Topics in Gravitational Wave Physics

Aaron David Johnson

University of Arkansas, Fayetteville

Follow this and additional works at: https://scholarworks.uark.edu/etd

Part of the Algebra Commons, Cosmology, Relativity, and Gravity Commons, Instrumentation Commons, and the Plasma and Beam Physics Commons

Citation


This Dissertation is brought to you for free and open access by ScholarWorks@UARK. It has been accepted for inclusion in Theses and Dissertations by an authorized administrator of ScholarWorks@UARK. For more information, please contact ccmiddle@uark.edu.
Topics in Gravitational Wave Physics

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics

by

Aaron David Johnson
University of Central Arkansas
Bachelor of Science in Physics and Mathematics, 2014

July 2020
University of Arkansas

This dissertation is approved for recommendation to the Graduate Council.

Daniel Kennefick, PhD
Dissertation Director

Bret Lehmer, PhD
Yo’av Rieck, PhD
Committee Member
Committee Member

William Oliver III, PhD
Committee Member
Abstract

We begin with a brief introduction to gravitational waves. Next we look into the origin of the Chandrasekhar transformations between the different equations found by perturbing a Schwarzschild black hole. Some of the relationships turn out to be Darboux transformations. Then we turn to GW150914, the first detected black hole binary system, to see if the nonlinear memory might be detectable by current and future detectors. Finally, we develop an updated code for computing equatorial extreme mass ratio inspirals which will be open sourced as soon as it has been generalized for arbitrary inclinations.
Acknowledgements

I thank my advisor, Dan Kennefick for always being available and for always being encouraging. I thank Kostas Glampedakis for basically being a second advisor, and for helping me so much during my two weeks while in Spain. I also thank Dan, Julia Kennefick and their family and friends in Ireland: Andrew Sullivan, James O’Callaghan, Robert O’Leary, Eugene Hickey and Maura Kennefick and their families who all took care of me during my time abroad in Ireland. Further, I thank University College Cork and in particular Paul Callanan for giving me an office to work in while I was there. Finally, I thank the excellent graduate and undergraduate students that I have worked with while in graduate school at the University of Arkansas: Andrew Osborne, Daniel Oliver, Shasvath Kapadia, and Alex Hixon in no particular order.
Dedication

To my wife, Iris,

and my son, Arthur.
Contents

Overview of the Dissertation  1

1 Gravitational Waves: An Introduction  3

1.1 Line Elements and the Metric  3
1.2 Geodesic Equation and Geodesic Deviation  5
1.3 Linearized General Relativity  10

2 Darboux transformations in black hole perturbation theory  13

2.1 Introduction  13
2.2 Darboux transformations in physics  14
2.3 Classical Darboux transformation  14
  2.3.1 Deriving the Darboux transformation  14
  2.3.2 Alternate form for constraint equations  17
  2.3.3 Composition of Darboux transformations  19
  2.3.4 Simple examples of Darboux transformations  20
  2.3.5 Exact solutions by uniqueness  22
  2.3.6 Long ranged to short ranged potentials  24
2.4 Classical Darboux transformation in black hole perturbation theory  24
  2.4.1 Derivations of perturbation equations  25
  2.4.2 Useful results: perturbation equations  27
  2.4.3 Darboux Relations  28
2.5 Generalized Darboux transformation  29
  2.5.1 Coupled system form of constraint equations  31
  2.5.2 Special Darboux transformation  32
3 Prospects of Detecting the Nonlinear Gravitational Wave Memory

3.1 Abstract

3.2 Introduction

3.3 The nonlinear memory of GW150914

3.4 Finding the signal to noise ratio

3.4.1 SNR for GW150914-like events in current and future detectors

3.5 GW150914-like events with varying mass and distance

3.6 Conclusion

3.7 Acknowledgments

3.8 Calculation of the nonlinear memory

3.9 A note on the form of the memory and changing parameters

4 Signal Confusion Background from Extreme Mass Ratio Capture Sources

4.1 Introduction

4.1.1 Computation of gravitational radiation from EMRIs

4.1.2 Why Python?

4.2 Frequency based hypergeometric Kerr code
List of Figures

1.1 Two particles from a ring of particles as a plane, plus polarized gravitational wave passes perpendicular to the page. ........................................ 12

3.1 Two particles from a ring of particles as a plane, plus polarized gravitational wave passes perpendicular to the page. The nonlinear memory is shown here as a residual plus polarization after the wave has passed. ............................ 40

3.2 Numerical relativity waveform for GW150914 generated with PyCBC is shown in the upper plot. The memory is shown with optimal inclination in the lower plot with maximum amplitude even after the gravitational wave has passed. The “wiggliness” of the waveform has been discussed in Appendix C of [1] and is caused by disregarding the average after taking the time integral (see Appendix 3.8). .......................................................... 46

3.3 Comparison of the MWM in [2] with the approximation given in this paper. .......................................................... 46

3.4 Heuristic plot showing the memory signal and a noise curve using the characteristic strain convention. Arrows show the direction the memory moves when changing the total mass, M, and the distance from the source, d. .................. 56

3.5 Luminosity distance ($d_L$), total mass (M) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given ($d_L, M$) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Current and near future ground based detectors (2nd generation detectors) are not able to see memory from any released detection. ........................................ 57

3.6 Luminosity distance ($d_L$), total mass (M) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given ($d_L, M$) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Plots in the left column show SNR values out to 3000 Mpc, and plots on the right column show the edge of the $\rho = 3$ contour. Future ground based detectors (3rd generation detectors) offer significantly improved sensitivity and are able to see memory from several released detections. Although GW170817 is a BNS, Yang and Martynov [3] show that for various equations of state CE gives an SNR of about 10 and should be detectable. ........................................ 58
3.7 Luminosity distance \( (d_L) \), total mass \( (M) \) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given \( (d_L, M) \) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Plots in the left column show SNR values out to 3000 Mpc, and plots on the right column show the edge of the \( \rho = 3 \) contour or out to 30 Gpc. Space based detectors in the mHz frequency regime will be able to see SMBH binary memory with masses significantly higher than those that are visible in the ground based detectors. 

3.8 Luminosity distance \( (d_L) \), total mass \( (M) \) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given \( (d_L, M) \) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Plots in the left column show SNR values out to 3000 Mpc, and plots on the right column show out to 30 Gpc. Future space based detectors that are sensitive in the dHz regime have excellent sensitivity to the memory from all the released events. The BNS memory might be visible in these detectors. Further study is needed for varying equations of state as in [3].

3.9 Luminosity distance \( (d_L) \), total mass \( (M) \) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given \( (d_L, M) \) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Plots in the left column show SNR values out to 3000 Mpc, and plots on the right column show the edge of the \( \rho = 3 \) contour. PTAs are sensitive to the memory from the largest SMBH binary systems.

3.10 Luminosity distance \( (d_L) \), total mass \( (M) \) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given \( (d_L, M) \) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Plots in the left column show SNR values out to 3000 Mpc, and plots on the right column show the edge of the \( \rho = 3 \) contour or out to 30 Gpc. PTAs are sensitive to the memory from the largest SMBH binary systems.
Overview of the Dissertation

This work which has three somewhat disjoint topics is written in chronological order. We begin by discussing some mathematics which may be helpful to know in the introduction. Further, we discuss the linearized theory of general relativity and how that brings us to the mathematical prediction of gravitational waves.

In the second chapter we discuss the Darboux transformation, a transformation between second order ordinary differential equations which has been known to mathematicians for over a century. We begin by developing the theory mathematically, and then we look at it as it is used in black hole perturbation theory. The overall goal of this was to bring the computation of gravitational radiation from highly eccentric extreme mass ratio inspirals into a more tractable setting via direct integration of the Teukolsky equation after using a Darboux type transformation. While this ambitious goal was not achieved, we found a profound theory connected to almost all of modern physics.

In 2015 LIGO directly detected the first gravitational wave (GW150914). This has created the field of gravitational wave astronomy and caused a wave of discoveries almost every year since. The first binary black hole detection consisted of larger masses than several papers in the 2000s had discussed. This raised the question of whether a nonlinear effect that is about an order of magnitude smaller than the oscillatory gravitational wave could be detectable. This non-oscillatory effect is known as the nonlinear memory which is caused by the oscillating mass-energy of the primary wave and results in a nonzero displacement between freely falling particles after the wave has passed.

In the final chapter we return to the goal of the first chapter: making a code that can
compute highly eccentric orbits, but now we approach from a different direction. This time we worked to update our group’s old eccentric equatorial Kerr code with advances made over the past ten years. Instead of integrating the transformed homogeneous Teukolsky equation, we use a series solution of hypergeometric functions. This new code is more accurate, faster at high eccentricities, and able to be generalized to generic orbits.
Chapter 1

Gravitational Waves: An Introduction

General relativity (GR), like other theories in physics uses vectors and tensors to describe what happens in the world. The mathematical spaces which we use have been promoted to a manifold which can be thought of as being locally like the real space we are used to, but may have non-vanishing curvature globally.

For example, we may be on the surface of a large sphere, but believe that our sphere is like flat 2D space because the curvature is not visible in our everyday lives. So locally, this is like a Cartesian space, but globally, it clearly follows different rules as we eventually will intersect our path if we continue along far enough. However, in GR we will not have the privilege of being able to move exterior to our own spacetime to tell if it is (intrinsically or extrinsically) flat or curved as we would be able to when going outside the Earth to see that our own world is curved. Fortunately, while this is the easiest way to see the curvature, this is not the only evidence either on Earth or for the universe.

In this section, I will go over some of the mathematical concepts, peculiar to relativity theory, which will aid comprehension of this thesis, though I have made every attempt to make each individual chapter self-contained. This chapter follows discussions in [4–7]

1.1 Line Elements and the Metric

In standard three dimensional Euclidean space, the Cartesian coordinate ($L^3$ norm) distance between objects is

$$ds^2 = dx^2 + dy^2 + dz^2.$$  \hspace{1cm} (1.1)
We may rewrite this in terms of a metric

\[ ds^2 = \delta_{ij} dx^i dx^j \]  

(1.2)

where we are using Einstein summation notation in which we have an understood summation when an upper and lower index are repeated. The latin indices throughout this work will denote an index that runs from 1 to 3, while a greek index will denote an index that runs from 0 to 3. Further, 0 corresponds to the temporal index and 1 through 3 are the spatial indices. For Euclidean space, the metric is just the Kronecker delta function:

\[
\delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j.
\end{cases}
\]

In special relativity (SR), we have (with c=1),

\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \]  

(1.3)

which is just

\[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \]  

(1.4)

where \( \eta_{\mu\nu} = \text{diag}(-1,1,1,1) \). Finally, for general relativity, we no longer have a static spacetime. We now have a metric which may vary based on location or time. For this the line element is,

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \]  

(1.5)

where \( g_{\mu\nu} \) is the metric that satisfies the Einstein field equations (EFE). Metrics have a matrix representation that can be useful in some computations. Throughout this work, we will typically use the index notation as is common in GR. As we are going to be dealing
with massive objects traveling slower than the speed of light, we will often find use for the proper time rather than the proper distance:

\[ d\tau^2 = -ds^2 = -g_{\mu\nu}dx^\mu dx^\nu \]  

(1.6)

1.2 Geodesic Equation and Geodesic Deviation

For a particle traveling slower than the speed of light, we can use the variational principle and extremize the proper time

\[ S = -m \int_{\tau_A}^{\tau_B} d\tau \]  

(1.7)

to yield a classical trajectory of a particle with mass \( m \). Let \( d\tau = (-g_{\mu\nu}dx^\mu dx^\nu)^{1/2} \), then

\[ \delta S = -m \int_{\tau_A}^{\tau_B} \frac{1}{2} \left(-g_{\mu\nu}dx^\mu dx^\nu\right)^{1/2} \left(-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\right) d\lambda. \]  

(1.8)

We can treat the variation operation as a derivative operator that commutes with all other total derivatives and require that the variation in our action go to zero (as we would in any other calculus of variations problem). This means that

\[ \delta S = -m \int_{\tau_A}^{\tau_B} \frac{1}{2} \left(-g_{\mu\nu} dx^\mu dx^\nu\right)^{-1/2} \left(-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\right) d\lambda, \]  

(1.9)

by the chain rule. If we choose an affine parameterization that amounts to picking \( \lambda = \tau \) (in the particles own rest frame), we find

\[ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 1. \]  

(1.10)
Therefore,

\[
\delta S = -\frac{m}{2} \int_{\tau_A}^{\tau_B} \delta \left( -g_{\mu \nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) d\tau \\
= \frac{m}{2} \int_{\tau_A}^{\tau_B} \left( \partial_{\nu} g_{\mu \sigma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - g_{\mu \nu} \frac{d^2 x^\nu}{d\tau^2} - \frac{d g_{\mu \nu}}{d\tau} \frac{d x^\mu}{d\tau} \frac{d x^\nu}{d\tau} \right) d\tau \\
= \frac{m}{2} \int_{\tau_A}^{\tau_B} \left( \partial_{\nu} g_{\mu \sigma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu \nu} \frac{d^2 x^\nu}{d\tau^2} + g_{\mu \nu} \frac{d x^\mu}{d\tau} \frac{d x^\nu}{d\tau} \right) d\tau. 
\]

(1.11)

(1.12)

(1.13)

Now, we can use integration by parts on the last two terms. That is,

\[
g_{\mu \nu} \frac{d\delta x^\mu}{d\tau} \frac{d x^\nu}{d\tau} = \frac{d}{d\tau} \left( g_{\mu \nu} \frac{d x^\nu}{d\tau} \frac{d x^\mu}{d\tau} \right) - g_{\mu \nu} \frac{d x^\nu}{d\tau} \frac{d^2 x^\mu}{d\tau^2} - \frac{d g_{\mu \nu}}{d\tau} \frac{d x^\mu}{d\tau} \frac{d x^\nu}{d\tau} \\
= - \left( g_{\mu \nu} \frac{d^2 x^\nu}{d\tau^2} + \partial_{\sigma} g_{\mu \nu} \frac{d x^\sigma}{d\tau} \frac{d x^\nu}{d\tau} \right) \delta x^\mu
\]

(1.14)

(1.15)

where in the last line we have dropped the boundary term which will go to zero when integrated. This is not valid in all theories, but is fine in the situations that we will be dealing with in general relativity. For a discussion of this, see [5]. After simplifying further, we find

\[
\delta S = \frac{m}{2} \int_{\tau_A}^{\tau_B} \left( \partial_{\nu} g_{\mu \sigma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \left( g_{\mu \nu} \frac{d^2 x^\nu}{d\tau^2} + \partial_{\sigma} g_{\mu \nu} \frac{d x^\sigma}{d\tau} \frac{d x^\nu}{d\tau} \right) \delta x^\mu \right. \\
\left. - \left( g_{\mu \nu} \frac{d^2 x^\nu}{d\tau^2} + \partial_{\sigma} g_{\mu \nu} \frac{d x^\sigma}{d\tau} \frac{d x^\nu}{d\tau} \right) \delta x^\nu \right) d\tau \\
\delta S = \frac{m}{2} \int_{\tau_A}^{\tau_B} \left( \partial_{\nu} g_{\mu \sigma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \left( \partial_{\sigma} g_{\mu \nu} \frac{d x^\sigma}{d\tau} \frac{d x^\nu}{d\tau} \right) \delta x^\sigma \right. \\
\left. - \left( 2 g_{\mu \sigma} \frac{d^2 x^\nu}{d\tau^2} + \partial_{\nu} g_{\mu \sigma} \frac{d x^\sigma}{d\tau} \frac{d x^\nu}{d\tau} \right) \delta x^\sigma \right) d\tau \\
\delta S = \frac{m}{2} \int_{\tau_A}^{\tau_B} \left[ \left( \partial_{\nu} g_{\mu \sigma} - \partial_{\mu} g_{\sigma \nu} - \partial_{\nu} g_{\mu \sigma} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - 2 g_{\mu \sigma} \frac{d^2 x^\nu}{d\tau^2} \right] \delta x^\sigma d\tau \\
\delta S = -m \int_{\tau_A}^{\tau_B} \left[ \frac{1}{2} g^{\alpha \sigma} \left( \partial_{\nu} g_{\sigma \nu} + \partial_{\nu} g_{\mu \sigma} - \partial_{\sigma} g_{\mu \nu} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \frac{d^2 x^\nu}{d\tau^2} \right] \delta x^\sigma d\tau
\]

(1.16)

(1.17)

(1.18)

(1.19)
Which when the variation vanishes, requires that

\[
\frac{d^2 x^\alpha}{d\tau^2} + \frac{1}{2} g^{\alpha\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0.
\]

(1.20)

This equation is known as the geodesic equation. A massive particle that follows this equation will extremize the proper time, and it will therefore be on an extremized path through spacetime. We can identify a piece of this equation that is ubiquitous in canonical GR: the Christoffel connection,

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}).
\]

(1.21)

This is a part of the parallel transport operator, the covariant derivative,

\[
\nabla_\mu V^\rho = (\partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu).
\]

(1.22)

By rewriting the geodesic equation with the covariant derivative, the geodesic equation can be written as

\[
U^\mu \nabla_\mu U^\nu = 0,
\]

(1.23)

where \(U^\mu = dx^\mu/d\tau\). This is a parallel transport of the four velocity along the direction (by contraction with the covariant derivative – essentially a dot product) of the four velocity.

When we move a vector across the surface of our manifold, the way in which the vector moves depends on the path we take. For example, moving on the geodesics of a sphere we can move from the north pole to the equator, then follow the equator around to the diametrically opposite side, and then back to the north pole. By doing so, we end up with vectors facing different directions. The extent to which the different parts fail to commute
may be found by the commutator,

\[
[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho
\] (1.24)

So we have

\[
\nabla_\mu \nabla_\nu V^\rho = \partial_\mu (\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} (\nabla_\lambda V^\rho) + \Gamma^\rho_{\mu\sigma} (\nabla_\nu V^\sigma)
\] (1.25)

where the first term is

\[
\partial_\mu (\nabla_\nu V^\rho) = \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\rho + \Gamma^\rho_{\nu\sigma} \partial_\mu (V^\rho)
\] (1.26)

and the last term is

\[
\Gamma^\rho_{\mu\sigma} (\nabla_\nu V^\rho) = \Gamma^\rho_{\mu\sigma} \partial_\nu v^\rho + \Gamma^\rho_{\mu\sigma} \Gamma^\alpha_{\nu\sigma} V^\alpha.
\] (1.27)

Any terms that are symmetric \((A_{\mu\nu} = A_{\nu\mu})\) will be zero when we evaluate the commutator.

In canonical general relativity, the Christoffel connection is symmetric in its lower indices. This means the second term (which in other theories would be related to the torsion tensor) is zero in GR. Since partial derivatives commute, \(\partial_\mu \partial_\nu V^\rho\) is symmetric on the indices \(\mu\) and \(\nu\). So this term also cancels out. Once we simplify the commutator, we end up with

\[
\nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho = (\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\alpha} \Gamma^\alpha_{\nu\sigma} - \Gamma^\rho_{\nu\alpha} \Gamma^\alpha_{\mu\sigma}) V^\sigma
\] (1.28)

\[
= R^\rho_{\sigma\mu\nu} V^\sigma = -R_{\mu\nu\sigma}^\rho V^\sigma
\] (1.29)

where \(R^\rho_{\sigma\mu\nu}\) is the Riemann curvature tensor, and the last line follows from symmetry of swapping the first two and last two indices and antisymmetry of swapping either the first two indices or the last two indices (we have swapped the last two here).

The Riemann curvature can tell us how geodesics deviate from other nearby geodesics.
To do so, we consider two freely falling particles following nearby geodesics. We define two vector fields: the tangent to the geodesic parameterized by $t$,

$$T^\mu = \partial_t x^\mu,$$  \hspace{1cm} (1.30)

and the deviation vectors parameterized by $s$,

$$S^\mu = \partial_s x^\mu.$$  \hspace{1cm} (1.31)

Now we define a vector related to how fast we move away from being a geodesic,

$$V^\mu = T^\rho \nabla_\rho S^\mu,$$  \hspace{1cm} (1.32)

and a vector related to how fast we accelerate away from being a geodesic,

$$A^\mu = T^\rho \nabla_\rho V^\mu.$$  \hspace{1cm} (1.33)

$S$ and $T$ are basis vectors, and therefore their commutators must vanish

$$[S, T] = 0.$$  \hspace{1cm} (1.34)

Using the commutator and plugging the velocity into the acceleration definition (and using integration by parts along with the geodesic equation), we find

$$A^\mu = R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma.$$  \hspace{1cm} (1.35)

Returning to the Riemann tensor, we further can find the Ricci tensor and curvature scalar
by contracting on the indices of the Riemann tensor

\[ R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \quad (1.36) \]

\[ R = g^{\mu\nu} R_{\mu\nu}. \quad (1.37) \]

From this we can calculate the Einstein field equation

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} \quad (1.38) \]

where \( T_{\mu\nu} \) contains all mass-energy terms. On the left side of the equation, we have all the information about the curvature of spacetime. Particles will follow the path set by this curvature, and in turn modify it according to their own mass-energy movement. John Wheeler once said, “mass tells spacetime how to curve, and spacetime tells mass how to move.” This phrase embodies the content of the Einstein field equation.

### 1.3 Linearized General Relativity

Now we will use what we have discussed in the previous section to compute and simplify the linearized theory of general relativity. The end result will be a statement about what gravitational waves look like. We shall have no doubt that they are real as they have been directly detected by LIGO along with electromagnetic counterparts\(^1\). Far from the source of gravitational radiation, we treat the gravitational waves as perturbations on the flat spacetime of special relativity,

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (1.39) \]

\(^1\)Although perhaps we should have had no concerns anyways because of the Hulse-Taylor binary system for which a Nobel prize was awarded.
where \(|h_{\mu\nu}| \ll 1\). That is, the perturbation is a very small deviation in the flatness of the background spacetime. Our definition for the metric implies that the inverse metric is

\[
g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} + \mathcal{O}(h^2). \tag{1.40}
\]

It is clear that

\[
g_{\mu\nu}g^{\lambda\nu} = \delta^\lambda_\mu + \mathcal{O}(h^2) \tag{1.41}
\]

Now we compute the Christoffel connection, \(\Gamma^\rho_{\mu\nu}\).

\[
\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \tag{1.42}
\]

Upon plugging in the Minkowski metric (\(\eta_{\mu\nu}\)) plus perturbation,

\[
\Gamma^\rho_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}) + \mathcal{O}(h^2). \tag{1.43}
\]

Next, we can find the Riemann curvature tensor

\[
R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho} \Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma} \Gamma^\alpha_{\nu\rho}. \tag{1.44}
\]

From above, we can see that both \(\Gamma\Gamma\) terms are \(\mathcal{O}(h^2)\), and

\[
R_{\nu\rho\sigma} = \frac{1}{2}(\partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\nu h_{\mu\rho} - \partial_\rho \partial_\mu h_{\nu\sigma} - \partial_\sigma \partial_\nu h_{\mu\rho}) \tag{1.45}
\]

To simplify this further, we now move our equations into the Lorenz gauge. This is analogous to the transformation typically used in electricity and magnetism. We define \(h = h_{\mu\nu}\eta^{\mu\nu}\) which is the trace of the metric perturbation. Further, we define a trace reduced metric as

\[
\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{1.46}
\]
with the trace $h = \eta^{\mu\nu}h_{\mu\nu}$. Upon using this in the Einstein field equation and doing a straightforward calculation,

$$\square \tilde{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\lambda \tilde{h}_{\alpha\lambda} - \partial_\mu \partial^\lambda \tilde{h}_{\lambda\nu} - \partial^\lambda \partial_\alpha \tilde{h}_{\lambda\mu} = -16\pi G T_{\mu\nu}$$  (1.47)

which still looks pretty complicated! Fortunately, we have not used up all our gauge freedoms. First we’ll look far from any sources where $T_{\mu\nu} = 0$. Next, we transform to the transverse traceless gauge where $\tilde{h} = h = 0$ (traceless) and $h^0\mu = 0$, $h^i\mu = 0$, $\partial^i h_{ij} = 0$. This cannot be used in the presence of a source term. Finally, we use this in the Einstein equation to find

$$\square h_{ij}^{TT} = 0.$$  (1.48)

Such an equation clearly has a plane wave solution. If we take $z$ to be the direction of propagation, then

$$h_{ij}^{TT} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos (\omega t - kz)$$  (1.49)

where $h_+$ and $h_\times$ are the individual gravitational wave polarizations. These polarizations can be pictured by using the geodesic deviation on a ring of particles. Doing so yields the motion in Figure 1.1 below for the plus polarization. The cross polarization will be rotated 45 degrees compared to this motion.

Figure 1.1: Two particles from a ring of particles as a plane, plus polarized gravitational wave passes perpendicular to the page.
Chapter 2

Darboux transformations in black hole perturbation theory

This chapter is based in part on the work published in Physical Review D [8] along with significant additional work.

2.1 Introduction

While considering different perturbations of the spacetime of a supermassive black hole (SMBH), one finds different potentials appear for a spherically symmetric black hole depending on how it was perturbed. However, it turns out that there is a transformation between all of these potentials. This was first discovered by Subrahmanyan Chandrasekhar (usually known as “Chandra”) in the mid 1970s [9]. When Daniel Kennefick visited Chandrasekhar for a history project\(^1\), Chandrasekhar noted how odd the connection between these different types of perturbations was. These transformations have been shown to disappear when generalized to higher dimensions or many alternative theories. Instead of garnering interest in the underlying mathematical theory the transformations were often used to go from a potential that is hard to compute to one that can be computed more easily. Chandrasekhar saw this as a “cheap trick” that did not penetrate into what he felt could be mathematically profound. In this chapter, we will investigate the mathematical mystery while using the “cheap trick” of computational speed increase as motivation for such studies. We further consider all functions to be sufficiently well behaved that we can avoid rigor which may be necessary otherwise.

\(^1\)Unfortunately no recordings exist for this interview. Chandra was in a car accident the morning on which the interview took place, and Dan felt it would not be okay to put him under unnecessary stress.
2.2 Darboux transformations in physics

It turns out that this transformation is not new to physics! Darboux transformations go by many different names and have been rediscovered multiple times throughout the last 50 years. Some names that this transformation has taken include:

1. shape invariance (supersymmetry) (see [10])
2. ladder operators (see [11])
3. the factorization method (see [12])

among others. First and foremost, it is a relationship between ordinary differential equations of second order that allows us to transform from one potential to the next.

2.3 Classical Darboux transformation

Before we discuss its connection to black hole perturbation theory, let’s begin by looking at what a Darboux transformation looks like and what we can do with it. The form in which we will look at the Darboux transformation is somewhat different than the form seen by mathematicians, but the content is the same.

2.3.1 Deriving the Darboux transformation

Starting with a differential equation of the form

\[ y'' + (\lambda - U(x)) y = y'' + q(x)y = 0, \quad (2.1) \]
where a prime denotes differentiation with respect to $x$ and $\lambda$ is an eigenvalue. We aim to go to a new equation of the same type but with a different potential,

$$Y'' + (\lambda - V(x)) Y = Y'' + Q(x)Y = 0. \tag{2.2}$$

We start by letting

$$Y = y' + f(x)y. \tag{2.3}$$

This is the classical Darboux transformation (DT). Everything that follows is to find constraint equations. We now take a derivative,

$$Y' = y'' + f' y + fy', \tag{2.4}$$

and use Eq. (2.1) so that

$$Y' = -q y + f' y + fy'. \tag{2.5}$$

Taking a second derivative,

$$Y'' = -(qy)' + f'' y + 2f' y' + fy'', \tag{2.6}$$

and once again using Eq. (2.1),

$$Y'' = -qy' - qy' + f'' y + 2f' y' - fy'. \tag{2.7}$$

Collecting common coefficients in terms of $y$ and its derivatives

$$Y'' = (f'' - f q - q') y + (2f' - q) y'. \tag{2.8}$$
By using Eq. (2.3), we solve for $y'$ and plug it into this equation

$$Y'' = (f'' - fq - q')y + (2f' - q)(Y - fy) \quad (2.9)$$

$$= (f'' - q' - 2f'f)y + (2f' - q)Y \quad (2.10)$$

$$= (f' - q - f^2)'y + (2f' - q)Y \quad (2.11)$$

Moving the right side of the equation to the left side, we can compare this equation with Eq. (2.2). Therefore, we require that

$$Q = q - 2f'. \quad (2.12)$$

and that the $y$ term goes to zero,

$$0 = (f^2 + q - f')' \quad (2.13)$$

$$-c = f^2 + q - f', \quad (2.14)$$

which is a Riccati equation in terms of $f$. Even though there is no known solution method, there is a trick – known colloquially as the “logarithmic derivative trick” – that can be used to put this equation into a more familiar form. Let $f = -u'/u$,

$$0 = \left(\frac{u'}{u}\right)^2 + (q + c) - \left(\frac{u'}{u}\right)' = \left(\frac{u''}{u}\right) + q + c \quad (2.15)$$

Multiplying the last piece by $u$, we find the standard form,

$$u'' + (q + c)u = 0. \quad (2.16)$$

**Summary**

$$y'' + q(x)y = 0 \quad \overset{Y = y' + f(x)y}{\longrightarrow} \quad Y'' + Q(x)Y = 0 \quad (2.17)$$
is the classical Darboux transformation (DT) with constraint equations

\[
\begin{align*}
Q &= q - 2f'' \\
f^2 + q - f' &= -c \\
\text{if } f = -u'/u &\Rightarrow u'' + (q + c)u = 0
\end{align*}
\] (2.18)

### 2.3.2 Alternate form for constraint equations

The previous section’s equations are difficult to solve. Given one of the variables, the other two variables are completely specified. In the event that we suspect two potentials are Darboux related, we can put the constraint equations into a form that we can immediately check. We first solve Eq. (2.12) for \(Q\) in terms of \(f'\),

\[
f' = -\frac{1}{2}(Q - q).
\] (2.19)

Putting this equation into Eq. (2.14) and differentiating,

\[
2ff' + q' + \frac{1}{2}(Q - q)' = 2ff' + \frac{1}{2}(Q + q)' = 0,
\] (2.20)

Again using \(f'\), we can solve for \(f\),

\[
2f\left(\frac{1}{2}(Q - q)\right) + \frac{1}{2}(Q + q)' = 0
\Rightarrow f = \frac{(Q + q)'}{2(Q - q)}.
\] (2.21)
(2.22)

From these constraints, we can easily see (by interchanging \(q\) and \(Q\)) that to invert the DT, we just put a minus sign in front of \(f\). We can require compatibility between these constraint equations to find an equation only relating the initial and final potentials,

\[
(f)' = f' \\
\left(\frac{(Q + q)'}{2(Q - q)}\right)' = -\frac{1}{2}(Q - q)
\] (2.23)
(2.24)
An interesting property of these new constraint equations is that they do not depend on the eigenvalues of the original equations. That is,

\[ f' = -\frac{1}{2} (\lambda - V(x) - (\lambda - U(x))) = -\frac{1}{2} (V - U) \quad (2.25) \]

\[ f = \frac{(\lambda - V(x) + \lambda - U(x))'}{2(\lambda - V(x) - (\lambda - U(x)))} = \frac{(V + U)'}{2(V - U)}. \quad (2.26) \]

This means that as long as \( U \) and \( V \) are not functions of \( \lambda \), then the DT is **isospectral**, meaning that the equations before and after the transformations have the same eigenvalues.

**Summary**

\[ y'' + q(x)y = 0 \quad \Rightarrow \quad Y'' + Q(x)Y = 0 \quad (2.27) \]

is the classical Darboux transformation (DT) with constraint equations,

\[ f' = -\frac{1}{2} (Q - q) \quad (2.28) \]

\[ f = \frac{(Q + q)'}{2(Q - q)} \]

which clearly show the **isospectral** property of the DT when written in terms of \( U \) and \( V \),

\[ f' = -\frac{1}{2} (V - U) \quad (2.29) \]

\[ f = \frac{(V + U)'}{2(V - U)} \]

and where \( U \) and \( V \) are not functions of \( \lambda \). As can be seen from these constraint equations, the inverse of the DT is

\[ Y = y' - f(x)y \quad (2.30) \]

Requiring compatibility between Eq. (2.28) leads to

\[ \left( \frac{(Q + q)'}{2(Q - q)} \right)' = -\frac{1}{2} (Q - q) \quad (2.31) \]

which when satisfied shows a DT exists.
2.3.3 Composition of Darboux transformations

Going from

\[ y'' + q(x)y = 0 \quad \xrightarrow{Y = y' + f(x)y} \quad Y'' + Q(x)Y = 0 \quad \xrightarrow{\psi = Y' + g(x)Y} \quad \psi'' + \gamma(x)\psi = 0 \]

(2.32)

by means of the DT’s over the arrows suggests that perhaps we can go directly from the first to the last of these equations with a DT which is a composition of the two transformations. The composition can be found by plugging the first transformation into the second, thereby relating \( \psi \) to \( y \),

\[
\psi = (y' + fy)' + g(y' + fy) \\
= (y'' + f'y + fy') + gy' + gfy \\
= y'' + (f + g)y' + (f' + gf)y. 
\]

(2.33)

(2.34)

(2.35)

Using \( y'' = -q(x)y \),

\[
\psi = (f + g)y' + (f' + gf - q)y
\]

(2.36)

By deriving the composition of two DT’s, we find that the result is not a transformation of the Darboux type. Therefore, we can create a generalized Darboux transformation (GDT) which may not preserve isospectrality in general (unless it can be decomposed into DTs).

**Summary**  A composition of two classical Darboux transformations can be reduced to a single transformation of the form

\[
\psi = \alpha y' + \beta y 
\]

(2.37)
where
\[ \alpha = f + g \quad \beta = f' + gf - q \] (2.38)

2.3.4 Simple examples of Darboux transformations

Simplest case  This method can be used to generate a new potential from a potential with a known solution. Let \( q = \omega^2 \) be the initial potential, and let \( Q = \omega^2 - V \) where \( V \) is the eigenvalue independent potential function to be found. Plugging these into Eq. (2.31),

\[
\left( \frac{(2\omega^2 - V)'}{2(-V)} \right)' = -\frac{1}{2}(-V) \tag{2.39}
\]
\[
\left( \frac{V'}{V} \right)' = V \tag{2.40}
\]
\[
\frac{VV'' - (V')^2}{V^2} = V \tag{2.41}
\]
\[
V'' - \frac{(V')^2}{V} - V^2 = 0. \tag{2.42}
\]

The solution to this equation can be found with Mathematica as

\[
V(x) = -\frac{1}{2}c_1 \text{sech}^2 \left( \frac{1}{2} \sqrt{c_1} (x + c_2) \right). \tag{2.43}
\]

From this,

\[
f(x) = -\sqrt{c_1} \tanh \left( \frac{1}{2} \sqrt{c_1} (x + c_2) \right), \tag{2.44}
\]

which completely specifies the transformation. This transformation takes us from

\[
y'' + \omega^2 y = 0 \tag{2.45}
\]

to

\[
y'' + \left( \omega^2 + \frac{1}{2}c_1 \text{sech}^2 \left( \frac{1}{2} \sqrt{c_1} (x + c_2) \right) \right) y = 0. \tag{2.46}
\]

20
These two equations have the same set of eigenvalues, $\omega^2$.

**Quantum harmonic oscillator** Another interesting case is that of the quantum harmonic oscillator. For the initial potential,

$$q = \frac{2m}{\hbar^2} \left( E - \frac{1}{2} m \omega^2 x^2 \right). \quad (2.47)$$

The final potential must be

$$Q = \frac{2m}{\hbar^2} \left( E + c - \frac{1}{2} m \omega^2 x^2 \right) = q + \frac{2m}{\hbar^2} c. \quad (2.48)$$

This can only work provided that $c$ is a constant. Otherwise, the equations become too complicated to solve. Throwing this into Eq. (2.31),

$$\left( \frac{2q'}{2(2mc/\hbar^2)} \right)' = -\frac{1}{2} \left( \frac{2m}{\hbar^2} c \right) \quad (2.49)$$

$$2q'' = \left( \frac{2m}{\hbar^2} c \right)^2 \quad (2.50)$$

$$\hbar^2 \omega^2 = c^2 \quad (2.51)$$

$$c = \pm \hbar \omega. \quad (2.52)$$

For the compatibility condition Eq. (2.31) to be satisfied, we require that $c = \pm \hbar \omega$. This is exactly the constant that the ladder operators increment and decrement. This is not terribly surprising given the connection between the Darboux transformation and the factorization method [13].
2.3.5 Exact solutions by uniqueness

**Riccati constant** After performing a classical Darboux transformation from

\[ y'' + q(x)y = 0 \quad Y = y' + fy \quad Y'' + Q(x)Y = 0, \tag{2.53} \]

we end up with two constraint equations. The Riccati equation, after letting \( f = -u'/u \), now looks similar to the equations which we are transforming between

\[ u'' + (q(x) + c)u = 0. \tag{2.54} \]

Solving for \( u \) in terms of \( f \),

\[ u = e^{-\int f(x) \, dx}. \tag{2.55} \]

This is the solution of the \( u \) equation which is not necessarily related to either of the original equations. However, if we let \( c = 0 \), then by uniqueness, the \( u \) equation must coincide (up to a multiplicative constant) with the initial potential equation. Therefore, \( u \) is a solution (up to a multiplicative constant) of the original equation. That is,

\[ y = y_0 e^{-\int f(x) \, dx}. \tag{2.56} \]

**Quantum harmonic oscillator ground state** From the example in Section (2.3.4),

\[ q = \frac{2m}{\hbar^2} \left( E - \frac{1}{2} m\omega^2 x^2 \right) \quad Q = q \pm \frac{2m\omega}{\hbar}. \tag{2.57} \]

The positive sign corresponds to positive energy levels. From Eq. (2.28),

\[ f = \frac{(Q + q)'}{2(Q - q)} = \frac{(2q + 2m\omega/\hbar)'}{2(2m\omega/\hbar)} = \frac{m\omega}{\hbar} x \tag{2.58} \]
Using this in Eq. (2.14),
\[ c = f' - q - f^2 = \frac{h \omega - 2E m}{h^2}. \]  
(2.59)

Setting \( c = 0 \) gives the eigenvalue (energy) and \( f \) will give the solution. Therefore,
\[ E = \frac{1}{2} \hbar \omega \quad y = y_0 e^{-\frac{m \omega}{2\hbar}}. \]  
(2.60)

\( y_0 \) can be determined by plugging \( y \) into the differential equation.

**Special solution method**  Again, we seek a solution by method similar to the first. That is, we will use uniqueness to make the constraint and potential equations look the same.

Using Eq. (2.28),
\[ f = \frac{(Q + q)' \left(2(q + c)' \right)}{2(Q - q)} = \frac{q'}{c} \]  
(2.61)

and from the second equation,
\[ f' = \left(\frac{q'}{c}\right)' = \frac{1}{2} (q - Q) = -\frac{1}{2} c. \]  
(2.62)

This method of solution is restricted by this equation. It can be solved for \( q \) to give a specific family of potentials for which we can use this method. The 2 parameter family of potentials is
\[ q(x) = -\frac{1}{4} c \omega x^2 + c_1 x + c_2. \]  
(2.63)

Note that with this solution method, we cannot have \( c = 0 \). Such a case would correspond to a null transformation \( (Q = q) \).
2.3.6 Long ranged to short ranged potentials

The classical Darboux transformation is insufficient to find a more suitable potential for computation. We would like to go from a long ranged potential of $O(x^{-1})$ to $O(x^{-2})$ or even shorter ranged. This cannot happen. Let $q(x)$ be an initial potential such that $U(x) \to O(x^{-1})$ as $x \to \infty$ and $Q(x)$ be the final potential such that $V(x) \to O(x^{-2})$ as $x \to \infty$.

Expanding these potentials at infinity (and keeping all orders),

$q(x) = \lambda - U(x) = \lambda - \left(\frac{a_1}{x} + \frac{a_2}{x^2} + \ldots\right)\quad Q(x) = \lambda - V(x) = \lambda - \left(\frac{b_2}{x^2} + \frac{b_3}{x^3} + \ldots\right)$

Requiring compatibility condition Eq. (2.31) in the form

$$(V + U)' = (V - U) \int (V - U) \, dx \quad (2.64)$$

be satisfied requires that all orders of $x$ go to zero. However, the first term in the integral term on the right hand side of the equation results in a $\ln(x)$ term:

$$\frac{x(5a_1x + 6a_2 - 2b_2)}{x(a_1x + a_2 - b_2) + a_3 - b_3} - b_4 + a_1 \ln(x) + \frac{b_2 - a_2}{x} + \frac{b_3 - a_3}{2x^2} + \frac{b_4}{3x^3} - \frac{4}{x} + O(x^{-4}) = 0 \quad (2.65)$$

This piece has no equivalent term on the left hand side and therefore will not be cancelled out. Thus this equation will only be satisfied if there $a_1 = 0$ or there is an equivalent $b_1$ term.

This amounts to both $q$ and $Q$ having the same long or short ranged behavior as $x \to \infty$

2.4 Classical Darboux transformation in black hole perturbation theory

Three different potentials for perturbations of Schwarzschild black holes were found through different methods. One was found for each even and odd parity perturbations (by using a
parity inversion on the Schwarzschild spacetime that amounts to flipping the signs of the angular coordinates). A third can be found from a formalism put forward by Newman and Penrose [14]. These potentials, known as the Regge-Wheeler, Zerilli, and Bardeen-Press potentials were found to be related by Chandrasekhar in the early 1970s [14]. In a later paper, it was also found that the charged black hole potentials resulting from a perturbation scheme are also related [14].

2.4.1 Derivations of perturbation equations

Derivations of the different perturbation equations can be found in Maggiore’s second volume on gravitational waves [15] and in Chandrasekhar’s book on the mathematical theory of black holes [14] as well as in the original papers by Regge and Wheeler [16], Bardeen and Press [17], and Zerilli [18]. Maggiore’s book follows the papers by Regge, Wheeler, and Zerilli by means of a decomposition of the metric into tensor spherical harmonics, while Chandra’s book follows his own work with John Friedman [19] in which they perturb a general axisymmetric spacetime. Both computations are straightforward, if a bit tedious. However, they would be quite lengthy to type out here and are shown in great detail in both texts. Instead we will cite the results of these calculations and get to the part that we are concerned with.

The serious student in gravitational physics should go through these computations at least once in his/her career, because they are critical to understanding black hole perturbation theory. These computations show that black holes ring like a bell when hit with any kind of field. They are not very good at ringing however, and damp out exponentially. These damped modes are known as quasi-normal modes (instead of normal modes which would not dampen) and are an important component of the ringdown phase of the black hole that
contain information about the mass and angular momentum in the form of frequency and
damping time. Finding these pieces of information from a black hole’s ringdown spectrum
is known as black hole spectroscopy. Spectroscopy has yet to be useful for LIGO, because it
is difficult for the instrument to distinguish between the modes with current signal to noise
ratios. LISA should have greater ability to find and decompose these modes [20].

Another part of black hole perturbation theory that we are more concerned with involves
a delta function source moving around a more massive black hole as a perturbation on
its spacetime. This will be important in the next chapter in which we consider such a
source moving around a rotating black hole and compute the gravitational radiation that it
causes. Such a situation happens in the center of the galaxy where compact objects orbit the
supermassive black hole there. Computations in the Schwarzschild (spherically symmetric)
case illuminate the process while also keeping some moderate simplicity which will not be
shared in the Kerr (rotating axisymmetric) case. Here we will look at the implications for
the spherically symmetric case and discuss what may happen in the more general case.

In the Kerr case, there are more relationships between potentials; however, these tend
to be much more complicated. For example, the transformation that was typically used in
computation of the Teukolsky equation is known as the Sasaki-Nakamura transformation.
This leads to a more complicated expression with better computational behavior. The pages
of math that come along with the Sasaki-Nakamura transformation can be seen in [21], for
example.
2.4.2 Useful results: perturbation equations

The end result of the perturbation calculation in Maggiore’s book is two partial differential equations. One of these equations is satisfied by the tensorial generalizations of the spherical harmonics which are dependent only on angles $\theta, \phi$. The other is dependent on both time and radius. However, once we use a Fourier transform to go from the time to the frequency domain, the radial, temporal equation becomes an ordinary differential equation in terms of the radius only. In the frequency domain, all perturbation equations for the Schwarzschild case are of the form (with $c = G = 1$):

$$\frac{d^2}{dr_*^2} R_{lm} + \left( \omega^2 - V_l \right) R_{lm} = T_{lm}$$

(2.66)

where

$$r_* = r + R_S \ln \left( \frac{r - R_S}{R_S} \right).$$

(2.67)

Note here that $R_{lm}$ will be different for each potential, and the set of eigenfrequencies, $\omega$, may also be different. Also note that every derivative that we take must be converted from $r$ to $r_*$ via the chain rule when computing equations for the DT. In the next few sections we will consider $T_{lm} = 0$. Later we will handle the more general case and find out whether we can move to a “better” source term (what better means will be discussed later). By perturbing our system axially, we find the Regge-Wheeler potential in the above equation [15],

$$V_l^{RW}(r) = \left( 1 - \frac{2M}{r} \right) \left[ \frac{l(l+1)}{r^2} - \frac{6M}{r^3} \right].$$

(2.68)
Perturbing the system polarly, we find the Zerilli potential,

\[ V_Z^l(r) = \left(1 - \frac{2M}{r}\right) \frac{2\lambda^2(\lambda + 1)r^3 + 6\lambda^2 Mr^2 + 18\lambda M^2 r + 18M^3}{r^3(\lambda r + 3M)^2}, \]  

(2.69)

where \( \lambda = (l - 1)(l + 2)/2 \). Finally, perturbing the geometry by using a basis of null vectors through the Newman Penrose formalism \([14]\), we find the Bardeen-Press potential,

\[ V_{BP}^l(r) = -\frac{4i\omega}{r^2}(r - 3M) + \frac{1}{r^3} [2(\lambda + 1)(r - 2M) + 2M]. \]  

(2.70)

### 2.4.3 Darboux Relations

Chandrasekhar found that these three potentials are related via what has become known among physicists as the Chandrasekhar transformation (what we have labelled a generalized Darboux transformation),

\[ Y = A(r)y' + B(r)y. \]  

(2.71)

However, as shown above, if the potentials are such that they satisfy Eq. (2.31), then we can find a DT that takes us from one potential to the next, and they are therefore isospectral. We find that the two equations with the same asymptotic behavior are related by a DT. That is, the Zerilli and Regge-Wheeler equations are related by a DT determined by

\[ f_{Z\to RW} = -\frac{-18M^3 + 9M^2 r + 3Mn(n + 1)r^2 + n^2(n + 1)r^3}{3Mr^2(3M + nr)} \]  

(2.72)

The inverse transformation may be found by inserting a minus sign in front of \( f \). What about the Bardeen-Press potential? It turns out that the relationship between the Bardeen-Press potential and the other two potentials is of the Chandrasekhar type. The transformation
from the Regge-Wheeler to Bardeen-Press potential is

\[
A_{\text{RW} \to \text{BP}} = \frac{2r(-3M + ir^2\omega + r)}{r - 2M} \tag{2.73}
\]

\[
B_{\text{RW} \to \text{BP}} = -\frac{2r^3\omega^2}{r - 2M} + \frac{2ir\omega(r - 3M)}{r - 2M} - \frac{6M}{r} + 2n + 2 \tag{2.74}
\]

A composition of these two transformations will yield another Chandrasekhar transformation from the Zerilli equation to the Bardeen-Press equation. The more general type of transformation was expected here, because the Bardeen-Press equation goes as \(1/r\) as \(r \to \infty\) while the other two potentials go as \(1/r^2\) as \(r \to \infty\). We will now develop the theory as we have the DT for this more general type of transformation.

### 2.5 Generalized Darboux transformation

Once again, we are going from

\[
y'' + q(x)y = 0 \tag{2.75}
\]

to

\[
Y'' + Q(x)Y = 0 \tag{2.76}
\]

but now our transformation is modified to

\[
Y = \beta(x)y' + f(x)y \tag{2.77}
\]
where $\beta(x)$ is an extra degree of freedom in the transformation. Following the same steps from the first section, we come to the constraint equations

$$\beta^2 q + f^2 + \beta f - \beta f' = c \quad (2.78)$$

$$Q = q - \frac{2f'}{\beta} - \frac{\beta''}{\beta} \quad (2.79)$$

which are generally difficult to work with. By following the same steps from Section (2.3.2), we come to an alternate form for these equations which will allow us to check (provided we know what $\beta$ is) whether the equations are related by a generalized Darboux transformation. The result is

$$f' = -\frac{1}{2} \left[ \beta(Q - q) + \beta'' \right] \quad (2.80)$$

$$f = \frac{\beta(Q + q)' + \beta'(Q + 3q) + \beta'''}{2(Q - q)} \quad (2.81)$$

Requiring compatibility between these equations

$$\frac{\beta(Q + q)' + \beta'(Q + 3q) + \beta''}{(Q - q)} = \int dx \left[ \beta(Q - q) + \beta'' \right] \quad (2.82)$$

results in a more complicated equation than the above sections. However, it is now possible to go between long and short ranged potentials as has been shown several times by direct factorization and brute force computation by Chandrasekhar [14]. Unless we know the exact form of the target potential $Q$ or $\beta$ ahead of time, we have little chance of finding a transformation from this formulation. If the target potential is known, a systematic procedure can be used to find a relationship between them via a series expansion of the $\beta$ and $f$ terms as has been done with the spin weighted spheroidal harmonics [11].
2.5.1 Coupled system form of constraint equations

Taking Eq. (2.78) and Eq. (2.79), we can turn both into a coupled second order ODE system. Starting with Eq. (2.79),

\[ Q = q - \frac{2f'}{\beta} - \frac{\beta''}{\beta} \quad (2.83) \]

\[ \beta(Q - q) = -2f' - \beta'' \quad (2.84) \]

\[ \beta'' + (Q - q)\beta = -2f'. \quad (2.85) \]

Now Eq. (2.78),

\[ \beta^2 q + f^2 + \beta f - \beta f' = c \quad (2.86) \]

\[ (\beta^2 q)' + 2ff' + \beta''f - \beta f'' = 0 \quad (2.87) \]

\[ (\beta^2 q)' + 2ff' - (\beta(Q - q) + 2f')f - \beta f'' = 0 \quad (2.88) \]

\[ (\beta^2 q)' - \beta(Q - q)f - \beta f'' = 0 \quad (2.89) \]

\[ f'' + (Q - q)f = \frac{(\beta^2 q)'}{\beta} \quad (2.90) \]

where a derivative was taken in the second line and in the third line \( \beta'' \) has been replaced by Eq. (2.85). Our coupled second order differential equation system is

\[ \beta'' + (Q - q)\beta = -2f' \quad (2.91) \]

\[ f'' + (Q - q)f = \frac{(\beta^2 q)'}{\beta}. \quad (2.92) \]

These equations tend to be easy to work with when dealing with special cases.
2.5.2 Special Darboux transformation

The idea here is to find a Darboux-like transformation that relies on $\beta$ rather than $f$. That is, instead of the DT, $Y = y' + f(x)y$, we let $f = \beta$ in the general transformation and use $Y = \beta(y' + y)$ as the transformation instead. Let $f = \beta$ in Eq. (2.91) and Eq. (2.92)

\[
\beta'' + (Q - q)\beta = -2\beta'
\]

(2.93)

\[
\beta'' + (Q - q)\beta = \frac{(\beta^2q)'}{\beta}
\]

(2.94)

Subtracting the second from the first,

\[
\frac{(\beta^2q)'}{\beta} + 2\beta' = 0
\]

(2.95)

\[
(\beta^2q)' + 2\beta'\beta = 0
\]

(2.96)

\[
(\beta^2q + \beta^2)' = 0
\]

(2.97)

\[
\beta^2(1 + q) = c.
\]

(2.98)

We solve this for $\beta$,

\[
\beta = \sqrt{\frac{c}{1 + q}}.
\]

(2.99)

Before plugging this into Eq. (2.91), we’ll solve it for $Q$

\[
Q = q - \frac{2\beta' + \beta''}{\beta}.
\]

(2.100)

The result is an equation for a new potential $Q$ entirely in terms of the initial potential

\[
Q = q - \left(\sqrt{\frac{1 + q}{c}}\right) \left(2 \left(\sqrt{\frac{c}{1 + q}}\right)' + \left(\sqrt{\frac{c}{1 + q}}\right)''\right).
\]

(2.101)
That is, given any potential \( q \), we can find a new (more complicated) potential just by using the special Darboux transformation \( Y = \beta(y' + y) \). Currently, it is not clear that this has any practical application. The potentials are clearly not always isospectral, and they do not usually have better computational behavior.

### 2.5.3 Exact Solutions by uniqueness

Using the standard logarithmic derivative trick from Section 2.3.1 (modified because of a different form of Riccati equation), we let \( f(x) = -\beta(u'/u) \) in the general transformation constraint Eq. (2.78). After simplification,

\[
 u'' + \left( q + \frac{c}{\beta^2} \right) u = 0. \tag{2.102}
\]

when we let \( c = 0 \), we can find an exact solution by the same procedure as in Section 2.3.5 but with a modification by the \( \beta \) term,

\[
 y(x) = y_0 e^{-\int f(x)/\beta(x) dx}. \tag{2.103}
\]

Such solutions correspond to the so called algebraically special solutions of black hole perturbation theory. These solutions correspond to a transmission coefficient of unity. Such solutions will be enumerated in a later section.

### 2.5.4 Solutions by picking \( \beta(x) \)

In this section, we will again use uniqueness to derive a solution. This time we start with Eq. (2.102). Instead of letting \( c = 0 \) to result in uniqueness, we could instead let our final
potential

\[ Q = q + \frac{c}{\beta^2}. \]  \hspace{1cm} (2.104)

We now know a solution to the final equation, \( Y'' + QY = 0 \):

\[ Y = Y_0 e^{-\int f(x)/\beta(x)dx} \]  \hspace{1cm} (2.105)

The constraint equations will completely specify the transformation since we have chosen our free parameter by picking \( Q \). Inverting the generalized Darboux transformation by solving the linear differential equation \( Y = \beta y' + fy \),

\[ y = e^{-\int f/\beta dx} \int Y/\beta e^{\int f/\beta dx} \]  \hspace{1cm} (2.106)

\[ y = Y_0 e^{-\int f/\beta dx} \int \frac{1}{\beta} dx \]  \hspace{1cm} (2.107)

It now remains to find \( f \) and \( q \) in terms of \( \beta \). Unfortunately, this is the best we can do here. It does not seem likely that we will be able to invert the \( q \) equation in terms of \( \beta \) thereby getting a solution to all second order differential equations of the type \( y'' + qy = 0 \). Using \( Q \) in Eq. (2.91) and Eq. (2.92),

\[ \beta'' + \left( \frac{c}{\beta^2} \right) \beta = -2f' \]  \hspace{1cm} (2.108)

\[ f'' + \left( \frac{c}{\beta^2} \right) f = \frac{(\beta^2 q)'}{\beta} \]  \hspace{1cm} (2.109)

Solving the first for \( f \), plugging into the second, and solving for \( q \) yields

\[ f = -\frac{1}{2} \int \left( \beta'' + \frac{c}{\beta} \right) dx \]  \hspace{1cm} (2.110)

\[ q = -\frac{1}{2\beta^2} \int \left[ \beta \left( \beta'' + \frac{c}{\beta} \right)' + \frac{c}{\beta} \int \left( \beta'' + \frac{c}{\beta} \right) dx \right] dx \]  \hspace{1cm} (2.111)
This means that by picking a $\beta(x)$, we get a solution to the equation $y'' + q(x)y = 0$. This
is interesting but not terribly useful, because we can only guess at what kind of $q$ we will
get. However, the situation seems better than it was before when we had no hope of getting
solutions to this kind of equation.

2.6 Application to black hole perturbation theory

With the help of an undergraduate student, Andrew Osborne, we have collected several
transformations between black hole perturbation potentials by using Mathematica. In the
following section $\lambda$ from above will now be labelled as $n$ where $n = (l - 1)(l + 2)/2$. In the
following transformations, all derivatives use the chain rule:

$$\frac{d}{dr_*} = \frac{dr}{dr_*} \frac{d}{dr}$$  \hspace{1cm} (2.112)

2.6.1 Schwarzschild Perturbations

For the Schwarzschild case,

$$\frac{dr}{dr_*} = \left(1 - \frac{2M}{r}\right)$$  \hspace{1cm} (2.113)

Regge-Wheeler to Zerilli

Initial potential:

$$U_{\text{RW}} = \left(1 - \frac{2M}{r}\right) \frac{2(n+1)r - 6M}{r^3}$$  \hspace{1cm} (2.114)

Final potential:

$$V_Z = \left(1 - \frac{2M}{r}\right) \frac{18M^3 + 18M^2nr + 6Mn^2r^2 + 2n^2(n+1)r^3}{r^3(3M+nr)^2}$$  \hspace{1cm} (2.115)
Reflectionless frequency

\[ \omega = \frac{in(n+1)}{3M} \]  (2.116)

Algebraically special solution:

\[ y(r) = \frac{1}{r} \left( e^{-\frac{2n(n+1)r}{3M}} (r - 2M) - \frac{2}{3} n(n+1)(3M + nr) \right) \]  (2.117)

Transformation details:

\[ \beta = 1 \]  (2.118)

\[ f = \frac{-18M^3 + 9M^2r + 3Mn(n+1)r^2 + n^2(n+1)r^3}{3Mr^2(3M + nr)} \]  (2.119)

\[ c = \frac{n^4 + 2n^3 - (n+1)^2n^2 + n^2}{9M^2} \]  (2.120)

Regge-Wheeler to Bardeen-Press

Initial potential:

\[ U_{\text{RW}} = \left( 1 - \frac{2M}{r} \right) \frac{2(n+1)r - 6M}{r^3} \]  (2.121)

Final potential:

\[ V_{\text{BP}} = \frac{2(n+1)(r - 2M) + 2M}{r^3} - \frac{4(n^2 + n)(r - 3M)}{3Mr^2} \]  (2.122)

Reflectionless frequency

\[ \omega = \frac{i(n^2 + n)}{3M} \]  (2.123)

Algebraically special solution:

\[ y(r) = \frac{1}{r} \left( e^{-\frac{2n(n+1)r}{3M}} (r - 2M) - \frac{2}{3} n(n+1)(3M + nr) \right) \]  (2.124)
Transformation details:

\[
\beta = \frac{2r(9M^2 - 3Mr + n(n + 1)r^2)}{3M(2M - r)} 
\]

(2.125)

\[
f = -\frac{2(54M^4 - 9M^3(2n + 5)r + 9M^2(n + 1)^2r^2 - 3Mn(n + 1)r^3 + n^2(n + 1)^2r^4)}{9M^2r(2M - r)} 
\]

(2.126)

\[
c = 8n(n + 1) 
\]

(2.127)

### 2.6.2 Reissner-Nordström perturbations

Reissner-Nordström perturbations are not listed here, but they are related by a classical Darboux transformation which can be found in Chandra’s book [14].

### 2.6.3 Kerr perturbations

Due to the complicated nature of these transformations, we will not include them here. When evaluating GDT constraints for the Kerr potentials in Mathematica, the simplification takes a significant amount of time (I have never gotten one to finish computing). However, we can do a series expansion to any given order as shown in [8]. More information about transformations and their type can be found in Table 1 of [8].

### 2.7 GDT with source term

Previously, the generalized Darboux transformation (GDT) was considered. We found that we had a free parameter \(\beta\). This allowed us to go from a long ranged to a short ranged potential, but we had no easy way to find out what this free parameter should be. However, if we consider an equation with a source term

\[
y'' + q(x)y = T(x),
\]

(2.128)
and we use the generalized Darboux transformation, then we end up with a new equation

\[ Y'' + Q(x)Y = S(x) \]  \hspace{1cm} \text{(2.129)}

where the transformation satisfies Eq. (2.78) and Eq. (2.79). Additionally, we have a new relationship between the source terms,

\[ S = fT + \beta T' + 2T\beta'. \]  \hspace{1cm} \text{(2.130)}

This can be solved for \( \beta \) as

\[ \beta = \frac{c_1}{\sqrt{T}}\left(1 + \frac{1}{2c_1} \int \frac{S - fT}{\sqrt{T}} \, dx\right). \]  \hspace{1cm} \text{(2.131)}

This means that the free parameter discussed earlier is no longer a free parameter, but will be specified by the source terms. However, to some extent we choose the source term. So it is an interesting prospect to see what sort of source terms are allowed while also yielding a GDT. This avenue has largely been unexplored. The method of using the source term to constrain the transformation seems to be the method used by Chandrasekhar, Detweiler, and Sasaki and Nakamura to generate their transformations from the Teukolsky equation to other potentials with more desirable computational properties (convergent source terms or purely real or imaginary source terms). Further, it would be interesting if this could be used in a purely mathematical way to generate generalized transformations between the spin weighted spheroidal harmonics which currently are only known to first order in \( \omega \) [11].
3.1 Abstract

In GW150914, approximately $3M_\odot$ were radiated away as gravitational waves from the binary black hole system as it merged. The stress energy of the gravitational wave itself causes a nonlinear memory effect in the detectors here on Earth called the Christodoulou memory. We use an approximation that can be applied to numerical relativity waveforms to give an estimate of the displacement magnitude and the profile of the nonlinear memory. We give a signal to noise ratio for a single GW150914-like detection event, and by varying the total mass and distance parameters of the event, we find distances and source masses for which the memory of an optimally oriented GW150914-like event would be detectable in aLIGO and future detectors.
3.2 Introduction

Recently reported observations [22–29] of gravitational waves in aLIGO consist of black hole binaries and one binary neutron star (BNS) system. Binary systems which lose components are known to produce a memory [30–32]. Gravitational bremsstrahlung results in a linear memory in a detector far from the source: a permanent displacement between freely falling test masses that grows as the wave passes and persists even after the wave has passed.

Christodoulou (by using the full nonlinear theory of relativity) [33] and Blanchet and Damour (by using a post-Minkowskian scheme) [34] independently discovered that the difference in relative position between ideal (freely falling) test masses long before and long after any gravitational wave has passed a detector is nonzero. This memory effect is known as the Christodoulou or nonlinear memory.

This is contrary to the standard picture of a gravitational wave that one usually imagines: a ring of particles subject to a plane gravitational wave will oscillate either in a plus or cross polarization pattern and then come back to its original orientation as a ring. In reality, the ring does not return to its original position but is instead left in a residual polarization state as in Figure 3.1. Two particles on the ring will either be closer or further apart depending on the sign of the memory. Knowledge of this sign clearly requires knowledge of the polarization state of the oscillatory part of the gravitational wave.

Figure 3.1: Two particles from a ring of particles as a plane, plus polarized gravitational wave passes perpendicular to the page. The nonlinear memory is shown here as a residual plus polarization after the wave has passed.
Thorne found that the oscillatory part of the gravitational wave causes the nonlinear memory by considering the wave to be made of gravitons causing a linear memory as they escape from the system [35]. Indeed, a compact binary system loses components in the form of gravitational radiation (about three solar masses in GW150914) thereby causing a nonlinear memory. Since the stress energy of the oscillating gravitational wave as it escapes to infinity causes the nonlinear memory, it can be thought of as the “wave of the wave” [36].

One may worry that LIGO is insensitive to permanent or DC changes in its arms. How then might we see such an effect? LIGO contains stationary masses held in place, but the detectors will still be able to detect the changing strain caused by the buildup of the memory [2]. Provided the compact binary system is close enough and the inclination angle is optimal, the nonlinear memory could be directly detectable in ground based detectors if enough of the change associated with the memory occurs on a timescale \( \tau \approx 1/f_{\text{opt}} \) where \( f_{\text{opt}} \) is the frequency of the detector’s peak sensitivity [35].

When Dan Kennefick worked on this in the 1990s, he looked at the detectability of the Christodoulou memory in binary black hole (BBH) systems with masses of at most 10 M\(_{\odot}\) [37]. Recent detections show most of the BBH systems detected have components with masses above 10 M\(_{\odot}\) [29]\(^1\). Now that the oscillatory part of the gravitational wave has been detected, the question has arisen whether the nonlinear memory could be detectable with current sensitivities in ground based detectors [2].

After GW150914, Lasky, Thrane, Levin, Blackman, and Chen (LTLBC) [2] used Favata’s minimal waveform model (MWM) [39] to find a nonlinear memory waveform. They found a

\(^1\)Prior to the first detection some physicists believed that the total mass of the black hole binary system would be around 5 - 15 M\(_{\odot}\). Even after the detection, Broadhurst, Diego, and Smoot argue that the sizes of detected BBHs have been exaggerated through gravitational lensing [38].
signal to noise ratio (SNR) of 0.42 for an optimally oriented source modeled by the MWM in aLIGO at design sensitivity. A signal with this SNR is not detectable without a clever scheme of adding subthreshold signals as discussed in their paper.

Computation of the nonlinear memory from numerical waveforms has proved difficult, but some calculations have been done for equal mass binary systems [40]. Extraction of the memory waveform from numerical data to leading order requires two numerical integrations of the $l = 2$, $m = 0$ part of the Weyl scalar $\psi_4$, the typical output of a numerical simulation. Each integration increases the amplitude of the numerical error until it swamps small, low frequency effects in the signal such as the nonlinear memory [41]. Current attempts use Cauchy characteristic matching (see Sec 6.2 in [42]) to attempt to get more accurate modes containing the memory. However, there is always a piece that is missing from these calculations due to the computation extending only to finite times in the past (we miss the entire inspiral phase before our simulation starts).

While not likely to be detectable by aLIGO, advanced Virgo (AdV), or KAGRA (a Japanese, cryogenic, underground detector which will be operational around 2018 [43]), one may wonder what the odds of detection are with future detectors. Might we be able to detect the memory as strain sensitivity increases in ground based detectors? Third generation, ground based detectors including the Einstein Telescope (ET) reduce seismic noise by being set up deep underground. The arms of ET are planned to be 10 km long in a triangular geometry with three detectors each comprised of a low frequency, cooled detector and a high frequency detector [44]. On the same timescale is the Cosmic Explorer (CE) which has an “L” shape like aLIGO and has 40 km long arms [45].

Current ground based gravitational wave detectors hit a wall of seismic noise at about
10 Hz. Given that the memory is primarily a low frequency effect, perhaps space based detectors sensitive in the decihertz frequency regime could detect the memory. A Japanese, space based detector, DECIGO, is proposed to launch on a timescale similar to LISA [46]. This gravitational wave detector has three 1000 km arms set up in a triangular pattern and uses differential Fabry-Perot interferometry. However, this timescale includes launching DECIGO pathfinder in 2015, which did not happen. It is sensitive to signals around the decihertz frequency range, filling the frequency gap between the LISA and LIGO detectors. Space based transponder type detectors such as the planned LISA mission are sensitive in the mHz regime and will detect memories of larger binary systems such as extreme mass ratio inspirals (EMRIs) [39]. Many decades from now, we may see the Big Bang Observer (BBO) launched. BBO consists of smaller LISA type detectors situated in specific “constellations” around the sun [47]. DECIGO and BBO have strong sensitivity in regions that make them suitable for detecting the memory from a GW150914-like event.

Pulsar Timing Arrays (PTA) are sensitive to even lower frequencies than space based interferometers and are checked for memory signals from supermassive black hole binaries (SMBH) [48]. Among the currently operating PTA groups are the Parkes Pulsar Timing Array (PPTA), the European Pulsar Timing Array (EPTA), and the North American Nanohertz Observatory for Gravitational Waves (NANOGrav). For a recent review of these collaborations, see [49]. The International Pulsar Timing Array group aims to combine the observational data from each group to get even better sensitivity and a greater number of pulsars [50]. In the future the Square Kilometer Array could detect many more pulsars to be used for data analysis and push sensitivity curves even deeper into the noise [51].

The nonlinear memory is interesting as a purely strong-field gravitational effect. As such,
its effects are dependent on the form of Einstein’s equations and therefore are useful in theory testing. For example, theories which include scalar fields also contain extra memory modes [52]. Further, since the rise time of the memory is related to the radii of the compact binary constituents, detection of neutron star binary memory could give independent insight into the equation of state by picking a mass-radius relation and calculating the memory [37].

In this paper, we aim to give an approximation of the memory and its profile. We apply Thorne’s formula (Equation (3) from [35]) to a numerical relativity waveform (Section 3.3). Using this model results in a calculation that is easy to use and computationally cheap. We calculate the memory for GW150914 (Section 3.3). Next, the memory obtained from this calculation can be used to give a signal to noise ratio for a given detector (Section 3.4). Finally, we vary the mass and distance parameters on a GW150914-like event to find the distance and total source mass for which an event would be detectable in several current and future detectors (Section 3.5). Throughout this paper we use geometric units ($G = c = 1$).

### 3.3 The nonlinear memory of GW150914

Thorne gives a formula for “practical computations” of the memory in [35],

$$h_{\text{mem}}(t) = \frac{2}{r} \int_{-\infty}^{\infty} dt' \int d\Omega' \frac{d^2 E}{dt'd\Omega'} (1 + \cos \theta') e^{2i\phi'},$$

(3.1)

where $r$ is the distance from source to detector, and $d\Omega'$ is the solid angle. By expanding the energy flux in terms of spherical harmonics and integrating out the angular dependence, we can approximate the flux as a prefactor multiplied by the Isaacson stress energy. The full calculation is given in Appendix 3.8 and gives an approximation for the memory in optimal
orientation,

\[ h_{\text{max}}(t) = \frac{r}{4\pi} \int_{-\infty}^{t} dt' \dot{h}_+^2, \quad (3.2) \]

where \( h_+ \) is the plus polarization of the oscillatory gravitational waveform from a numerical relativity simulation. The memory amplitude scales depending on inclination \([37]\) as

\[ \Phi(\iota) = \frac{18}{17} \sin^2 \iota \left( 1 - \frac{\sin^2 \iota}{18} \right). \quad (3.3) \]

This prefactor shows larger memory effects for edge-on binary systems in contrast to the primary oscillatory wave which is strongest from face-on binaries. In systems that exhibit maximum memory (\( \iota = \pi/2 \)), we will see only half of the maximum \( h_+ \) polarization and none of the \( h_\times \) polarization from the oscillatory part of the gravitational wave. The memory effect is not present in face-on systems (\( \iota = 0 \)). Using \( \Phi(\iota) \), we can find the memory amplitude at any inclination:

\[ h_{\text{mem}}(t) = \Phi(\iota) \frac{r}{4\pi} \int_{-\infty}^{t} dt' \dot{h}_+^2. \quad (3.4) \]

By using the method summarized above on the numerical relativity data given by a waveform generated with SEOBNRv4 \([53]\) using the PyCBC python package \([54]\) and the LIGO Algorithm Library (LAL), the nonlinear memory can be calculated for GW150914 as shown in Figure 3.2. This waveform uses the averaged parameters given in the spin precessing parameter estimation paper \([55]\).

The memory calculated has the same profile as that which LTLBC found with the MWM \([2]\), as can be seen in Figure 3.3. There are two likely reasons for the difference in amplitude. First, our estimate is an overestimate (see Appendix 3.8). Second, the memory we have calculated uses the parameters from \([55]\) while LTLBC use parameters from \([56]\). The maximum
Figure 3.2: Numerical relativity waveform for GW150914 generated with PyCBC is shown in the upper plot. The memory is shown with optimal inclination in the lower plot with maximum amplitude even after the gravitational wave has passed. The “wiggliness” of the waveform has been discussed in Appendix C of [1] and is caused by disregarding the average after taking the time integral (see Appendix 3.8).

memory has been adjusted to $\iota = 140^\circ$ to directly compare between the two models.

Figure 3.3: Comparison of the MWM in [2] with the approximation given in this paper.
3.4 Finding the signal to noise ratio

The signal to noise ratio (SNR) denoted $\rho$ can be found by using a Fast Fourier Transform (FFT) routine on the memory waveform. Assuming our template is accurate,

$$\rho^2 = 4 \int_0^\infty \frac{\vert \tilde{h}(f) \vert^2}{S_n(f)} df,$$

where $\tilde{h}(f)$ is the memory signal in the frequency domain, and $S_n(f)$ is the one sided noise power spectral density (PSD) for the detector used to detect the memory signal. These noise curves can be found in the web application listed in the abstract of [57] with the exception of NANOGrav, which may be found in the 11 year data set [58]. Outside of the frequency range for which any gravitational wave detector noise curve is given, we take the SNR to be zero. For the primary oscillatory wave an SNR of 5 - 8 is sufficient to be confident in detection. Since the memory will accompany the oscillatory part of the wave, papers that have considered memory detection have claimed that an SNR of 3 - 5 would be sufficient to be confident in detection [2] [37]. Following these other papers, we plot SNR values of 3 and 5 in all figures. In the event that we are looking for memory by itself, we also plot an SNR value of 8.

3.4.1 SNR for GW150914-like events in current and future detectors

<table>
<thead>
<tr>
<th>Detector</th>
<th>SNR</th>
<th>Detector</th>
<th>SNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adv</td>
<td>0.238</td>
<td>eLISA</td>
<td>0.025</td>
</tr>
<tr>
<td>aLIGO</td>
<td>0.450</td>
<td>LISA</td>
<td>0.214</td>
</tr>
<tr>
<td>KAGRA</td>
<td>0.243</td>
<td>DECIGO</td>
<td>96.53</td>
</tr>
<tr>
<td>ET</td>
<td>9.726</td>
<td>BBO</td>
<td>177.2</td>
</tr>
<tr>
<td>CE</td>
<td>21.73</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Current and future detectors’ SNR for optimally oriented GW150914-like memory events
As has been discussed in the paper by LTLBC [2], the memory is not likely to be detectable in aLIGO without adding several detections together. We agree with their results with a calculated SNR of 0.45 for an optimally oriented memory source (compared with an SNR of 0.42 found in [2]). Therefore, we need a closer event to detect the memory with aLIGO’s current sensitivity.

Using the distance that gives $\rho = 3 \ (5)$ and the median event rate from [59] of $17$ Gpc$^{-3}$yr$^{-1}$ events like GW150914, we can find how long we can expect to wait for a detectable GW150914-like memory event in current ground based detectors. A detectable event would occur at 65 (39) Mpc. With current event rates, we expect this to happen once in about 51 (237) years. However, as can be seen in Table 3.1, both third generation ground based detectors will be able to detect the nonlinear memory effect from optimally oriented GW150914-like events.

From Table 3.1, it is apparent that eLISA and LISA will do no better than ground detectors in detecting an event like GW150914. Both space based decihertz detectors, DECIGO and BBO, will certainly be able to see the memory. This is probably due to the frequency band being optimal for this signal. Additionally, these detectors have impressive projected sensitivities. Even with sensitivity reduced by a factor of 10, DECIGO would still see the memory from a GW150914-like event. Neither decihertz detector has a production timeline set, but DECIGO has an optimistic launch date as early as 2027 [46]. This launch date expected DECIGO pathfinder to launch in mid 2015 as a precursor to DECIGO. Given that this did not take place, the timeline should be adjusted accordingly.
3.5 GW150914-like events with varying mass and distance

Since current detectors have little chance of detecting any single GW150914-like event memory, we now vary parameters on the event to find what luminosity distance $d_L$ and total detector frame mass $M$ would be needed for a given detector to achieve a detection\(^2\). All points on the grid use the same template with varied mass and distance yielding a similar system in a different frequency regime or reducing the SNR as distance increases. Figures 3.5, 3.6, 3.7, 3.8, 3.9, 3.10 show the SNR for a given $(d_L, M)$. In the cases where it is relevant, the plots show a nearby plot extending to 3000 Mpc in distance on the left, and a plot that extends to the edge of the $\rho = 3$ contour or up to a luminosity distance of 30 Gpc on the right. There are contours for $\rho = 3$, $\rho = 5$, and $\rho = 8$ on these plots to show different standards for detection in each detector. Reported events are shown as marks at their estimated $(d_L, M)$ parameters [29].

However, GW170817 is a neutron star binary while the SNR values are given based on a binary black hole system. Therefore, we expect the actual memory SNR to be less than projected on the plot and moved further to the right. Yang and Martynov looked at the detectability of binary neutron star mergers with four different equations of state [3]. They found that two 1.325 $M_\odot$ neutron stars at a distance of 50 Mpc in aLIGO produce an SNR of about $\rho = 0.1$ and an SNR of about $\rho = 10$ in CE. These values are consistent with values given in [29] for GW170817. At the same values, we find $\rho = 0.483$ for aLIGO and $\rho = 18.479$ for CE. We assume that the memory amplitude is off by the same multiplicative factor at all frequencies. Here we pick the worse of the two factors of 4.83. At the median\(^2\)

\(^2\)Here we use the standard $\Lambda$CDM cosmology [60]. This results in a frequency shift which is equivalent to a mass change between the source and detector frames given by $m_{\text{detector}} = (1 + z)m_{\text{source}}$ where $z$ is the redshift [29].
distance and mass, \((d_L, M) = (40, 2.8)\), given in [29] for GW170817, we find an SNR of around \(\rho = 4.772\) in DECIGO and \(\rho = 17.68\) in BBO. These results deserve further study with different equations of state.

All SNR values here are calculated assuming a comparable mass binary system. As the frequency band gets lower, the mass ratio of the two black holes may become more extreme. However, for higher mass ratios, the memory is less prominent [39].

Figure 3.5 shows current and near future detectors unable to detect the nonlinear memory further away than 250 Mpc. Given that the majority of black hole related events have occurred further away than that, it seems unlikely that we will see the memory with current ground based detectors. Outlook for future ground based detection of the memory is positive. Both proposed third generation detectors improve sensitivity and visibility distance significantly.

Even if we don’t see the nonlinear memory from current ground based detectors, it should be visible in a different regime with LISA. Supermassive black hole mergers with total mass on the order of \(10^7 M_\odot\) give a promising source for memory detections as can be seen in Figure 3.7. DECIGO and BBO will be able to see optimally oriented memory from all of the recently reported sources. These detectors also open the exciting possibility of detecting neutron star binary gravitational nonlinear memory.

Pulsar timing arrays NANOGrav, EPTA, and IPTA have not reported detection yet. There is thought to be an upper bound of supermassive black hole mass at around \(10^{10} M_\odot\) [61]. The fact that no memory has been seen in such detectors supports this as shown in Figure 3.9 where we can see that comparable mass binary systems with this mass would produce a memory which should be visible to current pulsar timing arrays.
3.6 Conclusion

Gravitational wave astronomy is in its infancy. Now that the primary oscillatory wave has been detected, we look toward nonlinear parts of the gravitational wave. Current ground based detectors (see Figure 3.5) are unlikely to see the nonlinear memory without a clever stacking scheme as in [2] and [3]. The outlook for future detectors is positive: third generation ground based detectors (see Figure 3.6), space based detectors (see Figures 3.7, 3.8), and pulsar timing arrays (see Figures 3.9, 3.10) can all detect the nonlinear memory in different frequency regimes. DECIGO and BBO in particular yield interesting prospects for detecting the nonlinear memory from a neutron star binary system. This should be considered further with varying equations of state similar to what has been done with CE in [3]. Given that the rise time of the memory could be related to the radius in some way [37], the memory yields an independent method of constraining the neutron star equation of state. Future funding applications for newer, better gravitational wave detectors should include detecting the nonlinear memory and its applications as a science goal.

3.7 Acknowledgments

We thank Kip Thorne for suggesting this project to us, and Stephen Hawking, Malcolm Perry, and Andy Strominger for their query to Kip regarding the memory’s detectability. We thank Lasky, Thrane, Levin, Blackman, and Chen for kindly providing their data for straightforward comparison. We would also like to thank Kostas Glampedakis, Scott Hughes, Mark Hannam, Mark Scheel, Xavier Siemens, Jolien Creighton, and Alan Wiseman for useful discussion. AJ and DK would like to thank University College Cork for their hospitality.
during the preparation of this manuscript. AJ worked as a Sturgis International Fellow funded by the Roy and Christine Sturgis Educational Trust while abroad.

This work has made use of the LIGO Algorithm Library (LAL) and the PyCBC Python package.

3.8 Calculation of the nonlinear memory

Following the derivations for the nonlinear memory in [36, 62], we begin with the relaxed Einstein field equation in the harmonic gauge,

$$\square \bar{h}^{\mu \nu} = -16\pi \tau^{\mu \nu}. \quad (3.6)$$

where $\tau^{\mu \nu}$ contains the stress-energy $T^{\mu \nu}$, the Landau-Lifshitz pseudotensor $t_{LL}^{\mu \nu}$, and some pieces of $O(h^2)$. Assuming a flat background spacetime and ignoring the other pieces of $\tau^{\mu \nu}$, we focus on this part that sources the memory,

$$T_{jk}^{gw} = \frac{1}{r^2} \frac{d^2E}{dt'd\Omega'} \xi_j \xi_k, \quad (3.7)$$

where $\xi^j$ is a unit vector pointing from the source to solid angle $d\Omega'$ and $d^2E/dt'd\Omega'$ is the gravitational wave flux. Solving equation (3.6) by using the retarded Green’s function yields the standard result

$$\bar{h}^{jk} = 4 \int \frac{T_{jk}^{gw}(t' - |x - x'|, x')}{|x - x'|}. \quad (3.8)$$

Next, we use a method of direct integration of the relaxed Einstein equations (DIRE) [63]. This method changes the coordinates from Cartesian to spherical and the radial coordinate $r'$ to retarded time $u' = t' - r'$. We then integrate with respect to retarded time in the wave
This process yields

\[ h_{jk} = \int_{-\infty}^{u} du' \int \frac{d^2E}{d\Omega'dt'} \frac{\xi_j \xi_k}{t - u' - \mathbf{x} \cdot \mathbf{\xi}'} \ d\Omega' \]  

(3.9)

where \( u = t - r \) and \( \xi' = \mathbf{x}' / r' \). This is Equation (4) of [36]. We now specialize to a gravitational wave burst passing a detector at fixed \( r \). Using the limiting procedure in [36] and transforming to transverse traceless gauge,

\[ h_{jk}^{\text{TT}} = \frac{4}{r} \int_{-\infty}^{t} dt' \int \frac{d^2E}{d\Omega'dt'} \left( \frac{\xi_j \xi_k}{1 - \cos \theta'} \right)^{\text{TT}} d\Omega'. \]  

(3.10)

From here Thorne suggested a system of coordinates for “practical computations” and found a formula for the nonlinear memory. Equation (3) from [35] modified by replacing \( dE/d\Omega' \) with \( \int_{-\infty}^{t} (d^2E/dt'd\Omega') dt' \) as in [37] is

\[ h_{\text{mem}} = \frac{2}{r} \int_{-\infty}^{t} dt' \int d\Omega' \frac{d^2E}{d\Omega'dt'} (1 + \cos \theta') e^{2i\phi'} \]  

(3.11)

where \( r \) is the distance from the source to the detector, \( t \) is some time after the wave has passed, and \( \Omega \) is the solid angle. Notice that we are integrating over the entire history of the wave until now, hence the name “memory.” An expansion in terms of the spin-weighted spherical harmonics,

\[ \frac{d^2E}{d\Omega'dt'} = \frac{r^2}{16\pi} \sum_{l,l',m,m'} \left\langle h_{lm} h_{l'm'}^{*} \right\rangle -2Y_{lm} -2Y_{l'm'}^{*} \]  

(3.12)

where the brackets, \( \langle \rangle \), denote a time average over several wavelengths of the wave. This allows one to separate the angular piece from the temporal piece. Here the \(-2Y_{lm}\) are the spin-weighted spherical harmonics, and the * denotes complex conjugation. Using Eq. (3.12) in Eq. (3.11), we find the selection rule \( m + m' = -2 \). After performing the angular
integration,

\[ h_{\text{mem}}(t) = \frac{r}{8\pi} \int_{-\infty}^{t} dt' \left( \sqrt{\frac{2}{3}} \langle \dot{h}_{20} \dot{h}_{22}^* \rangle + \sqrt{\frac{1}{6}} \langle \dot{h}_{2-2} \dot{h}_{20}^* \rangle + \frac{2}{3} \langle \dot{h}_{2-1} \dot{h}_{21}^* \rangle \right). \tag{3.13} \]

Because of the issues with directly calculating \( h_{20} \) modes of the gravitational wave discussed in Section 3.2, we aim for an approximation. We change all smaller modes to the dominant \( h_{22} \) piece of the waveform. What we would like to use here is the \((l, m) = (2, 2)\) mode that exists at the source. Instead, we are given the mode in the detector’s frame. Then,

\[ h_{\text{mem}}(t) = \frac{r}{4\pi} \int_{-\infty}^{t} dt' \langle \dot{h}_{22}^2 \rangle. \tag{3.14} \]

This equation can be compared to Equation (21) in [62] from which we find the prefactor off by a factor of about 2 (after the inclination has been put in terms of \( \sin^2 \iota \)). The origin of this factor is unknown, but it does not significantly alter the results in a side-by-side comparison. For a direct approximation from the gravitational waves given by a numerical relativity waveform, we can replace the dominant mode with the Isaacson stress-energy,

\[ h_{\text{mem}}(t) = \frac{r}{4\pi} \int_{-\infty}^{t} dt' \langle \dot{h}_{22}^2 \rangle + \langle \dot{h}_{22}^2 \rangle. \tag{3.15} \]

By doing so, we’re adding in higher order terms to connect our approximation to the waveform given by a numerical simulation. Here we focus our attention to linearly polarized waves at \( \iota = \pi/2 \). At this inclination, the oscillatory wave only consists of half the amplitude of the \( h_+ \) polarization. The memory is at a maximum, but this amplitude will be based on the maximum amplitude of the oscillatory part of the wave. So as a kludgy model, we use

\[ h_{\text{max}}(t) = \frac{r}{4\pi} \int_{-\infty}^{t} dt' \dot{h}_{+}^2. \tag{3.16} \]
where we have dropped the average since the time integral is effectively taking the average by integrating over the entire history of the wave. This is the cause of the “wiggliness” seen in Figure 3.2 as discussed in [1].

3.9 A note on the form of the memory and changing parameters

The reader may have noticed that the memory looks like a Heaviside step function,

\[ \theta(t) = \begin{cases} 
  0 & t < 0 \\
  1 & t \geq 0 
\end{cases} \]

This is one of the functions used in the approximation in [37]. A step function has a Fourier transform that is proportional to \(1/f\). As mass increases in the system, we expect this signal in the frequency domain to increase as it comes into our detector’s band and then decrease as it goes out. However, this is not what was found in the ground and space based detectors (Figures 3.5, 3.6, 3.7, 3.8).

The signal in the frequency domain is only well approximated by a step function at low frequencies. The slow step response before merger happens on the radiation reaction timescale and the fast step response during merger happens on the merger timescale. These correspond to features that show up in the frequency domain.

Here we use a characteristic strain convention [57],

\[ [h_c(f)]^2 = 4f^2 |\tilde{h}(f)|^2 \]

and

\[ [h_n(f)]^2 = fS_n(f). \]
This is a useful convention for plotting the memory signal over the noise curve, because the area between the two curves on the plot is now proportional to the SNR. However, we must be careful to remember that now a $1/f$ curve on the plot will be a constant. Looking at Figure 3.4, we can see that lower frequencies do behave as $1/f$, but higher frequencies do not! Instead there is a local minimum in the plot and then it falls off completely shortly afterward.

As the mass increases, we find that the curve moves up and to the left. The SNR will increase until the local minimum hits the lowest frequency the detector can see and then it will increase rapidly and fall to zero shortly after. This is the behavior that is seen in Figures 3.5, 3.6, 3.7, 3.8.

Figure 3.4: Heuristic plot showing the memory signal and a noise curve using the characteristic strain convention. Arrows show the direction the memory moves when changing the total mass, $M$, and the distance from the source, $d$. 
Current and Near Future Ground Based Detectors

Figure 3.5: Luminosity distance \((d_L)\), total mass \((M)\) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given \((d_L, M)\) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Current and near future ground based detectors (2nd generation detectors) are not able to see memory from any released detection.
Future Ground Based Detectors

Figure 3.6: Luminosity distance ($d_L$), total mass ($M$) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given ($d_L, M$) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Plots in the left column show SNR values out to 3000 Mpc, and plots on the right column show the edge of the $\rho = 3$ contour. Future ground based detectors (3rd generation detectors) offer significantly improved sensitivity and are able to see memory from several released detections. Although GW170817 is a BNS, Yang and Martynov [3] show that for various equations of state CE gives an SNR of about 10 and should be detectable.
Figure 3.7: Luminosity distance \((d_L)\), total mass \((M)\) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given \((d_L, M)\) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Plots in the left column show SNR values out to 3000 Mpc, and plots on the right column show the edge of the \(\rho = 3\) contour or out to 30 Gpc. Space based detectors in the mHz frequency regime will be able to see SMBH binary memory with masses significantly higher than those that are visible in the ground based detectors.
Figure 3.8: Luminosity distance ($d_L$), total mass (M) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given ($d_L$, M) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Plots in the left column show SNR values out to 3000 Mpc, and plots on the right column show out to 30 Gpc. Future space based detectors that are sensitive in the dHz regime have excellent sensitivity to the memory from all the released events. The BNS memory might be visible in these detectors. Further study is needed for varying equations of state as in [3].
Figure 3.9: Luminosity distance ($d_L$), total mass (M) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given ($d_L, M$) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Plots in the left column show SNR values out to 3000 Mpc, and plots on the right column show the edge of the $\rho = 3$ contour. PTAs are sensitive to the memory from the largest SMBH binary systems.
Pulsar Timing Arrays

Figure 3.10: Luminosity distance ($d_L$), total mass ($M$) parameter space for current and near future ground based detectors. SNR values are shown on a logarithmic color scale for a given ($d_L, M$) pair. Contours show SNR values of 3, 5, and 8. Events are marked with a “+” to indicate where they fall on the SNR scale as a BBH, GW150914-like system at optimal orientation. Plots in the left column show SNR values out to 3000 Mpc, and plots on the right column show the edge of the $\rho = 3$ contour or out to 30 Gpc. PTAs are sensitive to the memory from the largest SMBH binary systems.
Chapter 4

Signal Confusion Background from Extreme Mass Ratio Capture Sources

4.1 Introduction

The proposed laser interferometer space antenna (LISA) mission should see many types of gravitational wave sources in the mHz frequency regime. One of the sources that we hope to see are extreme mass ratio inspirals (EMRI). These sources consist of a supermassive black hole (SMBH) and a stellar mass black hole (or other compact object – we use a black hole for simplicity). By watching a compact object that behaves as a perturbation on the larger black hole’s spacetime, we can learn about possible deviations from the Kerr metric [64]. However, long before these sources reach a detectable position in their orbits, they constitute a noise source.

SMBHs exist in the centers of most galaxies\(^1\). During their time there, they typically clear a region of stars and gas nearby. However, occasionally a scattering event will occur outside this region that throws a compact object into a highly eccentric orbit around the SMBH. Over time, these orbits decay until they finally have orbits that are likely to be detectable by LISA. While in this initial stage of high eccentricity, compact objects are not detectable by LISA, but they do emit a short-lived burst of radiation in the LISA band upon their closest approach. This burst of radiation will not be resolvable by LISA, because it is too short lived. However, an ensemble of such sources that are highly eccentric constitute a confusion noise source that would cover signals that might otherwise be detectable [66]. We call such sources capture sources as they have been captured by the black hole and will

\(^1\)A SMBH merger has a chance to eject the merger result out of the galaxy. This is known as a superkick [65].
eventually be consumed by it.

The compact binaries that we are interested in have certain properties that will be useful in future discussion. First, we take these objects to be black holes for simplicity. Second, they will be considered perturbations on the spacetime of the SMBH they are orbiting. This simplifies the computation significantly. We will not need weeks of massive computer clusters’ time to compute the radiation emitted from these sources. Instead, each mode can be computed in under a minute. Finally, we work under the adiabatic approximation. This means that our orbit evolves slowly from one orbit to the next. While this clearly would not be a good approximation in the situation that the compact object is plunging into the SMBH, at such early stages of the evolution, it is an excellent approximation.

Following Barack and Cutler [66], the parameter space which we are interested in is shown in Table 4.1. Ideally, we would also like to include inclined orbits ($\theta \neq \pi/2$), but this requires

<table>
<thead>
<tr>
<th>parameter</th>
<th>min</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$0.7M$</td>
<td>$0.95M$</td>
</tr>
<tr>
<td>$p$</td>
<td>$15M$</td>
<td>$200M$</td>
</tr>
<tr>
<td>$e$</td>
<td>0.999</td>
<td>0.999999</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$\pi/2$</td>
<td>$\pi/2$</td>
</tr>
</tbody>
</table>

Table 4.1: Parameter space for computation of capture source signal confusion noise

a rework of parts of the code which has only just begun. Some of these modifications will be discussed later. Current investigations with our code are in the direction of finding the most interesting parts of the waveform and only computing those modes. Otherwise, at high eccentricities we may be computing hundreds of thousands or even millions of modes that are negligible.
4.1.1 Computation of gravitational radiation from EMRIs

Using a point particle perturbation on a Kerr spacetime, we can simplify computation from something that needs weeks to run on a cluster of processors to something that can be completed in seconds (or minutes) on a personal computer. There are two types of codes that people often run for this situation. The first is a time domain code, where the Teukolsky master equation is solved directly using a partial differential equation solver. However, this approach requires a coordinate grid that becomes larger depending on the eccentricity of the orbit, and it therefore becomes much more expensive as the particle’s apastron becomes larger. The second type of code uses a Fourier transform to solve the Teukolsky equation in the frequency domain thereby breaking the time domain signal into its constituent modes. These modes are then summed in a particular way to return the time domain waveform. While this has the advantage of being relatively computationally inexpensive at lower eccentricities, it has the distinct disadvantage of having an increasingly large number of modes that need to be computed as the eccentricity increases.

Previous iterations of our frequency based Teukolsky solving code have existed for years. Two members of our group, Dan Kennefick and Kostas Glampedakis had one of the first working equatorial, eccentric EMRI orbit codes [21]. This new code is significant for several reasons:

- The new code is in Python which is dynamically typed, portable, and readable. This makes the code more accessible to new students.

- It is now multi-threaded (which is easily done in Python) and can be used on a cluster to speed up computations.
• Newer routines from the past ten years of research into such codes have been implemented. This puts us in a position to generalize the code to generic orbits.

4.1.2 Why Python?

Older codes are written primarily in Fortran or C, because these languages are fast. In Python the development process is streamlined with dynamical typing. This gets rid of the need to distinguish between variables being integers, floats, etc. Further, many packages in Python already have exactly what one would need for computation. Several robust packages such as the Scipy ecosystem [67] exist for doing linear algebra, root finding, minimization, and more. These packages have a Fortran backend wrapped in Python to increase the speed of computation significantly, closing the gap between faster languages and Python. Additionally, constructs such as pointers and containers, found in C and C++, can confuse beginners and obfuscate code.

Several other languages were considered throughout the code’s production. Partially or fully functional code in Julia, C, and C++ exists. Julia, an up and coming language, uses a syntax similar to Python and MATLAB. While it has a simple syntax, the existing scientific code is often not as robust as in Python. Also, like MATLAB, the indexing starts at one. On the upside, the entire Julia language is just in time compiled to its core. This means the code almost as fast as C or C++ after it is compiled the first time. While this sounds great, often the inner workings of how you can get a speed increase are difficult to understand. For example \((1 + x)^8\) is more quickly computed by using \(((1 + x)^2)^2)^2\).

Upon attempting the code in all of these other languages, we considered what would be best for future students who were attempting to join the relativity group at the University of
Arkansas, and more broadly, what tools would interest the scientific community. How could we write things so that other people could understand what was happening in the code? While Julia is readable, we decided that it was not as effective as Python, because it is still a relatively new language. The parts that need speed are programmed in C++. Those C++ functions can then be called from Python via Cython, a sublanguage that interfaces Python and C or C++. The end result is a function which is both fast and has the ease of use of Python functions.

A large number of functions for black hole perturbation theory already exist in the black hole perturbation toolkit (BHPT) [68]. The languages used vary widely from Python or SageMath [69], to the proprietary computer algebra software (CAS), Mathematica. Several of the currently used functions in my own code are translations of these Mathematica functions. Mathematica has some fast, easy to use functions, but the numerical algorithms used are hidden from the public.

So for the present, we use Python so that future students and collaborators may more easily be involved with the project, because the code is fairly easy to read and document, and because if speed is necessary it is easy to pull in C or C++ code. We will keep an eye on Julia for years to come as a possible language to port the code to.

4.2 Frequency based hypergeometric Kerr code

In this section I will attempt to show what the code does without the complications of the mathematics of black hole perturbation theory. For those who are interested, they may refer to a later section where some of the math has been worked out or for more information to references that have worked out the math while discussing their codes in detail [70, 71].
4.2.1 User interface

At the moment, the entire code runs from the terminal (or command line/Powershell for Windows users) in a Python file. The input required by the equatorial code consists of three of the four possible parameters: black hole spin ($a$), semi-latus rectum ($p$), eccentricity ($e$). The fourth parameter we hold fixed for equatorial orbits which is the orbital inclination ($\theta$).

4.2.2 Computation of Radiation to Infinity

The first section consists of several files related to computing geodesic motion most of which are translated directly from the BHPT [68]. Note that the computation for equatorial orbits is separated into two sections. Currently, the monodromy method used to calculate the $\nu$ value used for renormalization requires high precision to converge. So for all functions leading up to this computation, we use a high precision Python package called mpmath [72]. After calculating $\nu$, the code reverts to machine precision to compute everything else so that the speed of the code is faster.

First, we compute the adiabatic constants of motion: the energy ($E$), the $z$ component of angular momentum ($L_z$), and the Carter constant ($Q$) which corresponds to $L_z^2$ in the Schwarzschild black hole case. We call them adiabatic constants because they are really only constant in the adiabatic approximation where orbits evolve slowly. Although the Carter constant is not needed for equatorial orbits, the coordinates are computed using Mino time and then translated into angles that allow integration to proceed more easily. Because of this, we are able to compute geodesics of motion that are off axis which have non-zero Carter constant.
Next we compute the frequencies of motion $\Omega_r, \Omega_\theta, \Omega_\phi$. To do so, we compute the roots of the radial and polar geodesic equations, and then we find the frequencies of each direction of motion using Mino time. Finally, we may use these to compute the Boyer-Lindquist coordinate frequencies of motion. This allows us to compute the gravitational wave frequency as a sum of azimuthal ($m$) and radial ($k$) modes

$$\omega = m\Omega_\phi + k\Omega_r.$$  

Now we can use what we have computed so far in the angular Teukolsky equation. We first compute the eigenvalue at high precision using Leaver’s method [73]. This is the last thing required to prepare for the computation of $\nu$. We further use a spectral decomposition method as described in [74] at machine precision to compute the eigenvectors. The spectral method is quite fast at low precision by using Scipy’s Sparse matrix methods for matrix algebra, but it takes longer than Leaver’s method at higher precisions due to increased matrix size. For equatorial orbits it is not necessary to compute the eigenvalues and eigenvectors more than once per parameter set. This is not the case for off-axis orbits. The code that has been completed is prepared to be used for this inclined orbit case, and it can be modified to do so by allowing for complex results in this function.

$\nu$ is the last part computed at high precision. We use the monodromy method that is used in the BHPT [75]. In the future, to eliminate the need for high precision, we are considering wrapping the continued fraction solver that finds $\nu$ from the Gremlin code, developed by Scott Hughes and students [75]. This code could be called from Python by compiling the C++ code in Cython thereby eliminating the need for high precision in the code with the exception of the hypergeometric function routines.
Next, the asymptotic amplitudes are computed. The current code only computes $B_{in}$ for the radiation emitted to infinity. In all situations, the energy radiated down the large black hole is small compared to the energy radiated to infinity [66]. To compute $B_{in}$, we first need to compute the minimal solution to the continued fraction equation [75] using $\nu$.

The homogeneous solution for the Teukolsky radial equation can now be constructed. This involves computation of the Gauss hypergeometric functions, $\, _2F_1(a, b, c; z)$. Standard algorithms to compute such a function require us to worry about the relative size of the arguments. By using extra precision we can bypass a lot of these worries. In practice, Frederik Johannsen has done an excellent job of putting together algorithms that will compute these functions in the mpmath package for Python [72]. Previously, many iterations of the hypergeometric functions needed to be computed in our code. However, by using Gauss’s contiguous relations [76] (basically by taking derivatives) we can find identities between different hypergeometric functions with arguments differing by some integer values. This allows us to compute the required hypergeometric functions by multiplication and division instead of a sum which will include a large number of terms and many more multiplications and divisions.

The source term for the Teukolsky equation is found using an equation from the code in [21] for equatorial orbits. We have code for the general case, but have not debugged or tested it to see that it matches in the equatorial limit. Further, without any code to check against, we will need to do the integration and check the code against the literature to find out whether it gives good values for inclined orbits.

Finally, with the homogeneous solution and source in hand, we can compute the gravitational wave luminosity at infinity for an equatorial orbit. To do so we perform the integration
in [21] but with a modification given in [71] by which the integral is regularized so numerical integration routines do not have trouble with highly eccentric orbits.

The code as it stands cannot handle highly eccentric orbits yet, because the modification from [71] overflows as the eccentricity becomes too high without higher precision. However, a previous iteration of the code can handle this, because every step of the code works to 100 digits of accuracy which is enough to prevent overflow.

4.2.3 The Need for Speed

While individual modes may be cheap to compute, at higher eccentricities we will need to compute a large number of these modes on the order of thousands or hundreds of thousands to even get to an accuracy of one part in a million. Therefore, there is some necessity in speeding up the code as much as possible. One way which we may limit the computation time on the code is to *only* compute the modes which are non-negligible. This means that we are looking for modes that are within one part in a million of the largest mode.

There are three integer parameters that specify the modes: two from the gravitational wave frequency \((m, k)\) and one from the angular Teukolsky equation \((\ell)\). The largest modes is always found to be in the \((\ell, m) = (2, 2)\) mode. \(k\) is much harder to predict though. All we can say is that at low eccentricities the peak \(k\) value can be found near 0.

At high eccentricities, we can not predict, in general, at what \(k\) value we will find the peak value. Not all hope is lost! There are a couple of situations that may allow us to find the peak value.

For the \((\ell, m) = (2, 2)\) mode we can use a formula from the Peters and Matthews paper [77] as shown in [71]. This comes from a Newtonian order approximation of eccentric orbits.
For higher \((\ell, m)\) and off modes (where \(\ell \neq m\)) we will need to find a generalization so that we can predict where the peak \(k\) mode is.

The other situation is that we are close to the separatrix, the parameters that separate an adiabatic orbit from a plunge orbit. In this situation, we can use a formula in a paper by Oohara [78] that relates the orbital frequency at the last stable orbit and the component frequencies of the gravitational wave. Once we know the peak \(k\) for a semilatus rectum near the separatrix, then we could move slowly out in semilatus rectum so that the peak \(k\) moves by only 1 or 2 and is easy to find. In this way, we could eventually predict how far away we may need to move in \(k\) for a specific semilatus rectum value.

### 4.2.4 Future Efforts

Much of the code is sufficiently general to be used for generic orbits. After completing documentation and test cases, we plan on uploaded the entire code as a package to PyPI (the Python Package Index) so it can be installed by the public. Unlike other codes which people have made over the years, this code has never used any proprietary code locked under strict copyrights.

The end goal is to have a code on Github and PyPI that is open source, runs for generic orbits, and has many collaborators who are interested in this type of code. Eventually it would be nice to also have a time domain code too, but these tend to be more computationally expensive and may require using C++ or another statically typed language exclusively. Julia has an excellent differential equation solving package which is on par with anything you could find elsewhere and is worth investigating.
4.2.5 Semirelativistic Approximation

A faster alternative to computing thousands to hundreds of thousands of modes exists. By using the semirelativistic approximation, we can generate luminosities and gravitational waveforms quickly. Such waveforms are accurate to a closest approach of about $5M$ [79]. Given that we are interested in a closest approach of about that distance anyways, this should be sufficient for our needs. An additional advantage is that this approximation can handle inclined orbits.

We begin the semirelativistic approximation by computing the geodesic coordinates for the orbit

$$x^\mu = (t, r, \theta, \phi). \quad (4.2)$$

Current use of the approximation uses the BHPT to do this. We have to be careful, however, because the BHPT finds these coordinates in terms of the Mino time parameter $\lambda$. This will require a change of coordinates in the next part.

Now we project the geodesic coordinates onto flat space spherical polar coordinates:

$$x(\lambda) = r(\lambda) \sin \theta(\lambda) \cos \phi(\lambda) \quad (4.3)$$

$$y(\lambda) = r(\lambda) \sin \theta(\lambda) \sin \phi(\lambda) \quad (4.4)$$

$$z(\lambda) = r(\lambda) \cos \theta(\lambda) \quad (4.5)$$

Essentially we are requiring the orbit to be confined to a string in flat space that follows the orbit that the geodesic would map out in the Kerr spacetime. This causes certain mathematical inconsistencies [79], but otherwise gives an excellent approximation even with a periapsis of $5M$. 

73
Now we define

\[ s^i = (x, y, z), \]  

(4.6)

and then we compute the trace reduced quadrupole tensor

\[ Q_{ij} = s_is_j - \frac{1}{3}s_ks_k\delta_{ij}. \]  

(4.7)

The quadrupole formula requires the third time derivative of this equation. We find

\[ \dot{Q}_{ij} = \frac{d}{dt} (s_is_j - \frac{1}{3}s_ks_k\delta_{ij}) = \frac{d\lambda}{dt} \frac{d}{d\lambda} (s_is_j - \frac{1}{3}s_ks_k\delta_{ij}) \]  

(4.8)

where \( t_r(\lambda) = t(\lambda) - r(\lambda) \) is the retarded time and we use

\[ \frac{d\lambda}{dt} = \left( \frac{dt_r}{d\lambda} \right)^{-1} \]  

(4.9)

The second and third derivative follow in the same manner and we allow it to be computed by Mathematica. Upon computing the quadrupole's third time derivative, we can then use it in the quadrupole formula

\[ \frac{dE}{dt} = \frac{1}{5} \langle \cdots Q_{ij} \cdots \rangle. \]  

(4.10)

The angle brackets, \(<>\), denote a time average over more than one orbit. This can be done by integrating the time coordinate over the radial period of motion from 0 to \( T_r \) and then dividing by \( T_r \). Integration in time here is not straightforward. Instead, we must find \( \lambda \) that corresponds to \( t(\lambda) = T_r \) and then integrate over \( \lambda \).

4.3 Conclusion

Our new hypergeometric code is currently limited to equatorial orbits. However, the code is ready to make the jump to arbitrary orbits with only a few modifications. Equatorial orbits
have been tested up to eccentricities of 0.9999 with every set of values tested with both the old integration code and the new hypergeometric code agreeing to at least 5 digits (this is the accuracy requested by the integration code) up to $e = 0.9$ (see Table 4.2). At $e = 0.9$, the integrator gets confused by artificially converging integrands as the radiation computed for part of the orbit becomes small as apastron becomes very large. This is discussed and the solution which is used in our new code is suggested in [71]. It is important to note, however, that even though a solution is suggested for high eccentricity computations, there is no data in existing literature for eccentricities above $e = 0.9$. This is due to the large number of modes with no published way to predict where the largest $k$ modes are going to be for an arbitrary $(l, m)$ combination.

<table>
<thead>
<tr>
<th>$e$</th>
<th>Hypergeometric $\dot{E}_\infty$</th>
<th>Integration $\dot{E}_\infty$</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$5.035087397 \times 10^{-4}$</td>
<td>$5.035008872 \times 10^{-4}$</td>
<td>$1.559585113 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$2.383021958 \times 10^{-4}$</td>
<td>$2.382906092 \times 10^{-4}$</td>
<td>$4.862387706 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.7</td>
<td>$2.260540909 \times 10^{-5}$</td>
<td>$2.260688669 \times 10^{-5}$</td>
<td>$6.536052768 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$1.313202677 \times 10^{-5}$</td>
<td>$1.313197097 \times 10^{-5}$</td>
<td>$4.248852096 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$1.146469280 \times 10^{-6}$</td>
<td>$1.146467017 \times 10^{-6}$</td>
<td>$1.974447138 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.99</td>
<td>$1.663879141 \times 10^{-12}$</td>
<td>$1.665485271 \times 10^{-12}$</td>
<td>$9.643615435 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.999</td>
<td>$1.255464503 \times 10^{-17}$</td>
<td>$1.256281384 \times 10^{-17}$</td>
<td>$6.502371935 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.9999</td>
<td>$2.168059304 \times 10^{-22}$</td>
<td>$2.141161338 \times 10^{-22}$</td>
<td>$1.256232538 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 4.2: Comparison between high precision hypergeometric code and the older integration code used in [21] for $p = 3, a = 0.998, (\ell, m, k) = (2, 2, 0)$

This code which is written in the more modern Python language is as fast or faster for these computations than the integration routine written in Fortran. We also have a method of determining (with some computation and extrapolation), where the non-negligible modes are going to be. Now we are in a position to advance this code further than the Fortran code could have possibly gone. From the small subsection of the runs we have done so far
shown in the table above, we are convinced that the code works for at least the equatorial orbits that the old code can do, and we have some evidence that it is doing better on high eccentricities \((e > 0.9)\) than the old code can do. By using up to date methods, a modern programming language that is easy for new students to learn, and a modular programming paradigm, and the ability to run several modes simultaneously via multithreading, we are poised to attack the problem at hand!

4.4 Frequency Based Teukolsky Code: Equations and Methods

Using the Newman Penrose formalism, one arrives (after considerable calculation) at the Teukolsky master equation [21, 70, 71] for the Weyl scalar \(\Psi_4\). As \(r \to \infty\),

\[
\Psi_4 \to \frac{1}{2} \left( \tilde{h}_+ - i \tilde{h}_\times \right) \quad (4.11)
\]

The master equation is separable into parts which only depend on \(r\) and \(\theta\) by using a Fourier series

\[
\rho^{-4} \Psi_4 = \sum_{\ell m} \int_{-\infty}^{\infty} d\omega e^{-i\omega t + im\phi} -2S_{\ell m}^{\text{aw}}(\theta) R_{\ell m\omega}(r) \quad (4.12)
\]

where \(\rho = (r - ia \cos \theta)^{-1}\). The angular equation is solved by the spin weighted spheroidal harmonics \(-2S_{\ell m}^{\text{aw}}(\theta)\), and the radial equation is solved by \(R_{\ell m\omega}(r)\). The radial equation is

\[
\Delta^2 \frac{d}{dr} \left( \frac{1}{\Delta} \frac{dR_{\ell m\omega}}{dr} \right) - V(r) R_{\ell m\omega} = T_{\ell m\omega} \quad (4.13)
\]

where \(\Delta = r^2 - 2Mr + a^2\) and

\[
V(r) = -\frac{K^2 - 2is(r - M)K}{\Delta} - 4is\omega r + \lambda \quad (4.14)
\]
with \( K = (r^2 + a^2) \omega - ma \). \( \lambda \) is the eigenvalue of the angular equation,

\[
\begin{align*}
\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) + a^2 \omega^2 \cos^2 \theta - \frac{m^2}{\sin^2 \theta} + 4 \omega \cos \theta \\
+ \frac{4m \cos \theta}{\sin^2 \theta} - 4 \cot^2 \theta - 2 + E_{lm} \right] S_{lm}^{\omega} = 0.
\end{align*}
\]

(4.15)

. To find \( \lambda \), we use \( \lambda = E_{lm} + a^2 \omega^2 - 2am \omega \). We normalize the spin weighted spheroidal harmonics as

\[
\int_0^\pi |S_{lm}^{\omega}|^2 \sin \theta d\theta = 1.
\]

(4.16)

Solving the radial equation via Green’s function results in

\[
R_{\ell m \omega}(r) = \frac{1}{2i \omega C_{\ell m \omega}^{\text{trans}} B_{\ell m \omega}^{\text{inc}}} \left\{ R_{\ell m \omega}^{\text{up}}(r) \int_{r^+}^r dr' R_{\ell m \omega}^{\text{in}} T_{\ell m \omega} \Delta^{-2} \\
+ R_{\ell m \omega}^{\text{in}}(r) \int_r^\infty dr' R_{\ell m \omega}^{\text{up}} T_{\ell m \omega} \Delta^{-2} \right\}
\]

(4.17)

where

\[
\begin{align*}
R_{\ell m \omega}^{\text{in}} &\rightarrow \begin{cases} 
B_{\ell m \omega}^{\text{trans}} \Delta^2 e^{-iPr_*} & \text{for } r \rightarrow r^+ \\
\frac{3}{r^2} C_{\ell m \omega}^{\text{ref}} e^{iPr_*} + r^{-1} B_{\ell m \omega}^{\text{inc}} e^{-iPr_*} & \text{for } r \rightarrow \infty
\end{cases} \\
R_{\ell m \omega}^{\text{up}} &\rightarrow \begin{cases} 
C_{\ell m \omega}^{\text{up}} e^{iPr_*} + \Delta^2 C_{\ell m \omega}^{\text{ref}} e^{-iPr_*} & \text{for } r \rightarrow r^+ \\
\frac{3}{r^3} C_{\ell m \omega}^{\text{trans}} e^{iPr_*} & \text{for } r \rightarrow \infty
\end{cases}
\end{align*}
\]

(4.18)

(4.19)

\( P = \omega - ma/2Mr_+ \) and \( r_* \) is found by

\[
r_* = r + \frac{2Mr_+}{r^+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r^+ - r_-} \ln \frac{r - r_-}{2M}
\]

(4.20)

where \( r_\pm = M \pm \sqrt{M^2 - a^2} \). The solution at the horizon is thus

\[
R_{\ell m \omega}(r \rightarrow r^+) = \frac{B_{\ell m \omega}^{\text{trans}} \Delta^2 e^{-iPr_*}}{2i \omega C_{\ell m \omega}^{\text{trans}} B_{\ell m \omega}^{\text{inc}}} \int_{r^+}^\infty dr' R_{\ell m \omega}^{\text{up}} T_{\ell m \omega} \Delta^{-2} \equiv Z_{\ell m \omega}^H \Delta^2 e^{-iPr_*}
\]

(4.21)

and the solution at infinity is

\[
R_{\ell m \omega}(r \rightarrow \infty) = \frac{r^3 e^{iPr_*}}{2i \omega B_{\ell m \omega}^{\text{inc}}} \int_{r^+}^\infty dr' R_{\ell m \omega}^{\text{up}} T_{\ell m \omega} \Delta^{-2} \equiv Z_{\ell m \omega}^\infty r^3 e^{iPr_*}
\]

(4.22)
Note that this is a different set of definitions than usual. Here we follow [70] with \( Z^H_{\ell m \omega} \) at the horizon and \( Z^\infty_{\ell m \omega} \) at infinity. This is in contrast to the way they were originally defined which is somewhat confusing. The original definitions can be seen in Eq. (2.9) and (2.10) of [21]. We consider a point particle perturbation on the background spacetime. Therefore,

\[
T^{\mu \nu} = \mu \frac{u^\mu u^\nu}{\sum \sin \theta u^t} \delta(r - r(t)) \delta(\theta - \theta(t)) \delta(\phi - \phi(t)) \tag{4.23}
\]

where \( u^\mu = dx^\mu / d\tau \). Next, this is projected onto the null tetrad [21] and the pieces (labeled by \( A_{ab} \)) are used in the solution

\[
Z^H_{\ell m \omega} = \frac{\mu B^\text{trans}_{\ell m \omega}}{2i\omega B^\text{inc}_{\ell m \omega}} \int_{-\infty}^{\infty} dt e^{i\omega t - i\phi(t)} T^H_{\ell m \omega}[r(t), \theta(t)] \tag{4.24}
\]

\[
Z^\infty_{\ell m \omega} = \frac{\mu}{2i\omega B^\text{inc}_{\ell m \omega}} \int_{-\infty}^{\infty} dt e^{i\omega t - i\phi(t)} T^\infty_{\ell m \omega}[r(t), \theta(t)].
\]

where

\[
T^H_{\ell m \omega} = \left[ R^\text{in}_{\ell m \omega} \{ A_{n0} + A_{\bar{m}0} + A_{\bar{m}m0} \} \right. \\
\left. - \frac{dR^\text{in}_{\ell m \omega}}{dr} \{ A_{\bar{m}1} + A_{\bar{m}1} \} + \frac{d^2 R^\text{in}_{\ell m \omega}}{dr^2} A_{\bar{m}m2} \right]_{r=r(t), \theta=\theta(t)} \tag{4.25}
\]

and

\[
T^\infty_{\ell m \omega} = \left[ R^\text{in}_{\ell m \omega} \{ A_{n0} + A_{m0} + A_{\bar{m}m0} \} \right. \\
\left. - \frac{dR^\text{in}_{\ell m \omega}}{dr} \{ A_{mn1} + A_{mn1} \} + \frac{d^2 R^\text{in}_{\ell m \omega}}{dr^2} A_{mm2} \right]_{r=r(t), \theta=\theta(t)} \tag{4.26}
\]

In the following, we use \( \rho = (r - i \cos \theta)^{-1} \), and a bar denotes complex conjugation. Further, for simplicity, we use \( S(\theta) \) for the spin-weighted spheroidal harmonics, and a prime to denote a derivative with respect to the argument. In the following, we use \( \rho = (r - i \cos \theta)^{-1} \), and a bar denotes complex conjugation. Further, for simplicity, we use \( S(\theta) \) for the spin-weighted spheroidal harmonics, and a prime to denote a derivative with respect to the argument.

\[
A_{n0} = -\frac{1}{\Lambda^2} C_{nn} \sqrt{\frac{2}{\pi \rho^2 \bar{\rho}}} \left( FS(\theta) + GS'(\theta) + \rho^{-1} S''(\theta) \right) \tag{4.27}
\]
where

\[ F = 2 \cos(\theta)(-2i\omega \cos(\theta) + a\omega(2r + im) - 2i) - 2m(ar\omega - i) \]
\[ + a\omega \sin^2(\theta)(-i\omega \cos(\theta) + ar\omega - 2i) + (m^2 - 2)r \csc^2(\theta) - i \cot(\theta) \csc(\theta) \left( m^2 - 2imr - 2 \right) \]
\[ + 2 \cot^2(\theta)(-i\cos(\theta) + im + r) \quad (4.28) \]

and

\[ G = \csc(\theta)\rho^{-1}(-a\omega \cos(2\theta) + a\omega + 3\cos(\theta) - 2m) - 2i \sin(\theta) \quad (4.29) \]

\[ A_{mn0} = \frac{2C_{mn\rho^{-3}}}{\sqrt{\pi \Delta}} \left( HS(\theta) + \left( \frac{2r}{\cos^2(\theta) + r^2} + \frac{iK}{\Delta} \right) S'(\theta) \right) \quad (4.30) \]

where

\[ H = \left( \frac{2r}{\cos^2(\theta) + r^2} + \frac{iK}{\Delta} \right)(a\omega \sin(\theta) + 2 \cot(\theta) - m \csc(\theta)) + \frac{2iaK \sin(\theta) \cos(\theta)}{\Delta \cos^2(\theta) + \Delta r^2} \quad (4.31) \]

and

\[ K = \omega \left( a^2 + r^2 \right) - am \quad (4.32) \]

\[ A_{mn1} = \frac{2C_{mn}}{\sqrt{\pi \Delta \rho^3}} \left( S(\theta) \left( a\omega \sin(\theta) + \frac{a\sin(2\theta)}{\cos^2(\theta) + r^2} + 2 \cot(\theta) - m \csc(\theta) \right) + S'(\theta) \right) \quad (4.34) \]

\[ A_{mn2} = \frac{\sqrt{2}C_{mn\rho}}{\sqrt{\pi \Delta}(-\rho^2)} (S(\theta)(\Delta + K \cos(\theta) + iKr)) \quad (4.35) \]

\[ A_{mn2} = \frac{1}{\sqrt{2\pi}} \rho^{-3} \rho C_{mn} S(\theta) \quad (4.36) \]

Now we need the \( C_{ab} \) values that are used in these pieces.

\[ C_{nn} = \frac{d\lambda}{dt} \frac{\mu}{4\Sigma^2} \left[ E \left( r^2 + a^2 \right) - aL_z + \frac{dr}{d\lambda} \right]^2 \quad (4.37) \]
\[
C_{\tilde{m}\tilde{n}} = \frac{d\lambda}{dt} \mu \rho^2 \left[ i \left( aE - \frac{L_z}{\sin^2 \theta} \right) \sin \theta + \frac{d\theta}{d\lambda} \right]^2 
\] (4.38)

\[
C_{\tilde{m}\tilde{n}} = \frac{d\lambda}{dt} \frac{\mu \rho}{2\sqrt{2}\Sigma} \left[ E (r^2 + a^2) - aL_z + \frac{dr}{d\lambda} \right] \left[ i \left( aE - \frac{L_z}{\sin^2 \theta} \right) \sin \theta + \frac{d\theta}{d\lambda} \right] 
\] (4.39)

Note that this uses the Mino time parameter \( \lambda \) for the geodesic values which is related to proper time as

\[
\frac{d\tau}{d\lambda} = \Sigma 
\] (4.40)

where \( \Sigma = r^2 + a^2 \cos^2 \theta \).
Bibliography


[54] S. A. Usman et al., Classical and Quantum Gravity 33, 215004 (2016).


[60] P. A. Ade et al., Astronomy and Astrophysics (2016), 10.1051/0004-6361/201525830,


[71] S. Drasco and S. A. Hughes, Physical Review D 73 (2005), 10.1103/Phys-


