)$	$\mathbb{R}^2$
S^2	$(3, 3, 3)$	$\mathbb{R}^2$
S^2	$(2, 2, 2, 2)$	$\mathbb{R}^2$
T^2	None	$\mathbb{R}^2$

Table 1: Closed Orientable 2-Orbifolds (2-Orbifolds not listed are hyperbolic.)

sphere. This is the case when $(p, q, r) = (2, 2, n), (2, 3, 3), (2, 3, 4)$, or $(2, 3, 5)$. If we have $(2, 2, n)$, then the Euler characteristic is $\frac{1}{n}$, and so this has the sphere as the universal cover with multiplicity $2n$. If we have $(2, 3, 3)$, then the Euler characteristic is $1/6$ and so this has the sphere as the universal cover with multiplicity 12. $(2, 3, 4)$ has Euler characteristic $1/12$ so this has the sphere as the universal cover with multiplicity 24. Lastly, $(2, 3, 5)$ has Euler characteristic $1/30$ and so it is covered by the sphere with multiplicity 60. Below are possible maps of spherical 2-orbifolds.

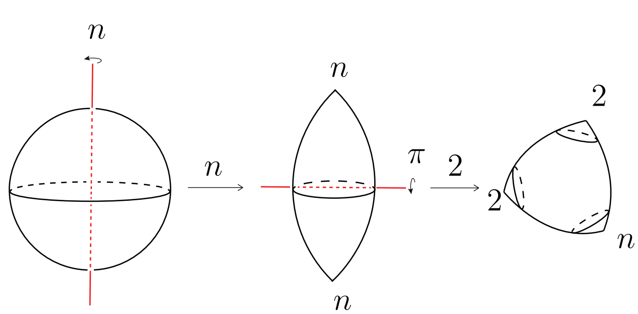
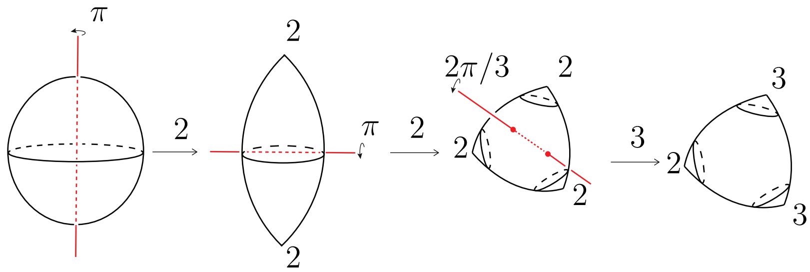


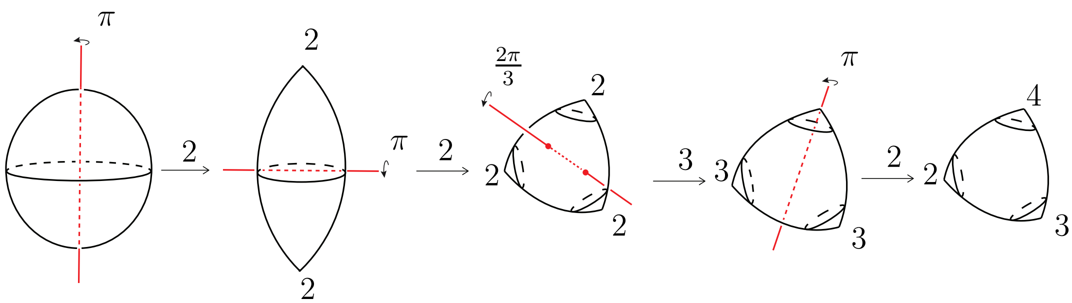
Figure 1: The map from the sphere to the 2-orbifold (2,2,n).



Tetrahedron: 4 faces (rotation by 2π/3), 6 edges (rotation by 2π/2),

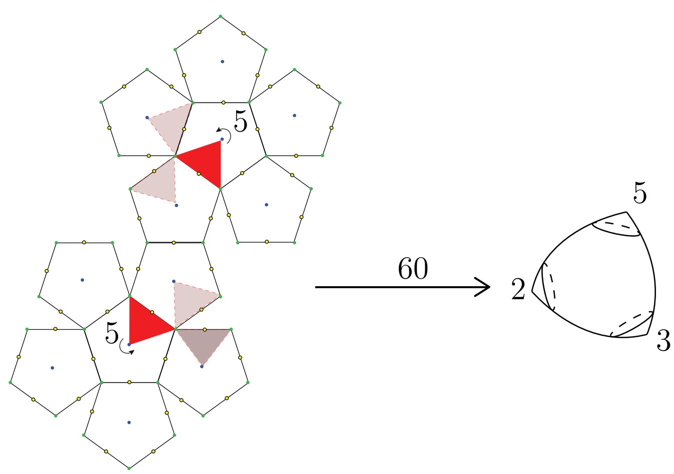
4 vertices (rotation by 2π/3)

Figure 2: The map from the sphere to the 2-orbifold (2,3,3).



Cube: 6 faces (rotation by 2π/4), 12 edges (rotation by 2π/2),
8 vertices (rotation by 2π/3)

Figure 3: The map from the sphere to the 2-orbifold (2,3,4).



Dodecahedron: 12 faces (rotation by 2π/5), 30 edges (rotation by 2π/2),
20 vertices (rotation by 2π/3)

Figure 4: The map from the sphere to the 2-orbifold (2,3,5).

Thurston-Perelman Geometrization Theorem says that any 3-manifold may be decomposed first by spheres and then by tori so that each of the resulting manifolds has one of the eight geometric structure. Now, Thurston's Orbifold Theorem is an analogous statement for our setting of closed, orientable 3-orbifolds. That is, any 3-orbifold with non-empty singular set can first be decomposed along spherical 2-orbifolds and then toroidal 2-orbifolds such that the remaining pieces are either geometric or contain a bad 2-orbifold. The latter condition is what will be of interest, as not many have studied these objects.

Notation. Most of the notation we use is standard, for information regarding basic understanding of closed, orientable 2 and 3 dimensional orbifolds refer to [2] or [1].

We denote the singular set of an orbifold $\mathcal{O}$ as Σ. We denote the underlying topological space of $\mathcal{O}$ by $|\mathcal{O}|$. The teardrop, $S^2(a)$, is the orbifold with $|S^2(a)|$ a 2-sphere S^2 and a single orbifold point of weight $a > 1$. The bad-football, $S^2(a, b)$, is the orbifold with $|S^2(a, b)|$ a 2-sphere S^2 and exactly two orbifold points of weights a and b, where we assume $b > a > 1$.

Definitions. A *3-orbifold* is a topological space, equipped with an atlas $\{(u_i, \varphi_i)\}$ where each local model is a quotient $u_i = \widetilde{u_i}/\Gamma_i$, where $\widetilde{u_i} \subseteq \mathbb{R}^3$ and Γ_i is a finite group action. The *underlying topological space* is the manifold covered by $\{\widetilde{u_i}\}$. By *orientable* we mean $\widetilde{u_i} \to \widetilde{u_i}/\Gamma_i$ is an orientation preserving finite group action and the underlying topological space is orientable. An *orbifold 3-ball*, $B^3(-)$, is B^3/Γ, where Γ is a finite orientation preserving group action. *Capping the boundary* is filling a spherical boundary component with an oribifold 3-ball. We define the singular set, Σ, of $\mathcal{O}$ to be the set $\{x \in |\mathcal{O}| : \Gamma_x \neq \text{id}_x\}$, where Γ_x is the stabilizer at x. An orbifold is *good* if it admits a manifold cover, and *bad* if it does not.

With The Orbifold Theorem we decompose along spherical and toroidal 2-orbifolds. A *spherical 2-orbifold* is a 2-orbifold whose universal cover is S^2, and a *toroidal 2-orbifold* is

2-orbifold that is covered by a 2-torus.

We know that the neighborhood of every point in a 3-orbifold is obtained by a finite orientation preserving group action on a 3-ball, B^3. Therefore by the classification of finite group actions on B^3, we have that *the standard neighborhood* of a point is either B^3, B^3 with a single arc component of Σ, or B^3 containing a vertex of valence three, and the weight of the edges are denoted as a_1, a_2, a_3, where the vertex and edges are components of Σ, and $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} > 1$ [2, Theorem 2.5.].

Proposition 2.1. *A closed, orientable, bad 3-orbifold admits an embedded teardrop or bad-football.*

Proof. Let $\mathcal{O}$ be a closed orientable bad 3-orbifold. We have by Thurston's Orbifold Theorem that $\mathcal{O}$ admits a closed, orientable, bad 2-suborbifold, S. By The Classification of Bad 2-orbifolds [5, Theorem 2.3. page 425] we have the following:

1. $S = S^2(a)$, where the cone point $a > 1$,
2. $S = S^2(a, b)$, where the cone points $b > a > 1$,
3. $S = D^2(a)$, where the corner reflector $a > 1$, or
4. $S = D^2(a, b)$, where the corner reflectors $b > a > 1$.

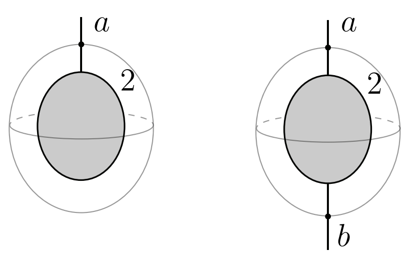


Figure 5: Figure for Proposition 2.1

If $S = S^2(a)$ or $S^2(a, b)$ we are done. Suppose that $S = D^2(a)$ or $D^2(a, b)$. Recall that the vertices of the singular set, Σ, are of valence three [2, Theorem 2.5.], and therefore the corner reflector points of S must have valence three. The underlying topological space of a neighborhood of $|D|$ is a 3-ball, the boundary is a 2-sphere, and in the 3-orbifold this 2-sphere intersects the singular set in either one or two points. The points correspond to the edges coming out of the corner reflector, where $a > 1$ and $a \neq b$ by assumption. Thus the boundary of the neighborhood of D is either a teardrop or a bad-football, respectively. $\square$

3 A Bound on the Number of Teardrops

As we discussed in the Introduction $\mathcal{O}$ is bad if and only if it contains an embedded teardrop or bad-football. If our goal is to remove all bad components from a 3-orbifold we must first know that there exists a bound on the number of these bad 2-orbifold components.

Theorem 3.1. *Given a closed, orientable 3-orbifold, there exists a bound on the number of disjointly embedded, pairwise non-parallel teardrops. Furthermore, if there are no teardrops then there exists a bound on the number of disjointly embedded, pairwise non-parallel bad-footballs.*

Remark 2. The assumption in the above theorem may be weakened to: If there are no two teardrops of different weights, then there exists a bound on the number of disjointly embedded, pairwise non-parallel bad-footballs.

In the proof of this theorem we will use the exterior of the singular set, $E(\Sigma) := \mathcal{O}\backslash N(\Sigma)$ where $N(\Sigma)$ is the open tubular neighborhood of Σ. In the first two lemmas we will focus on teardrops T, in particular we show that essential discs give rise to essential teardrops. Once we prove this we may use [4] to say that there exists a bound on the number of teardrops.

Lemma 3.2. *$T \cap E(\Sigma)$ is essential in $E(\Sigma)$.*

Proof. Let $D = T \cap E(\Sigma)$. Note that D is a disc. Discs are already incompressible and boundary-incompressible by definition so we only need to show that D is not boundary-parallel in $E(\Sigma)$. Notice that if $D \subseteq E(\Sigma)$ is isotopic to the boundary of $N(\Sigma)$, then the entire teardrop can be isotoped into $N(\Sigma)$. For a contradiction, suppose that T is isotopic into $N(\Sigma)$. Recall that manifolds are handlebodies, and handlebodies are irreducible, so in $|N(\Sigma)|$ we have that $|T|$ bounds a 3-ball, $|B|$, on one side. Furthermore recall that Σ is a trivalent graph with possible simple closed curve components. Consider

$\Sigma \cap B$ in $\mathcal{O}$, and observe that $\pi_1(\Sigma) \hookrightarrow \pi_1(N(\Sigma))$. If $\Sigma \cap B$ contains a cycle, c, then $\pi_1(c) \not\hookrightarrow \pi_1(B)$ as B is simply connected, which is a contradiction as $c \subseteq \Sigma \cap B \subseteq \Sigma$ and $B \subseteq N(\Sigma)$. If $\Sigma \cap B$ does not contain a cycle then it must be a tree, and every tree has at least two leaves. Furthermore, there is only one leaf on the boundary, the orbifold point of T. So, there is at least one more leaf in the interior of B, but this would cause this leaf to not have a standard neighborhood. This contradiction proves the lemma. □

For the following three lemmas we will use the base argument from Lemma 3.2, and thus we reiterate it in the lemma below.

Lemma 3.3. *If $B \subseteq N(\Sigma)$, where B is an orbifold 3-ball, then every component of $\Sigma \cap B$ is a tree and all the leaves are on the boundary.*

Proof. Observe that $\pi_1(\Sigma) \hookrightarrow \pi_1(N(\Sigma))$. If $\Sigma \cap B$ contains a cycle, c, then $\pi_1(c) \not\hookrightarrow \pi_1(B)$ as B is simply connected, which is a contradiction as $c \subseteq \Sigma \cap B \subseteq \Sigma$ and $B \subseteq N(\Sigma)$. If $\Sigma \cap B$ does not contain a cycle then it must be a tree, and every tree has at least two leaves. Furthermore if the interior of B contained a leaf then this would cause this leaf to not have a standard neighborhood, which is a contradiction. So, every leaf in $\Sigma \cap B$ is on ∂B. □

For the next lemma let $T_p = S^2(a)$ and $T_q = S^2(b)$ in $\mathcal{O}$, and let $D_p = T_p \cap E(\Sigma)$ and $D_q = T_q \cap E(\Sigma)$. Note that D_p and D_q are discs with orbifold points p and q with weights a and b, respectively.

Lemma 3.4. *If D_p and D_q are parallel in $E(\Sigma)$, then T_p and T_q are parallel in $\mathcal{O}$.*

Proof. Suppose that D_p and D_q are parallel in $E(\Sigma)$. Hence this space is homeomorphic to $D^2 \times I$, with boundary $D(p) \coprod (S^1 \times I) \coprod D(q)$, where $D(p), D(q) \subseteq N(\Sigma)$. This boundary is $S^2(a,b)$, where $S^2(a,b)$ may be isotoped into $N(\Sigma)$. Now, in $|N(\Sigma)|$ we have $|S^2(a,b)|$ is a sphere and thus bounds a ball $|B|$ on one side. Therefore, $S^2(a,b)$ bounds $B \subseteq N(\Sigma)$. Thus, by Lemma 3.3, $\Sigma \cap B$ is a tree where all the leaves are on the boundary. We know the leaves on ∂B are the two points p and q, so $\Sigma \cap B$ is a single arc, therefore $a = b$. Thus, $D(p)$ and $D(q)$ are parallel, and so T_p and T_q must be parallel as well. □

By the first two lemmas we have that essential discs in $E(\Sigma)$ give rise to essential T in $\mathcal{O}$. So by [4] we have that the number of disjointly embedded, pairwise non-parallel T are bounded.

4 A Bound on the Number of Bad-Footballs

In the next two lemmas we will focus on bad-footballs, F. We will show that in the absence of teardrops there exists a bound on the number of disjointly embedded, pairwise non-parallel bad-footballs contained in $\mathcal{O}$. In order to do so we will first show that only non-boundary parallel annuli in $E(\Sigma)$ give rise to non-boundary parallel bad-footballs.

If the annulus in $F \cap E(\Sigma)$ is compressible, then F compresses to two teardrops. The following example shows that when this happens you cannot expect a bound on bad-footballs.

Example 4.1. We consider a 3-orbifold with an embedded bad-football, $S^2(a, b)$, such that $S^2(a, b) \cap E(\Sigma)$ contains a compressing disc, D. When we compress along D we are left with two teardrops: $S^2(a)$ and $S^2(b)$, see Figure 6.

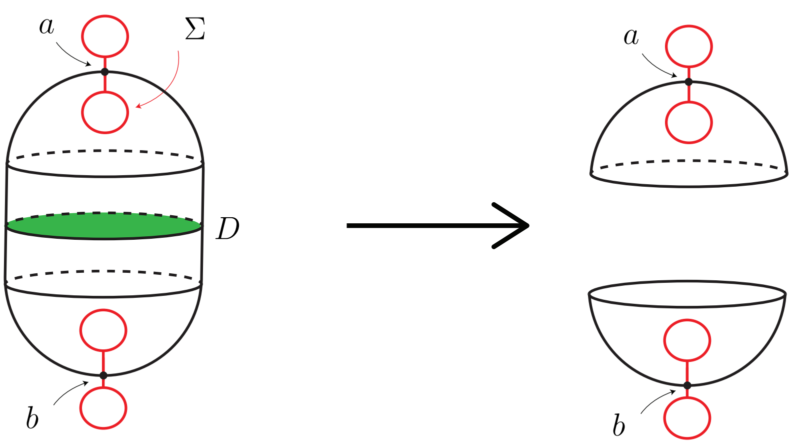


Figure 6: The bad-football containing a compression disc.

In Figure 6 the tubing between $S^2(a)$ and $S^2(b)$ is very simple, but it is possible to complicate the tubing in order to show that there may exists infinitely many, disjoint, non-parallel bad-footballs in the presence of a compression disc. Consider Figure 7, which shows an unbounded sequence of non-parallel bad-footballs obtained by a repeated iteration of knotting the tube between the two teardrops.

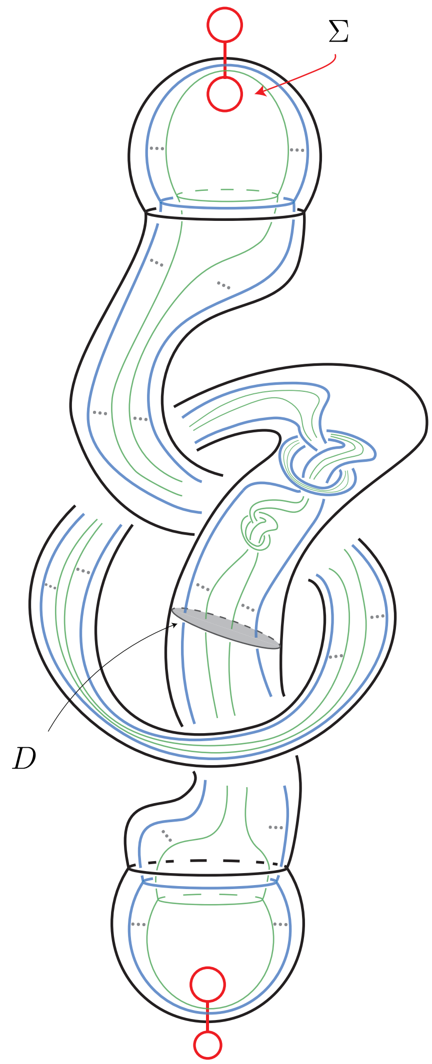


Figure 7: We may obtain an unbounded sequence of non-parallel, compressible bad-footballs.

Therefore, given any $\mathcal{O}$ with two disjointly embedded teardrops: $S^2(a)$, $S^2(b)$ where $a \neq b$, we may construct infinitely many, disjoint, non-parallel bad-football by doing the following: take any arc, α, connecting $S^2(a)$ and $S^2(b)$ with $\alpha \cap \Sigma = \emptyset$, tube along α, then take a repeated iteration of knotting the inner-tube along α as in Figure 7. Note that compression discs do exists in this set up. Thus, to bound the number of disjointly embedded, non-parallel footballs we must assume that $\mathcal{O}$ does not admit two embedded teardrops of distinct weights.

Lemma 4.2. *$F \cap E(\Sigma)$ is not boundary parallel.*

Proof. Analogous to the teardrop case, we let $A = F \cap E(\Sigma)$. Note that A is an annulus. Notice that if $A \subseteq E(\Sigma)$ is isotopic to the boundary of $N(\Sigma)$, then we may isotope the entire bad-football into $N(\Sigma)$. For a contradiction, suppose that F is isotopic into $N(\Sigma)$. In $|N(\Sigma)|$ we have $|F|$ which is a 2-sphere and thus bounds a 3-ball $|B|$ on one side. Therefore, F bounds $B \subseteq N(\Sigma)$ in the orbifold. By Lemma 3.3, $\Sigma \cap B$ is a tree where the leaves are on the boundary. We know the leaves on ∂B are the two points p and q, so $\Sigma \cap B$ is a single arc. Therefore $a = b$, but this gives rise to a good-football, which is a contradiction. This contradiction proves the lemma. □

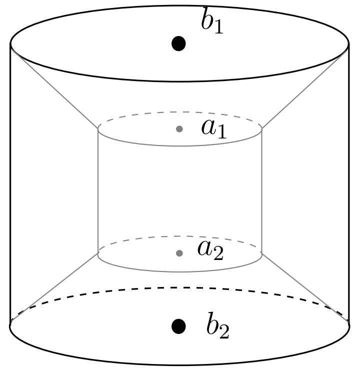
For the next lemma let $F_1 = S^2(a_1, b_1)$ and $F_2 = S^2(a_2, b_2)$ in $\mathcal{O}$, where p_i, q_i are the orbifold points of F_i of weight a_i and b_i, respectively for $i = 1, 2$. Let $A_1 = F_1 \cap E(\Sigma)$ and $A_2 = F_2 \cap E(\Sigma)$.

Lemma 4.3. *If A_1 and A_2 are parallel in $E(\Sigma)$, then F_1 and F_2 are parallel in $\mathcal{O}$.*

Proof. Suppose A_1 and A_2 are parallel in $E(\Sigma)$. Hence, this space is homeomorphic to $A \times I$, with the boundary containing two footballs $S^2(a_1, a_2), S^2(b_1, b_2)$, where each may be isotoped into $N(\Sigma)$. Analogous to the argument before, in the underlying manifold $|S^2(a_1, a_2)|$ and $|S^2(b_1, b_2)|$ are spheres, and thus must each bound a ball $|B_a|$ and $|B_b|$ on one side. Therefore as $B_a, B_b \subseteq N(\Sigma)$ and by Lemma 3.3, $\Sigma \cap B_a$ and $\Sigma \cap B_b$ are both trees where all the leaves are on the boundary. We know the leaves on ∂B_a are the two points p_1, p_2, so $\Sigma \cap B_a$ is a single arc, therefore $a_1 = a_2$. Similarly, $b_1 = b_2$. Therefore, the two bad-footballs $F_1 = S^2(a_1, b_1)$ and $F_2 = S^2(a_2, b_2)$ must be parallel.

□

Therefore, we have that non-boundary parallel annuli give rise to non-boundary parallel bad-footballs, and we are ready to prove Theorem 3.1.



Graphic for Lemma 4.3.

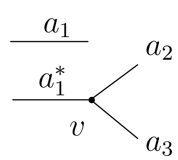
Proof of Theorem 3.1. We have shown in the first two lemmas of Section 3 that essential discs in $E(\Sigma)$ give rise to essential T in $\mathcal{O}$. So by [4] we have that disjointly embedded, pairwise non-parallel T are bounded. Now, recall F is incompressible in the absence of T. In this section we showed that non-boundary parallel annuli give rise to non-boundary parallel F. Since we are only considering one type of surface, i.e., annuli there is certainly a bound on the betti numbers of these surfaces, and hence we may apply [3] to show that there exists a bound on incompressible non-boundary parallel F. That is, we have a constant c such that if $F_1, \ldots, F_k, k > c$, then at least two members F_i, F_j are parallel. □

5 Local Pictures about a Bad-Football

In Theorem 1.1 we state that $\mathcal{O}_i$ is obtained from $\mathcal{O}_{i-1}$ by cutting and capping X_i. When $1 \leq i \leq n$ we look at the X_i members containing an embedded teardrop, and when $n+1 \leq i \leq n+m$ we look at the X_i members containing bad-footballs. So, even though in the theorem we deal with the teardrop components before the bad-football components, in this paper we present the families of bad 3-orbifolds containing bad-footballs first for simplicity.

In Theorem 5.2 we will discover the 3 families of possible compact 3-orbifolds X containing a bad-football, observe that the boundary of each X contains either 1 or 2 spherical 2-orbifolds, and see that $|X|$ is either $S^2 \times I$ or $(S^2 \times S^1)\backslash B^3$.

The following lemma will be used for both Theorems 5.2 and 6.2. We will let e_1 and e_1^* be two edges of the singular set of an orientable 3-orbifold with weights a_1 and a_1^*, respectively. We will let $a_1^* > a_1$, and let v be the vertex where e_1^* terminates. We denote the three edges that branch out of v are e_1^*, e_2, and e_3 of weights a_1^*, a_2, and a_3, respectively. By relabelling we may assume that $a_2 \leq a_3$. Refer to Figure 8 for an image of this set up.



where $a_1^* > a_1$ *and* $a_3 \geq a_2$

Figure 8: Edges of the singular set in Lemma 5.1

Lemma 5.1. *If the edges of the singular set in an orientable 3-orbifold are as described above, then* $S^2(a_1, a_2, a_3)$ *is a spherical 2-orbifold.*

Proof. As $S^2(a_1^*, a_2, a_3)$ is a spherical 2-orbifold we know that $\frac{1}{a_1^*} + \frac{1}{a_2} + \frac{1}{a_3} > 1$, and so since we assumed $a_1 * > a_1$, we have that $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} > 1$. Therefore, we must have that $S^2(a_1, a_2, a_3)$ is a spherical 2-orbifold. $\square$

In Theorem 5.2 we will show that every embedded bad-football in a closed, orientable 3-orbifold is contained in a compact 3-suborbifold of a very specific type.

Theorem 5.2 (Local Pictures about a Bad-Football)**.** *If $\mathcal{O}$ is a closed, orientable 3-orbifold with an embedded bad-football, $F = S^2(a,b)$ where $b > a > 1$, then there exists a compact 3-suborbifold X containing F with the following properties:*

1) the underlying topological space of X is either:

i) $S^2 \times I$ (corresponding to Form 1), or

ii) $(S^2 \times S^1)\backslash B^3$ (corresp. to Form 2 or Smooth Form below),

2) the boundary, ∂X, consists of one or two spherical 2-orbifolds,

3) the orbifold structure is one of the forms in Figures 9, 10, and 11.

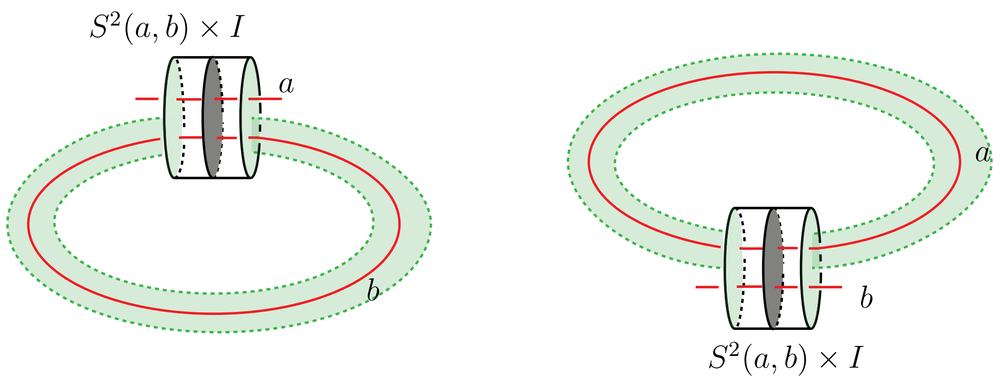


Figure 9: Smooth Form.

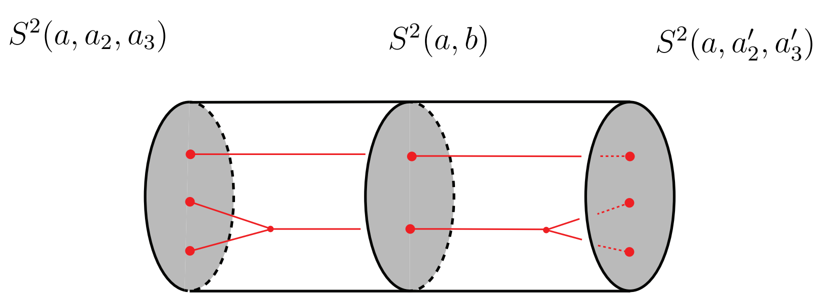


Figure 10: Form 1.

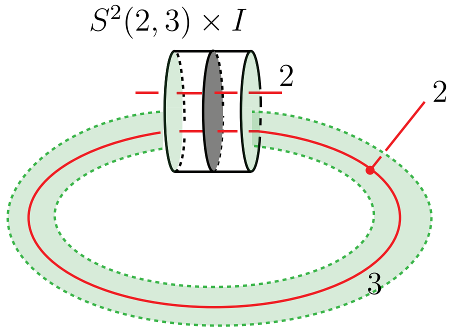


Figure 11: Form 2.

Proof. A flow chart for the proof of this theorem can be found in Figure 13. Let us consider a bad-football $S^2(a,b)$ and Σ the singular set in the 3-orbifold. So, $S^2(a,b)\cap\Sigma=\{p,q\}$ are the singular points on the bad-football, where the weights of p and q are a and b, resp., and $b>a>1$. We will let $e,f\in\Sigma$ be the edges [1] containing $\{p,q\}$, respectively. Denote v_+ and v_- as the vertices where f terminates, and we say that v_+ and v_- are the *inner-vertices*. Call the space described above, χ, and consider the weight preserving map $\omega:\chi\to\Sigma$.

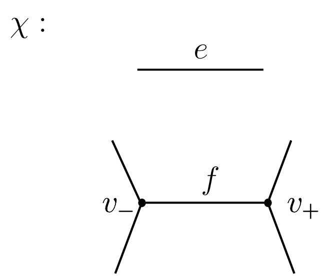


Figure 12: Edges and vertices of the space χ

Case 1. Either the edge e or f does not contain a vertex: Let us consider the edge that does not contain a vertex. This edge is a smooth simple closed curve, and so in the Smooth Form the 3-submanifold X is the union of $S^2(a,b)\times I$ with a tubular neighborhood of the segment of this edge joining $S^2(a,b)\times\{0\}$ to $S^2(a,b)\times\{1\}$ in the

[1] In general e or f may be smooth simple closed curves or edges containing at least one vertex. Furthermore, if e (or f) does contain vertices, then the vertices that e (or f) terminates at could either be distinct or not. When e (or f) does not contain a vertex is covered in case 1.

complement of $S^2(a,b)\times I$. In the underlying topological space you can see that we have $(S^2\times S^1)\backslash B^3$. Therefore $\partial X = S^2(a,a)$ if f does not contain a vertex, or $S^2(b,b)$ if e does not contain a vertex. In either case we have that the boundary of X has one component: a good-football, which is a spherical 2-orbifold. In the case that both e and f do not contain a vertex we take the tubular neighborhood about the heavier weighted edge f.

Case 2. The inner-vertices are distinct: If the inner-vertices are distinct, then the map $\omega:\chi\to\Sigma$ is injective. We construct Form 1 in a similar fashion to how we construct Form 3 in Theorem 6.2; the 3-suborbifold X is the space $N(\text{left})\cup(S^2(a,b)\times I)\cup N(\text{right})$. To construct $N(\text{left})$, first we take the tubular neighborhood of the path to the left of $S^2(a,b)\times I$ along the edge f past the first vertex. Then we let the two edges that branch out of this vertex have weights denoted by a_2 and a_3. Then, $N(\text{left})$ is $S^2(a,b)\times\{0\}$ union this tubular neighborhood. The neighborhood $N(\text{right})$ is constructed in the same fashion, but by traveling to the right of $S^2(a,b)\times I$ along f until we pass the first vertex. We let the two edges that branch out of this vertex have weights denoted by a_2' and a_3'. So, $N(\text{right})$ is $S^2(a,b)\times\{1\}$ union the tubular neighborhood of the path to the right. Therefore, the underlying topological space of X is $S^2\times I$, and ∂X has two components: $S^2(a,a_1,a_2)$ and $S^2(a,a_1',a_2')$. By Lemma 5.1 and since we assume that the weight of f is heavier than the weight of e, i.e., $b>a$, we know that $S^2(a,a_1,a_2)$ and $S^2(a,a_1',a_2')$ are spherical.

Case 3. The inner-vertices are identified: If the inner-vertices are identified, then the map $\omega:\chi\to\Sigma$ is not injective. Similar to the Smooth Form in case 1 above we construct Form 2 to be the 3-suborbifold X as the union of $S^2(a,b)\times I$ with a tubular neighborhood of the segment of f joining $S^2(a,b)\times\{0\}$ to $S^2(a,b)\times\{1\}$ in the complement of $S^2(a,b)\times I$. Therefore, in the underlying topological space we have $(S^2\times S^1)\backslash B^3$. We let the edge that branches out of $\omega(v_+)$ have weight a_3. So, the weights that branch out of $\omega(v_+)$ are (b,b,a_3). By The Classification of 2-orbifolds, the only possibilities are either $(2,2,n\geq 2)$ or $(3,3,2)$. We have assumed $b>a>1$, so $b=3$ and $a=2$, and therefore must have $(b, b, a_3) = (3, 3, 2)$. Furthermore, as the boundary of the 3-suborbifold is $S^2(a, a, a_3)$ we have that $\partial X = S^2(2, 2, 2)$, a spherical 2-orbifold.

$\square$

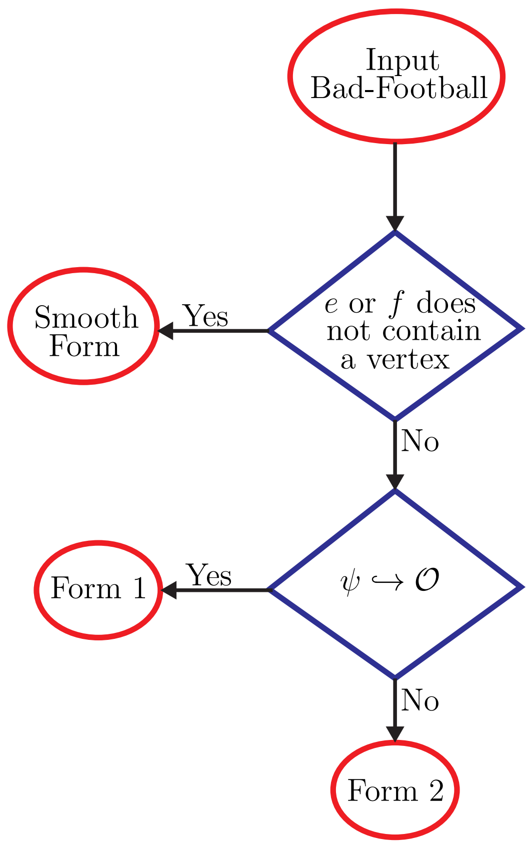


Figure 13: The flowchart diagraming the proof of Theorem 5.2

6 Local Pictures about a Teardrop

In Theorem 6.2 we will show that every embedded teardrop in a closed, orientable 3-orbifold is contained in a compact 3-suborbifold of a very specific type. We call this specific 3-suborbifold X. In particular, we will show that there are 7 families of possible X, and the boundary of each contains either 1 or 2 spherical 2-orbifolds, and $|X|$ is either $S^2 \times I$ or $(S^2 \times S^1)\backslash B^3$. In Example 6.1 we first show an explicit possibility for X.

Example 6.1. Let us consider a teardrop $S^2(3)$ and Σ the singular set in the 3-orbifold. So, $S^2(3) \cap \Sigma = \{p\}$ is the singular point on the teardrop with weight 3. We will let $e \in \Sigma$ be the edge containing $\{p\}$. Now, $S^2(3)$ has a neighborhood homeomorphic to $S^2(3) \times I$, where $(S^2(3) \times I) \cap \Sigma = \{p\} \times I$ is contained in e. We travel to the right of $S^2(3) \times I$ along e until we reach a vertex that branches out into two edges weights 2 and 4. Then we travel along the heavier weighted edge of weight 4 until we meet another vertex which branches out into two edges. The two edges that branch out are of weights 2 and 3. Then we consider a tubular neighborhood of the path traveled and denote this tubular neighborhood by N(right). Next, we travel to the left of $S^2(3) \times I$ along e until we reach a vertex that branches out into two edges of the same weight, 2. Again we will consider a tubular neighborhood of the path traveled and will denote this tubular neighborhood by N(left). The space X shown in Figure 14 is $N(\text{left}) \cup (S^2(3) \times I) \cup N(\text{right})$. In particular, this

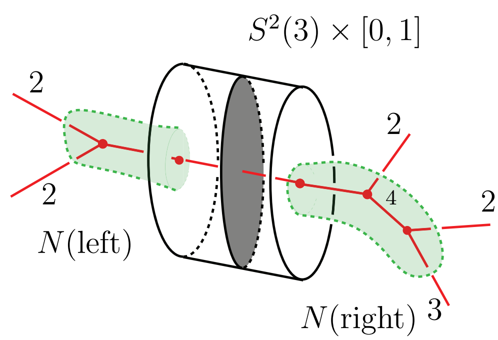


Figure 14: $X = N(\text{left}) \cup (S^2(3) \times I) \cup N(\text{right})$

space is an explicit example of Form 2 from Theorem 6.2. We notice that the boundary of

X has two components $S^2(2,2)$ and $S^2(2,2,3)$, which are spherical 2-orbifolds. Furthermore the underlying topological space is $S^2 \times I$.

Notation. *For the proof of Theorem 6.2 we denote the tubular neighborhood of the path to the right and left of $S^2(a) \times I$ as $N(right)$ and $N(left)$, respectively.*

Theorem 6.2 (Local Pictures about a Teardrop)**.** *If $\mathcal{O}$ is a closed, orientable 3-orbifold with an embedded teardrop, $T = S^2(a)$ where $a > 1$, then there exists a compact 3-suborbifold X containing T with the following properties:*

1) the underlying topological space of X is either:

i) $S^2 \times I$ (corresp. to Form $1, 2,$ or 3 below), or

ii) $(S^2 \times S^1)\backslash B^3$ (corresp. to Form $4, 5, 6$ or Smooth Form below),

2) the boundary, ∂X, consists of one or two spherical 2-orbifolds, and

3) the orbifold structure is one of the forms in Figures 15, 16, 17, 18, 19, 20, and 21.

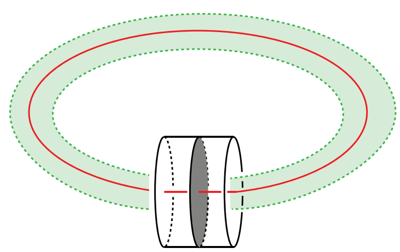


Figure 15: Smooth Form.
$(S^2(a) \times S^1)\backslash B^3$, where B^3 is a smooth 3-ball

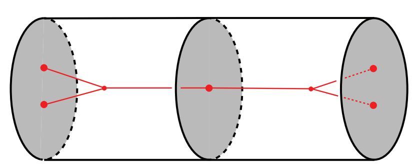


Figure 16: Form 1.

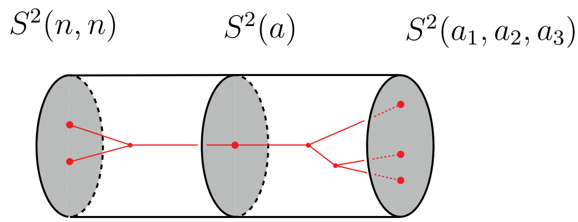


Figure 17: Form 2.

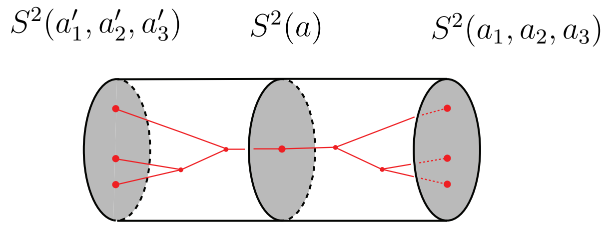


Figure 18: Form 3.

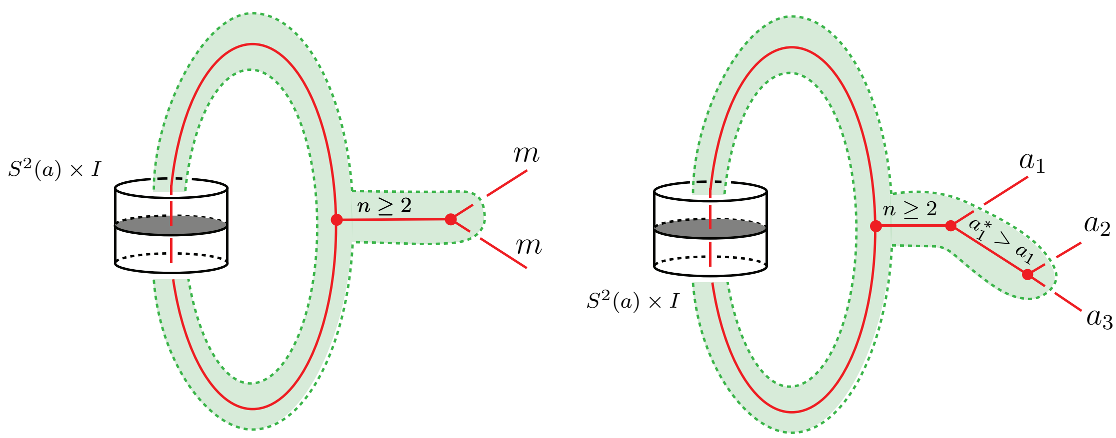


Figure 19: Form 4.

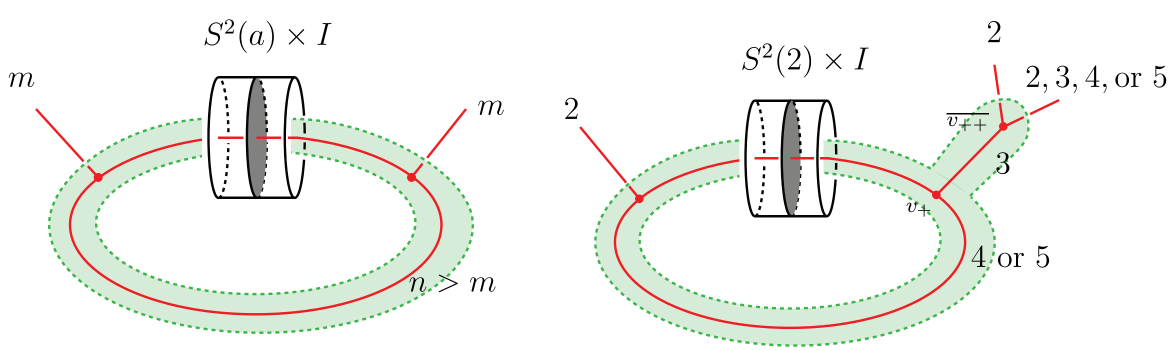
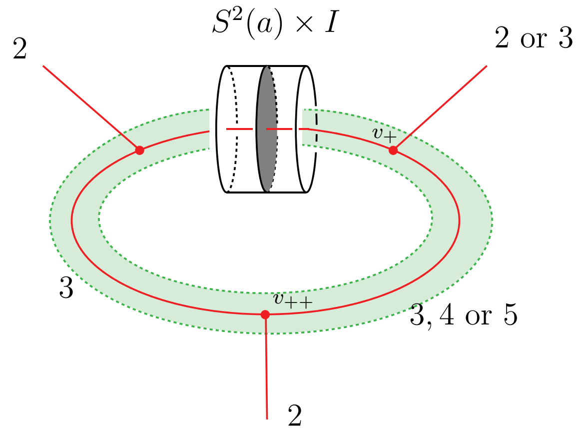


Figure 20: Form 5.



The boundary of the standard neighborhood of the vertex v_+ is not $S^2(2,3,3)$. If so, we would have Form 2. If $\overline{a_-} \neq \overline{a_+}$, then $a = 2$.

Figure 21: Form 6.

Proof. A flow chart for the proof of this theorem can be found in Figure 29. Let us consider a teardrop $S^2(a)$ and Σ the singular set in the 3-orbifold. So, $S^2(a) \cap \Sigma = \{p\}$ is the singular point on the teardrop with weight $a > 1$. We will let $e \in \Sigma$ be the edge[2] containing $\{p\}$, where $(S^2(a) \times I) \cap \Sigma = \{p\} \times I$ is contained in e. We will denote v_+ and v_- as the vertices where e terminates, e_+ and $\overline{e_+}$ as the edges that branch out of v_+, and v_{++} and $\overline{v_{++}}$ as the vertices where e_+ and $\overline{e_+}$ respectively terminate. Similarly, we denote e_- and $\overline{e_-}$ as the edges that branch out of v_-, and v_{--} as the vertex where e_- terminates. Furthermore, we let $a_-, a_+, \overline{a_-}, \overline{a_+}$ be the weight of the edges $e_-, e_+, \overline{e_-}, \overline{e_+}$, respectively. The aforementioned vertices and edges are not assumed to be distinct, and we say that v_+ and v_- are the *inner-vertices* while v_{++}, v_{--}, and $\overline{v_{++}}$, are the *outer-vertices*.

By relabelling we assume that $a_- \geq \overline{a_-}$ and $a_+ \geq \overline{a_+}$. We consider the following three subspaces of the space shown in Figure 22: χ_1, χ_2, and χ_3. Furthermore, we refer to the vertices that are contained in each χ_i as the *active vertices*[3] and χ_i as *active spaces*. We note that the subspaces contain parts of edges, for example χ_1 and χ_2 only contain part of

[2] In general e may be a smooth simple closed curve or an edge containing at least one vertex. Furthermore, if e does contain vertices, then the vertices that e terminates at could either be distinct or not. When e does not contain a vertex is covered in case 1.

[3] If there is only one active outer-vertex, by relabelling we assume it is v_{++}.

e_-, while χ_3 contains all of e_-. After case 1 which covers the Smooth Form, we determine the other forms of X by considering the weight preserving map $\omega_i : \chi_i \to \Sigma$.

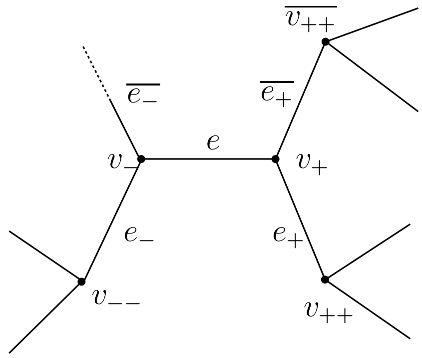


Figure 22: The vertices and edges above are not necessarily distinct.

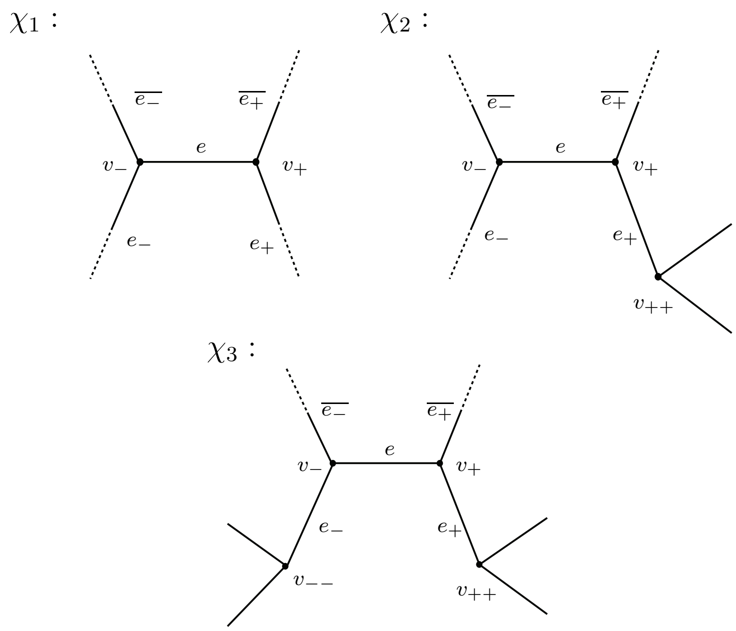


Figure 23: The active spaces χ_1, χ_2, and χ_3.

- $\overline{a_+} = a_+$ and $\overline{a_-} = a_-$
    - Active Space: χ_1
    - Active Vertices: v_+, v_-
        * $\omega_1 : \chi_1 \hookrightarrow \Sigma \Rightarrow$ Form 1

    - $\omega_1 : \chi_1 \not\hookrightarrow \Sigma \Rightarrow$ Form 4

- $\overline{a_+} \neq a_+$ and $\overline{a_-} = a_-$
  - Active Space: χ_2
  - Active Vertices: v_+, v_-, v_{++}
    - $\omega_2 : \chi_2 \hookrightarrow \Sigma \Rightarrow$ Form 2
    - $\omega_2 : \chi_2 \not\hookrightarrow \Sigma \Rightarrow$ Form 5

- $\overline{a_+} \neq a_+$ and $\overline{a_-} \neq a_-$
  - Active Space: χ_3
  - Active Vertices: v_+, v_-, v_{++}, v_{--}
    - $\omega_3 : \chi_3 \hookrightarrow \Sigma \Rightarrow$ Form 3
    - $\omega_3 : \chi_3 \not\hookrightarrow \Sigma \Rightarrow$ Form 6

Case 1. The edge e does not contain a vertex:

If the edge e does not contain a vertex, then e is a smooth simple closed curve. In the Smooth Form the 3-submanifold X is the union of $S^2(a) \times I$ with a tubular neighborhood of the segment of e joining $S^2(a) \times \{0\}$ to $S^2(a) \times \{1\}$ in the complement of $S^2(a) \times I$, which is diffeomorphic to $(S^2(a) \times S^1)\backslash B^3$, where $a > 1$ and B^3 is a smooth 3-ball. That is, B^3 contains no component of the singular set. Therefore, the underlying topological space of X is $(S^2 \times S^1)\backslash B^3$ and $\partial X = S^2$.

Case 2. The vertices in χ_i are distinct:

If all of the active vertices in χ_1 are distinct then we consider the weight preserving map $\omega_1 : \chi_1 \to \Sigma$. Since all of the active vertices are distinct we have that ω_1 is injective and in particular, $\omega_1(\chi_1)$ has edges of weight $\overline{a_+} = a_+$ and $\overline{a_-} = a_-$. We construct X to be $N(\text{left}) \cup (S^2(a) \times I) \cup N(\text{right})$, similar to Example 6.1, where $N(\text{right})$ (and $N(\text{left})$, respectively) is the tubular neighborhood of the path traveled along the singular set to the right (and left, respectively) of $S^2(a) \times I$ until we reach a vertex. For each path, to the right and to the left of $S^2(a) \times I$ these vertices branch out into two edges of the same weight. When X is constructed in this way we say it has Form 1.

We construct Form 2 and Form 3 in a similar fashion. If all of the active vertices in χ_2 are distinct we consider $\omega_2 : \chi_2 \to \Sigma$, where ω_2 is injective and so $\omega_2(\chi_2)$ has edges of weight $\overline{a_+} \neq a_+$ and $\overline{a_-} = a_-$. The 3-suborbifold X is the space $N(\text{left}) \cup (S^2(a) \times I) \cup N(\text{right})$, which we saw in Example 6.1. The path along the singular set to the right of $S^2(a) \times I$ meets a vertex that branches out to two edges of different weights, and then travels past the first vertex along the heavier weighted edge until it meets another vertex. The path along the singular set to the left of $S^2(a) \times I$ stops at a vertex that branches out to two edges of the same weight. Then $N(\text{right})$ and $N(\text{left})$ are the tubular neighborhoods of these paths, respectively. When X is constructed in this way we say it has Form 2.

If all of the active vertices in χ_3 are distinct we consider $\omega_3 : \chi_3 \to \Sigma$, where ω_3 is injective and so $\omega_3(\chi_3)$ has edges of weight $\overline{a_+} \neq a_+$ and $\overline{a_-} \neq a_-$. The 3-suborbifold X is the space $N(\text{left}) \cup (S^2(a) \times I) \cup N(\text{right})$, where both paths along the singular set to the right and to the left of $S^2(a) \times I$ meet a vertex of different weights. In particular, $N(\text{right})$ and $N(\text{left})$ is the tubular neighborhood of the path that travels past the first vertex along the heavier weighted edge until it stops at another vertex. When X is constructed in this way we say it has Form 3.

In Forms 1, 2, and 3, this results in the underlying topological space of X to be $S^2 \times I$, and the boundary of X to have two components. A component of ∂X is a good-football $S^2(n, n)$ when the path along Σ to the right or left of $S^2(a) \times I$ branches out to two edges of the same weight, n. That is, $\overline{a_+} = a_+ = n$ or $\overline{a_-} = a_- = n$. If the path to the right or left of $S^2(a) \times I$ branches out to two edges of different weights, a_1 and a_1^* where $a_1 < a_1^*$, then we traveled along the edge of weight a_1^* until this edge branches out into two edges of weights a_2 and a_3. Therefore, one component of ∂X will be $S^2(a_1, a_2, a_3)$ and by Lemma

5.1 this is a spherical 2-orbifold. So, in either case the two components of ∂X are spherical 2-orbifolds.

We have covered the cases for when the edge e does not contain any vertices and when all of the active vertices in χ_i are distinct. We now discuss the possibilities of when ω_i does not inject into Σ, that is the active vertices may be identified.

Case 3. The active vertices v_+ and v_- are identified:

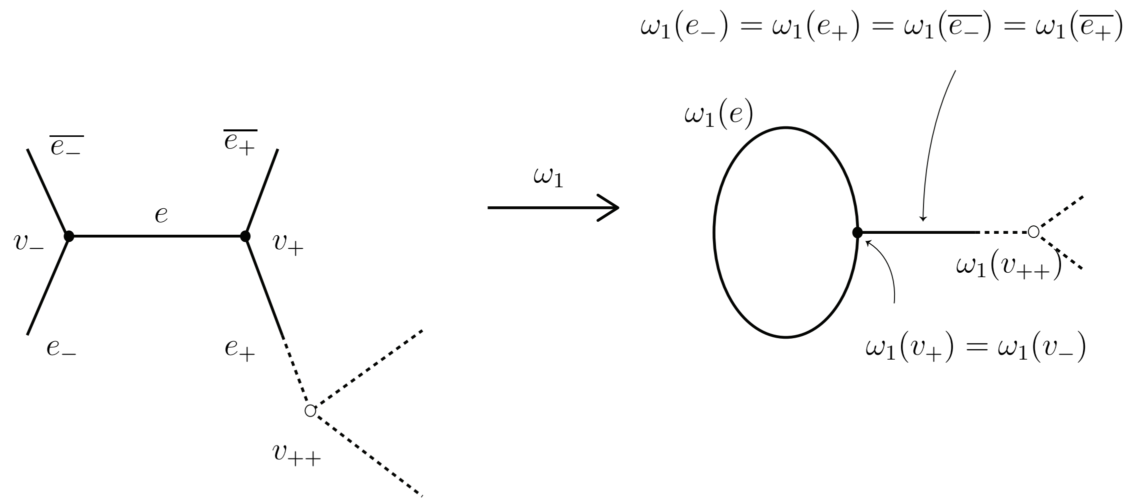


Figure 24: $\omega_1 : \chi_1 \to \Sigma$

If the active vertices v_+ and v_- are identified, then $e_-, \overline{e_-}, e_+$, and $\overline{e_+}$ are all identified and therefore $a_- = \overline{a_-} = a_+ = \overline{a_+}$. See Figure 24. So, χ_1 is the active space we need to consider. The map $\omega_1 : \chi_1 \to \Sigma$ is not injective. Furthermore, the singular point on the teardrop can only have weight $a = 2$ or 3 by The Classification of 2-orbifolds. If $a = 3$, then the singular set branches out from the vertex $\omega_1(v_+)$ to an edge of weight 2, and if $a = 2$, then the singular set branches out to an edge of weight greater than or equal to 2. We construct X to be the space $N(\text{left}) \cup (S^2(a) \times I) \cup N(\text{right})$, where both paths along the singular set to the right and to the left of $S^2(a) \times I$ meet at the same vertex, $\omega_1(v_+)$. We travel along the right and left paths simultaneously until we reach $\omega_1(v_+)$. Then both paths travel along the singular arc that branches out from this vertex (that has not been traversed) until we reach the next vertex. The next vertex will branch out into two edges if they are of the same weight, we stop here, and if they are of different weights, we travel further along the heavier weighted edge until we reach a vertex. Then we take the tubular neighborhood of this path. When X has this construction we say it has Form 4.

This results in the underlying topological space of X to be $(S^2 \times S^1)\backslash B^3$, and the boundary of X to have one component. If the two edges that branch out of $\omega_1(v_{++})$ are of the same weight, say weight n, then this component is a good football $S^2(n,n)$. If the two edges are of different weights a_1 and a_1^* where $a_1 < a_1^*$, then we traveled along the edge of weight a_1^* until it branches out to two edges of weights a_2 and a_3. Then the boundary component would be $S^2(a_1, a_2, a_3)$ and by Lemma 5.1 this is a spherical 2-orbifold. Hence, in either case ∂X has one component which is a spherical 2-orbifold.

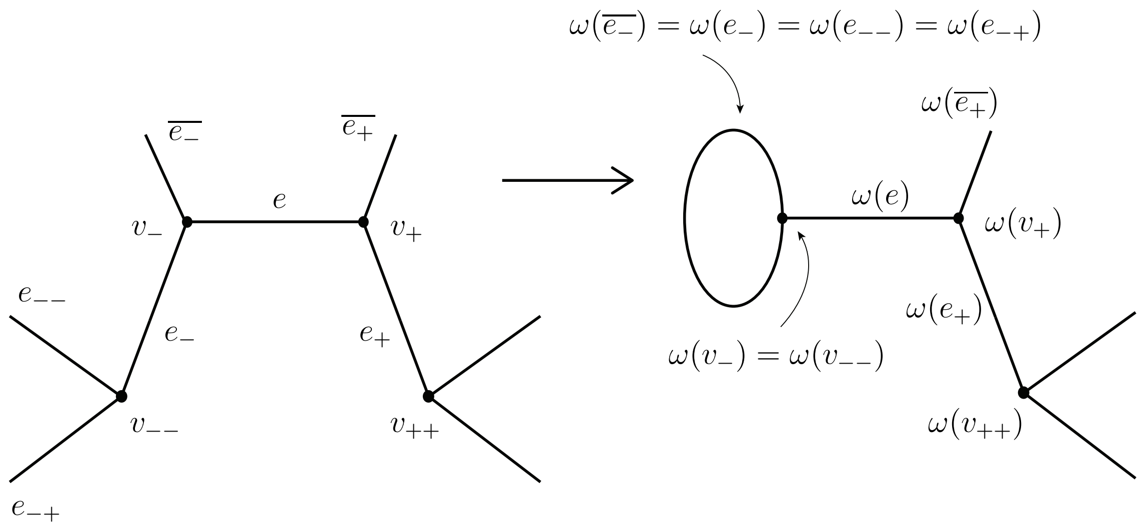


Figure 25: When v_- and v_{--} are identified we have $\overline{a_-} = a_-$ in Σ.

If v_- and v_{--} are identified then $\overline{e_-}, e_-, e_{--}$ and e_{-+} are all identified, and hence $\overline{a_-} = a_-$. So, if $\overline{a_+} = a_+$, then $\omega_1 : \chi_1 \hookrightarrow \Sigma$ and X has Form 1. If $\overline{a_+} \neq a_+$, then $\omega_2 : \chi_2 \hookrightarrow \Sigma$ and so X has Form 2. If v_- and v_{++}, where $\overline{a_-} = a_-$ and $\overline{a_+} = a_+$, then $\omega_1 : \chi_1 \hookrightarrow \Sigma$ and X has Form 1, see Figure 25. So we need not consider these as separate cases.

Case 4. The active vertices v_- and v_{++} are identified:

If the active vertices v_- and v_{++} are identified, we have that an inner-vertex is identified with an outer-vertex. See Figure 26. We could consider either χ_2 or χ_3 for the active space, as both spaces fail to inject into Σ. In particular, we could choose the identifications so that either $\overline{a_-} = a_-$ or $\overline{a_-} \neq a_-$, but $\omega_3 : \chi_3 \to \Sigma$ would have significantly more identifications compared to $\omega_2 : \chi_2 \to \Sigma$. Therefore, for simplicity we consider χ_2 as the active space. The map $\omega_2 : \chi_2 \to \Sigma$ is not injective. We could have either $\overline{a_-} \neq a_-$ or $\overline{a_+} \neq a_+$ by relabelling we assume that $\overline{a_+} \neq a_+$. We construct X in the following fashion and say that this is Form 5. As we travel along the singular set to the right of $S^2(a) \times I$, we will reach a vertex that branches out to two edges of different weights. Then we travel along the heavier weighted edge[4] until we reach another vertex (which is $\omega_2(v_{++})$), and then stop. For the left, we travel along the singular set to the left of $S^2(a) \times I$ until we reach a vertex (which is $\omega_2(v_-) = \omega_2(v_{++})$). This vertex is the same vertex that the right path terminated at, and it branches out to two edges of different weights. The heavier weighted edge is the edge that the right path traversed along, by assumption.

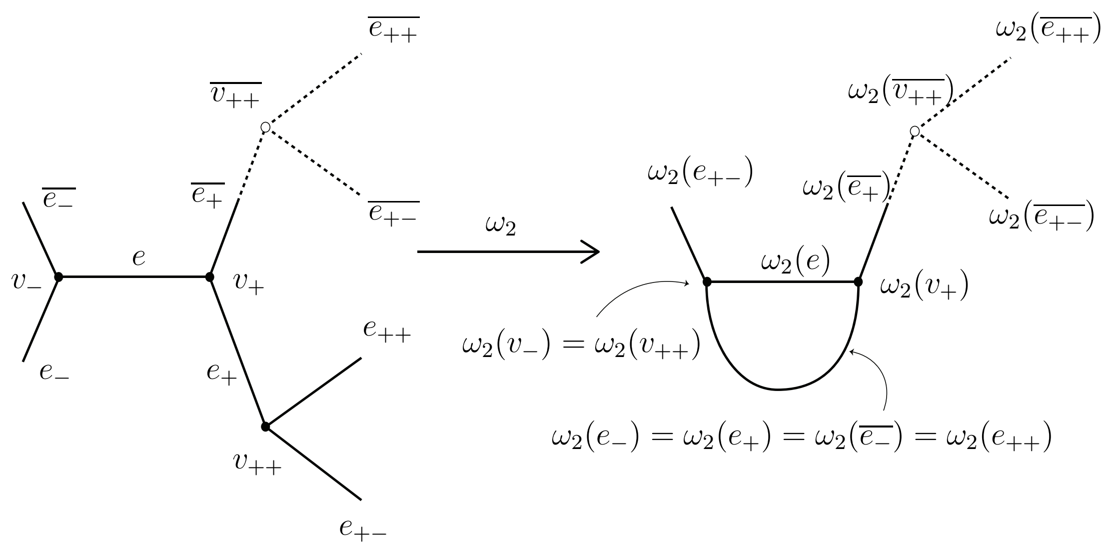


Figure 26: $\omega_2 : \chi_2 \to \Sigma$

If $a_{+-} = \overline{a_+}$, then the lighter weighted edges from the first vertices that both paths met

[4]Recall that by relabelling we assume $\overline{a_+} \leq a_+$, thus the heavier weighted edge is $\omega_2(e_+)$.

are of the same weight. If $a_{+-} \neq \overline{a_+}$, then the lighter weighted edges from the first vertices that both paths met are of different weights. So, we need to union another tubular neighborhood which we will denote as the path "up". Assume that $\overline{a_+} > a_{+-}$, and traverse from $\omega_2(v_+)$ along the heavier weighted edge $\omega_2(\overline{e_+})$ until we reach a vertex, as in Figure 20: Form 5. In both situations we take the tubular neighborhoods of these paths and X is the space $N(\text{left}) \cup (S^2(a) \times I) \cup N(\text{right})$, or $N(\text{left}) \cup (S^2(a) \times I) \cup N(\text{right}) \cup N(\text{up})$. For the former, ∂X is a good-football, and for the latter ∂X is a spherical turnover. Furthermore, the underlying topological space of X is $(S^2 \times S^1) \backslash B^3$.

For this particular case we can give specific values for the weights of the singular set in X when $a_{+-} \neq \overline{a_+}$. That is, the only possible teardrop is $S^2(2)$, and $a_{+-} = 2$, $\overline{a_+} = 3$, and $a_+ = 4$ or 5, assuming that $a_{+-} < \overline{a_+} < a_+$.

We recall that $\omega_2(\overline{e_+})$ terminates at the vertex $\omega_2(\overline{v_{++}})$, which branches out to two edges, where one edge must be 2 (as the sum of the reciprocals is strictly greater than 1), and the other edge is either 2, 3, 4, or 5. Therefore if we did not have $a_{+-} = 2$, and instead had $a_{+-} > 2$, then $\overline{a_+} > 3$, and so $a_+ > 4$, but there is no spherical 2-orbifold such that $S^2(a \geq 2, \overline{a_+} > 3, a_+ > 4)$. Therefore, the vertex $\omega_2(v_+)$ would not have a standard neighborhood. Thus, the above specified weights are the only possibilities for the space X when $a_{+-} \neq \overline{a_+}$.

Case 5. The active vertices v_{++} and v_{--} are identified:

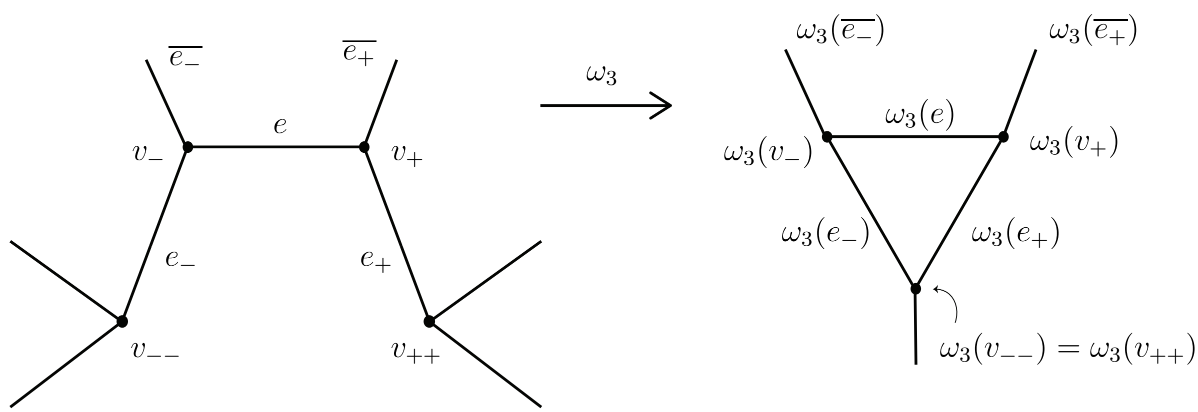
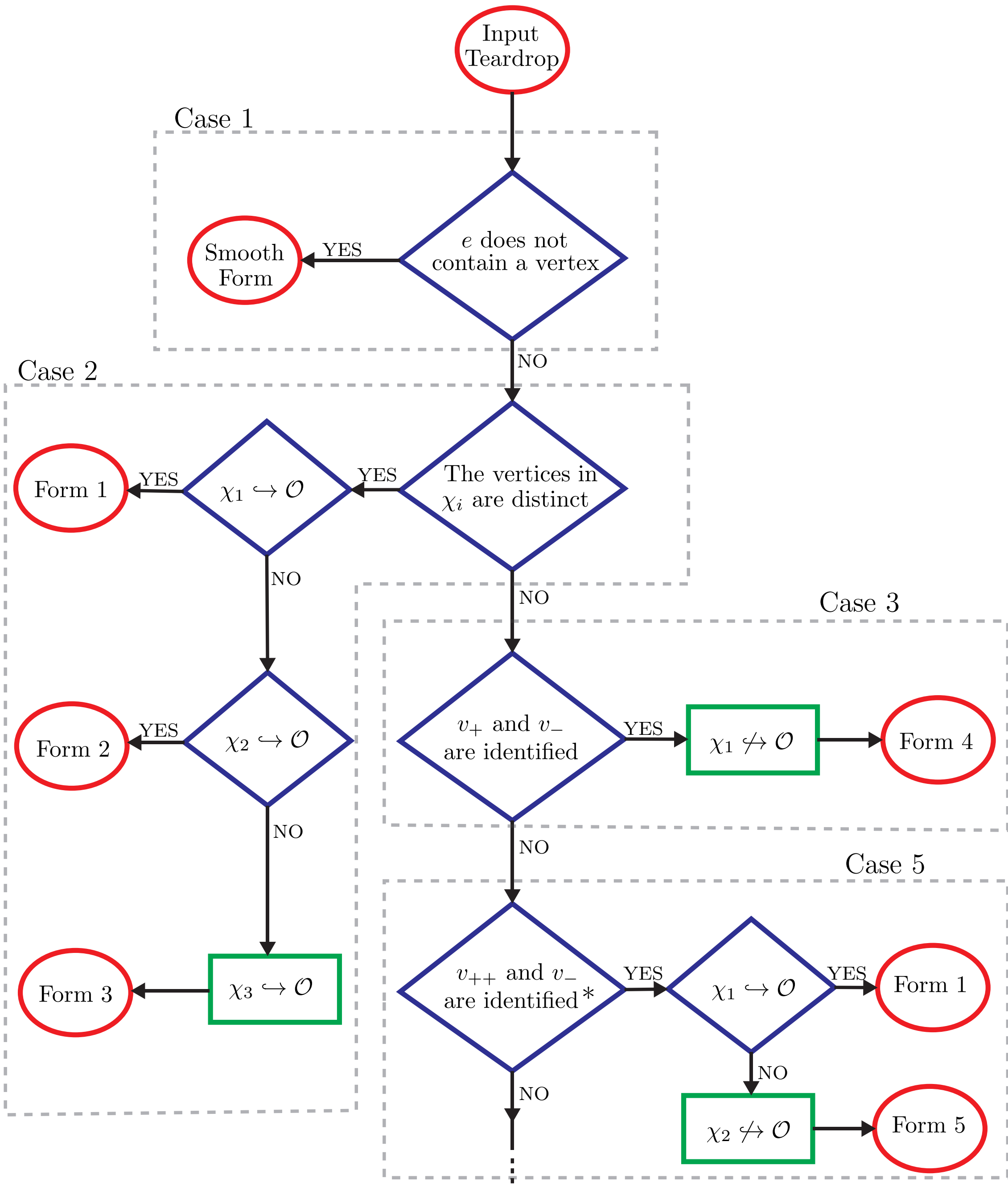


Figure 27: $\omega_3 : \chi_3 \to \Sigma$

If the active vertices v_{++} and v_{--} are identified, then χ_3 is the active space we need to consider. See Figure 27. The map $\omega_3 : \chi_3 \to \Sigma$ is not injective. Therefore, we have the weights $a_+ \neq \overline{a_+}$ and $a_- \neq \overline{a_-}$. Recall that we assume $a_+ > \overline{a_+}$ and $a_- > \overline{a_-}$. We construct X in the following fashion and say it has Form 6. If $\overline{a_+} \neq \overline{a_-}$, then without loss of generality, we assume $\overline{a_-} < \overline{a_+}$, and so $\overline{a_-} < \overline{a_+} < a_+$ and $\overline{a_-} < a_-$. So, there are only two possibilities for this bad orbifold $a_+ = 4$ or 5, where $a_- = 3, \overline{a_-} = 2, \overline{a_+} = 3$. If $\overline{a_-} > 2$ and $\overline{a_+} > 3$, then $a_+ > 4$, but this would cause v_+ to not have a standard neighborhood. Hence, this leaves only $S^2(2)$ as the possible teardrop. Now, if $\overline{a_-} = \overline{a_+}$, then we must have $\overline{a_-} = \overline{a_+} = 2, a_- = 3$, and $a_+ = 3, 4,$ or 5. Hence, if $a_+ = 3$, then the singular point on the teardrop could have weights $a = 2, 3, 4,$ or 5. If $a_+ = 4$ or 5, then $a = 2$ or 3. These are the only possible weights, because if $\overline{a_-} = \overline{a_+} > 2$, then $a_- > 3$ and $a_+ > 3$, but then $\omega_3(v_{++})$ would not have a standard neighborhood. Note that for this case we do not have $\overline{a_+}$ equal to a_+. Therefore, at vertex $\omega_3(v_+)$ the boundary of the standard neighborhood is not $S^2(2, 3, 3)$. This is indeed a spherical 3-suborbifold, but it would result in Form 2.

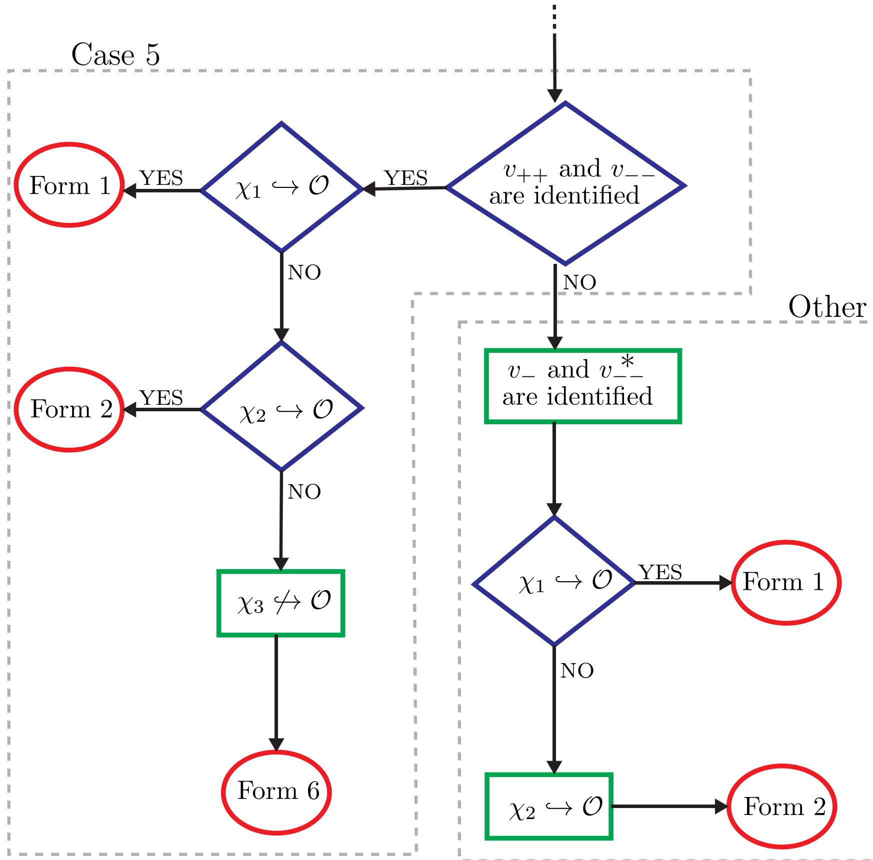
In Form 6 we travel along the singular set to the right of $S^2(a) \times I$. We will reach a vertex that branches out to two edges of different weights. The lighter edge will be of weight 2 or 3, and the heavier edge will be of either 3, 4, or 5. Now, we travel along the heavier weight until we reach another vertex and stop here for the right path. For the left, we travel along the singular set to the left of $S^2(a) \times I$, which has weight 3, until you reach a vertex. This vertex is the same vertex that the right path terminated at, and it branches out to two edges of different weights. The heavier weighted edge is the edge that the right path traversed along, and the lighter edge has weight 2. Then we take the tubular neighborhoods of these paths, and X is the space $N(\text{left}) \cup (S^2(a) \times I) \cup N(\text{right})$. Thus, the underlying topological space is $(S^2 \times S^1) \backslash B^3$ and ∂X is $S^2(2, 2, \overline{a_+})$, a spherical 2-orbifold where $\overline{a_+} = 2$ or 3.

□



**Since an inner-vertex is identified to an outer-vertex we need not consider* χ_3.

Figure 28: The flowchart (part 1) diagraming the proof of Theorem 6.2



**Since an inner-vertex is identified to an outer-vertex we need not consider* χ_3.

Figure 29: The flowchart (part 2) diagraming the proof of Theorem 6.2

7 Proof of Main Theorem

In this section we prove the main theorem, but before we prove that theorem we need to prove some preliminary statements that justify why this process causes the number of bad 2-suborbifolds to decrease.

Lemma 7.1. *Let Y be a 3-orbifold with spherical boundary and let $\hat{Y}$ be the 3-orbifold obtained from capping the boundary of Y. Then any collection of disjointly embedded 2-orbifolds in $\hat{Y}$ can be isotoped into Y.*

Proof. Consider only one boundary component at a time. The boundary component is one of S^2, $S^2(m,m)$, or $S^2(a_1,a_2,a_3)$, and the cap is the cone of the boundary component. Recall that 2-orbifolds cannot intersect the vertices of Σ. Therefore, if a collection of 2-orbifolds exists after filling we may isotope it out of the orbifold ball, and hence it must have existed before. $\square$

Let $\mathcal{O}_{i-1}$ be a closed, orientable, bad 3-orbifold (the index is chosen to coincide with indices below), and let $\mathcal{T}_{i-1} \subset \mathcal{O}_{i-1}$ be a maximal collection of disjointly embedded, non-parallel teardrops. Assume that $\mathcal{T}_{i-1}$ is non-empty. We know by Theorem 6.2, for any $T \in \mathcal{T}_{i-1} \subset \mathcal{O}_{i-1}$ there exists $X_i \supset T$ contained in $\mathcal{O}_{i-1}$. Let X_i be as in Theorem 6.2 and then cut and cap X_i from $\mathcal{O}_{i-1}$ to obtain $\mathcal{O}_i$.

Lemma 7.2. *The maximal number of disjointly embedded, non-parallel teardrops in $\mathcal{O}_i$ is strictly less than the maximal number of disjointly embedded, non-parallel teardrops in $\mathcal{O}_{i-1}$.*

Proof. Let $t_{\max}$ be the maximal number of disjointly embedded, non-parallel teardrops in $\mathcal{O}_{i-1}$. Suppose towards contradiction that $\mathcal{T}_i \subset \mathcal{O}_i$ is a collection of $t_{\max}$ disjointly embedded, non-parallel teardrops.

We may assume by Lemma 7.1, after isotopy $\mathcal{T}_i$ is disjoint from the cap(s). So, we have $\mathcal{T}_i \subset \mathcal{O}_{i-1} \backslash X_i$, and hence since $T \subset X_i$ we have

$$\mathcal{T}_i \cap T = \emptyset.$$

Therefore, $\mathcal{T}_i \cup T$ form $t_{\max} + 1$ disjointly embedded teardrops in $\mathcal{O}_{i-1}$. So either:

1. There exists $T^{'}, T^{''} \in \mathcal{T}_{i-1}$ that are parallel in $\mathcal{O}_{i-1}$.

   Therefore, $T^{'}$ and $T^{''}$ co-bound a product region, $P \subset \mathcal{O}_{i-1}$. Since $T^{'} \cap X_i, T^{''} \cap X_i = \emptyset$, either $X_i \subset P$ or $X_i \cap P = \emptyset$. We consider both cases.

   (a) $X_i \subset P$:

   By considering all of the forms of X_i from Theorem 6.2, we see that $\Sigma \cap X_i$ contains a simple closed curve (Smooth Form) or a vertex (all other forms). Therefore $X_i \not\subset P$, as these do not exists in a product region.

   (b) $X_i \cap P = \emptyset$:

   Then $P \subset \mathcal{O}_i$ showing that $T^{'}$ and $T^{''}$ are parallel in $\mathcal{O}_i$, contrary to our assumption.

   Therefore, case (1) cannot happen.

2. $T^{'}$ is parallel to T, for some $T' \in \mathcal{T}_{i-1}$.

   Let the component of Σ (i.e. the edge or simple closed curve) containing the orbifold point of T be denoted as e. By considering all the forms of X_i in Theorem 6.2 we know that $e \subset X_i$. As $T^{'}$ is parallel to T, the orbifold point of $T^{'}$ must also lie on e, but from Lemma 7.1, after isotopy we must have $\mathcal{T}_i \cap T = \emptyset$. Therefore, case (2) cannot happen.

So, we must have that the number of disjointly embedded, non-parallel teardrops in $\mathcal{T}_i$ is strictly less than those in $\mathcal{T}_{i-1}$. □

Suppose $\mathcal{O}_{n+i-1}$ does not contain a teardrop (the index is chosen to coincide with indices below) and let $\mathcal{F}_{n+i-1} \subset \mathcal{O}_{n+i-1}$ be a maximal collection of disjointly embedded, non-parallel bad-footballs. Assume that $\mathcal{F}$ is non-empty. We know by Theorem 5.2, for any $F \in \mathcal{F}_{n+i-1} \subset \mathcal{O}_{n+i-1}$ there exists $X_{n+i} \supset F$ contained in $\mathcal{O}_{n+i-1}$. Let X_{n+i} be as in Theorem 5.2 and then cut and cap X_{n+i} from $\mathcal{O}_{n+i-1}$ to obtain $\mathcal{O}_{n+i}$.

Lemma 7.3. *The maximal number of disjointly embedded, non-parallel bad-footballs in $\mathcal{O}_{n+i}$ is strictly less than the maximal number of disjointly embedded, non-parallel bad-footballs in $\mathcal{O}_{n+i-1}$. Furthermore, there does not exist a teardrop in $\mathcal{O}_{n+i}$.*

Proof. By Theorem 3.1 the number of disjointly embedded, non-parallel teardrops is finite, and by Lemma 7.2 as you cut and cap X_i the number of teardrops is strictly decreasing. Therefore, at some point there exists a 3-orbifold containing no teardrops. We suppose that $\mathcal{O}_{n+i-1}$ does not contain a teardrop and $\mathcal{O}_{n+i}$ is obtained from $\mathcal{O}_{n+i-1}$ by cut and cap X_i. Therefore, $\mathcal{O}_{n+i}$ also contains no embedded teardrops, for any $i \geq 0$.

For the remainder of the proof a similar argument to the proof of Lemma 7.2 may be used. Let $f_{\max}$ be the maximal number of disjointly embedded, non-parallel bad-footballs in $\mathcal{O}_{n+i-1}$. Suppose towards contradiction that $\mathcal{F}_{n+i-1} \subset \mathcal{O}_{n+i-1}$ is a collection of $f_{\max}$ disjointly embedded, non-parallel bad-footballs.

We may assume by Lemma 7.1, after isotopy $\mathcal{F}_{n+i}$ is disjoint from the cap(s). So, we have $\mathcal{F}_{n+i} \subset \mathcal{O}_{n+i-1} \backslash X_{n+i}$, and hence since $F \subset X_{n+i}$ we have

$$\mathcal{F}_{n+i} \cap F = \emptyset.$$

Therefore, $\mathcal{F}_{n+i} \cup F$ form $f_{\max} + 1$ disjointly embedded teardrops in $\mathcal{O}_{n+i-1}$. So either:

1. There exists $F', F'' \in \mathcal{F}_{n+i-1}$ that are parallel in $\mathcal{O}_{n+i-1}$.

   Therefore, F' and F'' co-bound a product region, $P \subset \mathcal{O}_{n+i-1}$ Since $F' \cap X_{n+i}, F'' \cap X_{n+i} = \emptyset$, either $X_{n+i} \subset P$ or $X_{n+i} \cap P = \emptyset$. We consider both cases.

(a) $X_{n+i} \subset P$:

By considering all of the forms of X_{n+i} from Theorem 5.2, we see that $\Sigma \cap X_{n+i}$ contains a simple closed curve (Smooth Form) or a vertex (all other forms). Therefore $X_{n+i} \not\subset P$, as these do not exists in a product region.

(b) $X_{n+i} \cap P = \emptyset$:

Then $P \subset \mathcal{O}_{n+i}$ showing that $F^{'}$ and $F^{''}$ are parallel in $\mathcal{O}_{n+i}$, contrary to our assumption.

Therefore, case (1) cannot happen.

2. $F^{'}$ is parallel to F, for some $F' \in \mathcal{F}_{i-1}$.

Let the component of Σ (i.e. the edges or simple closed curves) containing the orbifold points of F be denoted as e and f. By considering all of the forms of X_{n+i} in Theorem 5.2 we know that either e or f is contained in X_{n+i} (or both). As $F^{'}$ is parallel to F, the orbifold points of $F^{'}$ must also lie on e and f, but from Lemma 7.1, after isotopy we must have $\mathcal{F}_i \cap F = \emptyset$. Therefore, case (2) cannot happen.

So, we must have that the number of disjointly embedded, non-parallel bad-footballs in $\mathcal{F}_i$ is strictly less than those in $\mathcal{F}_{i-1}$. □

Proof of the Main Theorem. We construct n and $\mathcal{O}_i$ $(i = 0, \ldots, n)$ as follows:

Set $\mathcal{O}_0 = \mathcal{O}$ and assume we have constructed $\mathcal{O}_{i-1}$.

1. If $\mathcal{O}_{i-1}$ does not contain a teardrop, set $n = i - 1$.

2. If $\mathcal{O}_{i-1}$ does contain a teardrop, let X_i be as in Theorem 6.2 and then cut and cap X_i from $\mathcal{O}_{i-1}$ to obtain $\mathcal{O}_i$.

We claim that this process terminates. By Theorem 3.1 and Lemma 7.2 we see that this is true as there are finitely many teardrops, and that the number of teardrops is strictly decreasing after applying cut and cap of each X_i containing a teardrop. Therefore, $\mathcal{O}_n$ does not contain a teardrop.

We next construct m and $\mathcal{O}_{n+i}$ $(i = 1, \ldots, m)$ as follows:

Assume we have constructed $\mathcal{O}_{n+i-1}$.

1. If $\mathcal{O}_{n+i-1}$ does not contain a bad-football, set $m = i - 1$.

2. If $\mathcal{O}_{n+i-1}$ does contain a bad-football, let X_{n+i} be as in Theorem 5.2 and then cut and cap X_{n+i} from $\mathcal{O}_{n+i-1}$ to obtain $\mathcal{O}_{n+i}$.

We claim that this process terminates. By Lemma 7.3 and induction $\mathcal{O}_{n+i}$ does not contain any teardrops. By Theorem 3.1 in the absence of teardrops there are finitely many bad-footballs. By Lemma 7.3 we know that the number of bad-footballs is strictly decreasing after applying cut and cap of each X_{n+i} containing a bad-football. Therefore, $\mathcal{O}_{n+m}$ does not contain a teardrop or a bad-football, and thus is a good 3-orbifold.

$\square$

8 References

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