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Finite Dimensional Approximation and Pin(2)-equivariant Property for Rarita-Schwinger-Seiberg-Witten Equations

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Finite Dimensional Approximation and $Pin(2)-$equivariant Property for Rarita-Schwinger-Seiberg-Witten Equations

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Abstract

The Rarita-Schwinger operator Q was initially proposed in the 1941 paper by Rarita and Schwinger to study wave functions of particles of spin 3/2, and there is a vast amount of physics literature on its properties. Roughly speaking, $3/2-$spinors are spinor-valued 1-forms that also happen to be in the kernel of the Clifford multiplication. Let X be a simply connected Riemannian spin $4-$manifold. Associated to a fixed spin structure on X, we define a Seiberg-Witten-like system of non-linear PDEs using Q and the Hodge-Dirac operator $d^* + d^+$ after suitable gauge-fixing. The moduli space of solutions $\mathcal{M}$ contains (3/2-spinors, purely imaginary 1-forms). Unlike in the case of Seiberg-Witten equations, solutions are hard to find or construct. However, by adapting the finite dimensional technique of Furuta, we provide a topological condition of X to ensure that $\mathcal{M}$ is non-compact; and thus, $\mathcal{M}$ contains infinitely many elements.

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1 Introduction

Gauge theory is the study of the analysis of certain partial differential equations stemmed from particle physics. Its applications to low-dimensional topology are of significance. Ever since the 80s, from the work of Atiyah, Donaldson, and Witten, gauge theory has proved to be a powerful tool. Its advances have lead to some of the powerful results in problems of smooth 4-dimensional manifolds such as Donaldson's *Diagonalization Theorem, Furuta's 10/8th theorem,* etc.

One of the classical invariants of 4-manifolds is intersection forms. Given a 4-manifold X, for simplicity assume that X is closed and simply connected, embedded oriented surfaces in X can intersect in interesting ways. If the surfaces intersect transversally, one would only have a finite number of intersections. The intersection form Q_X evaluated at the surfaces is exactly the sign count of the intersections. Algebraically, since every embedded oriented surface in X can be represented by a homology class in $H^2(X;\mathbb{Z})$, Q_X can also be thought of as the composition

$$H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to H^4(X;\mathbb{Z}) \to H_4(X;\mathbb{Z}) \cong \mathbb{Z},$$

where the first map is given by the cup product and the second map is Poincare duality. As a result, Q_X is a *unimodular symmetric form* defined on finite-generated free abelian group. The invariants of Q_X are tied directly to invariants of X. For example, $\operatorname{rank} Q_X = b_2(X)$, where b_2 is the usual second betti number of X; and the signature of Q_X is exactly the signature of the 4$-$manifold. Thus, intersection form is a homotopy invariant of 4$-$manifolds, i.e, two manifolds which have inequivalent intersection forms cannot be homotopically equivalent. However, what is even more remarkable is that in 1982, Michael Freedman proved a theorem that suggests that the homeomorphism type of the manifold depends *only* on the intersections forms [8]. Effectively, Freedman's theorem affirms the 4$-$dimensional topological Poincare conjecture.

Coming to the smooth category, the situation can be a bit trickier. Since for every smooth $4-$manifold, its Kirby-Siebenmann class must vanish. Thus, it is reasonable to suspect that the intersection form may carry some information about smooth structure of the manifold. Specifically, there are two natural questions: Given a symmetric unimodular form Q

Question 1.1. *Can Q be realized as the intersection form of a closed simply connected smooth $4-$manifold?*

Question 1.2. *How many non-diffeomorphic $4-$manifolds can realize Q?*

These two questions constitute what is called *geography problems of simply connected smooth 4-manifolds.*

Regarding question 1, we now almost have a complete answer. If Q is definite, Donaldson in 1983 proved that Q can be realized on a smooth manifold if and only if $Q \cong \pm I$ [4]. The proof of this result is the first striking and unexpected application of gauge theory to the geography problems. The key ingredient in Dondaldson's proof is that the moduli space of solutions to the Yang-Mill equation $F_A^+ = 0$ associated to a certain principal $SU(2)-$bundle $P \to X$, as a smooth oriented $5-$dimensional manifold, is a cobordism between $\#_k \mathbb{C}P^2$ (or $\#_k \overline{\mathbb{C}P^2}$) and X.

Another proof of Donaldson's theorem can also be obtained by gauge theory of the Seiberg-Witten equations [7]. Given a spin-c structure on X, the Seiberg-Witten equations are

$$D_A^+ \psi = 0, \quad F_A^+ = \rho^{-1}(\psi\psi^*)_0 + \eta,$$

where D_A^+ is the twisted Dirac operator given by a choice of a unitary connection A on the determinant line bundle associated to a spin-c structure of X, ψ is a positive spinor and $(\psi\psi^*)_0$ is the traceless part of the endomorphism on the spinor bundle, η is some generic self-dual 2-forms on X. Fitted into abelian $U(1)-$gauge theory, the analysis of the Seiberg-Witten equations are much simpler than the Yang-Mill equation in Donaldson theory. What is even more 'miraculous' is that the moduli space of solutions to the Seiberg-Witten equation is *a priori* compact! Armed with this key fact, the proof followed from a contradiction argument: When the intersection form Q is negative definite, then with respect to a spin-c structure whose first Chern class of its determinant line bundle is equal to an arbitrarily given characteristic vector of Q, the virtual dimension $d+1$ of the moduli space of solutions cannot be positive, i.e,

$$d+1 = \frac{1}{4}(Q(w,w) - 3\sigma(X) - 2\,\chi(X)) = \frac{1}{4}(Q(w,w) + b_2(X)) \leq 0,$$

where $\sigma(X)$ is the signature of the $4-$manifold X. This is because, otherwise, truncating around the reducible point of the moduli space would obtain a contradiction to the Pontrjagin's theorem. Now if Q is non-standard, Elkies showed that there is a characteristic vector w such that

$$-Q(w,w) = |Q(w,w)| < \operatorname{rank} Q = b_2(X).$$

This is a contradiction to the previous claim, which implies that $Q \cong -I$.

In the case Q is indefinite, Q could be either even or odd. A classification theorem of Hasse-Minkowski if Q is odd, it is equivalent to a diagonal matrix whose entries are ± 1, otherwise, $Q \cong p\,E8 \oplus q\,H$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We already know that $\#_k \mathbb{C}P^2 \#_l \overline{\mathbb{C}P^2}$ can realize Q when Q is odd indefinite. As a result, what remains is the case of even indefinite intersection forms.

Since a smooth $4-$manifold is *spin* if and only if its intersection form is even, spin is the right category to look for manifolds that can realize even indefinite forms. Furthermore, Rokhlin's theorem tells us that signature of a spin manifold must be divisible by 16 [22]. As a result, its intersection form must be equivalent to $Q \cong 2k\,E8 \oplus q\,H$. Without loss of generality, we may assume $k \geq 0$. In 1982, Matsumoto proposed the following conjecture

Conjecture 1.3. *The form $2k\,E8 \oplus q\,H$ can be realized on a simply connected closed smooth spin $4-$manifold if and only if $q \geq 3k$.*

One direction is rather trivial. If $q \geq 3k$, $2k\,E8 \oplus q\,H$ can always be realized on $\#_k\,K3\#_{q-3k}\,S^2 \times S^2$. Thus, the other direction would be equivalent to the following version

Conjecture 1.4. *If X is a simply connected closed smooth spin $4-$manifold, then*

$$b_2(X) \geq \frac{11}{8}\,|\sigma(X)|.$$

As of right now, the conjecture is still open, though substantial progress has been made over the years. Donaldson used Yang-Mill equation to show that if $H_1(X;\mathbb{Z})$ has no 2-torsions and $k = 1$, then $q \geq 1$ [5]. After the introduction of Seiberg-Witten equations, Kronheimer improved the Donaldson's result by removing the torsion condition. In 2001, Furuta announced a result which is 'closest' to the 11/8th conjecture thus far

Theorem 1.5 ([10]). *If X is a simply connected closed smooth spin $4-$manifold, then*

$$b_2(X) \geq \frac{10}{8}\,|\sigma(X)| + 2.$$

Furuta's proof also relies on the analysis of the Seiberg-Witten equations. However, what is different about Furuta's approach is that he considered the Seiberg-Witten functional $\mathcal{F}$, rather than the moduli space of solutions. The Seiberg-Witten functional is $Pin(2)-$equivariant. Exploiting this extra symmetry, Furuta showed that there exists a finite dimensional approximation of $\mathcal{F}$ that also satisfies a certain Borsuk-Ulam hypothesis, which induces a map between representation spheres of $Pin(2)$. Then an application of equivariant K-theory deduces the result.

Recently, Hopkins, Lin, Shi, and Xu extended the method of Furuta further and combined with Mahowald invariants to obtain their $10/8 + 4$-Theorem. But they also showed that these are best coefficients we can get by the current approaches via Seiberg-Witten theory.

Theorem 1.6 ([15])**.** *If X is a simply connected closed smooth spin $4-$manifold that is not homeomorphic to either $S^4, S^2 \times S^2$, or $K3$, then*

$$b_2(X) \geq \frac{10}{8} |\sigma(X)| + 4.$$

The two crucial points in the proofs of Theorem 1.5 and Theorem 1.6 lie in the fact that the moduli space of solutions to the Seiberg-Witten equations is already compact, and the presence of extra symmetry coming from $Pin(2)$. The finite dimensional technique is necessary to put every ingredient in finite dimension context, which is really the most technical part of the proof. The majority of this thesis focuses on this technique but applies to the setting where the *Rarita-Schwinger operator* replaces the usual Dirac operator in the Seiberg-Witten equations.

The Rarita-Schwinger operator $\mathcal{Q}$ was first introduced by Rarita and Schwinger in a 1941 paper to describe the wave functions of the so-called *3/2-spinors* [20]. These higher spinors that live in kernel of $\mathcal{Q}$ are called *Rarita-Schwinger fields*. In Physics, Rarita-Schwinger fields are important in the study of supergravity and superstring theory; thus, its literature in this area of science is vast. In Mathematics, this operator has not been studied as much with the exception in the classical context of Clifford analysis.

Within geometric setting, Branson and Hijazi showed that $\mathcal{Q}$ is conformally invariant; they also wrote down a Weitzenbock-type formula for $\mathcal{Q}^2$ [3]. Based on its Weitzenbock formula, $\mathcal{Q}^2$ has a lower order term whose analysis is not immediate under usual geometric hypothesis, which could imply that the relation between the kernel of $\mathcal{Q}$ with the geometry of the underlying space is harder to achieve. M. Wang also studied the role the solutions to $\mathcal{Q}$ in the deformation theory of Einstein metrics that admits parallel spinors [25]. Recently in 2018, Homma and Semmelmann considered the problem of "counting solutions" to $\mathcal{Q}$

[14]. In their work, they gave a complete classification of positive quaternion-Kahler manifolds and spin symmetric spaces that have *non-trivial* Rarita-Schwinger fields. In 2021, a paper of Bar and Mazzeo showed the existence of a sequence of closed simply connected negative Kahler-Einstein spin manifolds for which the dimension of Ker $\mathcal{Q}$ tends to infinity [2]. These results could allude to the fact that $\mathcal{Q}$ in the setting of a Seiberg-Witten-type theory may have something new to say about smooth simply connected spin 4−manifolds.

In this thesis, we study the *Rarita-Schwinger-Seiberg-Witten equations* (RS-SW). From appearance, the RS-SW equations do not look that much different from the usual Seiberg-Witten equations. We shall save a proper introduction later. But for now, given a closed spin 4−manifold X and a spin structure $\mathfrak{s}$, associated to a line bundle $L \to X$, the RS-SW equations read as

$$\mathcal{Q}_A^+\psi = 0, \quad F_A^+ = \rho^{-1}(\psi\psi^*)_0 + \eta, \tag{1.1}$$

where A is a unitary connection on L, ψ is a twisted positive 3/2−spinor, η is some fixed self-dual 2−form, and $(\psi\psi^*)_0$ is a section of $i\mathfrak{su}(\mathfrak{s}^+)$ that is a traceless self-adjoint endomorphism. The equations still fit into abelian gauge theory with structure group $U(1)$.

Let X be a 4−dimensional smooth spin manifold that is closed, and $\mathfrak{s}_{1/2}$ be a fixed spinor bundle over X. Since X is smooth, one has a Riemannian metric g on X. We will also fix this metric. Denote $\rho : TX \otimes \mathfrak{s}_{1/2} \to \mathfrak{s}_{1/2}$ by the Clifford multiplication. As usual, since the manifold is even dimensional, there is an orthogonal decomposition of $\mathfrak{s}_{1/2} = \mathfrak{s}_{1/2}^+ \oplus \mathfrak{s}_{1/2}^-$ such that the Clifford multiplication ρ exchanges the chirality.

Consider $\ker \rho$, which is another vector bundle over X. This bundle is often denoted by $\mathfrak{s}_{3/2}$ and its sections are called 3/2−spinors. With respect to the chirality of $\mathfrak{s}_{1/2}$, we also have the so-called positive and negative 3/2−spinor bundles $\mathfrak{s}_{3/2}^\pm$. Note that there is an orthogonal decomposition $TX \otimes \mathfrak{s}_{1/2}^\pm = \mathfrak{s}_{1/2}^\mp \oplus \mathfrak{s}_{3/2}^\pm$.

On the bundle $\mathfrak{s}_{1/2}$, there are two natural first order differential operators. The Dirac operator $D : \Gamma(\mathfrak{s}_{1/2}) \to \Gamma(\mathfrak{s}_{1/2})$ is defined by $D = \rho \circ \nabla^{\mathfrak{s}_{1/2}}$; and the twistor operator $P : \Gamma(\mathfrak{s}_{1/2}) \to \Gamma(\mathfrak{s}_{3/2})$ is given by $P = \pi_{3/2} \circ \nabla^{\mathfrak{s}_{1/2}}$. Here, $\nabla^{\mathfrak{s}_{1/2}}$ is the canonical spinor connection with respect to the fixed metric g on X ,and $\pi_{3/2}$ is the projection of $T^*X \otimes \mathfrak{s}_{1/2} = TX \otimes \mathfrak{s}_{1/2} \to \mathfrak{s}_{3/2}$. Taking into account of the chirality, we note that $D^{\pm} : \Gamma(\mathfrak{s}^{\pm}_{1/2}) \to \Gamma(\mathfrak{s}^{\mp}_{1/2})$. Whereas, $P^{\pm} : \Gamma(\mathfrak{s}^{\pm}_{1/2}) \to \Gamma(\mathfrak{s}^{\pm}_{3/2})$.

Next, consider the twisted Dirac operator $\mathcal{D}^{\pm} : \Gamma(\mathfrak{s}^{\pm}_{1/2} \otimes TX) \to \Gamma(\mathfrak{s}^{\mp}_{1/2} \otimes TX)$. The compatible connection on this Clifford bundle is constructed by twisting the spinor connection $\nabla^{\mathfrak{s}_{1/2}}$ by the Levi-Civita connection ∇^{LC} in the usual way. Let ι be the embedding of $\mathfrak{s}^{\mp}_{1/2}$ into $\mathfrak{s}^{\pm}_{1/2} \otimes TX$. Then the twisted Dirac operator $\mathcal{D}^{\pm}$ takes the matrix form, with respect to the decomposition $TX \otimes \mathfrak{s}^{\pm}_{1/2} = \mathfrak{s}^{\mp}_{1/2} \oplus \mathfrak{s}^{\pm}_{3/2}$, as

$$D^{\pm}_{TX} = \begin{pmatrix} -\frac{1}{2}\iota \circ D^{\pm} \circ \iota^{-1} & 2\iota \circ P^{+*} \\ \frac{1}{2}P^{+} \circ \iota^{-1} & \mathcal{Q}^{\pm} \end{pmatrix}.$$

The operator $\mathcal{Q}^{\pm} : \Gamma(\mathfrak{s}^{\pm}_{3/2}) \to \Gamma(\mathfrak{s}^{\mp}_{3/2})$ given by $\mathcal{Q}^{\pm} = \pi^{\mp}_{3/2} \circ D^{\pm}_{TX}|_{\Gamma(\mathfrak{s}^{\pm}_{3/2})}$ is called the Rarita-Schwinger operator. Of course, the whole construction carries out when we want to define twisted version of the Rarita-Schwinger operator for $3/2-$spinors with coefficients in some line bundle L.

Let L be any complex line bundle over X and A be any unitary connection on L. We denote $\mathcal{Q}^{\pm}_A$ by the twisted Rarita-Schwinger operator associated to the bundles $\Gamma(\mathfrak{s}^{\pm}_{3/2} \otimes L)$, and P_A by their corresponding twistor operators. Besides looking for twisted Rarita-Schwinger fields, we impose a curvature condition for A. What follow is referred as (unperturbed) Rarita-Schwinger-Seiberg-Witten equations (RS-SW),

$$\mathcal{Q}^{+}_A\psi = 0, \quad F^{+}_A = \rho^{-1}(\psi\psi^*)_0 := \rho^{-1}(\mu(\psi)) \tag{1.2}$$

If A_0 is a fixed reference unitary connection of L, then every other connection $A = A_0 + ia$, where $a \in \Omega^1(X)$. We introduce some notations that are going to be used throughout the notes.

Definition 1.1. We define $\mathcal{C} = \Gamma(\mathfrak{s}^+_{3/2} \otimes L) \oplus \Omega^1(X)$ to be the configuration space of the RS-SW equations. $\mathcal{R} = \Gamma(\mathfrak{s}^-_{3/2} \otimes L) \oplus \Omega^+(X)$ is denoted by its range space.

The gauge group $\mathcal{G} = \{\text{Smooth maps } X \to S^1\}$ acts on $\mathcal{C}$ and $\mathcal{R}$ by pulling-back the connections and left multiplying with conjugation on the twisted $3/2-$spinors. Not much different from the standard Seiberg-Witten theory, we note that if (ψ, A) is a solution to (1.1), then elements of its orbit by the group action of $\mathcal{G}$ are also solutions to the RS-SW equations. Furthermore, we also have a local slice theorem, i.e, there is a representative $A_0 + ia$ in the orbit of A such that $d^*a = 0$. Therefore, when taking into account of the gauge symmetry, the solutions of the following equations are what we study in this thesis

$$\mathcal{Q}^+_{A_0}\psi + \pi^-_{3/2} \circ \rho(a)\psi = 0, \quad d^*a = 0, \quad d^+a + F^+_{A_0} = \rho^{-1}(\mu(\psi)). \tag{1.3}$$

Question 1.7. *Is there any non-trivial solution to equations (1.2)?*

To answer part of this question, we provide a necessary condition which ensure that there is always non-trivial solution to equations (1.2) when L is a trivial line bundle. In this situation, we may take A_0 to be trivial, then (1.2) can be rewritten as

$$\mathcal{Q}^+\psi + \pi^-_{3/2} \circ \rho(a)\psi = 0, \quad d^*a = 0, \quad d^+a = \rho^{-1}(\mu(\psi)). \tag{1.4}$$

Let $\mathcal{D} = \mathcal{Q}^+ \oplus (d^* \oplus d^+)$ be a map from $\mathcal{C} \to \mathcal{R} \oplus \Omega^0$ and $\mathcal{Q} : \mathcal{C} \to \mathcal{R} \oplus \Omega^0$ be quadratic map given by $\mathcal{Q}(\psi, a) = [(\pi^-_{3/2}(a)\psi)] \oplus [-\rho^{-1}(\mu(\psi))]$. It turns out that the so-called $3/2-$monopole $\mathcal{F} = \mathcal{D} \oplus \mathcal{Q}$ is also $Pin(2)-$equivariant. Therefore, when the finite dimensional technique is applied to $\mathcal{F}$, with computations in $Pin(2)-$equivariant K-Theory, we proved that

Theorem 1.8. *Suppose that the moduli space of solutions $\mathcal{M} := \mathcal{F}^{-1}(0)$ for the Rarita-Schwinger-Seiberg-Witten equations on a closed, simply connected, spin $4-$manifold*

X with $b_2^+(X) \geq 1$ is compact. Then we must have

$$b_2(X) \geq \frac{15}{4}\sigma(X) + 2.$$

The above result obviously says something stronger, i.e, we have a topological necessary condition for an analytic result. In particular if $b_2(X) < 15/4\,\sigma(X) + 2$, $\mathcal{M}$ is never compact regardless of an *a priori* choice of Riemannian metric imposed on X. And thus, it must contain solutions of RS-SW equations other than the trivial one. Moreover, Theorem 1.8 tells us that the moduli space of solutions of the RS-SW equations is not expected to always be compact like the usual Seiberg-Witten equations.

Question 1.9. *Is there any closed, simply connected, spin $4-$manifold with compact $\mathcal{M}$?*

This would be the next natural question to ask in consideration of Theorem 1.8. It is not hard to manufacture manifolds with non-compact $\mathcal{M}$. For example, for any $n < 20$, $\overline{K3}\#_n S^2 \times S^2$ would not have compact moduli space of solutions with respect to all Riemannian metrics. Similarly, $\mathcal{M}$ associated to $\#_n\overline{K3}$ is also never compact for all n. More interesting examples of manifolds that always posses non-compact $\mathcal{M}$ are those constructed by Akhmedov, Park, and Urzua [1]. The sequence of manifolds $\{M_k\}_{k\geq 2}$ they constructed consists only of simply connected spin symplectic $4-$manifolds that are near the BMY line.

On the other hand, any smooth spin $4-$manifold that has indefinite intersection form with negative signature would automatically satisfy the $15/4-$bound above by direct calculations. Furthermore, a "perturbed" version of $\mathcal{M}$ associated to such manifold is actually empty (hence obviously compact)! In particular, we also will prove the following transversality result

Theorem 1.10. *Suppose X is a simply connected smooth spin $4-$manifold such that $b_2^+(X) \geq 1$. Furthermore assume that for every (a, ψ) an solution to the RSSW equations,*

$\mathcal{H}^2_{(a,\psi)} = 0$. *Then there is a generic self-dual* $2-$*form* ω *on* X *such that the following holds. The gauge equivalence classes of pairs* $[a, \psi]$ *that solves the perturbed RSSW equations:*

$$Q^+\psi + \pi^-(a \cdot \psi) = 0, \qquad d^+a = \rho^{-1}(\mu(\psi)) + \omega$$

forms a smooth manifold of dimension

$$d = \frac{19}{4}\sigma(X) - b_2^+(X) - 1.$$

With respect to $\mathcal{M}_\eta$ being compact, one can also obtain the $15/4-$bound as in Theorem 1.8. Thus it is interesting to know if there is a manifold with $d \geq 0$ and $\mathcal{M}_\eta$ is also compact non-trivially.

Question 1.11. *Is there a Seiberg-Witten-type invariant associated to the RS-SW equations?*

After the transversality result above, this should be a natural question to consider. Obviously when $\mathcal{M}$ (or $\mathcal{M}_\eta$) is *a priori* compact, the same program to define the usual Seiberg-Witten invariant can be applied. However it is rather complicated when the moduli space of solutions is not compact. There is a possible direction one can take to deal with non-compactness issue.

One approach is via higher index theory and analytic K-homology theory. It can be shown that $\mathcal{M}_\eta$ is a complete Riemannian manifold. On top of that, there is a line bundle $\mathcal{L} \to \mathcal{M}_\eta$ given by the space of solutions to RS-SW equations quotient out by the base gauge group $\mathcal{G}_0$. Then the twisted signature operator $\mathcal{D}_\mathcal{L}$ with coefficients in $\mathcal{L}$ defined a $K-$homology class $[\mathcal{D}_\mathcal{L}] \in K_*(\mathcal{M}_\eta)$. A notion of *coarse equivalence* should ensure that this class $[\mathcal{D}_\mathcal{L}]$ should be invariant under any generic path of perturbation in $H^+(X;\mathbb{R})$ up to a "universal" $K-$homology. See [12] and [13] for more details.

We shall save the developments of the ideas discussed above in our future work.

Remark 1. On $\mathbb{R}^4$, (1.2) is just a system of Seiberg-Witten equations for multi-monopoles $\{\psi_i\}_{i=0}^3$ that Taube and several other people have considered with an extra condition that $-\psi_0 + I\psi_1 + J\psi_2 + K\psi_3 = 0$, where I, J, K are the usual Pauli spin matrices.

Remark 2. In general, by Theorem 1.8, $X\#S^2 \times S^2$ might still have non-compact moduli space of solutions under a certain topological data assumed about X. If there is a smooth invariant associated to the RS-SW equations, then it is possible that such invariant of $X\#S^2 \times S^2$ is non-zero. In contrast to Seiberg-Witten theory, Taubes showed that the Seiberg-Witten invariant of such manifold would always be zero [16].

Remark 3. Recall that in Seiberg-Witten theory, solutions to the Seiberg-Witten equations have a magical *a priori* C^0-bound which is crucial in the proof of compactness of the moduli space of solutions. Theorem 1.8 tells us compactness of solutions is not expected for RS-SW under a certain topological condition. However, it is interesting to know if there is still an analogous statement that can be made about solutions to RS-SW equations.

In particular, what sufficient conditions can be said about the Riemannian metric g such that $\mathcal{M}_g$ (perturbed or not) is compact? Due to two technicality that need to be resolved, standard techniques cannot be applied directly in this setting as one would hope: A, there is a "divergence" term that appears in $\mathcal{Q}_A^2$. B, $|\mu(\psi)|^2 \neq |\psi|^4/2$ just like in the multi-monopole setting.

The organization of the thesis is as follows:

In section 2, we briefly introduce some notions about differential geometric aspect of connections and curvatures associated to vector bundles. Basic ideas about characteristic classes and Hodge theory will also be discussed.

In section 3, we discuss some elementary facts about Clifford algebra and how it ties into spin geometry and the construction of Dirac operators.

In section 4, we work out some details about basic facts of the generalized Dirac operators. A discussion about Sobolev spaces in the context of analysis of sections of vector bundles is included.

In section 5, we recall the setup of finite dimensional approximation via global Kuranishi model. Such models were used by Furuta in the proof of his famous 10/8th–Theorem. We closely adapt Furuta's technique in our setting.

In section 6, we re-introduce the construction of Rarita-Schwinger operators in greater details and calculate the index of a twisted Rarita-Schwinger operator. Then we re-state our variant of the Seiberg-Witten equations using the Rarita-Schwinger operator. We shall show that the equations are invariant under the gauge group action $\mathcal{G}$, and in fact there is a gauge fixing condition just like in the usual Seiberg-Witten setting. We will also prove that the equations have an extra $Pin(2)-$symmetry when $L = \underline{\mathbb{C}}$. Note that this $Pin(2)-$symmetry is a special phenomena for only "trivial" $spin^c$ structure over X, the technique that we use for the proof of Theorem 1.8 cannot be extended to a general $spin^c$ bundle.

In section 7, we prove various analytical facts about the functional $\mathcal{F}$ of our RS-SW equations to set up for its finite dimensional approximation. Specifically, we show that the linearization $d\mathcal{Q}$ of the quadratic part $\mathcal{Q}$ of (1.4) is a $Pin(2)-$equivariant compact operator at any configuration whose norm is less than a prefixed $R_0 > 0$. Moreover, the union over $B(0, R_0)$ of all the images of the closed unit ball via $d\mathcal{Q}$ has compact closure.

In section 8, we briefly recall some facts about equivariant $K-$theory and provide a proof Theorem 1.8. The use of equivariant $K-$theory in our proof is similar to Furuta's with a minor difference: we show that *if* $\mathcal{M}_g$ *is compact*, then there is a finite dimensional approximation for $\mathcal{F}$ that can be viewed as $Pin(2)-$equivariant map between $Pin(2)-$representation spheres. And such a map cannot exist *unless* $b_2(X) \geq 15\,\sigma(X)/4 + 2$.

In section 9, we discuss the transversality of the moduli space of solutions to the perturbed version of equation (1.4). The technique of proof is standard, and it almost follows verbatim in the case of classical Seiberg-Witten equations. The crucial feature of the proof is the Rarita-Schwinger operator also enjoys analytic continuation just like the Dirac operator.

2 Background

The majority of the content of this section is based on Chapter 1-2 of [21], and Chapter 1 of [18]. More detailed exposition on Hodge theory and elliptic complexes can also be found in many excellent resources, for example, Chapter 1.5 of [11] and Appendix III of [6].

2.1 Connections on vector bundles

Suppose M is a smooth manifold of dimension n, a vector bundle V over M is defined by $(V, \pi, M, \mathbb{F})$, where $\mathbb{F}$ is a $\mathbb{C}$ or $\mathbb{R}-$ finite dimensional vector space, V is a smooth manifold, $\pi : V \to M$ is a smooth surjective map such that for any point $p \in M$ and a coordinate patch U containing p, $\pi^{-1}(U) \cong_{\text{diff}} U \times \mathbb{F}$. Naturally, by this definition, we immediately deduce that $V_p := \pi^{-1}(p) \cong_{\text{diff}} \{p\} \times \mathbb{F}$; such thing is called a fiber of the vector bundle V. A section of the vector bundle V is a smooth map $s : M \to V$ such that $\pi \circ s = \text{Id}_M$. The collection of all sections of the vector bundle V will be denoted $\Gamma(V)$.

Example 2.1. One of the first elementary examples of vector bundle is the tangent bundle. Let M be a smooth manifold of dimension n. The tangent bundle $TM := \bigsqcup_{p\in M} T_pM$ is a real vector bundle over M of dimension n. π will be the map that takes any vector v who belongs to some tangent space T_pM to p. Consequently, each fiber would be the tangent space T_pM. The sections of TM are just vector fields on M. Hence, $\Gamma(TM) = \mathfrak{X}(M)$.

Example 2.2. Another interesting example of a vector bundle is the cotangent bundle. Let M be a smooth manifold of dimension n. Note that for each $p \in M$, the tangent space T_pM is a fiber of the tangent bundle, thus can be identified with a $n-$dimensional real vector space over $\mathbb{R}$. Then the cotangent bundle is defined to be $T^*M := \bigsqcup_{p\in M} T_p^*M$, where T_p^*M is the dual of the vector space T_pM. So each fiber of the cotangent bundle is the cotangent space T_p^*M, and the space of sections $\Gamma(T^*M) = \Omega^1(M)$, which is exactly all the $1-$form on M.

Example 2.3. In a similar manner like the above example, for any given vector bundle $\pi : V \to M$, we can also construct another vector bundle $\pi^* : V^* \to M$. A more interesting construction is the vector bundle of the exterior algebra Λ^*V over M. We decree that each fiber of the exterior algebra bundle is Λ^*V_p so that the map $\pi : \Lambda^*V \to M$ is given by $\alpha \mapsto p$, if $\alpha \in \Lambda^*V_p$, for some $p \in M$.

It is often helpful to think of vector bundle as a family of vector spaces, parametrized by a smooth underlying manifold. Besides the definition presented above, vector bundles can be constructed via *transition functions*. Let $\{U_\alpha : \alpha \in A\}$ be an open cover of M and $F = \mathbb{C}$ or $\mathbb{R}$. Suppose for each $\alpha, \beta \in A$, we have the following smooth transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m, F)$$

that sastisify the *cocylce condition* $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Denote $\tilde{V}$ by the set of all triples $(\alpha, p, v) \in A \times M \times F^m$ where $p \in U_\alpha$. We define an equivalence relation $\sim$ on $\tilde{V}$ as follows:

$$(\alpha, p, v) \sim (\beta, q, w) \text{ if and only if } p = q \in U_\alpha \cap U_\beta, \quad v = g_{\alpha\beta}(p)w.$$

Let V be the set of equivalence classes $[\alpha, p, v]$ and let $\pi : V \to M$ be the projection map that takes $[\alpha, p, v]$ to p. There is a unique manifold structure on V such that (V, π, M, F^m) also satisfies the definition of vector bundle above.

Definition 2.1. Suppose M is a smooth $n-$dimensional manifold, and $\pi : V \to M$ is a vector bundle over M. A connection on V over M is a linear map $\nabla : \mathfrak{X}(M) \otimes \Gamma(V) \to \Gamma(V)$ such that for all vector field $X \in \mathfrak{X}(M)$ and $s \in \Gamma(V)$,

$\nabla_X s \in \Gamma(V)$; and for any smooth function $f \in \Gamma(M)$, we have

$$\nabla_{fX} s = f \nabla_X s; \text{ and } \nabla_X (fs) = f \nabla_X s + (Xf)s.$$

It is most beneficial to think about connection as a natural generalization of the Lie derivative and Lie bracket. In the case of Lie derivative, $\mathcal{L}_X(f) = Xf$ is a connection defined for the line bundle over the manifold M; while in the case of Lie bracket, $[X, Y] = XY - YX$ is a connection on the tangent bundle TM. Thus we say that what $\nabla_X s$ calculates is a *covariant derivative* of s in the X direction.

For a fixed $s \in \Gamma(V)$, the map $X \mapsto \nabla_X s$ defined on $\mathfrak{X}(M)$ is $\Gamma(M)-$linear; hence it can be considered as a homomorphism between $\Gamma(M)-$modules. Note that then the values of $\nabla_X s$ at a point $p \in M$ depends on the value of X at p. Therefore at p, we define an element $(\nabla s)(p) \in End(T_pM, V_p) \cong_{\text{isom}} V_p \otimes T_p^* M$ by $(\nabla s)(p)(X_p) = \nabla_{X_p} s$. Thus connection can be considered as a linear map

$$\nabla : \Gamma(V) \to \Gamma(V \otimes T^* M)$$

such that $\nabla(fs) = f\nabla s + s \otimes df$, for any smooth function $f \in \Gamma(M)$ and $s \in \Gamma(V)$. In other words, connection is a vector-bundle valued $1-$form on M.

Now let's specify the the rank of the vector bundle $\pi : V \to M$ to be l. We choose a coordinate patch U on M so that V has a local trivialization. Then we have a frame field $(e_1, e_2, \cdots, e_l)$, where $e_i(p) = \phi^{-1}(p, \mathbf{e_i})$, ϕ is a paramatrization of $U \times \mathbb{F}$ and $(\mathbf{e_i})$ is the standard basis for $\mathbb{F}$. Furthermore, since $\mathfrak{X}(U) = \text{span}_{\Gamma(U)}\{\partial_1, \partial_2, \cdots, \partial_n\}$, where $\partial_i := \partial / \partial_{x_i}$, we have:

$$\nabla_{\partial_i} e_j = \sum_{k=1}^{l} \omega_j^k(\partial_i) e_k.$$

We denote $\omega_j^k(\partial_i) := \Gamma_{ji}^k$, and we call this the *Christoffel symbols* of the connection. If

$s \in \Gamma(V|_U)$, we can write $s = \sum_{j=1}^{l} s^j e_j$, where $s^j \in \Gamma(U)$. By linearity, we have:

$$\nabla_{\partial_i} s = \sum_{j=1}^{l} \nabla_{\partial_i} s^j e_j = \sum_{j=1}^{l} (s_j \nabla_{\partial_i} e_j + (\partial_i s^j) e_j) \tag{2.1}$$

$$= \sum_{j,k=1}^{l} s^j \Gamma_{ji}^k e_k + \sum_{j=1}^{l} (\partial_i s^j) e_j \tag{2.2}$$

$$= \sum_{k=1}^{l} \left(\sum_{j=1}^{l} \Gamma_{ji}^k s^j \right) e_k + \sum_{j=1}^{l} (\partial_i s^j) e_j. \tag{2.3}$$

If we identify s as a smooth map $(s^1, s^2, \cdots, s^l)^T : U \to \mathbb{F}$, then (3) can be re-interpreted as

$$\nabla_{\partial_i} = \Gamma_i + \partial_i, \tag{2.4}$$

where $\Gamma_i := (\Gamma_{ji}^k)$, which is an $(l \times l)-$matrix of $\Gamma(U)-$valued entries belonging to $\mathrm{End}(\Gamma(V|_U)) \cong_{\mathrm{isom}} \Gamma(\mathrm{End}(V|_U))$. Equation (2.4) is informative in the sense that it gives us the necessary and sufficient condition for the existence of connection locally: If a connection exists on a vector bundle over a manifold, locally we have these matrices of Christoffel symbols. Conversely, given a local trivialization of a vector bundle over a manifold, a connection is defined locally by a choice of n matrices $\Gamma_1, \Gamma_2, \cdots, \Gamma_n \in \Gamma(\mathrm{End}(V|_U))$.

Proposition 2.1. (Existence of global connection) *If M is a paracompact smooth manifold of dimension n, and $\pi : V \to M$ is a vector bundle over M, then there exists a global connection on V over M.*

Proof. The key here is paracompactness. Let $\{U_\alpha\}$ be an open cover of M that consists of coordinate patches and $V|_{U_\alpha}$ is a local trivialization of the vector bundle. Then we have a partition of unity $\{\rho_\alpha\}$ on M that is subordinated to such open cover. By the above observation, we know that from a choice of n matrices $\Gamma_1, \cdots, \Gamma_n \in \Gamma(\mathrm{End}(V|_{U_\alpha}))$, we obtain a local connection ∇^{U_α} on $V|_{U_\alpha}$. Then define

$$\nabla := \sum_\alpha \rho_\alpha \nabla^{U_\alpha},$$

which is a desired global connection on V. □

Definition 2.2. A connection ∇ on the tangent bundle TM over a smooth manifold M is said to be *symmetric* if and only if for every vector field X and Y, we have

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Definition 2.3. A connection ∇ on TM over a Riemannian manifold (M, h) is *compatible* with the metric if and only if for any three vector fields $X, Y_1, Y_2 \in \mathfrak{X}(M)$, we have

$$\langle \nabla_X Y_1, Y_2 \rangle + \langle Y_1, \nabla_X Y_2 \rangle = X \langle Y_1, Y_2 \rangle.$$

What we would like to do next is defining the "canonical" connection in the setting of Riemannian manifold. Recall that M is a *Riemannian manifold* if it is a smooth manifold equipped a map $g : p \mapsto \langle \, , \rangle_p$ defined on M that is a smoothly varying inner-product on the fibers of TM. $g(p) = \langle \, , \rangle_p$ is what we call a Riemannian metric on M.

Theorem 2.2. (Levi-Civita) *A paracompact Riemannian manifold M possesses a unique connection that is symmetrical and compatible with the metric. This is called the Levi-Civita connection, denoted by ∇^{LC}.*

Proof. Suppose that such a connection ∇ exists on the tangent bundle TM exists. We shall show that it is is unique. In terms of local coordinate, by compatibility, we have:

$$\langle \nabla_{\partial_i} \partial_j, \partial_k \rangle + \langle \partial_j, \nabla_{\partial_i} \partial_k \rangle = \partial_i \langle \partial_j, \partial_k \rangle.$$

Denote $g_{jk} := \langle \partial_j, \partial_k \rangle$ by a smooth function on M. Then the metric is represented by the

$n \times n$ symmetric matrix $G = (g_{jk})$. Furthermore,

$$\partial_i g_{jk} = \left\langle \sum_{r=1}^{n} \Gamma^r_{ji} \partial_r, \partial_k \right\rangle + \left\langle \partial_j, \sum_{q=1}^{n} \Gamma^q_{ki} \partial_q \right\rangle \tag{2.5}$$

$$= \sum_{r=1}^{n} \Gamma^r_{ji} \langle \partial_r, \partial_k \rangle + \sum_{q=1}^{n} \Gamma^q_{ki} \langle \partial_j, \partial_q \rangle \tag{2.6}$$

$$= \sum_{r=1}^{n} \Gamma^r_{ji} g_{rk} + \sum_{q=1}^{n} \Gamma^q_{ki} g_{jq} \tag{2.7}$$

$$= \sum_{a=1}^{n} (\Gamma^a_{ji} g_{ak} + \Gamma^a_{ki} g_{aj}). \tag{2.8}$$

Similarly, we also have:

$$\partial_j g_{ki} = \sum_{a=1}^{n} (\Gamma^a_{kj} g_{ai} + \Gamma^a_{ij} g_{ak}). \tag{2.9}$$

$$\partial_k g_{ij} = \sum_{a=1}^{n} (\Gamma^a_{ik} g_{aj} + \Gamma^a_{jk} g_{ai}). \tag{2.10}$$

Now using the fact that ∇ is symmetric, combine (2.8), (2.9) and (2.10), we deduce

$$\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} = 2 \sum_{a=1}^{n} \Gamma^a_{ij} g_{ak}.$$

The derived equation above uniquely determines the Christoffel symbols, for G is an invertible matrix. This shows that if such a connection ∇ exists on TM, then it has to be unique. Conversely, to construct the Levi-Civita connection, locally choose an appropriate collection of symbols Γ^a_{ij} that satisfy the above equations (This can always be done because the matrix of coefficients of the linear system of equations like above is G). Such choice gives us a local formula for symmetric and compatible connection. Then use partition of unity to obtain ∇^{LC}. □

2.2 Curvature of a connection

In the previous subsection, we see that in local coordinate, a connection ∇ on a vector bundle $V \to M$ can be written as $\nabla = d + A$, where $A \in \Gamma(T^*M \otimes End(V))$. Often, A is called a connection matrix. One can introduce the subscript A to ∇ written as ∇_A to emphasize the dependence of a connection and its connection matrix.

The map $\nabla_A : \Gamma(V) \to \Gamma(T^*M \otimes V)$ can be extended to a map

$$d_A : \Gamma(\Lambda^k T^*M \otimes E) \to \Gamma(\Lambda^{k+1} T^*M \otimes E)$$

by defining $d_A(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|}\omega \otimes \nabla_A s$, where $\omega \in \Omega^k(M)$ and $s \in \Gamma(V)$. One should note that d_A is a generalization of the usual exterior derivative of the de Rham complex. Unlike the exterior derivative, $d_A^2 : \Gamma(E) \to \Gamma(\Lambda^2 T^*M \otimes E)$ is not generally zero. However,

$$d_A^2(fs) = d_A(\nabla_A(fs)) = d_A(df \otimes s + f\nabla_A s) \tag{2.11}$$

$$= d^2 f \otimes s - df \otimes \nabla_A s + df \otimes \nabla_A s + f d_A d_A s = f d_A d_A s. \tag{2.12}$$

Therefore, d_A^2 is linear over functions and can be regarded as a tensor field! We call d_A^2 the *curvature of a connection* and we shall use F_A to denote it by a section of $\Lambda^2 T^*M \otimes End(V)$.

In local coordinate if we write $\nabla_A = d + A$, then

$$d_A^2 s = (d + A)(ds + As) = d(As) + A(ds) + (A \wedge A)s \tag{2.13}$$

$$= (dA)s - A(ds) + A(ds) + (A \wedge A)s = (dA + A \wedge A)s. \tag{2.14}$$

Furthermore, similar calculations show that $d_A F_A = 0$. This is called the *Bianchi identity*.

Remark 4. If V is a vector bundle with structure group $O(l)$, (2.14) tells us locally F_A is a

2-form with value in skew-symmetric matrices. If V is a complex vector bundle with structure group $U(l)$, then F_A is a 2-form with value in skew-hermitian matrices. In particular, when V is a complex line bundle ($l = 1$), the transition functions transform local representations F_A in such a way that they all agree on the overlaps. Thus, we arrive at a globally defined purely imaginary 2-form F_A on M. Note that Bianchi identity would imply that $dF_A = 0$, which means that $[F_A] \in iH^2_{dR}(M)$.

2.3 Characteristic classes

In the previous subsection, we see that given a connection ∇_A on a vector bundle, there is a connection matrix A that is locally defined. The local data of F_A is glued together according to a certain transformation

$$F_{A_\alpha} = g_{\alpha\beta} F_{A_\beta} g_{\alpha\beta}^{-1} \qquad \text{on} \qquad U_\alpha \cap U_\beta.$$

The transformation above allows us to extract topological invariant of vector bundles from connection. To see this, for now we assume that our vector bundle has structure group $U(m)$ and ∇_A is a compatible connection. This means that locally F_A is skew-adjoint matrices-valued two forms. As a result,

$$\left(\frac{i}{2\pi} F_{A_\alpha}\right)^k$$

is self-adjoint matrices-valued two forms for any natural number k. Taking its trace, we get a differential form that is real valued. Note that trace is invariant under conjugation. So we have a real-valued $2k-$form denoted by $\tau_k(A)$. Using Bianchi indetity, one can show that

Lemma 2.3. *For each k, $d\tau_k(A) = 0$. Therefore, $[\tau_k(A)] \in H^{2k}_{dR}(M)$.*

What is even more remarkable about the cohomology class $[\tau_k(A)]$ is the following

Proposition 2.4. $[\tau_k(A)]$ *is independent of the choice of compatible connection* ∇_A *and indpendent of the choice of hermitian metric on the* $U(m)-$*vector bundle.*

Proof. First, it is not hard to realize that since pull back of a curvature of a connection is exactly the curvature of a pull back connection, $\tau_k(f^*A) = f^*\tau_k(A)$. Now let V be a Hermitian bundle over M and ∇_A, ∇_B be two compatible connections of V. Consider the bundle $V \times [0,1] \to M \times [0,1]$. Then under a local trivialization $U \times [0,1]$ of $V \times [0,1]$ where U is a local trivialization of V, the pull-backs of ∇_A and ∇_B are given by

$$\nabla_{\pi^*A} = d + \pi^*A, \qquad \nabla_{\pi^*B} = d + \pi^*B.$$

Here $\pi : M \times [0,1] \to M$ is a projection onto the first component. Note that ∇_{π^*A} and ∇_{π^*B} are now compatible connections of $V \times [0,1]$. Next, define $d + (1-t)\pi^*A + t\pi^*B$ to be the local presentation of a compatible connection ∇_C of $V \times [0,1]$. If F_0, F_1 are embedding of $M \to M \times [0,1]$ given by $F_0(p) = (p,0)$ and $F_1(p) = (p,1)$, then

$$\nabla_{F_0^*C} = \nabla_A, \qquad \nabla_{F_1^*C} = \nabla_B.$$

As a result, we have

$$[\tau_k(A)] = F_0^*[\tau_k(C)] = F_1^*[\tau_k(C)] = [\tau_k(B)].$$

Similar argument for different choices of hermitian metric on V and note that space of Hermitian metrics on V is convex, we would arrive at the same conclusion. □

Proposition 2.4 tells us that, to each Hermitian vector bundle $V \to M$, we can associate to it a cohomology class in $H^{2k}_{dR}(M)$ given by $\tau_k(V) := [\tau_k(A)]$, where ∇_A is any compatible connection of V. Such classes are called *characteristic classes* of the vector bundle. Being a de Rham cohomology class, it is not difficult to see that if $F : N \to M$ is a smooth map between manifolds and $V \to M$ is a Hermitian vector bundle, then $\tau_k(F^*V) = F^*\tau_k(V)$.

Characteristic classes can be put together to give rise to other invariant of Hermitian vector bundle over manifolds. For example, by putting them together as an infinite series to express $tr(\exp(iF_A/2\pi))$, we have *Chern character*:

$$ch(V) = \text{rank}(V) + \tau_1(V) + \frac{1}{2!}\tau_2(V) + \cdots$$

But since M is finite dimensional, $ch(V)$ is an actual polynomial of finite terms living in the ring $H^*_{dR}(M)$. Chern character satisfies the following identities

Proposition 2.5. *If V_1 and V_2 are two Hermitian bundles over M, then* $ch(V_1 \oplus V_2) = ch(V_1) + ch(V_2)$ *and* $ch(V_1 \otimes V_2) = ch(V_1)ch(V_2)$.

Proof. Let ∇_{A_1} and ∇_{A_2} are two compatible connections on V_1 and V_2. Then a compatible connection on $V_1 \oplus V_2$ is simply $\nabla_{A_1} \oplus \nabla_{A_2}$. On the other hand, a connection on $V_1 \otimes V_2$ would be $\nabla_{A_1} \otimes 1 + 1 \otimes \nabla_{A_2}$. As a result, it is not hard to see that the associated curvature of $\nabla_{A_1} \oplus \nabla_{A_2}$ is $F_{A_1} \oplus F_{A_2}$, while the curvature of $\nabla_{A_1} \otimes 1 + 1 \otimes \nabla_{A_2}$ is $F_{A_1} \otimes 1 + 1 \otimes F_{A_2}$. From here, it is not hard to derive the above formulae. □

Characteristic classes can also be combined differently to give rise to *chern classes*. For example in $4-$dimension topology, we define

$$c_1(V) = \tau_1(V), \qquad c_2(V) = \frac{1}{2}(\tau_1(V)^2 - \tau_2(V)).$$

Higher chern classes could also be defined but they would all vanish because we live in $4-$dimensional topology. From Proposition 2.5, one arrive at similar formulae for chern classes with respect to basic operations on vector bundles.

If V is a real vector bundle, we can complexify it to obtain a hermitian bundle $V \otimes \mathbb{C}$. In this case, since the transition functions of V take value in some symmetric matrices, the transition functions of $V \otimes \mathbb{C}$ is exactly the same as their conjugations. Thus $V \otimes \mathbb{C}$ and

$(V \otimes \mathbb{C})^*$ are isomorphic. This means that $c_1(V \otimes \mathbb{C}) = 0$. However, its second chern class might be non-zero. For a real vector bundle V, this second chern class is called the *Pontrjagin class* and is denoted by

$$p_1(V) = -c_2(V \otimes \mathbb{C}).$$

In 4-dimensional topology, a special case of real vector bundle we would be interested in is the tangent bundle TM. From the Pontrjagin class of TM, we construct another characteristic class called the $\widehat{\mathcal{A}}-$genus

$$\widehat{\mathcal{A}}(TM) = 1 - \frac{1}{24} p_1(TM) \in H^*_{dR}(M)$$

The $\widehat{\mathcal{A}}-$genus with the chern character are important in the formulae to calculate the index of Dirac operators as we shall see in later sections.

2.4 Hodge theory

For each smooth compact manifold M, we can associate to it a de Rham complex

$$0 \to \Omega^0(M) \to \Omega^1(M) \to \cdots \to \Omega^n(M) \to 0.$$

Each map in the middle is given by the exterior derivative d. From the de Rham complex, we have the de Rham cohomology $H^*_{dR}(M)$, which is an invariant of M. In certain situation, we from each cohomology, we would like to pinpoint a "preferred" representative. Hodge theory enables us to do exactly that via analysis.

Definition 2.4. Let α be a $k-$form. We define $*\alpha$ to be the unique $(n-k)-$form such that for all $k-$form β, we have

$$\langle \alpha | \beta \rangle \text{vol} = \beta \wedge *\alpha.$$

The $*$ operator is linear and has the property that $**\alpha = (-1)^{nk+k}\alpha$. The inner product between forms of the same degree is thought of as the usual inner product of finite dimensional vector spaces once an orthonormal frame field is specified, see subsection 3.1 for more details.

Definition 2.5. If α is a $k-$form, we define

$$d^*\alpha = (-1)^{nk+k+1} * d(*\alpha).$$

Thus $d^*\alpha$ is a $(k-1)-$form and $(d^*)^2 = 0$.

Remark 5. Once the Hodge $*$ operator is defined, we can define a global L^2-inner product on forms. With respect to such L^2-inner product, d^* is actually the adjoint of the exterior derivative. Furthermore, $d + d^* : \Omega^*(M) \to \Omega^*$ is a first order elliptic operator, and is called the *Hodge-Dirac operator*. Consequently, the Hodge-Laplacian $\Delta = (d + d^*)^2 = dd^* + d^*d : \Omega^p(M) \to \Omega^p(M)$ is an elliptic operator of order 2. These facts will be revisited again in greater details in subsection 3.5 and section 4.

Being elliptic, as we shall see later in section 4, $ker\,\Delta \subset \Omega^p(M)$ is finite dimensional. A $p-$form that is in the kernel of Δ is called a *harmonic p-form*, and often we denote $ker\,\Delta = \mathcal{H}^p(M)$.

Theorem 2.6. (Hodge Theorem) *Every de Rham cohomology class on a compact oriented Riemannian manifold M has a unique harmonic representative, i.e, $H^p_{dR}(M) \cong \mathcal{H}^p(M)$. Moreover, there is an L^2-orthogonal decomposition*

$$\Omega^p(M) = d\Omega^{p-1} \oplus \mathcal{H}^p(M) \oplus d^*\Omega^{p+1}(M).$$

For a proof of the above theorem, we refer readers to [reference].

One application of Hodge Theorem is to topology. Since $*$ takes harmonic forms to harmonic forms, this means that $\mathcal{H}^p(M) \cong \mathcal{H}^{n-p}(M)$. Therefore, we also have an isomorphism for de Rham cohomology, $H^p_{dR}(M) \cong H^{n-p}_{dR}(M)$. This is exactly the *Poincare duality.* Let b_p denotes the rank of $H^p_{dR}(M)$, b_p is known to be the $p^{th}-$betti number of M. Then we immediately have $b_p = b_{n-p}$.

In particular, for $4-$dimensional topology, the Euler characteristic of M is determined only by b_0, b_1, and b_2. Note that b_2 even has further decomposition. Since $*^2 = 1$, $\Omega^2(M) = \Omega^+ \oplus \Omega^-$ orthogonally, where $\Omega^\pm$ is the eigenspace associated to the eigenvalue ± 1 of $*$. Thus if $\{e_1, \cdots, e_4\}$ is a local orthonormal frame on M, then Ω^+ is generated by

$$dx_1 \wedge dx_2 + dx_3 \wedge dx_4, \quad dx_1 \wedge dx_3 + dx_4 \wedge dx_2, \quad dx_1 \wedge dx_4 + dx_2 \wedge dx_3,$$

while Ω^- is generated by

$$dx_1 \wedge dx_2 - dx_3 \wedge dx_4, \quad dx_1 \wedge dx_3 - dx_4 \wedge dx_2, \quad dx_1 \wedge dx_4 - dx_2 \wedge dx_3,$$

Sections of $\Omega^\pm$ are called self-dual and anti-self-dual two-forms. Any $2-$form ω can be orthogonally projected onto $\Omega^\pm$ as follows

$$\omega^\pm = P_\pm(\omega) = \frac{1}{2}(\omega \pm *\omega).$$

Since $*$ exchanges the kernel of d and d^*, self-dual and anti-self-dual part of a harmonic $2-$forms are also harmonic. As a result, $\mathcal{H}^2(M) = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^\pm$ is the space of all harmonic (anti)self-dual $2-$forms. Hence, $b_2 = b_+ + b_-$ where $b_\pm = dim\, \mathcal{H}^\pm$. The *signature* of M is then given by $\sigma(M) = b_+ - b_-$.

On a compact oriented $4-$manifold M, we also have the following elliptic complex

$$0 \to \Omega^0(M) \to \Omega^1(M) \to \Omega^+ \to 0,$$

where the second map is the exterior derivative d, and the third map is $d^+ = P_+ \circ d$. Hodge

Theorems allows us to calculate the Euler characteristic of such complex. Indeed if $\omega \in \Omega^+$ is in the orthogonal complement of image of d, then $d^*\omega = 0$. But ω is self-dual, this means $d\omega = 0$. Thus ω is harmonic. Now if $\omega \in ker\, d^+$ but is in the orthogonal complement of image of d, then $d^*\omega = 0$ and $d\omega \in \Omega^+$. Hence $*d\omega = d\omega$, which implies that $d\omega = 0$. As a result, $\Delta\omega = (d^*d + dd^*)\omega = 0$. Therefore, if M is connected, the cohomology groups of the above complex are exactly $\mathcal{H}^0(M) \cong \mathbb{R}, \quad \mathcal{H}^1(M) \cong H^1_{dR}(M) \cong \mathbb{R}^{b_1}, \quad \mathcal{H}^+ \cong \mathbb{R}^{b_+}$. Thus, the its Euler characteristic is given by $1 - b_1 + b_+$. This in turns tells us that the $index\,(d^+ + d^*) = -1 + b_1 - b_+$.

3 Spin geometry

The majority content of this sections is based on Chapter 3-4 of [21]. Other excellent resources on Clifford algebras, Spin groups, Pin groups, Spin geometry can also be found in Chapter 1-2 of [17], Chapter 2-3 of [19], Chapter 2 of [18], Chapter 1-2 of [9].

3.1 Clifford algebra

Suppose V is an $n-$dimensional vector space over $\mathbb{R}$ equipped with a non-degenerate, positive definite symmetric bilinear form $q : V \times V \to \mathbb{R}$. Consider the tensor algebra over V,

$$\mathcal{T}(V) = \mathbb{R} \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \cdots$$

Convention-wise, we denote $\mathcal{T}^k(V) := \bigotimes_{i=1}^k V$, where $\mathcal{T}^0(V) := \mathbb{R}$. Then the tensor algebra can be rewritten as $\mathcal{T}(V) = \bigoplus_{k=1}^\infty \mathcal{T}^k(V)$. Now suppose $(e_1, e_2, \cdots, e_n)$ is an orthonormal basis of V with respect to q. Hence, as a vector space over $\mathbb{R}$, we have:

$$\mathcal{T}(V) = \text{span}_{\mathbb{R}}\{1, e_1, e_2, \cdots, e_n, \cdots, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, \cdots\},$$

where $i_j \in \{1, 2, \cdots, n\}$ and i_j's are not necessarily distinct. Next consider the two-sided ideal I generated by $v \otimes v + q(v, v) \cdot 1$, where $v \in V$. Note that since $\mathcal{T}(V)$ is a unital associative algebra over $\mathbb{R}$, $\mathcal{T}(V)/I$ is also a unital associative algebra over $\mathbb{R}$.

Definition 3.1. The *Clifford algebra* over a real finite dimensional vector space V equipped with a non-degenerate, positive definite symmetric bilinear form q is $Cl(V, q) := \mathcal{T}(V)/I$.

Note that with the set-up above, in the algebra $Cl(V, q)$, $e_i e_j + e_j e_i = -2\delta_{ij}$, for all $1 \leq i, j \leq n$. Therefore, as a vector space over $\mathbb{R}$, we have:

$$Cl(V, q) = \text{span}_{\mathbb{R}}\{1, e_1, e_2, e_3, \cdots, e_n, \cdots, e_{i_1} e_{i_2} \cdots e_{i_k}, \cdots, e_1 e_2 \cdots e_n\},$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 < k < n$. With simple counting argument, we have $\dim_{\mathbb{R}} Cl(V, q) = 2^n$. If we use the multi-index notation, a typical element $a \in Cl(V, q)$ can be written as $a = \sum_{|I|=k} a_I e_I$, where $1 \leq k \leq n$. It is clear then that $Cl(V, q)$ can be decomposed into $Cl^+(V, q) \oplus Cl^-(V, q)$, where here we denote:

$$Cl^+(V, q) := \{a = \sum_{|I|=k} a_I e_I : k \text{ is even.}\},$$

$$Cl^-(V, q) := \{a = \sum_{|J|=m} a_J e_J : m \text{ is odd.}\}.$$

By simple computations, we immediately obtain the following proposition:

Proposition 3.1. $Cl^+ \cdot Cl^+ \subseteq Cl^+$, $Cl^- \cdot Cl^+ \subseteq Cl^-$, *and* $Cl^- \cdot Cl^- \subseteq Cl^+$.

Example 3.1. Take $V := \mathbb{R}$ and q to be the normal real number multiplication. Then $Cl(\mathbb{R}, \times) = \text{span}_{\mathbb{R}}\{1, e_1 : e_1^2 = -1\}$. It is not hard to see that as algebras, $Cl(\mathbb{R}, \times) \cong \mathbb{C}$.

Example 3.2. If $V := \mathbb{R}^2$ and q is the standard Euclidean inner product $\langle\, , \rangle_{\text{Eu}}$, then $Cl(\mathbb{R}^2, \langle\, , \rangle) = \text{span}_{\mathbb{R}}\{1, e_1, e_2, e_1 e_2\}$. Denote $i := e_1, j := e_2$, and $k := e_1 e_2$. By elementary calculations, we have $i^2 = j^2 = k^2 = -1$, $ij = k, ki = j$, and $jk = i$. Therefore as algebras, $Cl(\mathbb{R}^2, \langle\, , \rangle) \cong \mathbb{H}$, where $\mathbb{H}$ denotes the quarternion algebra.

Example 3.3. If q is trivial, then $Cl(V,0) = \Lambda^*V$, the exterior algebra over V. Furthermore, in general, given a bilinear form q, $Cl(V,q) \cong \Lambda^*V$ as vector spaces (not algebras!); the isomorphism here is straight-forward: $e_{i_1}e_{i_2}\cdots e_{i_j} \mapsto e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}$. There is a deeper connection between $Cl(V,q)$ and a particular sub-algebra of $\mathrm{End}(\Lambda^*V)$, which we will explore shortly in subsequent subsection.

Next, we would like to introduce a norm on $Cl(V,q)$ that generalizes the way norm is defined for complex numbers: The reversion is given by $e_{i_1}\widetilde{e_{i_2}\cdots} e_{i_j} = e_{i_j}e_{i_{j-1}}\cdots e_{i_2}e_{i_1}$. And the conjugation is given by $\overline{e_{i_1}e_{i_2}\cdots e_{i_j}} = (-1)^j e_{i_j}e_{i_{j-1}}\cdots e_{i_2}e_{i_1}$. We also decree that the reversion and the conjugation are $\mathbb{R}-$linear endomorphisms of $Cl(V,q)$. However, $\overline{e_I e_J} = (-1)^{|I|+|J|}\widetilde{e_J}\cdot\widetilde{e_I} = \overline{e_J}\cdot\overline{e_I}$. In general, for any $x, y \in Cl(V,q)$, $\overline{xy} = \overline{y}\cdot\overline{x}$. The same is true for the reversion map.

For $x, y \in Cl(V,q)$, write $x = \sum_I x_I e_I$ and $y = \sum_J y_J e_J$. Then $\overline{x}\cdot y = \sum_{I,J} x_I y_J \overline{e_I} e_J$. Note that when $I \equiv J$, $\overline{e_I}e_I = (-1)^{|I|}(-1)^{|I|} = 1$. Hence, we can re-write $\overline{x}y$ as:

$$\overline{x}\cdot y = \sum_{I\equiv J} x_I y_I + \ \{\text{terms of degree greater or equal to } 1\}.$$

We define the scalar part of $\overline{x}\cdot y$ to be $\sum_{I\equiv J} x_I y_I$, denoted by $\mathrm{sc}(\overline{x}y)$. If we view the scalar part as a map on $Cl(V,q)\times Cl(V,q) \to \mathbb{R}$, it is easy to see that it is bilinear, positive definite, non-degenerate, and symmetric. Hence, it defines for us an inner product on $Cl(V,q)$: $\mathrm{sc}(\overline{x}y) := (x,y)$. It follows then that the norm will be naturally defined as $||x|| := (x,x)^{1/2}$.

If we think of $Cl(V,q)$ as a left module over itself, and the left action is defined by the Clifford multiplication from the left. Denote L_v to be the left multiplication by v, where $v \in V \subset Cl(V,q)$. Note that $L_v \in \mathrm{End}(Cl(V,q))$. Then we have the following proposition:

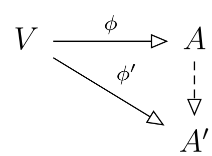
Lemma 3.2. *L_v is adjoint to its negative with respect to the inner product $(\ ,\ )$.*

Proof. For every $x, y \in Cl(V,q)$, we have

$$(L_v(x), y) = (vx, y) = \mathrm{sc}(\overline{vx}y) = \mathrm{sc}(\overline{x}(-vy)) = (x, -vy) = (x, -L_v(y)).$$ □

The above Lemma and Example 3.3 will be relevant for the next discussion.

Now we present another definition of the Clifford algebra in terms of its universal property. Suppose again that V is an $n-$dimensional vector space over $\mathbb{R}$ equipped with a non-degenerate, positive semi-definite, symmetric bilinear form q. A Clifford algebra over V is a unital associative $\mathbb{R}-$algebra A together with a map $\phi : V \to A$ such that $\phi(v)^2 = -q(v, v)1$, and which is universal among all algebras equipped with such maps; that is, if there is another map $\phi' : V \to A'$, where A' is another unital associative algebra such that $\phi'(v)^2 = -q(v, v)1$, there exists a unique algebra homomorphism $A \to A'$ such that the follow diagram commutes:



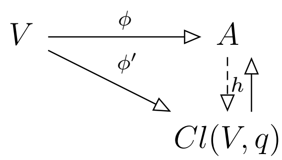
We shall show that this definition of Clifford algebra is equivalent to the Definition 3.1 via the following theorem.

Theorem 3.3. *A Clifford algebra over any vector space V equipped with a bilinear, symmetric, non-degenerate, positive semi-definite form q exists, and unique up to isomorphism.*

Proof. Suppose $(e_1, e_2, \cdots, e_n)$ is an orthonormal basis of V with respect to q. Let A be a unital associative $\mathbb{R}-$algebra spanned by 2^n possible products of formal symbols $\mathbf{e}_1^{k_1}, \cdots,$ $\mathbf{e}_n^{k_n}$, where k_j's are either 0 or 1 with multiplication determined by the rule

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}\mathbf{1}.$$

The linear $\mathbb{R}-$linear map $\phi : V \to A$ given by $\phi(e_i) = \mathbf{e}_i$ will then satisfy $\phi(v)^2 = -q(v, v)\mathbf{1}$. The uniqueness up to isomorphism is obtained from the abstract nonsense of the universal property: Recall the Clifford algebra over V in Definition 3.1 is defined as $Cl(V,q) := \mathcal{T}(V)/\{v \otimes v + q(v,v)1\}$. Then there is a natural $\mathbb{R}-$linear map $\phi' : V \to Cl(V,q)$ defined by $\phi'(e_i) = e_i$ that also satisfies $\phi'(v)^2 = -q(v,v)1$. Now define a map $h : Cl(V,q) \to A$ by $\sum a_I e_I \mapsto \sum a_I \mathbf{e}_I$. By universal property, we obtain the following commutative diagram



This shows immediately that $A \cong Cl(V,q)$ as algebras. $\square$

The universal property definition is not so easy to compute the Clifford algebras, but it is useful in proving things. The advantage of this picture of the Clifford algebra will be demonstrated shortly by showing how it is deeply related to the exterior algebra. First, we introduce a few new definitions.

From now on, to make things less lengthy, every $n-$dimensional vector space V over $\mathbb{R}$ will be equipped with a bilinear, symmetric, non-degenerate, positive semi-definite form q, unless stated otherwise. And (e_i) is always referred to as an orthonormal bases with respect to q.

Definition 3.2. Consider the exterior algebra $\Lambda^* V$, for each $v \in V$, we define the contraction by v as a linear map:

$$\iota_v : \Lambda^k V \to \Lambda^{k-1} V \text{ given by } e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mapsto \sum_{j=1}^{k} (-1)^j q(v, e_{i_j}) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}.$$

To the de-clutter the notations, we will write $\hat{e}_{i_j} := e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}$ and $\hat{e}_{i_l}\hat{e}_{i_j} := e_{i_1} \wedge \cdots \wedge \hat{e}_{i_l} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}$. Note that by some computations, we have:

$$\iota_v^2(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = \iota_v \iota_v (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) \tag{3.1}$$

$$=\iota_v\left(\sum_{j=1}^{k}(-1)^j q(v,e_{i_j})\hat{e}_{i_j}\right) \tag{3.2}$$

$$=\sum_{j=1}^{k}(-1)^j q(v,e_{i_j})\iota_v(\hat{e}_{i_j}) \tag{3.3}$$

$$=\sum_{j=1}^{k}(-1)^j q(v,e_{i_j})\sum_{l=1,l\neq j}^{k}(-1)^l q(v,e_{i_l})\hat{e}_{i_l}\hat{e}_{i_j} \tag{3.4}$$

$$=\sum_{j,l=1,j\neq l}^{k}(-1)^{l+j}q(v,e_{i_j})q(v,e_{i_l})\hat{e}_{i_l}\hat{e}_{i_j} \tag{3.5}$$

$$=\sum_{j<l}^{k}(-1)^{l+j}q(v,e_{i_j})q(v,e_{i_l})\hat{e}_{i_j}\hat{e}_{i_l}+\sum_{j>l}^{k}(-1)^{l+j-1}q(v,e_{i_j})q(v,e_{i_l})\hat{e}_{i_l}\hat{e}_{i_j}. \tag{3.6}$$

Swapping j and l in the latter summation of the right hand side of equation (16) does not change it. Thus we obtain $\iota_v^2(e_I)=0$, which implies that $\iota_v^2\equiv 0$ on Λ^*V. Furthermore from Example 3.3, we know that $\Lambda^*V=Cl(V,0)$. Therefore by the universal property, we have the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\ \phi\ } & \Lambda^*V \\ & \searrow^{\iota} & \downarrow{\scriptstyle\psi} \\ & & \mathrm{End}(\Lambda^*V) \end{array}$$

In other words, we have a unique algebra homomorphism $\psi:\Lambda^*V\to\mathrm{End}(\Lambda^*V)\cong_{\text{isom}}\Lambda^*V\otimes\Lambda^*V^*$. Now by commutativity, we have $\psi(e_{i_1}\wedge\cdots\wedge e_{i_j})=\iota_{e_{i_1}}\cdots\iota_{e_{i_j}}$. Hence, ι induces a binary operator on Λ^*V, and it can be thought of as a linear map

$$\psi':\Lambda^*V\otimes\Lambda^*V\to\Lambda^*V$$

such that $\psi'(e_{i_1}\wedge\cdots\wedge e_{i_j},\omega)=\iota_{e_{i_1}}\cdots\iota_{e_{i_j}}(\omega):=\iota_{e_I}(\omega)$, where in general, we have $\psi'(\alpha,\omega)=\psi'(\sum a_Ie_I,\omega)=\sum a_I\iota_{e_I}(\omega)$. Most of the time, we will write $\psi'(\alpha,\omega):=\psi'_\alpha\omega$. Another natural binary operation on Λ^*V is the wedge product: $\epsilon_\alpha(\omega):=\alpha\wedge\omega$. Now for each $\alpha \in \Lambda^* V$, we define $c_\alpha := \epsilon_\alpha + \psi'_\alpha \in \mathrm{End}(\Lambda^* V)$. This induces a left action $\Lambda^* V \otimes \Lambda^* V \to \Lambda^* V$.

Proposition 3.4. *Let A be a subalgebra of End$(\Lambda^* V)$ generated by $\{c_\alpha : \alpha \in \Lambda^* V\}$. Then A is a Clifford algebra for (V, q). In other words, $A \cong Cl(V, q)$ as algebras.*

Proof. It suffices to show that, for each $v \in V \subset \Lambda^* V$, $c_v^2 = -q(v,v)\mathbf{1}$, where $\mathbf{1}$ denotes the identity map on $\Lambda^* V$. Note that $c_v^2 = (\epsilon_v + \psi'_v)^2 = \epsilon_v^2 + \epsilon_v \psi'_v + \psi'_v \epsilon_v + \psi'^2_v$. However, $\epsilon_v^2 \equiv 0$ for $v \wedge v = 0$ and $\psi'^2_v = \iota_v^2 \equiv 0$ because of previous computations. Therefore, $c_v^2 = \epsilon_v \iota_v + \iota_v \epsilon_v$ so that for each $e_I := e_{i_1} \wedge \cdots \wedge e_{i_k}$, we have:

$$c_v^2(e_I) = \epsilon_v \iota_v(e_I) + \iota_v \epsilon_v(e_I) \tag{3.7}$$

$$= \epsilon_v \left(\sum_{j=1}^{k} (-1)^j q(v, e_{i_j}) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k} \right) + \iota_v (v \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}) \tag{3.8}$$

$$= \sum_{j=1}^{k} (-1)^j q(v, e_{i_j}) v \wedge \hat{e}_{i_j} - q(v,v) e_I + \sum_{j=1}^{k} (-1)^{j+1} q(v, e_{i_j}) v \wedge \hat{e}_{i_j} = -q(v,v) e_I. \tag{3.9}$$

Therefore by the abstract nonsense of the universal property argument, we immediate obtain $A \cong Cl(V, q)$ as algebras. □

Let $\lambda : \Lambda^* V \to Cl(V, q)$ be the inverse of the vector space isomorphism in Example 3.3. Consider $e_1 \in V \subset \Lambda^* V$ and $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. We then have $c_{e_1}(e_I) = \epsilon_{e_1}(e_I) + \psi'_{e_1}(e_I)$. So if $i_1 = 1$, $\lambda(c_{e_1}(e_I)) = -e_{i_2} \cdots e_{i_k} = e_1 e_{i_1} \cdots e_{i_k} = \lambda(e_1)\lambda(e_I)$. However if $i_1 > 1$, $\lambda(c_{e_1}(e_I)) = e_1 e_I = \lambda(e_1)\lambda(e_I)$. Regardless, we always have $\lambda(c_{e_1}(e_I)) = \lambda(e_1)\lambda(e_I)$. And by linearity, we deduce that $\lambda(c_\alpha(\omega)) = \lambda(\alpha)\lambda(\omega)$, for all $\alpha, \omega \in \Lambda^* V$. This tells us that the binary operation c on the the exterior algebra over V corresponds to the Clifford multiplication. In the literature, the operators ϵ_α and ψ'_α , respectively, are called the exterior and the interior product by α.

Furthermore, recall that we have an inner product for $Cl(V, q)$, this in turn should give us an approach to define an inner product on $\Lambda^* V$ in the following way:

$(\alpha|\omega) := \mathrm{sc}(\lambda(c_{\overline{\alpha}}(\omega))) = (\lambda(\alpha), \lambda(\omega))$, where $\overline{\alpha} := \sum(-1)^{|I|}\alpha_I e_{i_k} \wedge \cdots \wedge e_{i_1}$. Similar to Lemma 3.2, for each $v \in V$ we have an analogous result:

Lemma 3.5. *c_v is adjoint to its negative with respect to the inner product $(\ |\ )$.*

Proof. Let $\alpha, \omega \in \Lambda^* V$. By the above observation, we have

$$(c_v(\alpha)|\omega) =(\lambda(c_v(\alpha)), \lambda(\omega)) = (\lambda(v)\lambda(\alpha), \lambda(\omega)) = \mathrm{sc}(\overline{\lambda(v)\lambda(\alpha)}\lambda(\omega)) \tag{3.10}$$

$$=\mathrm{sc}(\overline{\lambda(\alpha)}(-\lambda(v)\lambda(\omega))) = (\lambda(\alpha), \lambda(-c_v(\omega))) = (\alpha| - c_v(\omega)). \tag{3.11}$$

□

The main point of this subsection is the discussion of a proper way to define a left action of the $Cl(V, q)$ on the exterior algebra of the dual space. With all the machineries set up for us previously, such procedure should not be difficult. Recall that if (V, q) is an $n-$dimensional vector space over $\mathbb{R}$, $V \cong V^*$. The isomorphism is naturally defined as $v \mapsto q(v, .)$. In fact, it is also an isometry, if we define an inner product on V^* in the following way: $\langle f, g\rangle := q(v_f, v_g)$, where v_f, v_g are unique vectors in V such that $f = q(v_f, .)$ and $g = q(v_g, .)$. The identification of $V \cong V^*$ gives us an isomorphism of $\Lambda^* V \cong \Lambda^* V^*$ (by taking the k^{th} exterior power of the map $v \mapsto q(v, .)$). It is then reasonable that we should expect an analogous action like c_v on $\Lambda^* V^*$, for each $v \in V$.

For the contraction ι_v, let $\Phi : \Lambda^* V \to \Lambda^* V^*$ be the isomorphism defined above; naturally the contraction by v on the dual should be defined as $\Phi\iota_v\Phi^{-1}$. Denote $(f_1, \cdots, f_n)$ to be an orthonormal bases of V^* with respect to $\langle\, ,\rangle$, where $f_i = q(e_i, .)$. Consider a $k-$covectors $f_{i_1} \wedge \cdots \wedge f_{i_k}$, we have

$$\hat{\iota}_v(f_I) := \Phi\iota_v\Phi^{-1}(f_{i_1} \wedge \cdots \wedge f_{i_k}) =\Phi\iota_v(e_{i_1} \wedge \cdots \wedge e_{i_k}) \tag{3.12}$$

$$=\Phi\left(\sum_{j=1}^{k}(-1)^j q(v, e_{i_j}) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}\right) \tag{3.13}$$

$$=\sum_{j=1}^{k}(-1)^j q(e_{i_j},v) f_{i_1}\wedge\cdots\wedge \hat{f}_{i_j}\wedge\cdots\wedge f_{i_k} \tag{3.14}$$

$$=\sum_{j=1}^{k}(-1)^j f_{i_j}(v) f_{i_1}\wedge\cdots\wedge \hat{f}_{i_j}\wedge\cdots\wedge f_{i_k}. \tag{3.15}$$

With a little bit of multi-linear algebra, one can show that the RHS of (3.15) applying to $(v_1, v_2, \cdots, v_{k-1})$, where $v_i \in V$, gives us $-(f_{i_1}\wedge\cdots\wedge f_{i_k})(v, v_1, \cdots, v_k)$. In the literature, they write $(f_{i_1}\wedge\cdots\wedge f_{i_k})(v, v_1, \cdots, v_k) := v \lrcorner f_I$. So we shall also denote $\hat{\iota}_v(.) := -v\lrcorner(.)$. Similar to the computations from (3.1) – (3.6), we obtain $\hat{\iota}_v^2 = 0$. Hence, $\hat{\iota}$ induces a left action $\hat{\psi} : \Lambda^* V \otimes \Lambda^* V^* \to \Lambda^* V^*$.

On the other hand, for the exterior product ϵ_v, the correspondent linear map from $\Lambda^* V^* \to \Lambda^* V^*$ should be $\hat{\epsilon}_v(f_{i_1}\wedge\cdots\wedge f_{i_k}) := f_v \wedge f_{i_1}\wedge\cdots\wedge f_{i_k}$, where f_v is the dual of v. Therefore, we also have a left action by the exterior product $\hat{\epsilon} : \Lambda^* V \otimes \Lambda^* V^* \to \Lambda^* V^*$. Now define a linear map from $\Lambda^* V \otimes \Lambda^* V^* \to \Lambda^* V^*$ by $\hat{c}_\alpha := \hat{\epsilon}_\alpha + \hat{\psi}_\alpha$, where $\alpha \in \Lambda^* V$. Moreover, mirroring the computations from (3.7) – (3.9), we have:

$$\hat{c}_v^2(f_I) =(\hat{\epsilon}_v^2 + \hat{\epsilon}_v\hat{\psi}_v + \hat{\psi}_v\hat{\epsilon}_v + \hat{\psi}_v^2)(f_I) = \hat{\epsilon}_v\hat{\psi}_v(f_I) + \hat{\psi}_v\hat{\epsilon}_v(f_I) \tag{3.16}$$

$$=\hat{\epsilon}_v\left(\sum_{j=1}^{k}(-1)^j f_{i_j}(v)\hat{f}_{i_j}\right) + \hat{\psi}_v(f_v\wedge f_{i_1}\wedge\cdots\wedge f_{i_k}) \tag{3.17}$$

$$=\sum_{j=1}^{k}(-1)^j f_{i_j}(v) f_v\wedge\hat{f}_{i_j} - f_v(v) f_{i_1}\wedge\cdots\wedge f_{i_k} + \sum_{j=1}^{k}(-1)^{j+1} f_{i_j}(v) f_v\wedge\hat{f}_{i_j} \tag{3.18}$$

$$=- f_v(v) f_{i_1}\wedge\cdots\wedge f_{i_k} = -q(v,v) f_I. \tag{3.19}$$

In other words, $\hat{c}_v^2 = -q(v,v)\mathbf{1}$, where $\mathbf{1}$ is the identity on the exterior algebra of the dual. Therefore, the algebra $\hat{A}$ generated by $\{\hat{c}_\alpha : \alpha \in \Lambda^* V\}$ is also isomorphic to $Cl(V,q)$. This shows that $\Lambda^* V^*$ has a left $Cl(V,q)-$module structure, where the left action of $Cl(V,q)$ on $\Lambda^* V^*$ is defined by $\alpha\cdot\eta := \hat{c}_\alpha(\eta)$. In the literature, they call such action the left Clifford multiplication; in particular, we are interested in the left Clifford multiplication by

v, where $v \in V$. Note that $v \cdot \eta = f_v \wedge \eta - v \lrcorner \eta$.

Finally it should be noted that $\Lambda^* V^*$ has an inner product structure. And such construction should not be surprising: For $\eta, \mu \in \Lambda^* V^*$, we define $\langle \eta | \mu \rangle := (\Phi^{-1}(\eta) | \Phi^{-1}(\mu))$. With respect to the this inner product, the left Clifford multiplication by a vector v behaves in the way that is analogous to Lemma 3.5, Lemma 3.2.

Lemma 3.6. *For any $\eta, \mu \in \Lambda^* V^*$ and $v \in V$, we have $\langle v \cdot \eta | \mu \rangle + \langle \eta | v \cdot \mu \rangle = 0$.*

Proof. Note that with respect to V, $\hat{c}_v = \Phi c_v \Phi^{-1}$. We then have:

$$\langle v \cdot \eta | \mu \rangle = (\Phi^{-1} \hat{c}_v(\eta) | \Phi^{-1}(\mu)) = (c_v \Phi^{-1}(\eta) | \Phi^{-1}(\mu)) = (\Phi^{-1}(\eta) | - c_v \Phi^{-1}(\mu)) = \tag{3.20}$$

$$= - (\Phi^{-1}(\eta) | \Phi^{-1} \hat{c}_v(\mu)) = -\langle \eta | \hat{c}_v(\mu) \rangle = -\langle \eta | v \cdot \mu \rangle. \tag{3.21}$$

As a result, it is indeed true that $\langle v \cdot \eta | \mu \rangle + \langle \eta | v \cdot \mu \rangle = 0$. □

3.2 Clifford bundle and the generalized Dirac operator

Suppose (V, q) is a finite dimensional vector space over $\mathbb{R}$. We say S is a *Clifford module with respect to $Cl(V, q)$* if and only if it is a finite dimensional vector space over $F = \mathbb{R}$, or $\mathbb{C}$ equipped with a linear map $c : V \to \mathrm{End}(S)$, where $c_v^2 = -q(v, v)\mathbf{1}_S$. Because of the universal property of the Clifford algebra, c induces a unique linear map

$$c' : Cl(V, q) \otimes S \to S$$

such that $c'|_V(.) \equiv c$. The map c' can be understood as a left action of $Cl(V, q)$ on S. This in turn tells us that S also enjoys a left module structure over $Cl(V, q)$. We actually have encountered several examples of Clifford modules in the previous sections. In particular, $Cl(V, q)$, $\Lambda^* V$, and $\Lambda^* V^*$ are all such modules with respect to $Cl(V, q)$.

Definition 3.3. Let (M, g) be a Riemannian manifold. A bundle of Clifford modules over

M is a vector bundle $\pi : S \to M$, where each fiber S_p $(p \in M)$ is a Clifford module with respect to $Cl(T_pM, g_p)$.

Example 3.4. For a given $n-$dimensional Riemannian manifold (M, g). Consider the smooth manifold $Cl(TM, g) := \bigsqcup_{p\in M} Cl(T_pM, g_p)$ and the map $\pi : Cl(TM, g) \to M$ is defined by $\pi(\alpha) = p$ if α belongs to some $Cl(T_pM, g_p)$. Then $Cl(TM, g)$ is a vector bundle over M where each fiber is exactly $Cl(T_pM, g_p)$. Since any $Cl(V, q)$ is a Clifford module with respect to itself, where c is given by the left multiplication in the Clifford algebra, $Cl(TM, g)$ is a bundle of Clifford modules over M.

Example 3.5. The vector bundle $\pi : \Lambda^*T^*M \to M$ is a also a bundle of Clifford modules over (M, g). Indeed, each fiber is $\Lambda^*T_p^*M$, which is exactly a Clifford module with respect to $Cl(T_pM, g_p)$, where the linear map c is uniquely induced by the linear map $\hat{c}(p) : Cl(T_pM, g_p) \otimes \Lambda^*T_p^*M \to \Lambda^*T_p^*M$ that is given be $\hat{c}(p)_v(\eta) = \hat{c}_{v_p}(\eta)$, for every vector field $v \in \mathfrak{X}(M)$ as in subsection 3.1.

It should be noted that, for each $p \in M$, there is a natural identification between $Cl(T_pM, g_p) \cong \Lambda^*T_p^*M$ (again, not as algebras!). The bundle of Clifford module structure from one is carried over to the other. However most of the times, we choose to work with Λ^*T^*M, for the sections of such vector bundles are exactly all of the differential forms $\Omega^*(M)$. This set-up is always more convenient when discussing about integrations.

Now recall that for each p, $Cl(T_pM, g_p)$ and $\Lambda^*T_p^*M$ have inner product structures that are defined as in subsection. And if M is paracompact, the inner products can be glued together by partition of unity so that these bundles of Clifford modules have smoothly varying inner products across the fibers.

The behavior of the left Clifford multiplication by tangent vectors on these bundles is exactly as described as in Lemma 3.6, Lemma 3.5, Lemma 3.2. This relation, together with a particular condition on the bundle, give us a very special kind of bundle that we will discuss next.

Definition 3.4. (Clifford bundles) Suppose (M, g) is a Riemannian manifold and S is a bundle of Clifford modules over M that has a smooth varying inner product $\langle | \rangle$ across its fibers. We say that S is a Clifford bundle over M if and only if it satisfies the two following conditions:

1. The left Clifford multiplication by tangent vectors on the fibers of S is skew-adjoint, i.e, for every $v_p \in T_pM$ and $s_1, s_2 \in S_p$, we have $\langle c_{v_p}(s_1) | s_2 \rangle + \langle s_1 | c_{v_p}(s_2) \rangle = 0$.

2. S posses a connection ∇^S that is compatible with $\langle | \rangle$, and with the Levi-Civita connection in the sense that for every $X, Y \in \mathfrak{X}(M)$, $s \in \Gamma(S)$, we have

$$\nabla^S_X(c_Y(s)) = c_{\nabla^{LC}_X(Y)}(s) + c_Y(\nabla^S_X(s)).$$

Often times, the Clifford bundle S we will be considering are $\mathbb{Z}/2-$graded, i.e, S has direct-sum decomposition $S = S^+ \oplus S^-$. We will also require that the connection and the metric respect the decomposition. Furthermore, it is also needed that the left Clifford multiplication by tangent vectors is odd, that is, for each $v_p \in T_pM$, c_{v_p} will send S^+_p to S^-_p and vice versa. Compare this with Proposition 3.1.

Examples for Clifford bundles over a Riemannian manifold M are, unexpectedly, the familiar constructions we have encountered so far: $Cl(TM, g)$, $\Lambda^* T^* M$. The exact details why those two bundles satisfy the second condition in Definition 3.4 will be reserved for later sections can be found at [reference]. We are now ready to define the notion of Dirac operator on Riemannian manifolds.

Definition 3.5. (Dirac operator) The Dirac operator D of a Clifford bundle S over a Riemannian manifold (M, g) is the first order differential operator on $\Gamma(S)$ defined by the following composition of maps

$$\Gamma(S) \longrightarrow \Gamma(T^*M \otimes S) \longrightarrow \Gamma(TM \otimes S) \longrightarrow \Gamma(S),$$

where the first map is given by the compatible connection ∇^S, the second map is the identification by the metric g between TM and T^*M, and the last map is the left Clifford multiplication by tangent vectors. In the case of S being $\mathbb{Z}/2-$graded, because of the left Clifford multiplication action, D is an odd operator.

Let (U, x) be a coordinate patch on M so that TM has a local trivialization. We then have $\Gamma(TM|_U) = \mathfrak{X}(U) = \text{span}_{\Gamma(U)}\{\partial_1, \partial_2, \cdots, \partial_n\}$, where $\partial_i := \partial/\partial x_i$. Without loss of generality, we shall assume that (∂_i) is an orthonormal basis with respect to g. As a result, $\Gamma(T^*M|_U) = \Omega^1(U) = \text{span}_{\Gamma(U)}\{dx_1, \cdots, dx_n\}$, where $dx_i := g(\partial_i, .)$. Denote $\Phi : TM|_U \to T^*M|_U$ the isomorphism that is induced by the Riemannian metric, $\partial_i \mapsto dx_i$. Fix an $s|_U \in \Gamma(S)$, we shall follow the construction of the Dirac operator step by step to obtain the local formula for Ds:

Step 1, s is first mapped to $\nabla^S s$, which is a linear map from $TM|_U \to S|_U$. Because of Φ defined above, $\nabla^S s$ induces the linear map $\nabla^S_{\Phi^{-1}} s : T^*M|_U \to S_U$. Such a map then can be interpreted as an element of $\Gamma(TM|_U \otimes S|_U)$, this is step 2. Finally after step 2, we have a pairing of $\Phi^{-1}(.) \in TM|_U$ and $\nabla^S_{\Phi^{-1}(.)} s \in S|_U$, so the left Clifford multiplication by tangent vectors (vector fields evaluated at points in U) yields $c_{\Phi^{-1}(.)}(\nabla^S_{\Phi^{-1}(.)} s)$; this concludes step 3. As a result, for a fixed $s|_U \in \Gamma(S)$, $Ds(\Phi^{-1}(.)) = c_{\Phi^{-1}(.)}(\nabla^S_{\Phi^{-1}(.)} s)$. In particular, when we evaluate Ds at each ∂_i, we obtain $Ds(\partial_i) = Ds(\Phi^{-1}(dx_i)) = c_{\Phi^{-1}(dx_i)}(\nabla^S_{\Phi^{-1}(dx_i)} s) = c_{\partial_i}(\nabla^S_{\partial_i} s)$. Therefore, the local formula for D is

$$D = \sum_{i=1}^{n} c_{\partial_i}(\nabla^S_{\partial_i}). \tag{3.22}$$

As expected, we will illustrate that the above local formula for the Dirac operator D should be consistent with how we normally define the Dirac operator in the Euclidean case $\mathbb{R}^n$. However, note that in $\mathbb{R}^n$, the intricacy of yielding a local formula for D is not necessary for $\mathbb{R}^n$ naturally posses a global frame field and the Riemannian metric is the usual inner product. Though, it is beneficial to show that the definition of D is independent of the choice of a orthonormal basis.

Let $\{e_1, \cdots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$ so that for every point $p \in \mathbb{R}^n$, we have $p = \sum p_i e_i$. Let x_i be the coordinate functions on $\mathbb{R}^n$ given by $x_i(p) = p_i = \langle p, e_i \rangle$. Then the global frame field that spans $\mathfrak{X}(\mathbb{R}^n)$ is $\{\partial/\partial x_1, \cdots, \partial/\partial x_n\}$. And thus for each vector field $X \in \mathfrak{X}(\mathbb{R}^n)$, we have $X = \sum X^i \partial/\partial x_i$, where $X^i \in \Gamma(\mathbb{R}^n)$.

Lemma 3.7. *The map $\nabla : \mathfrak{X}(\mathbb{R}^n) \otimes \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)$ given by*

$$\nabla_X Y(p) \equiv \nabla_{X(p)} Y := \lim_{t \to 0} \frac{Y(p + tX(p)) - Y(p)}{t},$$

for each $p \in \mathbb{R}^n$, defines the Levi-Civita connection on $\mathbb{R}^n$.

Proof. First we show that ∇ is a well-defined connection on $\mathbb{R}^n$. We write $X = \sum X^i \partial/\partial x_i$ and $Y = \sum Y^i \partial/\partial x_i$; also denote $c(t) = p + tX(p)$, where we view $X(p)$ here as the point $(X^1(p), \cdots, X^n(p)) \in \mathbb{R}^n$ and $t \in (-\epsilon, \epsilon) \subset \mathbb{R}$. Then $\nabla_X Y(p) = (Y \circ c)'(0)$ so that

$$\nabla_X Y(p) = \sum_{i=1}^{n} \frac{d}{dt}(Y^i \circ c)\bigg|_{t=0} \frac{\partial}{\partial x_i}\bigg|_{c(0)} = \sum_{i=1}^{n} \frac{d}{dt}(Y^i \circ c)\bigg|_{t=0} \frac{\partial}{\partial x_i}\bigg|_{p}.$$

On the other hand, for each i we have

$$XY^i(p) = X_p Y^i = \lim_{t \to 0} \frac{Y^i(p + tX(p)) - Y^i(p)}{t} = \lim_{t \to 0} \frac{Y^i \circ c(t) - Y^i \circ c(0)}{t} = \frac{d}{dt}(Y^i \circ c)\bigg|_{t=0}.$$

Therefore,

$$\nabla_X Y(p) = \sum_{i=1}^{n} XY^i(p) \frac{\partial}{\partial x_i}\bigg|_{p}.$$

With the above formula, it is straight-forward to check that ∇ is $\Gamma(\mathbb{R}^n)-$bilinear in X, Y and obeys the Leibniz rule in Y. Furthermore, $\nabla_X Y$ indeed is a vector field on $\mathbb{R}^n$. Hence, it gives a well-defined connection. To finish the proof, we check if ∇ is symmetric and compatible with the Riemannian metric.

In coordinates, for every $f \in \Gamma(\mathbb{R}^n)$ we have

$$
\begin{aligned}
[X,Y]f =& X(Yf) - Y(Xf) = X\left(\sum_{i=1}^{n} Y^i \frac{\partial f}{\partial x_i}\right) - Y\left(\sum_{i=1}^{n} X^i \frac{\partial f}{\partial x_i}\right) && (3.23)\\
=& \sum_{i=1}^{n}(XY^i)\frac{\partial f}{\partial x_i} + \sum_{i,j=1}^{n} Y^i X^j \frac{\partial^2 f}{\partial x_j \partial x_i} - \sum_{i=1}^{n}(YX^i)\frac{\partial f}{\partial x_i} - \sum_{i,j=1}^{n} X^i Y^j \frac{\partial^2 f}{\partial x_j \partial x_i} && (3.24)\\
=& \sum_{i=1}^{n}(XY^i)\frac{\partial f}{\partial x_i} - \sum_{i=1}^{n}(YX^i)\frac{\partial f}{\partial x_i} = \nabla_X Y(f) - \nabla_Y X(f). && (3.25)
\end{aligned}
$$

Thus, $\nabla_X Y - \nabla_Y X = [X,Y]$, which means ∇ is symmetric. Lastly, for each $p \in \mathbb{R}^n$ and $Z = \sum Z^i \partial/\partial x_i$, we have

$$
\begin{aligned}
\langle \nabla_X Y(p), Z(p)\rangle + \langle Y(p), \nabla_X Z(p)\rangle =& \sum_{i=1}^{n}(XY^i)(p)Z^i(p) + \sum_{i=1}^{n} Y^i(p)(XZ^i)(p) && (3.26)\\
=& \left(X\sum_{i=1}^{n} Y^i Z^i\right)(p) = X\langle Y(p), Z(p)\rangle. && (3.27)
\end{aligned}
$$

Hence ∇ is compatible with the metric of $\mathbb{R}^n$. ∇ uniquely determines the Levi-Civita connection on $\mathbb{R}^n$. □

Consider the vector bundle $\pi : Cl(\mathbb{R}^n) \to \mathbb{R}^n$ given by $s \mapsto p$ if $s \in Cl(T_p\mathbb{R}^n) \approx Cl(\mathbb{R}^n)$. Each section $f \in \Gamma(Cl(\mathbb{R}^n)) \equiv \Gamma(\mathbb{R}^n, Cl(\mathbb{R}^n))$ will be of the form $f = \sum_I f^I e_I$, where $f^I \in \Gamma(\mathbb{R}^n)$. We define $\nabla^{Cl} : \mathfrak{X}(\mathbb{R}^n) \otimes \Gamma(Cl(\mathbb{R}^n)) \to \Gamma(Cl(\mathbb{R}^n))$ by

$$\nabla_X^{Cl} f = \sum_I (Xf^I)e_I := Xf.$$

It is not hard to see that ∇^{Cl} is $\Gamma(\mathbb{R}^n)-$linear in X and obey the Leibniz rule in f. So it defines for us a connection on the vector bundle $Cl(\mathbb{R}^n)$. Furthermore, note that $Cl(\mathbb{R}^n)$ satisfies all the criterion of a bundle of Clifford modules over $\mathbb{R}^n$, where the action of $Cl(T_p\mathbb{R}^n)$ on each fiber $Cl(T_p\mathbb{R}^n) \approx Cl(\mathbb{R}^n)$ is defined to be
$c_{v(p)}s = v(p)s := (\sum v^i(p)e_i)(\sum s^I e_I)$, the product here is the usual Clifford multiplication

in $Cl(\mathbb{R}^n)$. Finally, $Cl(\mathbb{R}^n)$ has a smoothly varying inner product $(\ ,\ )$ and by Lemma 3.2, for each $s_1, s_2 \in Cl(T_p\mathbb{R}^n) \approx Cl(\mathbb{R}^n)$ and $v(p) \in T_p\mathbb{R}^n$, we have
$(c_{v(p)}s_1, s_2) + (s_1, c_{v(p)}s_2) = 0$. We also claim that ∇^{Cl} is compatible with the Levi-Civita connection ∇ defined in Lemma 3.7, i.e, for each $X, Y \in \mathfrak{X}(\mathbb{R}^n)$ and $f \in \Gamma(Cl(\mathbb{R}^n))$, we have

$$\nabla^{Cl}_X(c_Y(f)) = c_{\nabla_X Y}(f) + c_Y(\nabla^{Cl}_X f).$$

Indeed, since $c_Y(f) = (\sum_{j=1}^n Y^j e_j)(\sum_I f^I e_I) = \sum_{j=1}^n \sum_I Y^j f^I e_j e_I$, and regardless of when $e_j e_I$ simplifies in the Clifford algebra, it still represents a basis element of the algebra, the left-hand side of the above equation yields

$$\nabla^{Cl}_X(c_Y(f)) = \sum_{j=1}^n \sum_I X(Y^j f^I) e_j e_I \tag{3.28}$$

$$= \sum_{j=1}^n \sum_I (XY^j) f^I e_j e_I + \sum_{j=1}^n \sum_I Y^j (X f^I) e_j e_I. \tag{3.29}$$

The first term of the right-hand side of (39) is exactly $(\sum_{j=1}^n (XY^j) e_j)(\sum_I f^I e_I) = c_{\nabla_X Y}(f)$. While the second term is $(\sum_{j=1}^n Y^j e_j)(\sum_I (X f^I) e_I) = c_Y(\nabla^{Cl}_X f)$. Finally, it is routine to check that ∇^{Cl} is compatible with the inner product on $Cl(\mathbb{R}^n)$ defined in subsection 3.1. With all of the above observations, we conclude:

Proposition 3.8. *$Cl(\mathbb{R}^n)$ is a Clifford bundle over $\mathbb{R}^n$.*

Now we are ready to write down the formula for the Dirac operator in $\mathbb{R}^n$ as promised. For each $f \in \Gamma(Cl(\mathbb{R}^n))$, from (32), we have

$$Df = \sum_{i=1}^n c_{\partial_i}(\nabla^{Cl}_{\partial_i} f) = \sum_{i=1}^n e_i \frac{\partial f}{\partial x_i}.$$

In literature, this is how we normally define the operator on the Euclidean space. Even if we start off defining the Dirac operator on $\mathbb{R}^n$ as $D = \sum_{i=1}^n e_i \partial/\partial x_i$, it should be straight-forward to show that it is independent of the choice of orthonormal basis.

Suppose $\{w_1, \cdots, w_n\}$ is another orthonormal basis of $\mathbb{R}^n$. Let $g = (g_{ij}) \in O(n)$ so that $ge_i = w_i$. If y_i are the coordinate functions on $\mathbb{R}^n$ with respect to $\{w_i\}$, then $y_i = \sum_j g_{ji}x_j$. Note that because $g^T g = gg^T = \mathbf{1}$, we also have $x_i = \sum_j g_{ij}y_j$. For each $f \in \Gamma(\mathbb{R}^n)$ written in x_i-coordinate, we obtain $h = f \circ \phi$, where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a change of coordinate from y_i to x_i. By chain rule, we yield

$$\sum_{i=1}^{n} w_i \frac{\partial h}{\partial y_i} = \sum_{i=1}^{n} ge_i \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial y_i} = \sum_{i=1}^{n} ge_i \sum_{j=1}^{n} g_{ji}\frac{\partial f}{\partial x_j} \tag{3.30}$$

$$= \sum_{i,j=1}^{n} ge_i g_{ji}\frac{\partial f}{\partial x_j} = \sum_{j=1}^{n} g\left(\sum_{i=1}^{n} g_{ji}e_i\right)\frac{\partial f}{\partial x_j} \tag{3.31}$$

$$= \sum_{j=1}^{n} gg^T e_j \frac{\partial f}{\partial x_j} = \sum_{j=1}^{n} e_j \frac{\partial f}{\partial x_j}. \tag{3.32}$$

(3.32) is still true when $s \in \Gamma(Cl(\mathbb{R}^n))$, for $s = \sum_I s^I e_I$ and $s^I \in \Gamma(\mathbb{R}^n)$. The above calculation tells us that D is independent of the choice of orthonormal basis.

3.3 Connection and curvature revisited

Recall our setting: Let M be a Riemannian manifold, and V be a vector bundle over M equipped with a connection ∇.

Definition 3.6. The curvature operator K associated with ∇ is given by:

$$K(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s,$$

where X, Y are any vector fields on M and $s \in \Gamma(V)$.

Lemma 3.9. *If we view K as a map from $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(V) \to \Gamma(V)$, then K is $\Gamma(M)-$linear in each variable, anti-symmetric in the first two variable. Thus for $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma(V)$, $K(X,Y)s$ at $p \in M$ depends only on the value of X, Y, and s at p, but not their values at nearby points.*

Proof. Let $f \in \Gamma(M)$ arbitrary, note that $[fX, Y] = f[X, Y] - (Yf)X$. Then we have:

$$K(fX, Y)s = \nabla_{fX}\nabla_Y s - \nabla_Y \nabla_{fX} s - \nabla_{[fX,Y]} s \tag{3.33}$$

$$= f\nabla_X \nabla_Y s - \nabla_Y (f\nabla_X s) - \nabla_{f[X,Y]-(Yf)X} s \tag{3.34}$$

$$= f\nabla_X \nabla_Y s - f\nabla_Y \nabla_X s - (Yf)\nabla_X s - f\nabla_{[X,Y]} s + (Yf)\nabla_X s \tag{3.35}$$

$$= f\nabla_X \nabla_Y s - f\nabla_Y \nabla_X s - f\nabla_{[X,Y]} s = fK(X, Y)s. \tag{3.36}$$

Similar calculations at Y and s. Next, note that $[Y, X] = -[X, Y]$; so by direct comparison, we have $K(Y, X)s = -K(X, Y)s$. Now for the last part of the lemma, let $p \in M$ and U be a neighborhood around p so that we have a local frame field $\{\partial/\partial x_i\}$ and trivialization $\{e_j\}$ on V. We then write $X = \sum_i X^i \partial/\partial x_i$, $Y = \sum_j Y^j \partial/\partial x_j$, and $s = \sum_k s^k e_k$. By $\Gamma(M)-$linearity that we proved above, we get:

$$K(X, Y)s = K\left(\sum_i X^i \frac{\partial}{\partial x_i}, \sum_j Y^j \frac{\partial}{\partial x_j}\right)\left(\sum_k s^k e_k\right) \tag{3.37}$$

$$= \sum_{i,j,k} K\left(X^i \frac{\partial}{\partial x_i}, Y^j \frac{\partial}{\partial x_j}\right)(s^k e_k) = \sum_{i,j,k} X^i Y^j s^k K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) e_k. \tag{3.38}$$

We observe that from (3.38), the value of $K(X, Y)s$ at p depends only on the values of X^i, Y^j, and s^k at p and not their nearby points. □

The above lemma tells us that even though the curvature operator K's definition does not involve any differentiation in its variable. Furthermore, because of $\Gamma(M)-$linearity and anti-symmetric property, we now can view K as a linear map from $\mathfrak{X}(M) \otimes \mathfrak{X}(M) \otimes \Gamma(V) \to \Gamma(V)$. Consequently, K induces a linear map from $TM \otimes TM \to \mathrm{End}(V)$ that is anti-symmetric. In other words, K is a $\mathrm{End}(V)-$valued $2-$form on M. In local coordinates, we can write

$$K = \sum_{i<j} K\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) dx_i \wedge dx_j.$$

In our context of Clifford bundle, if $S \to M$ is a Clifford bundle equipped with the connection ∇^S satisfying Definition 3.4 and K^S is the curvature operator associated with ∇^S, then we define the Clifford contraction $\mathbf{K}^S$ in the following way:

Definition 3.7. Let $\{\partial_i := \partial/\partial x_i\}$ be a local orthonormal frame of TM. The Clifford contraction $\mathbf{K}^S$ is an endomorphism on S given by

$$\mathbf{K}^S = \sum_{i<j} c_{\partial_i}(c_{\partial_j}(K^S(\partial_i, \partial_j))).$$

The definition of $\mathbf{K}^S$ does not depend on the choice of local frame.

Finally, we would like to make some comments about how to precisely obtain a special local orthonormal frame field on TM from the Levi-Civita connection ∇^{LC}. A smooth curve $\gamma : [0,1] \to M$ is called a geodesic if and only if $\nabla^{LC}_{\gamma'}\gamma' = 0$, or in other words γ' is parallel along γ. Note that the geodesic equation $\nabla^{LC}_{\gamma'}\gamma' = 0$ is a second-order differential equation, so it has a unique solution subjected to the initial value condition $\gamma(0)$ and $\gamma'(0)$. Therefore, for any point on M, there is a unique geodesic segment through that point in a given direction.

Note that if $t \mapsto \gamma(t)$ is a solution to the geodesic equation, then so is $t \mapsto \gamma(ct)$, for any constant $c \in \mathbb{R}$. Indeed, denote $\alpha(t) := \gamma(ct)$. By chain rule, we have $\alpha'(t) = c\gamma'(ct)$. Hence we obtain

$$\nabla^{LC}_{\alpha'(t)}\alpha'(t) = \nabla^{LC}_{c\gamma'(ct)}(c\gamma'(ct)) \tag{3.39}$$

$$= c(c\nabla^{LC}_{\gamma'(ct)}\gamma'(ct) + (\gamma'_{ct}(c))\gamma'(ct)) = c^2\nabla^{LC}_{\gamma'(ct)}\gamma'(ct) = 0. \tag{3.40}$$

Such property of a differential equation described above is called *isochronous*. It follows that for every point $p \in M$ and U is a star-shaped open subset of T_pM around the origin, the *exponential map* exp is defined in the following way:

$$\exp : U \to M,$$

where a vector $v \in U \subseteq T_pM$ is mapped to $\gamma(1)$ of the unique geodesic that satisfies $\gamma(0) = p$, $\gamma'(0) = v$. Since the total derivative of exp is nonsingular at the origin, by the inverse function theorem, exp is a local diffeomorphism from a neighborhood of the origin in T_pM onto a neighborhood of p in M. Then an orthonormal basis of T_pM induces a special coordinate system, called a *geodesic coordinate system*, around a neighborhood of p.

Proposition 3.10. *At the origin of a geodesic coordinate system, the Christoffel symbols of the Levi-Civita connection vanish.*

Proof. Let $\{\partial_i := \partial/\partial x_i\}$ be a local orthonormal frame field associated with the geodesic coordinate system around p. We would like to show that $\nabla^{LC}_{\partial_i}\partial_j = 0$ at p. But because of the symmetric property of the connection, we have $\nabla^{LC}_{\partial_i}\partial_j = \nabla^{LC}_{\partial_j}\partial_i$; hence it is sufficient to show that $\nabla^{LC}_X X = 0$ at p for all vector fields $X = \sum_k X^k\partial_k$, where X^k are constant functions. Now there is a unique geodesic γ that starts at p in the direction of $X(p) \in T_pM$, and thus $\nabla^{LC}_{\gamma'(t)}\gamma'(t) = 0$. In particular, when $t = 0$, we have $\nabla^{LC}_{X(p)}X(p) = 0$. □

One can also arrive at the same conclusion in Proposition 3.10 by utilizing the computations in Theorem 2.2. A local orthonormal frame field defined above always exists for an arbitrary $p \in M$; and such frame where all the Christoffel symbols vanish at p is called a *synchronous* at p. Furthermore as a consequence of the fact $\Gamma^k_{ij}(p) = 0$ and the symmetry of the connection, we obtain the Lie bracket $[\partial_i, \partial_j](p) = 0$ for all i and j. These facts will be of relevant later when we compute the operator D^2.

Recall our setting: Let (M,h) by a Riemannian manifold of dimension n and $S\to M$ is a Clifford bundle over M equipped with the Clifford multiplication c. Denote ∇^{LC} the Levi-Civita connection and ∇^S the compatible connection on S. Around any arbitrary point $p\in M$, we can find a open neighborhood $U\subseteq M$ so that we have an oriented local synchronous frame field $\{\partial_i := \partial/\partial x_i\}$ at p for $TM|_U$. Such local frame field induces an oriented local oriented coordinate systems $x_1,\cdots,x_n$ around p. Hence by (3.22) at p, the corresponding local formula for the Dirac operator is $Ds=\sum_i c_{\partial_i}(\nabla^S_{\partial_i}s)$ for each $s\in\Gamma(S|_U)$. Consequently at p, we have

$$
\begin{aligned}
D^2 s &= \sum_{i=1}^n D(c_{\partial_i}(\nabla^S_{\partial_i}s)) = \sum_{i=1}^n\sum_{j=1}^n c_{\partial_j}(\nabla^S_{\partial_j}(c_{\partial_i}(\nabla^S_{\partial_i}s))) && (3.41)\\
&= \sum_{i=1}^n\sum_{j=1}^n c_{\partial_j}(c_{\nabla^{LC}_{\partial_j}\partial_i}(s) + c_{\partial_i}(\nabla^S_{\partial_j}\nabla^S_{\partial_i}s)) && (3.42)\\
&= \sum_{i=1}^n\sum_{j=1}^n c_{\partial_j}(c_{\partial_i}(\nabla^S_{\partial_j}\nabla^S_{\partial_i}s)) = \sum_{i=j=1}^n c_{\partial_i}(c_{\partial_i}((\nabla^S_{\partial_i})^2 s)) + \sum_{i\neq j} c_{\partial_j}(c_{\partial_i}(\nabla^S_{\partial_j}\nabla^S_{\partial_i}s)) && (3.43)\\
&= -\sum_{i=1}^n (\nabla^S_{\partial_i})^2 s + \sum_{j<i} c_{\partial_j}(c_{\partial_i}(\nabla^S_{\partial_j}\nabla^S_{\partial_i} - \nabla^S_{\partial_i}\nabla^S_{\partial_j})s). && (3.44)
\end{aligned}
$$

We would like to provide some short justifications for some of the equations above: (3.42) is true because of the compatibility of ∇^S with the Clifford multiplication, (3.43) is true because of Proposition 3.10, and (3.44) is true because of the fact that $c_v^2 = -(v,v)\mathbf{1}$. Now we denote K^S the curvature operator associated with ∇^S so that by definition, at p, $K^S(\partial_j,\partial_i)s = \nabla^S_{\partial_j}\nabla^S_{\partial_i}s - \nabla^S_{\partial_i}\nabla^S_{\partial_j}s - \nabla^S_{[\partial_j,\partial_i]}s$. But from the remarks in previous subsection, for synchronous frame, $[\partial_j,\partial_i](p)=0$. Thus at p, $K^S(\partial_j,\partial_i)s = \nabla^S_{\partial_j}\nabla^S_{\partial_i}s - \nabla^S_{\partial_i}\nabla^S_{\partial_j}s$. Combining with the definition of the Clifford contraction $\mathbf{K}^S$, as a result, (3.44) simplified further yields

$$D^2 s = -\sum_{i=1}^{n} (\nabla^S_{\partial_i})^2 s + \mathbf{K}^S s := \nabla^{S*}\nabla^S s + \mathbf{K}^S s. \tag{3.45}$$

(3.45) is a very important equation called the *Weitzenbock formula.* The first term of D^2 is a second-order differential operator which is a generalized Laplacian on Riemannian manifold. The second term, the Clifford contraction, is an endomorphism on S. The notation $\nabla^{S*}\nabla^S$ has its reasons: Since $\nabla : \Gamma(S) \to \Gamma(T^*M \otimes S)$ can be regarded as a differential operator and these bundles are equipped with a metric so that the spaces of sections have L^2-inner products, ∇^S has a formal adjoint operator ∇^{S*}. We then shall show rigorously the following

Proposition 3.11. *With respect to any local orthonormal frame field on TM, we have*

$$\nabla^{S*}\nabla^S = -\sum_{i=1}^{n} (\nabla^S_{\partial_i})^2.$$

Before delving into the proof, we would like to first describe the precise manner which $\Gamma(T^*M \otimes S)$ has an global L^2-inner product. In general, suppose V and W are two finite dimensional vector spaces over $\mathbb{R}$ equipped with their own inner products. Let $\{v_i\}_{i=1}^n$ and $\{w_j\}_{j=1}^m$, respectively, be ordered basis of V and W so that every element $\omega \in V \otimes W$, we can write $\omega = \sum_{i,j} \omega^{ji} v_i \otimes w_j$. Note that by linearity,

$$\omega = \sum_{i=1}^{n}\sum_{j=1}^{m} \omega^{ji} v_i \otimes w_j = \sum_{i=1}^{n} v_i \otimes \sum_{j=1}^{m} \omega^{ji} w_j = \sum_{i=1}^{n} v_i \otimes h_i, \text{ where } h_i := \sum_{j=1}^{m} \omega^{ji} w_j. \tag{3.46}$$

Therefore, without even specifying an ordered basis for W, a general element of $V \otimes W$ can be written as $\sum_i v_i \otimes h_i$, where h_i is some element of W. Hence, an inner product on

$V \otimes W$ can be defined as follow:

$$\left\langle \sum_{i=1}^{n} v_i \otimes h_i, \sum_{j=1}^{n} v_j \otimes h'_j \right\rangle := \sum_{i=1}^{n}\sum_{j=1}^{n} \langle v_i, v_j\rangle_V \langle h_i, h'_j\rangle_W.$$

It should be routine to check that the above definition of the inner product on $V \otimes W$ is independent of the choice of basis on V and W. Furthermore, in the case where $\{v_i\}$ and $\{w_j\}$ are orthonormal bases, each element $\omega = \sum_{i,j} \omega^{ji} v_i \otimes w_j$ corresponds to an $m \times n-$matrix (ω^{ji}) so that we have an isomorphism of vector spaces $V \otimes W \cong M(m \times n, \mathbb{R})$[1]. The inner product defined above corresponds to the Frobenius inner product on $M(m \times n, \mathbb{R})$. Indeed,

$$\left\langle \sum_{i,j} \omega_1^{ji} v_i \otimes w_j, \sum_{k,l} \omega_2^{lk} v_k \otimes w_l \right\rangle = \left\langle \sum_{i=1}^{n} v_i \otimes \sum_{j=1}^{m} \omega_1^{ji} w_j, \sum_{k=1}^{n} v_k \otimes \sum_{l=1}^{m} \omega_2^{lk} w_l \right\rangle \tag{3.47}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} \langle v_i, v_k\rangle_V \left\langle \sum_{j=1}^{m} \omega_1^{ji} w_j, \sum_{l=1}^{m} \omega_2^{lk} w_l \right\rangle_W \tag{3.48}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \omega_1^{ji}\omega_2^{ji} \tag{3.49}$$

$$= \mathrm{tr}((\omega_1^{ji})^T(\omega_2^{lk})) = \langle (\omega_1^{ji}), (\omega_2^{lk})\rangle_F. \tag{3.50}$$

It is fairly easy to check that the Frobenius inner product is unchanged in orthonormal equivalent classes of matrices, and because of the above (3.47) – (3.50), the definition of the inner product on $V \otimes W$ is also unaffected by the change of orthonormal bases.

Definition 3.8. In our context of $T^*M \otimes S$, let's say $\langle\ ,\ \rangle$ is a smoothly varying inner product on the fibers of T^*M induced by the Riemannian metric and $(\ |\ )$ is one for S. Let $\{\partial_i := \partial/\partial x_i\}$ be a local frame field on TM so that $\{dx_i\}$ is a local basis for T^*M. The

[1]The space of $m \times n-$matrices.

inner product on each fiber of $T^*M \otimes S$ would be

$$\left(\sum_{i=1}^{n} dx_i \otimes s_i, \sum_{j=1}^{n} dx_j \otimes s'_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle dx_i, dx_j\rangle (s_i|s'_j), \tag{3.51}$$

where s_i, s'_j are some sections of S. In the case of local synchronous frame, the inner product on each fiber is simplified to be equal to $\sum_i (s_i|s'_i)$.

For the rest of this section, we shall only deal with local synchronous frame field. Recall for an orientable Riemannian manifold [2], given an ordered local frame field $\{\partial/\partial x_i\}$, we have the volume form $\mathrm{vol} \in \Omega^n(M)$ defined by $\mathrm{vol} = dx_1 \wedge \cdots \wedge dx_n$. Therefore, for some vector bundle $V \to M$ equipped with a metric $(\ , \ )$ that is smoothly varying on its fibers, a global L^2-inner product on $\Gamma(V)$ is given by

$$\langle s_1, s_2\rangle_{L^2} = \int_M (s_1, s_2)_p \mathrm{vol}, \text{ where } s_i \in \Gamma(V). \tag{3.52}$$

Definition 3.9. In particular, for the bundle $T^*M \otimes S$, the global L^2-inner product on $\Gamma(T^*M \otimes S)$ is given by

$$\langle \omega_1, \omega_2\rangle_{L^2} = \int_M (\omega_1, \omega_2)_p \mathrm{vol},$$

where $\omega_i \in \Gamma(T^*M \otimes S)$ and $(\ , \ )$ is defined as in (3.51).

As we have seen before in Subsection 3.1, given a local frame field $\{\partial/\partial x_i\}$ for TM and given the isomorphism Φ induced by the Riemannian metric $\Lambda^*TM \cong \Lambda^*T^*M$, we have an inner product on $\Gamma(\Lambda^*T^*M) \equiv \Omega^*(M)$ given by $\langle\alpha|\beta\rangle_p = (\Phi^{-1}(\alpha)|\Phi^{-1}(\beta))_p$ for $\alpha, \beta \in \Omega^*(M)$ and $p \in M$. In particular, when $\alpha, \beta \in \Omega^k(M)$, we can write $\alpha = \sum_{|I|=k} \alpha^I dx_I$ and $\beta = \sum_{|J|=k} \beta^J dx_J$, where α^I, $\beta^J \in \Gamma(M)$; so that $\langle\alpha|\beta\rangle_p = \sum_I \alpha^I(p)\beta^I(p)$.

In relation to the global L^2-inner product on $\Omega^*(M)$, for each $\alpha, \beta \in \Omega^k(M)$ with at

[2] In fact, here is the first time we mention the orientability condition, we would like to note that this condition should be implicitly understood whenever Riemannian manifold shows up in the paper.

least one of them is compactly supported, we have

$$\langle \alpha, \beta \rangle_{L^2} = \int_M \langle \alpha | \beta \rangle_p \text{vol} = \int_M \beta \wedge *\alpha = \int_M \alpha \wedge *\beta = \int_M \langle \beta | \alpha \rangle_p \text{vol} = \langle \beta, \alpha \rangle_{L^2}.$$

Proposition 3.12. *d^* is the formal adjoint of d with respect to the global L^2-inner product.*

Proof. Let $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$, at least one of them is compactly supported. By Stoke's theorem, we have

$$0 = \int_{\partial M} \beta \wedge *\alpha = \int_M d(\beta \wedge *\alpha) = \int_M d\beta \wedge *\alpha + (-1)^{k-1} \int_M \beta \wedge d(*\alpha) \tag{3.53}$$

$$= \langle \alpha, d\beta \rangle_{L^2} + (-1)^{k-1+(n-k+1)n+(n-k+1)} \int_M \beta \wedge * * d(*\alpha) \tag{3.54}$$

$$= \langle \alpha, d\beta \rangle_{L^2} - \int_M \beta \wedge *((-1)^{nk+k+1} * d(*\alpha)) = \langle \alpha, d\beta \rangle_{L^2} - \int_M \beta \wedge * d^* \alpha \tag{3.55}$$

$$= \langle \alpha, d\beta \rangle_{L^2} - \langle d^* \alpha, \beta \rangle_{L^2}. \tag{3.56}$$

□

Theorem 3.13. (Divergence theorem) *Let $\alpha = \sum \alpha^i dx_i$, where α^i's are smooth functions, be a one form on M that is compactly supported, then*

$$\int_M d^* \alpha \, vol = 0.$$

$d^*\alpha$ is a zero form on M, i.e smooth function and is often called the divergence of α.

Proof. Immediately from Proposition 3.12, we have

$$\int_M d^* \alpha_p \, \text{vol} = \int_M \langle d^* \alpha | 1 \rangle_p \text{vol} = \langle d^* \alpha, 1 \rangle_{L^2} = \langle \alpha, d(1) \rangle_{L^2} = \langle \alpha, 0 \rangle_{L^2} = 0.$$

□

It is beneficial to compute the divergence $d^*\alpha$ explicitly. Denote $g_{ij} = \langle dx_i, dx_j \rangle$ so that we have an invertible symmetric matrix (g_{ij}) and (g^{ij}) is its inverse. Let $g = \det(g_{ij})$. Note that in general, we have

$$*\alpha = \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{j+1}\sqrt{g}g^{ij}\alpha^i dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

But the local frame field is chosen to be orthonormal, the above expression simplifies to be

$$*\alpha = \sum_{k=1}^{n}(-1)^{k+1}\alpha^k dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n.$$

As a result,

$$d(*\alpha) = \sum_{k=1}^{n}(-1)^{k+1}\frac{\partial \alpha^k}{\partial x_k} dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n \tag{3.57}$$

$$= \sum_{k=1}^{n}(-1)^{k+1}(-1)^{k-1}\frac{\partial \alpha^k}{\partial x_k} dx_1 \wedge \cdots \wedge dx_n = \left(\sum_{k=1}^{n}\frac{\partial \alpha^k}{\partial x_k}\right) dx_1 \wedge \cdots \wedge dx_n. \tag{3.58}$$

Hence, $d^*\alpha = - * d(*\alpha) = -\sum_{k=1}^{n} \partial\alpha^k/\partial x_k$. This information will be relevant shortly.

Lemma 3.14. *The definition of the linear map* $\nabla^{S*} : \Gamma(T^*M \otimes S) \to \Gamma(S)$

$$\nabla^{S*}\left(\sum_{k=1}^{n} dx_k \otimes s_k\right) = -\sum_{k=1}^{n}\nabla^S_{\partial_k} s_k,$$

where s_k *are some sections of* S*, is indeed the formal* L^2-*adjoint of* ∇^S*.*

Proof. Note that for arbitrary $s \in \Gamma(S)$ that is compactly supported, because of the identification $T^*M \otimes S \cong \mathrm{End}(TM, S)$, we have
$\left(\sum_k dx_k \otimes \nabla^S_{\partial_k} s\right)(\partial_i) = \sum_k dx_k(\partial_i)\nabla^S_{\partial_k} s = \nabla^S_{\partial_i} s$. Therefore, $\nabla^S s = \sum_k dx_k \otimes \nabla^S_{\partial_k} s$. Now

for $\omega = \sum_j dx_j \otimes s_j \in \Gamma(T^*M \otimes S)$, we have

$$(\nabla^S s, \omega)_p = \left(\sum_{k=1}^n dx_k \otimes \nabla^S_{\partial_k} s, \sum_{j=1}^n dx_j \otimes s_j \right)_p = \sum_{k=1}^n (\nabla^S_{\partial_k} s | s_k)_p. \tag{3.59}$$

On the other hand,

$$(s | \nabla^{S*} \omega)_p = \left(s \middle| - \sum_{j=1}^n \nabla^S_{\partial_j} s_j \right)_p = - \sum_{j=1}^n (s | \nabla^S_{\partial_j} s_j)_p. \tag{3.60}$$

Hence, combining (69) and (70) yields

$$-(\nabla^S s, \omega)_p + (s|\nabla^{S*}\omega)_p = - \left(\sum_{k=1}^n (\nabla^S_{\partial_k} s | s_k)_p + (s | \nabla^S_{\partial_k} s_k)_p \right) = - \sum_{k=1}^n \partial_k (s|s_k)|_p. \tag{3.61}$$

But by the above observation about divergence, (3.61) is exactly $d^*\alpha_p$, where $\alpha = \sum_k (s|s_k) dx_k$. The Divergence theorem will then tell us that

$$\int_M (s|\nabla^{S*}\omega)_p \mathrm{vol} - \int_M (\nabla^S s, \omega)_p \mathrm{vol} = \int_M d^* \alpha_p \mathrm{vol} = 0.$$

Consequently, $\langle s, \nabla^{S*}\omega \rangle_{L^2} = \langle \nabla^S s, \omega \rangle_{L^2}$, which is what we need to show. □

Back to the proof of Proposition 3.11. Once we have established the formula for the formal L^2-adjoint of ∇^S in Lemma 3.14, it is just a matter of piecing things together. Let $s \in \Gamma(S)$, then at p we obtain

$$\nabla^{S*}\nabla^S s = \nabla^{S*} \left(\sum_{i=1}^n dx_i \otimes \nabla^S_{\partial_i} s \right) = - \sum_{i=1}^n \nabla^S_{\partial_i} \nabla^S_{\partial_i} s = - \sum_{i=1}^n (\nabla^S_{\partial_i})^2 s. \ \square$$

Note that $\nabla^{S*}\nabla^S$ is an L^2-self-adjoint differential operator. It is natural to ask whether $\mathbf{K}^S$ is also self-adjoint. Let s_1 and s_2 be in $\Gamma(S)$, at least one of them is compactly supported. For each k, we view $\nabla^S_{\partial_k} : \Gamma(S) \to \Gamma(S)$ as a differential operator. We claim that $\nabla^{S*}_{\partial_k} = -\nabla^{S}_{\partial_k}$. Indeed at p, by compatibility we obtain

$$(\nabla^S_{\partial_k} s_1|s_2)_p - (s_1| - \nabla^S_{\partial_k} s_2)_p = (\nabla^S_{\partial_k} s_1|s_2)_p + (s_1|\nabla^S_{\partial_k} s_2)_p = \partial_k(s_1|s_2)|_p = -d^*\eta_p, \tag{3.62}$$

where $\eta = (s_1|s_2)dx_k$. By the Divergence theorem, we immediately have

$$\langle \nabla^S_{\partial_k} s_1, s_2\rangle_{L^2} = \langle s_1, -\nabla^S_{\partial_k} s_2\rangle_{L^2}.$$

Combine with the fact that the left Clifford multiplication is skew-adjoint, then

$$\langle \mathbf{K}^S s_1, s_2\rangle_{L^2} = \sum_{j<i} \langle c_{\partial_j}(c_{\partial_i}((\nabla^S_{\partial_j}\nabla^S_{\partial_i} - \nabla^S_{\partial_i}\nabla^S_{\partial_j})s_1)), s_2\rangle_{L^2} \tag{3.63}$$

$$= \sum_{j<i} \langle s_1, (\nabla^S_{\partial_i}\nabla^S_{\partial_j} - \nabla^S_{\partial_j}\nabla^S_{\partial_i}) c_{\partial_i}(c_{\partial_j}(s_2))\rangle_{L^2} \tag{3.64}$$

$$= \sum_{j<i} \int_M (s_1|(\nabla^S_{\partial_i}\nabla^S_{\partial_j} - \nabla^S_{\partial_j}\nabla^S_{\partial_i}) c_{\partial_i}(c_{\partial_j}(s_2)))_p \mathrm{vol} \tag{3.65}$$

Now at p, by the compatibility with the Levi-Civita connection and Proposition 3.10

$$\nabla^S_{\partial_j}(c_{\partial_i}(c_{\partial_j}(s_2))) = c_{\nabla^{LC}_{\partial_j}\partial_i}(c_{\partial_j}(s_2)) + c_{\partial_i}(\nabla^S_{\partial_j} c_{\partial_j}(s_2)) \tag{3.66}$$

$$= c_{\partial_i}(c_{\partial^{LC}_{\partial_j}\partial_j}(s_2) + c_{\partial_j}(\nabla^S_{\partial_j} s_2)) = c_{\partial_i}(c_{\partial_j}(\nabla^S_{\partial_j} s_2)). \tag{3.67}$$

Therefore,

$$\nabla^S_{\partial_i}\nabla^S_{\partial_j} c_{\partial_i}(c_{\partial_j}(s_2)) = c_{\nabla^{LC}_{\partial_i}\partial_i}(c_{\partial_j}(\nabla^S_{\partial_j} s_2) + c_{\partial_i}(\nabla^S_{\partial_i} c_{\partial_j}(\nabla^S_{\partial_j} s_2)) \tag{3.68}$$

$$= c_{\partial_i}(c_{\nabla^{LC}_{\partial_i}\partial_j}(\nabla^S_{\partial_j} s_2) + c_{\partial_j}(\nabla^S_{\partial_i}\nabla^S_{\partial_j} s_2)) = c_{\partial_i}(c_{\partial_j}(\nabla^S_{\partial_i}\nabla^S_{\partial_j} s_2)). \tag{3.69}$$

Similarly, we obtain $\nabla^S_{\partial_j}\nabla^S_{\partial_i}c_{\partial_i}(c_{\partial_j}(s_2)) = c_{\partial_i}(c_{\partial_j}(\nabla^S_{\partial_j}\nabla^S_{\partial_i}s_2))$. Thus, (3.65) becomes

$$\langle \mathbf{K}^S s_1, s_2\rangle_{L^2} = \sum_{j<i}\int_M (s_1|c_{\partial_j}(c_{\partial_i}(\nabla^S_{\partial_j}\nabla^S_{\partial_i} - \nabla^S_{\partial_i}\nabla^S_{\partial_j})s_2))_p \mathrm{vol} = \langle s_1, \mathbf{K}^S s_2\rangle_{L^2}.$$

Proposition 3.15. $\mathbf{K}^S : \Gamma(S) \to \Gamma(S)$ *is an* L^2-*self-adjoint operator.* □

Corollary 3.16. *Consequently, the generalized Laplacian* $D^2 : \Gamma(S) \to \Gamma(S)$ *is also a self-adjoint differential operator.* □

One of the important properties of the Dirac operator D is that, unsurprisingly, it is an L^2-self-adjoint differential operator:

Proposition 3.17. *For any* s_1 *and* s_2 *are smooth sections of* S *where at least one of them is compactly supported, then*

$$\langle Ds_1, s_2\rangle_{L^2} = \langle s_1, Ds_2\rangle_{L^2}.$$

Proof. We evaluate the integrand in local coordinate at p

$$(Ds_1|s_2)_p - (s_1|Ds_2)_p = \sum_{i=1}^n (c_{\partial_i}(\nabla^S_{\partial_i}s_1)|s_2)_p - \sum_{j=1}^n (s_1|c_{\partial_j}(\nabla^S_{\partial_j}s_2))_p \tag{3.70}$$

$$= -\sum_{i=1}^n (\nabla^S_{\partial_i}s_1|c_{\partial_i}(s_2))_p - \sum_{j=1}^n (s_1|\nabla^S_{\partial_j}c_{\partial_j}(s_2))_p \tag{3.71}$$

$$= -\sum_{i=1}^n \partial_i(s_1|c_{\partial_i}(s_2))|_p = d^*\mu_p, \tag{3.72}$$

where $\mu = \sum_{k=1}^n (s_1|c_{\partial_k}(s_2))dx_k$. Then the Divergence theorem with (3.70) – (3.72) immediately yields what we need to show, i.e $\langle Ds_1, s_2\rangle_{L^2} = \langle s_1, Ds_2\rangle_{L^2}$. □

Theorem 3.18. (Bochner) *If the least eigenvalue of* $\mathbf{K}^S$ *at each point of a compact* M *is strictly positive, then there are no non-trivial solutions of the differential equation* $D^2 s = 0$.

Proof. By contradiction, suppose that there exists a non-trivial $s \in \Gamma(S)$ where $D^2 s = 0$. At each $p \in M$, denote $\lambda^i(p)$'s the eigenvalues of $\mathbf{K}^S$ at p; note that λ^i's are smooth real-valued functions on M. By Proposition 3.15, $\mathbf{K}^S$ is self-adjoint; hence by the Spectral theorem, we can write $\mathbf{K}^S_p = \sum_i \lambda^i(p) P_{i,p}$, where $P_{i,p}$'s are orthogonal projections onto $\lambda^i(p)-$eigenspaces. Since $P_{i,p}$'s are orthogonal projections, we have $P_{i,p}^2 = P_{i,p}$ and $P_{i,p}^* = P_{i,p}$. Now consider

$$(\mathbf{K}^S s|s)_p = \sum_i (\lambda^i(p) P_{i,p} s|s)_p = \sum_i \lambda^i(p)(P_{i,p}^2 s|s)_p \tag{3.73}$$

$$= \sum_i \lambda^i(p)(P_{i,p}s|P_{i,p}s)_p \geq \lambda(p) \sum_i ||P_{i,p}s_p||^2 > 0, \tag{3.74}$$

where $\lambda(p)$ is the smallest positive eigenvalue of $\mathbf{K}^S$ at p. By compactness of M, the function $p \mapsto (\mathbf{K}^S s|s)_p/||s||^2_{L^2}$ being continuous achieves an infimum. Furthermore, (84) shows us that it is bounded below by a strictly positive continuous function. Therefore, there exists a $c > 0$ where $(\mathbf{K}^S s|s)_p/||s||^2_{L^2} \geq c$ for all p. As a result,

$$\frac{\langle \mathbf{K}^S s, s\rangle_{L^2}}{||s||^2_{L^2}} = \int_M \frac{(\mathbf{K}^S s|s)_p}{||s||^2_{L^2}} \mathrm{vol} \geq \int_M c\,\mathrm{vol} = c \cdot \mathrm{vol}(M) := C,$$

which is a finite positive number. Equivalently, $\langle \mathbf{K}^S s, s\rangle_{L^2} \geq C||s||^2_{L^2}$. But by the Weitzenbock forumula, we have

$$\langle \mathbf{K}^S s, s\rangle_{L^2} = \langle D^2 s, s\rangle_{L^2} - \langle \nabla^{S*}\nabla^S s, s\rangle_{L^2} = -||\nabla^S s||^2_{L^2} \leq 0.$$

This is a contradiction to the previous assessment. Thus, there are no non-trivial solutions to the second order differential equation $D^2 s = 0$. □

3.5 The exterior algebra bundle, its canonical connection, and the Dirac operator for the exterior algebra bundle

Recall that we have $\Lambda^* T^* M \to M$ a bundle of Clifford modules which is equipped with an smoothly varying inner product across the fibers. The goal of this subsection is to prove precisely that the exterior algebra of the cotangent bundle is indeed a Clifford bundle. To do this, we shall follow this guideline:

I. We shall construct a connection on $\Lambda^* T^* M$ such that it is compatible with the metric on the bundle and the Levi-Civita connection.

II. We will show that the action of $T_p M$ on fiber $\Lambda^* T_p^* M$ is skew-adjoint with respect to the inner product on the fiber.

One would expect that the said connection in (I) is a natural unique extension of the Levi-Civita connection. This is indeed the case; the necessary tools to prove such claim is the principal $G-$bundle and principal connection. Let G be a Lie group, the *principal $G-$bundle over M* is (P, π, M, G) where P is a smooth manifold and $\pi : P \to M$ is a smooth surjective map. Furthermore, P is a locally trivial fiber bundle whose fiber is G and considered as right $G-$space. Thus we have a smooth right action of G on P such that $\pi(p \cdot g) = \pi(p)$ for any $p \in P$ and $g \in G$. Consequently, $M = P/G$ so that $\dim M = \dim P - \dim G$.

Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation of the Lie group G onto a finite dimensional vector space V over $\mathbb{R}$. Then G acts $P \times V$ in the following way $(p, v) \cdot g := (p \cdot g, \rho(g^{-1})v)$ where $p \in P$, $v \in V$ and $g \in G$. The quotient space $P \times_\rho V$ of $P \times V$ induced by the right action of G is a vector bundle over M whose fiber is isomorphic to V. Often, $P \times_\rho V$ is called the *associated vector bundle* of P with respect to the representation ρ.

Conversely if W is a vector bundle of rank k over M equipped with a smoothly varying inner product across its fibers, then the collection $O(W, k)$ of ordered orthonormal frames at each fiber of W is a principal $\mathrm{SO}(k)-$bundle over M.

Definition 3.10. The vertical space VP is a sub-bundle of TP given by the kernel of the map $d\pi : TP \to TM$. Here, $d\pi$ should be understood as the tangent map induced by π. $X \in \mathfrak{X}(P)$ is said to be a vertical tangent vector if and only if $X(p) \in V_p P$ for all $p \in P$.

Lemma 3.19. *For each $u \in \mathfrak{g}$—the Lie algebra of G and each $p \in P$, define*

$$X_u(p) = \frac{d}{dt}(p \cdot exp(tu))\bigg|_{t=0}.$$

Then the linear map $u \mapsto X_u(p)$ defines an isomorphism between $\mathfrak{g} \to V_pP$. As a result, $VP \cong P \times \mathfrak{g}$. Furthermore, X_u is called the killing field on P associated with u.

Proof. $u \mapsto X_u(p)$ is obviously linear by the Leibniz's rule of differentiation. Note that

$$d_p\pi(X_u(p)) = \frac{d}{dt}\pi(p \cdot \exp(tu))\bigg|_{t=0} = \frac{d}{dt}\pi(p)\bigg|_{t=0} = 0.$$

Thus $X_u(p) \in \operatorname{Ker} d_p\pi = V_pP$. Therefore, $u \mapsto X_u(p)$ is a well-defined linear map from $\mathfrak{g} \to V_pP$. Now suppose $X_{u_1}(p) = X_{u_2}(p)$ for some $u_i \in \mathfrak{g}$. By the Existence and Uniqueness theorem, we have $p \cdot \exp(tu_1) = p \cdot \exp(tu_2)$ on some neighborhood of t containing $[0, 1]$. However the assertion is true for any $p \in P$; hence $\exp(tu_1) = \exp(tu_2)$. Differentiating at $t = 0$, we obtain $u_1 = u_2$. This shows that the map is injective. Now since π is surjective, the induced map $d_p\pi$ is a surjective linear map. Then by Rank-Nullity theorem, we have

$$\dim T_pP = \dim \operatorname{Ker} d_p\pi + \dim \operatorname{Im} d_p\pi = \dim V_pP + \dim T_{\pi(p)}M.$$

As a result, $\dim V_pP = \dim P - \dim M = \dim G = \dim T_eG = \dim \mathfrak{g}$, where e is the identity of G. With all of the above observations, we conclude that $u \mapsto X_u(p)$ is indeed an isomorphism between $\mathfrak{g} \to V_pP$. □

Definition 3.11. The assignment $H : p \in P \mapsto H_pP \subseteq T_pP$ is called a principal connection on the fiber bundle (P, π, M, G) if and only if it satisfies all the following conditions:

1. $T_pP = V_pP \oplus H_pP$, for each $p \in P$. Consequently, we have $TP = VP \oplus HP$.

2. $d_pR_g(H_pP) = H_{p \cdot g}P$, where R_g is the diffeomorphism of P onto itself given by the right multiplication by a fixed $g \in G$.

3. H is smooth.

If the principal $G-$bundle $P \to M$ admits a principal connection, then we have the projection onto the vertical space $TP = VP \oplus HP \to VP$. In particular, the projection maps each vector field $X(p) + Y(p) \mapsto X(p)$. But by Lemma 3.19, $X(p)$ is associated uniquely with $u \in \mathfrak{g}$, where $X_u(p) = X(p)$. Hence, this projection induces a $\mathfrak{g}-$valued $1-$form ω on P, $\omega : TP \to \mathfrak{g}$. Such a $\mathfrak{g}-$valued $1-$form ω on P is called a *connection* $1-$*form* and has the following properties

Proposition 3.20.

1. $\omega(X_u) = u$ *for every* $u \in \mathfrak{g}$.

2. $(R_g)^*\omega = Ad(g^{-1})\omega$ *for* $g \in G$, *where* $Ad(g^{-1})$ *is the induced map from* $\mathfrak{g} \to \mathfrak{g}$ *by the conjugation automorphism on* G, $x \mapsto g^{-1}xg$.

Conversely, given $\mathfrak{g}-$*valued* $1-$*form* $\omega : TP \to \mathfrak{g}$ *satisfying* (1) *and* (2), *then* $HP = \{X \in TP : \omega(X) = 0\}$ *defines a principal connection on* (P, π, M, G).

Proof. (1) follows directly from the discussion above. For (2), recall that the exponential map commutes with the adjoint action on of Lie group on its Lie algebra. In particular for each $g \in G$ and $u \in \mathfrak{g}$, we have

$$g^{-1}\exp(u)g = \exp(\mathrm{Ad}(g^{-1})u) = \exp(g^{-1}ug).$$

Since ω is induced by the projection onto VP, without loss of generality, we can look at its application to the killing field X_u. At each $p \in P$, we have $\mathrm{Ad}(g^{-1})\omega_p(X_u(p)) = g^{-1}ug$. On the other hand, since $R_g(p \cdot g^{-1}) = p$, we evaluate the pull-back of ω at $p \cdot g^{-1}$:

$$\begin{aligned}(R_g)^*\omega_{p\cdot g^{-1}}(X_u(p \cdot g^{-1})) &= \omega_p(d_{p\cdot g^{-1}}R_g\, X_u(p \cdot g^{-1})) &(3.75)\\ &= \omega_p\left(\frac{d}{dt}(p \cdot g^{-1}\exp(tu)g)\bigg|_{t=0}\right) &(3.76)\end{aligned}$$

$$= \omega_p \left(\frac{d}{dt}(p \cdot \exp(tg^{-1}ug)\bigg|_{t=0} \right) = \omega_p(X_{g^{-1}ug}(p)) = g^{-1}ug. \quad (3.77)$$

Hence, the left hand side and the right hand side agree with each other.

Conversely suppose ω is a $\mathfrak{g}-$valued connection $1-$form on P satisfies (1) and (2). Then to show that the assignment H defines a principal connection, it is enough to check that $d_pR_g(H_pP) = H_{p\cdot g}P$. Indeed, let $X \in \Gamma(HP) \subseteq \mathfrak{X}(P)$, then

$$\omega_{p\cdot g}(d_pR_g\, X(p)) = (R_g)^*\omega_p(X(p)) = \mathrm{Ad}(g^{-1})\omega_{p\cdot g}(X(p\cdot g)) = \mathrm{Ad}(g^{-1})0 = 0.$$

By the definition of HP, we have $d_pR_g\, X(p) \in H_{p\cdot g}P$, which implies that $d_pR_g(H_pP) \subseteq H_{p\cdot g}P$. Since d_pR_g is an isomorphism, $d_pR_g|_{H_pP}$ is also an isomorphism onto its image. Thus, $\dim d_pR_g(H_pP) = \dim H_pP = \dim P - \dim G = \dim H_{p\cdot g}P$. Therefore, $d_pR_g(H_pP) = H_{p\cdot g}P$ as required. □

The above discussion tells us that the existence of a principal connection on (P, π, M, G) is equivalent to either a choice of sub-bundle HP satisfying Definition 3.11 or a $\mathfrak{g}-$valued $1-$form ω that has properties Proposition 3.20. We shall use whichever one that is most convenient depending on the situation.

Just as for connections on vector bundles, we shall calculate the local formula of connection $1-$form ω on (P, π, M, G). Let $s_\alpha : U_\alpha \subseteq M \to \pi^{-1}(U_\alpha) \subseteq P$ be a local section associated canonically with a local trivialization of P. Denote $g_\alpha : \pi^{-1}(U_\alpha) \to G$ the smooth $G-$equivariant map that is a fiber-wise diffeomorphism so that $g_\alpha(s_\alpha(m)) = e$ for all $m \in U_\alpha$. If $U_\alpha \cap U_\beta \neq \varnothing$, then for each $m \in U_\alpha \cap U_\beta$ there exists uniquely a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that

$$s_\alpha(m) = s_\beta(m) \cdot g_{\alpha\beta}(m).$$

The suspiciously ambiguous notation $g_{\alpha\beta}$ will be justified shortly. We define $A_\alpha := s_\alpha^*\omega$. Note that A_α is a $\mathfrak{g}-$valued $1-$form on U_α. Then consider a $\mathfrak{g}-$valued $1-$form on

$\pi^{-1}(U_\alpha) \subseteq P$

$$\omega^\alpha := \mathrm{Ad}(g_\alpha^{-1})\pi^* A_\alpha + g_\alpha^* \Theta, \tag{3.78}$$

where Θ is the *Maurer-Cartan form* on G. We claim that ω^α is ω restricted to $T\pi^{-1}(U_\alpha)$ and thus defines a connection 1−form on $\pi^{-1}(U_\alpha)$. Indeed, let $m \in U_\alpha$ so that $p := s_\alpha(m) \in \pi^{-1}(U_\alpha)_m$. It is sufficient to show that ω^α and ω agree on $T_p\pi^{-1}(U_\alpha)$. Now the existence of a principal connection on P is equivalent to a choice of splitting of the following short exact short sequence

$$0 \longrightarrow VP|_{\pi^{-1}(U_\alpha)} \hookrightarrow TP|_{\pi^{-1}(U_\alpha)} \xrightarrow{ds_\alpha \circ d\pi} ds_\alpha \circ d\pi\, TP|_{\pi^{-1}(U_\alpha)} \longrightarrow 0.$$

In particular, we have a decomposition $T_p\pi^{-1}(U_\alpha) = V_p\pi^{-1}(U_\alpha) \oplus d_m s_\alpha \circ d_p\pi\, T_p\pi^{-1}(U_\alpha)$ such that every $X(p) \in T_p\pi^{-1}(U_\alpha)$ can be written as $X(p) = X_u(p) + d_m s_\alpha \cdot d_p\pi\, X(p)$. Thus applying ω^α to $X(p)$ yields

$$\omega_p^\alpha(X(p)) = \mathrm{Ad}(e)(s_\alpha \circ \pi)^* \omega_p(X(p)) + \Theta_e(d_p g_\alpha\, X(p)) \tag{3.79}$$

$$= \omega_p(d_m s_\alpha \cdot d_p\pi\, X(p)) + \Theta_e(d_p g_\alpha X_u(p)) \tag{3.80}$$

(3.80) is true because $g_\alpha \circ s_\alpha$ is a constant map so that $dg_\alpha \cdot ds_\alpha$ is trivial. Now separately,

$$\Theta_e(d_p g_\alpha X_u(p)) = (L_{e^{-1}})_*(d_p g_\alpha\, X_u(p)) = d_p g_\alpha X_u(p) \tag{3.81}$$

$$= \frac{d}{dt} g_\alpha(p \cdot \exp(tu))\bigg|_{t=0} = \frac{d}{dt}(g_\alpha(p)\exp(tu))\bigg|_{t=0} = \frac{d}{dt}\exp(tu)\bigg|_{t=0} = u, \tag{3.82}$$

which is also equal to $\omega_p(X_u(p))$ by previous lemma. Thus (3.80) is simplified to

$$\omega_p^\alpha(X(p)) = \omega_p(d_m s_\alpha \cdot d_p\pi\, X(p)) + \omega_p(X_u(p)) = \omega_p(X(p)).$$

Now in the case where $U_\alpha \cap U_\beta \neq \varnothing$, for each $m \in U_\alpha \cap U_\beta$, $s_\alpha(m)$ also belongs to the fiber $\pi^{-1}(U_\beta)_m$. Thus we have a relation $s_\alpha(m) = s_\beta(m) \cdot g_\beta(s_\alpha(m))$. But by uniqueness, it follows that $g_\beta \circ s_\alpha = g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. Since $\omega = \omega^\alpha = \omega^\beta$ on $U_\alpha \cap U_\beta$,

$$A_\alpha = s_\alpha^*(\mathrm{Ad}((g_\beta \circ s_\alpha)^{-1})\pi^* A_\beta + g_\beta^* \Theta) = \mathrm{Ad}(g_{\alpha\beta}^{-1})(\pi \circ s_\alpha)^* A_\beta + (g_\beta \circ s_\alpha)^* \Theta \tag{3.83}$$

$$= \mathrm{Ad}(g_{\alpha\beta}^{-1}) A_\beta + g_{\alpha\beta}^* \Theta. \tag{3.84}$$

The previous discussion gives us an important recipe to establish a principal connection on a principal bundle. In fact

Theorem 3.21. (Existence of principal connection) *If M is a smooth manifold and (P, π, M, G) is a principal bundle where P is paracompact, then there exists a principal connection on the bundle.*

Proof. Let $\{U_\alpha\}$ be a collection of open cover of M that are local trivialization of P. Let s_α be a local section associated canonically with a local trivialization U_α of P. Denote $g_\alpha : \pi^{-1}(U_\alpha) \to G$ and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ the smooth maps defined as previously. Define $\{A_\alpha\}$ a collection of $\mathfrak{g}-$valued $1-$forms on U_α's such that (94) is satisfied and $\{\omega^\alpha\}$ is one for $\pi^{-1}(U_\alpha)$'s as in (88). Note that $M = \bigcup U_\alpha$; thus $P = \bigcup \pi^{-1}(U_\alpha)$. Since P is paracompact, there exists a partition of unity $\{f_\alpha\}$ subordinated to the open cover $\{\pi^{-1}(U_\alpha)\}$. Let $\omega := \sum_\alpha f_\alpha \omega^\alpha$ so that it is a global $\mathfrak{g}-$valued $1-$form on P. It is routine to check that ω^α satisfies Proposition 3.20 on each $\pi^{-1}(U_\alpha)$; hence ω also satisfies those properties globally. Therefore, ω defines a global connection $1-$form on P. □

Example 3.6. Let M be $n-$dimensional Riemannian manifold equipped with the Levi-Civita connection ∇^{LC}, $(O(M,n), \pi, M, \mathrm{SO}(n))$ be a principal $\mathrm{SO}(n)-$bundle. Note that $O(M,n)$ is paracompact; thus it admits a connection $1-$form. We construct such form in the following way:

In $\mathfrak{so}(n)$, choose the basis $\{X_{ij}\}_{i<j}$ where $X_{ij} := E_{ij} - E_{ji}$ [3]. Let

[3] E_{ij} is the $n \times n-$matrix where the $ij-$entry is 1 and the rest is 0.

$s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha) \subseteq O(M,n)$ be a local section associated canonically with a local trivialization of $O(M,n)$, $s_\alpha = (s_{\alpha,1}, \cdots, s_{\alpha,n})$. We define a family of $\mathfrak{so}(n)-$valued $1-$forms $\{A_\alpha\}$ on U_α's

$$A_\alpha = \sum_{i<j} \langle \nabla^{LC} s_{\alpha,i}, s_{\alpha,j} \rangle X_{ij}. \tag{3.85}$$

A_α's satisfy (3.84) and thus produce a family of $\mathfrak{so}(n)-$valued $1-$forms $\{\omega^\alpha\}$ on $\pi^{-1}(U_\alpha)$'s satisfying (3.78). Using partition of unity, we obtain the required connection $1-$form ω.

Suppose that (P, π, M, G) admits a principal connection. Let $X \in \mathfrak{X}(M)$ be a non-trivial vector field. Since $d\pi : TP \to TM$ is surjective and $TP = VP \oplus HP$, there exists an $\tilde{X} \in \Gamma(HP) \subseteq \mathfrak{X}(P)$ such that $d\pi\, \tilde{X} = X$. If $\tilde{X}'$ is another vector field that lives in $\Gamma(HP)$ satisfying $d\pi\, \tilde{X}' = X$, then $d\pi(\tilde{X} - \tilde{X}') = 0$. This means that $\tilde{X}(p) - \tilde{X}'(p) \in V_p P$ for each $p \in P$. But at the same time, $\tilde{X}(p) - \tilde{X}'(p) \in H_p P$; hence $\tilde{X} \equiv \tilde{X}'$ on P. The discussion prompts the following definition

Definition 3.12. For any $X \in \mathfrak{X}(M)$ that is non-trivial, the unique vector field $\tilde{X}$ on P is called the horizontal lift of X if and only if

1. $\tilde{X}(p) \in H_p P$ for any $p \in P$.

2. $d\pi\, \tilde{X} = X$; equivalently, $d_p\pi\, \tilde{X}(p) = X(\pi(p))$ for all $p \in P$.

Horizontal lift of a vector field has the following properties

Proposition 3.22. *Let* $X \in \mathfrak{X}(M)$

1. *The unique horizontal lift* $\tilde{X}$ *of* X *is a* $G-$*invariant vector field on* P.

2. *If* $Y \in \mathfrak{X}(M)$ *and* $f \in \Gamma(M)$, *then* $\tilde{X} + \tilde{Y} = \widetilde{X+Y}$ *and* $\widetilde{fX} = (f \circ \pi)\tilde{X}$.

Proof. (2) follows directly from the definition. For (1), we need to show that $(R_g)_* \tilde{X} = \tilde{X}$ where $g \in G$. Indeed, let p be any point in P, consider

$$d_{p \cdot g}\pi(d_p R_g \tilde{X}(p) - \tilde{X}(p \cdot g)) = d_{p \cdot g}\pi \cdot d_p R_g \tilde{X}(p) - d_{p \cdot g}\pi \tilde{X}(p \cdot g) \tag{3.86}$$

$$= d_p(\pi \circ R_g) \tilde{X}(p) - X(\pi(p \cdot g)) \tag{3.87}$$

$$= d_p\pi \tilde{X}(p) - X(\pi(p)) = X(\pi(p)) - X(\pi(p)) = 0. \tag{3.88}$$

But both $d_p R_g \tilde{X}(p)$ and $\tilde{X}(p \cdot g)$ live in $H_{p \cdot g}P$, hence $d_p R_g \tilde{X}(p) = \tilde{X}(p \cdot g)$. □

The following lemma is extremely useful in calculations in a coordinate-free way.

Lemma 3.23. *Suppose $P \times_\rho V$ is an associated vector bundle over M given by the representation $\rho : G \to GL(V)$. Denote $\Gamma(P, V)^G$ the space of $G-$equivariant smooth maps $f : P \to V$ [4]. There is a natural isomorphism between $\Gamma(P \times_\rho V)$ and $\Gamma(P, V)^G$, given by mapping each $f \in \Gamma(P, V)^G$ to f_M defined by*

$$f_M(m) = (p, f(p)),$$

where $m \in M$ and p is any element of P_m. $(p, f(p))$ should be understood as a representative of its equivalence class in $P \times_\rho V$.

Proof. First note that f_M is well-defined. Indeed, suppose we choose another $q \in P_m$, there exists a $g \in G$ such that $q = p \cdot g$. Then by $G-$equivariance of f, we have $(q, f(q)) = (p \cdot g, f(p \cdot g)) = (p \cdot g, \rho(g^{-1})f(p))$. This shows that $(q, f(q))$ is the same equivalent class of $(p, f(p))$ in $P \times_\rho V$. Now to show the $1:1$ correspondence, it is sufficient to show that each section s of $P \times_\rho V$ is of the from f_M for some $f \in \Gamma(P, V)^G$. To do this, suppose $s(m) = (p, v)$, where $p \in P_m$ and $v \in V$; we define $f(p)$ to be equal to the unique $v \in V$. As a map from $P \to V$, f should be smooth and $G-$equivariant by construction. □

[4] $G-$equivariant here means that $f(p \cdot g) = \rho(g^{-1})f(p)$ for each $p \in P$ and $g \in G$.

Let $X \in \mathfrak{X}(M)$ and $\gamma : [0,1] \to M$ be the integral curve of X where $\gamma(0) = m$. Then $\tilde{X}$ is the horizontal lift of X so that for each $p \in P_{\gamma(0)} = P_m$ we obtain the integral curve $\tilde{\gamma}_p : [0,1] \to P$ of $\tilde{X}$ with $\tilde{\gamma}_p(0) = p$. Note that

$$(\pi \circ \tilde{\gamma}_p)'(t) = d_t(\pi \circ \tilde{\gamma}_p) = d_{\tilde{\gamma}_p(t)}\pi \cdot d_t\tilde{\gamma}_p = d_{\tilde{\gamma}_p(t)}\pi \cdot \tilde{\gamma}_p'(t) = d_{\tilde{\gamma}_p(t)}\pi\, \tilde{X}(\tilde{\gamma}_p(t)) = X(\pi \circ \tilde{\gamma}_p(t)).$$

But $\pi \circ \tilde{\gamma}_p(0) = \pi(p) = m$; thus by the Existence and Uniqueness theorem, $\pi \circ \tilde{\gamma}_p = \gamma$ on a neighborhood containing $[0,1]$. Moreover, $\tilde{\gamma}_p'(t) \in H_{\tilde{\gamma}_p(t)}P$ for all $t \in [0,1]$. Hence, we call $\tilde{\gamma}_p$ the unique *horizontal lift* of γ starting at a fixed point $p \in P_{\gamma(0)}$.

Denote $W := P \times_\rho V$. Let $f_0 \in W_{\gamma(0)} = W_m$; and suppose $w_0 = (p, v)$, where (p, v) is a representative of its equivalence class in W. We define the *parallel translation* of f_0 along γ to be $f_1 = (\tilde{\gamma}_p(1), v) \in W_{\gamma(1)}$. The parallel translation should be well-defined, i.e, different representative elements of the same equivalence class in $W_{\gamma(0)}$ should be parallel translated along γ to the same equivalent class in $W_{\gamma(1)}$. Indeed, suppose $(p \cdot g, \rho(g^{-1})v)$ is another representative element of the equivalence class of w_0, then by the algorithm of parallel translation along γ, $(p \cdot g, \rho(g^{-1})v)$ should be sent to $(\tilde{\gamma}_{p\cdot g}(1), \rho(g^{-1})v) \in W_{\gamma(1)}$. Now by Proposition 3.22, consider the following

$$(\tilde{\gamma}_p(t) \cdot g)' = \frac{d}{dt} R_g \circ \tilde{\gamma}_p(t) = d_{\tilde{\gamma}_p(t)} R_g \cdot \tilde{\gamma}_p'(t) = d_{\tilde{\gamma}_p(t)} R_g\, \tilde{X}(\tilde{\gamma}_p(t)) = \tilde{X}(\tilde{\gamma}_p(t) \cdot g) \in H_{\tilde{\gamma}_p(t)\cdot g}P.$$

At the same time, $\tilde{\gamma}_p(0) \cdot g = p \cdot g$ and $\pi(\tilde{\gamma}_p(t) \cdot g) = \pi(\tilde{\gamma}_p(t)) = \gamma(t)$ on $[0,1]$. Therefore, $R_g \circ \tilde{\gamma}_p$ is the horizontal lift of γ starting at $p \cdot g \in P_{\gamma(0)}$. In other words, $\tilde{\gamma}_p(t) \cdot g = \tilde{\gamma}_{p\cdot g}(t)$ for all $t \in [0,1]$. In particular, we obtain $(\tilde{\gamma}_{p\cdot g}(1), \rho(g^{-1})v) = (\tilde{\gamma}_p(1) \cdot g, \rho(g^{-1})v)$.

In fact, for any $t \in [0,1]$, the parallel translation along γ induces a linear map from $W_{\gamma(0)} \to W_{\gamma(t)}$ given by $(p, v) \mapsto (\tilde{\gamma}_p(t), v)$. Such linear map is an isomorphism. To prove this, because each fiber of W can be identified with V so they have the same dimensions, we only need to check that $(p, v) \mapsto (\tilde{\gamma}_p(t), v)$ is injective: Suppose there are $p_1, p_2 \in W_{\gamma(0)}$

and $v_1, v_2 \in V$ such that $(\tilde{\gamma}_{p_i}(t), v_i)$'s are in the same equivalent class in $P_{\gamma(t)}$. Then there exists a $g_t \in G$ such that $\tilde{\gamma}_{p_2}(t) = \tilde{\gamma}_{p_1}(t) \cdot g_t$ and $v_2 = \rho(g_t^{-1})v_1$. In particular, when $t = 0$, we immediately see that (p_1, v_1) and (p_2, v_2) are in the same equivalent class in $W_{\gamma(0)}$, which implies the injection. Denote $\theta_t : W_{\gamma(t)} \to W_{\gamma(0)}$ the inverse isomorphism. For each $s \in \Gamma(W)$, we define

$$\nabla^W_{X(m)} s := \frac{d}{dt}\theta_t\, s(\gamma(t))\Big|_{t=0}. \tag{3.89}$$

With all of the notations above, (3.89) describes the following

Proposition 3.24. ∇^W *can be viewed as a linear map from* $\mathfrak{X}(M) \otimes \Gamma(W) \to \Gamma(W)$*; furthermore, it defines a connection on the vector bundle* W *over* M*.* ∇^W_X *can also be viewed as the directional derivative of a* $G-$*equivariant vector-valued function along* $\tilde{X}$*.*

Proof. To show that ∇^W defines a connection on the vector bundle W, we verify these two things: (a) It is linear and $\Gamma(M)-$linear in the X; and (b) It obeys the Leibniz rule in s.

(a) Suppose that X_1 and X_2 are two vector fields on M. For a fixed $m \in M$, let $\gamma_i : [0,1] \to M$ be the integral curve of X_i where $\gamma_i(0) = m$. Denote $\tilde{\gamma}^i_p$ the horizontal lift of γ_i starting at $p \in P_m$. Let $\theta^i_t : W_{\gamma_i(t)} \to W_{\gamma_i(0)}$ be isomorphisms induced by the parallel translation. By Lemma 3.23, we have an $\mathrm{x} \in \Gamma(P,V)^G$ such that $s(m) = (p, \mathrm{x}(p))$ for any $p \in P_m$. Now consider

$$\nabla^W_{X_i(m)} s = \frac{d}{dt}\theta^i_t s(\gamma_i(t))\Big|_{t=0} = \frac{d}{dt}\theta^i_t(\tilde{\gamma}^i_p(t), \mathrm{x}(\tilde{\gamma}^i_p(t)))\Big|_{t=0} = \frac{d}{dt}(p, \mathrm{x}(\tilde{\gamma}^i_p(t)))\Big|_{t=0} = (p, d_p\mathrm{x}\, \tilde{X}_i(p)).$$

With the same computations combined with Proposition 3.22,

$$\nabla^W_{(X_1+X_2)(m)} s = (p, d_p\mathrm{x}\,(\widetilde{X_1 + X_2})(p)) = (p, d_p\mathrm{x}(\tilde{X}_1(p) + \tilde{X}_2(p))).$$

From the two equations above, we immediately obtain

$\nabla^W_{X_1(m)+X_2(m)} s = \nabla^W_{X_1(m)} s + \nabla^W_{X_2(m)} s$. If f is a smooth function on M, then again by Proposition 3.22, $\nabla^W_{(fX)(m)} s = (p, d_p\mathrm{x} \cdot f(\pi(p))\tilde{X}(p))$. At the same time, $f(m)\nabla^W_{X(m)} s = f(m)(p, d_p\mathrm{x}\,\tilde{X}(p)) = (p, d_p\mathrm{x} \cdot f(m)\tilde{X}(p)) = (p, d_p\mathrm{x} \cdot f(\pi(p))\tilde{X}(p))$. Therefore, $f\nabla^W_X s = \nabla^W_{fX} s$.

(b) For $f \in \Gamma(M)$, we need to show that $\nabla^W_{X(m)} fs = f(m)\nabla^W_{X(m)} s + (X_m\, f)s(m)$. We have the right hand side equal to $(p, f(m)d_p\mathrm{x}\,\tilde{X}(p) + (X_m\, f)\mathrm{x}(p))$. On the other hand using product rule, the left hand side equals to

$$\frac{d}{dt}\theta_t(\tilde{\gamma}_p(t), f(\gamma(t))\mathrm{x}(\tilde{\gamma}_p(t)))\bigg|_{t=0} = \frac{d}{dt}(p, f(\gamma(t))\mathrm{x}(\tilde{\gamma}_p(t)))\bigg|_{t=0} = (p, (X_m\, f)\mathrm{x}(p) + f(m)d_p\mathrm{x}\,\tilde{X}(p)).$$

Comparing the two sides and we yield the equality. □

Let M be an $n-$dimensional Riemannian manifold. From all of the discussion above, what we have shown is that given a principal $G-$bundle $\pi : P \to M$ that admits a principal connection and a representation $\rho : G \to \mathrm{GL}(V)$, one can construct a connection on the associated vector bundle $P \times_\rho V$. This is possible via the parallel translations of the horizontal lifts. Furthermore, the process is completely reversible in the context of bundle of ordered orthonormal frames. Denote $W := O(M, n) \times_\iota \mathbb{R}^n$, where $\iota : \mathrm{SO}(n) \hookrightarrow \mathrm{GL}(\mathbb{R}^n)$. With the Example 3.6 comes into mind, we shall summarize this intimate dynamics in the following flowchart diagram

Consider the tensor bundle $O(M, n) \otimes O(M, n)$ over M, it still defines for a principal $\mathrm{SO}(n)-$bundle over M. The principal connection on $O(M, n) \otimes O(M, n)$ is inherited from $O(M, n)$ via the connection $1-$form $\omega \otimes \omega$ [5]. Let $\rho : \mathrm{SO}(n) \to \mathrm{GL}(\mathbb{R}^n \otimes \mathbb{R}^n)$ be the tensor product representation induced by ι. The associated vector bundle $(O(M, n) \otimes O(M, n)) \times_\rho (\mathbb{R}^n \otimes \mathbb{R}^n)$ now has a connection that is determined uniquely by $\omega \otimes \omega$; such connection in turns produces a connection ∇ on the vector bundle $TM \otimes TM \to M$. Then the induced connection, which we also call ∇, on the vector bundle

[5]Though $\omega \otimes \omega$ is a connection $1-$form on $O(M, n) \otimes O(M, n)$, it is a $\mathfrak{so}(n)-$valued $2-$form on $O(M, n)$.

$\Lambda^2 TM \to M$ can be understood as an extension of ∇^{LC}. The same process can be repeated to yield an extension of ∇^{LC} for $\Lambda^k TM$. Therefore, we obtain a type-preserving connection, which we again denote ∇, that is an extension of ∇^{LC} on the vector bundle $\Lambda^* TM \to M$. Note that ∇ is unique and satisfies the following properties

(a) $\nabla_X(\alpha \wedge \beta) = (\nabla_X \alpha) \wedge \beta + \alpha \wedge (\nabla_X \beta)$ for each $X \in \mathfrak{X}(M)$ and $\alpha, \beta \in \Gamma(\Lambda^* TM)$.

(b) ∇_X restricted to $\Gamma(M)$ is the directional derivative along X.

Since $\Lambda^* TM \cong \Lambda^* T^* M$ by the isomorphism Φ induced by the Riemannian metric, the connection ∇ on $\Lambda^* TM$ is naturally carried-over to define one on $\Lambda^* T^* M$ in the following way $\overline{\nabla}_X \mu := \Phi(\nabla_X \Phi^{-1}(\mu))$ for $\mu \in \Omega^*(M)$. Note that $\overline{\nabla}$ is compatible with the metric on $\Lambda^* T^* M$, the computation to show this is routine in local orthonormal frame field. Now consider left Clifford multiplication multiplication $\hat{c}_Y(.) = \eta_Y \wedge\ . - Y \lrcorner .$ on the bundle of Clifford modules $\Lambda^* T^* M$, where $Y \in \mathfrak{X}(M)$ and η_Y is the corresponding $1-$form of Y in $\Omega^1(M)$. For each $\mu \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$, we would like show

$$\overline{\nabla}_X(\hat{c}_Y(\mu)) = \hat{c}_{\nabla^{LC}_X Y}(\mu) + \hat{c}_Y(\overline{\nabla}_X \mu). \tag{3.90}$$

In a coordinate patch, let $\{\partial_i := \partial/\partial x_i\}$ be a local ordered orthonormal frame field. Without loss of generality, we shall prove (3.90) in the case $\mu = dx_I$, where $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and $\widehat{dx}_{i_j} := dx_1 \wedge \cdots \widehat{dx}_{i_j} \wedge \cdots \wedge dx_{i_k}$, $i_1 < i_2 < \cdots < i_k$. By Proposition 3.10, we have

$$\overline{\nabla}_X dx_i = \Phi(\nabla^{LC}_X \partial_i) = \Phi\left(\sum_{j=1}^{n} dx_j(X) \nabla^{LC}_{\partial_j} \partial_i\right) = \Phi(0) = 0.$$

Thus combining with (a) above yields $\overline{\nabla}_X dx_I = 0$. Furthermore, if we denote $Y = \sum_i Y^i \partial_i$

so that $\eta_Y = \sum_i Y^i dx_i$, we obtain

$$\overline{\nabla}_X \eta_Y = \sum_{i=1}^{n} \left(Y^i \overline{\nabla}_X dx_i + (XY^i) dx_i \right) = \sum_{i=1}^{n} (XY^i) dx_i.$$

Therefore, $\overline{\nabla}_X(\eta_Y \wedge dx_I) = \sum_{i=1}^{n} (XY^i) dx_i \wedge dx_I$.

Similarly, applying $\overline{\nabla}_X$ to $-Y \lrcorner\, dx_I$ gives us

$$\overline{\nabla}_X(-Y \lrcorner\, dx_I) = \sum_{j=1}^{k} (-1)^j (X\, dx_{i_j}(Y)) \widehat{dx}_{i_j}.$$

As a result, the left hand side of (3.90) is equal to

$$\overline{\nabla}_X(\eta_Y \wedge dx_I - Y \lrcorner\, dx_I) = \sum_{i=1}^{n} (XY^i) dx_i \wedge dx_I + \sum_{j=1}^{k} (-1)^j (X\, dx_{i_j}(Y)) \widehat{dx}_{i_j}. \tag{3.91}$$

On the other hand,

$$\nabla_X^{LC} Y = \sum_{j=1}^{n} dx_j(X) \nabla_{\partial_j}^{LC} Y = \sum_{j=1}^{n} dx_j(X) \left(\sum_{i=1}^{n} (\partial_j Y^i) \partial_i \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} dx_j(X)(\partial_j Y^i) \right) \partial_i.$$

Hence, using the fact that $\overline{\nabla}_X dx_I = 0$ again, the right hand side of (3.90) is equal to

$$\hat{c}_{\nabla_X^{LC} Y}(dx_I) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} dx_j(X)(\partial_j Y^i) \right) dx_i \wedge dx_I + \sum_{j=1}^{n} (-1)^j dx_{i_j}(\nabla_X^{LC} Y) \widehat{dx}_{i_j}. \tag{3.92}$$

Note that

$$\sum_{j=1}^{n} dx_j(X)(\partial_j Y^i) = \left(\sum_{j=1}^{n} dx_j(X) \partial_j \right) Y^i = \left(\sum_{j=1}^{n} \langle \partial_j, X \rangle \partial_j \right) Y^i = XY^i,$$

and at the same time

$$dx_{i_j}(\nabla_X^{LC} Y) = dx_{i_j} \left(\sum_{i=1}^{n} (XY^i) \partial_j \right) = \sum_{i=1}^{n} (XY^i) dx_{i_j}(\partial_i) = XY^{i_j} = X\, dx_{i_j}(Y).$$

Then direct comparison between (3.91) and (3.92) yields the result. (3.90) tells us us that $\overline{\nabla}$ is compatible with the Levi-Civita connection. We summarize our discussion about $\overline{\nabla}$ in the following proposition that completes the objective (I) stated in the beginning of the subsection.

Proposition 3.25. *The connection $\overline{\nabla} : \mathfrak{X}(M) \otimes \Omega^*(M) \to \Omega^*(M)$ on the bundle of Clifford modules $\Lambda^* T^* M$ is compatible with the metric on the bundle and with the Levi-Civita connection.* □

Objective (II) is automatically fulfilled, thus we conclude

Corollary 3.26. *The bundle of Clifford modules $\Lambda^* T^* M \to M$ indeed defines a Clifford bundle over M.* □

We investigate the local formula for the Dirac operator and the Laplacian associated with $\overline{\nabla}$. For each $\mu \in \Omega^k(M)$, in local geodesic coordinate system, without loss of generality assume $\mu = f^I dx_I$, where $f^I \in \Gamma(M)$. Then by (32) we have

$$D\mu = \sum_{i=1}^{n} c_{\partial_i}(\overline{\nabla}_{\partial_i}\mu) = \sum_{i=1}^{n} c_{\partial_i}((\partial_i f^I) dx_I) \tag{3.93}$$

$$= \sum_{i=1}^{n} (\partial_i f^I) dx_i \wedge dx_I + \sum_{i=1}^{n} (\partial_i f^I) \sum_{j=1}^{k} (-1)^j dx_{i_j}(\partial_i) \widehat{dx}_{i_j} \tag{3.94}$$

$$= \sum_{i=1}^{n} (\partial_i f^I) dx_i \wedge dx_I + \sum_{j=1}^{k} \sum_{i=1}^{n} (-1)^j dx_{i_j}(\partial_i)(\partial_i f^I) \widehat{dx}_{i_j} \tag{3.95}$$

$$= \sum_{i=1}^{n} (\partial_i f^I) dx_i \wedge dx_I + \sum_{j=1}^{k} (-1)^j (\partial_{i_j} f^I) \widehat{dx}_{i_j}. \tag{3.96}$$

Note that first term of the right hand side of (3.96) is exactly the exterior derivative $d\mu$, while the second term is $d^*\mu$. Therefore, $D = d + d^*$ locally. As a result, the Laplacian $D^2 = (d + d^*)^2 = d^2 + dd^* + d^*d + (d^*)^2 = dd^* + d^*d$. Recall that $D = d + d^*$ is the Hodge-Dirac operator in subsection 2.4.

3.6 Pin groups and Spin groups

The following discussion can be generalized to any finite dimensional vector space V over $\mathbb{R}$ equipped with a positive definite symmetric bilinear form; but for simplicity, we shall restrict to the case of $Cl(\mathbb{R}^n)$, where $\mathbb{R}^n$ is equipped with its usual inner product. Denote $\{e_1, \cdots, e_n\}$ an ordered orthonormal basis of $\mathbb{R}^n$. Let
$x = x^1 e_1 + x^2 e_2 + \cdots x^n e_n \in \mathbb{R}^n \subset Cl(\mathbb{R}^n)$, where $x^i \in \mathbb{R}$. Consider the action on $\mathbb{R}^n$ defined in the following way

$$e_i x e_i = e_i x^1 e_1 e_i + e_i x^2 e_2 e_i + \cdots + e_i x^i e_i e_i + \cdots + e_i x^n e_n e_i \tag{3.97}$$

$$= x^1 e_1 + x^2 e_2 + \cdots + (-x^i) e_i + \cdots x^n e_n. \tag{3.98}$$

Thus the action $e_i x e_i$ is a reflection of x in the e_i-direction, which is represented by the following $n \times n-$matrix R_{e_i} belonging to $\mathrm{O}(n)$

$$R_{e_i} = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & -1 & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 1 \end{pmatrix}.$$

Now suppose v and w are two orthogonal vectors in $\mathbb{R}^n$ so that if we write $v = \sum_i v^i e_i$ and $w = \sum_j w^j e_j$, then we have $\sum_k v^k w^k = 0$. We claim that $vw = -wv$ in the Clifford algebra $Cl(\mathbb{R}^n)$. Indeed, consider

$$vw = \left(\sum_{i=1}^n v^i e_i\right)\left(\sum_{j=1}^n w^j e_j\right) = \sum_{i,j=1}^n v^i w^j e_i e_j = \sum_{k=1}^n v^k w^k e_k^2 + \sum_{i \neq j} v^i w^j e_i e_j \tag{3.99}$$

$$= -\sum_{k=1}^n v^k w^k + \sum_{i \neq j} v^i w^j e_i e_j = \sum_{i \neq j} v^i w^j e_i e_j = -wv. \tag{3.100}$$

Based on the above observation, we investigate the action yxy on $\mathbb{R}^n$ where y is an element of the unit sphere $\mathbb{S}^{n-1}$ and x is an arbitrary vector in $\mathbb{R}^n$. We write $x = x_{||} + x_{\perp}$, here $x_{||}$ is the projection of x onto y and $x_{\perp}$ is orthogonal to y. Since $x_{||}$ is parallel with y, there is a constant $c \in \mathbb{R}$ such that $y = cx_{||}$. Thus we obtain

$$yxy = yx_{||}y + yx_{\perp}y = cx_{||}x_{||}cx_{||} + (-x_{\perp}yy) = (c^2x_{||}^2)x_{||} + x_{\perp} = -c^2||x_{||}||^2x_{||} + x_{\perp} = -x_{||} + x_{\perp}.$$

Therefore similar to e_ixe_i, the action yxy is also a reflection of x in the $y-$direction. And as an automorphism of $\mathbb{R}^n$, we shall denote it R_y. It is not hard to see that $R_y \in \mathrm{O}(n)$.

Theorem 3.27. *If A is an element of $O(n)$, then there are reflections $R_1, R_2, \cdots, R_p \in O(n)$ such that $A = R_1 \circ R_2 \circ \cdots \circ R_p$.*

Proof. We will proceed by induction. For $n = 1$, A is either ± 1, clearly it is a reflection across the origin or product of two reflections across the origin. Suppose that the statement of the theorem is true up to $\mathrm{O}(n-1)$, we would like to show that it is also true for $\mathrm{O}(n)$. Consider two different cases:

Case 1: Suppose there is a non-zero $x \in \mathbb{R}^n$ such that $A\,x = \pm x$. Let X be a subspace of $\mathbb{R}^n$ of dimension $n-1$ that is the orthogonal complement to $\{x\}$. Then for every $y \in X$, we have $\langle A\,y, x\rangle = \langle A\,y, \pm A\,x\rangle = \pm\langle y, x\rangle = 0$. This shows that $A\,y$ is orthogonal to x, which implies that $A\,y \in X$. Thus X is invariant under A. But $A|_X \in \mathrm{O}(n-1)$ and by our induction hypothesis, we have $A|_X = R_1 \circ R_2 \circ \cdots \circ R_p$, where R_i are reflections in certain directions.

Case 2: If there is no non-zero vectors in $\mathbb{R}^n$ that gets fixed or mapped to its antipodal, let $x \in \mathbb{R}^n \backslash \{0\}$ be fixed so that x and $A\,x$ are two distinct vectors that are not collinear. Consider the subspace W of $\mathbb{R}^n$ that is spanned by x and $A\,x$. Since $A\,x$ and x are not collinear, there is an angle φ between them. Let y be the vector in W of unit length that is orthogonal to the bisector of φ. Denote R the reflection in the $y-$direction. Note that $R(A\,x) = x$. Thus $R \circ A$ is an element of $\mathrm{O}(n)$ where it fixes a non-zero vector in $\mathbb{R}^n$. Case

1 tells us then that there exists reflections $R_1, \cdots, R_p$ such that $R \circ A = R_1 \circ \cdots \circ R_p$, which implies that $A = R^{-1} \circ R_1 \circ R_2 \circ \cdots R_p$ as required. □

Definition 3.13. The Pin group $\mathrm{Pin}(n)$ is defined as a subset of $Cl(\mathbb{R}^n)$ whose elements are invertible in the Clifford algebra and of the form $x_1\, x_2\, \cdots\, x_p$, where $x_i \in \mathbb{S}^{n-1}$.

The above definition is well-defined for each element $a = x_1 \cdots x_p$ of $\mathrm{Pin}(n)$ has the inverse as $(-x_p)\cdots(-x_1) = \overline{a}$ which also belongs to $\mathrm{Pin}(n)$, and $\mathrm{Pin}(n)$ is obviously closed under the Clifford multiplication. Furthermore for each element $a \in \mathrm{Pin}(n)$, $a\mathbb{R}^n\tilde{a}$ defines a group homomorphism $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$ in the following way
$\rho(a) = \rho(x_1 \cdots x_p) = R_{x_1} \circ \cdots \circ R_{x_p}$, where R_{x_i}'s are reflection in the x_i-directions. And as an immediate consequence of Theorem 3.27, we obtain

Corollary 3.28. *The homomorphism $\rho : Pin(n) \to O(n)$ defined above is surjective.* □

Theorem 3.29. *There is an exact sequence*

$$1 \longrightarrow \mathbb{Z}/2 \hookrightarrow Pin(n) \xrightarrow{\rho} O(n) \longrightarrow 1,$$

where we identify $\mathbb{Z}/2 = \{-1, 1\}$ as a subgroup of $Pin(n)$ that consists of the identity map and the antipodal map

Proof. It suffices to show that $\mathrm{Ker}\rho = \{-1, 1\}$. Let a be an arbitrary element of the kernel of ρ. Then for ever $x \in \mathbb{R}^n$, we have $\rho(a)\, x = ax\tilde{a} = x$. We write $a = x_1\, x_2\, \cdots\, x_p$ where $x_i \in \mathbb{S}^{n-1}$ not necessarily distinct. Note that then $\rho(a) = R_1 \circ \cdots \circ R_p$, where R_i is reflection in the x_i-direction. By our assumption, we actually have $\rho(a) \in \mathrm{SO}(n)$ so that $\det \rho(a) = 1$. Hence equivalently, $(-1)^p = \det R_1 \cdots \det R_p = 1$, which implies that p has to be even.

Now for each $x \in \mathbb{R}^n$ we have $ax\tilde{a} = x$, which implies that $ax = x\tilde{a}^{-1}$. However since $\tilde{a}^{-1} = (-1)^p x_1\, \cdots\, x_p = x_1 \cdots x_p = a$, we yield $ax = xa$ for all $x \in \mathbb{R}^n$. Furthermore because

$a \in Cl(\mathbb{R}^n)$, so we can write a as

$$a = a_0 + a_1 e_1 + a_2 e_2 + \cdots + a_{j_1 j_2 \cdots j_r} e_{j_1} \cdots e_{j_r} + \cdots + a_{12\cdots n} e_1 e_2 \cdots e_n, \tag{3.101}$$

where $a_{j_1 j_2 \cdots j_r} \in \mathbb{R}$ for each $0 \leq r \leq n$. We shall use (3.101) to compare ax and xa for specific values of x. If r is odd and $1 \leq r < n$, we take $x := e_{j_1}$. Observe that

$$(e_{j_1} e_{j_2} \cdots e_{j_r}) e_{j_1} = (-1)^r e_{j_1} e_{j_1} e_{j_2} \cdots e_{j_r} = (-1)^{r+1} e_{j_2} e_{j_3} \cdots e_{j_r} = e_{j_2} \cdots e_{j_r}.$$

While $e_{j_1}(e_{j_1} e_{j_2} \cdots e_{j_r}) = -e_{j_2} \cdots e_{j_r}$. Comparing coefficients yeild $a_{j_1 j_2 \cdots j_r} = -a_{j_1 j_2 \cdots j_r}$, which tells us that $a_{j_1 j_2 \cdots j_r} = 0$. If r is even and $1 \leq r < n$, we consider $x := e_{j_r}$ and apply similar computations to get

$$e_{j_r}(e_{j_1} \cdots e_{j_r}) = (e_{j_r} e_{j_1} \cdots e_{j_{r-1}}) e_{j_r} = (-1)^{r-1} e_{j_1} \cdots e_{j_r} e_{j_r} = (-1)^r e_{j_1} \cdots e_{j_{r-1}} = e_{j_1} \cdots e_{j_{r-1}},$$

and $(e_{j_1} e_{j_2} \cdots e_{j_r}) e_{j_r} = -e_{j_1} e_{j_2} \cdots e_{j_{r-1}}$. Thus $-a_{j_1 j_2 \cdots j_r} = a_{j_1 j_2 \cdots j_r}$ so that $a_{j_1 j_2 \cdots j_r} = 0$. Therefore in all cases of r, $1 \leq r < n$, we have $a_{j_1 j_2 \cdots j_r} = 0$. Then (3.101) simplifies to be $a = a_0 + a_{12\cdots n} e_1 \cdots e_n$. Note that $||a|| = 1$[6], so $a_0^2 + a_{12\cdots n}^2 = 1$. Hence there exists an φ such that $a = \cos\varphi + \sin\varphi \, e_1 e_2 \cdots e_n$. We substitute $x := e_n$ into the equation $ax\tilde{a} = x$ and compare the two sides

$$ae_n\tilde{a} = (\cos^2\varphi - \sin^2\varphi) e_n + \cos\varphi \, \sin\varphi \left((-1)^{\frac{(n-2)(n-1)}{2} + 1} - 1 \right) e_1 e_2 \cdots e_{n-1}.$$

As a result simultaneously we must have

$$\cos^2\varphi - \sin^2\varphi = 1, \quad \cos\varphi \sin\varphi \left((-1)^{\frac{(n-2)(n-1)}{2} + 1} - 1 \right) = 0.$$

Equivalently, $\cos 2\varphi = 1$ and $\sin 2\varphi \cdot C_n = 0$. Either-way φ has to be of the form $k\pi$, where

[6] We would like to emphasize that $||.||$ here is the norm of the Clifford algebra $Cl(\mathbb{R}^n)$.

$k \in \mathbb{Z}$. Thus $a = \cos k\pi = \pm 1$. We conclude that Kerρ is indeed $\mathbb{Z}/2$. □

From our discussion so far, we have found that Pin(n), as a group, is a double cover of O(n). Moreover by the definition of Pin(n), it is a subspace of S^{2^n-1}, which is the unit sphere in $Cl(\mathbb{R}^n)$ with respect to the norm of the Clifford algebra. Since S^{2^n-1} has a subspace topology inherited from $Cl(\mathbb{R}^n)$ where the topology here is induced by the norm, Pin(n) also has a subspace topology. Note that it is not hard to see the left (right) multiplication action of Pin(n) on itself is smooth; similarly, the inverted map Pin(n) $\to$ Pin(n) given by $a \mapsto a^{-1}$ is also smooth. Thus, Pin(n) is a Lie group.

Now consider the map $\psi : \text{Pin}(n) \to \{\pm 1\} = \mathbb{Z}/2$ defined by $\psi(a) = a\,\tilde{a}$. Such a map is well-defined for each $a \in \text{Pin}(n)$, $a = x_1\,x_2\,\cdots\,x_p$, where $x_i \in \mathbb{S}^{n-1}$, then $\psi(a) = x_1x_2\cdots x_{p-1}x_px_px_{p-1}\cdots x_2x_1 = (-1)^p$, which is ± 1 depending on the parity of p. Since ψ is continuous with respect to the topology of Pin(n) and the discrete topology of $\mathbb{Z}/2$, Pin(n) has at least 2 connected components. In fact

Proposition 3.30. *Pin(n) has exactly two connected components.*

Proof. Suppose $a = x_1 \cdots x_p$ is an element of the pin group, where $x_i \in \mathbb{S}^{n-1}$. If p is even, since $\mathbb{S}^{n-1} \subseteq \text{Pin}(n)$ is connected, for each i there is a path γ_i from x_i to $(-1)^{i+1}e_i$. Define $\gamma(t) := \gamma_1(t)\gamma_2(t)\cdots\gamma_p(t)$. Note that $\gamma : [0,1] \to \text{Pin}(n)$ is a path that connects a to 1. On the other hand if p is odd, similarly we can construct a path between a and $-e_1$ in Pin(n). Note that $-e_1$ and 1 are not in the same component of Pin(n). Indeed suppose otherwise there exists a path $\alpha_t \in \text{Pin}(n)$ for all $t \in [0,1]$ such that $\alpha_0 = 1$ and $\alpha_1 = -e_1$. ρ then induces a path between the identity map $\mathbf{1}$ and the reflection in the $-e_1-$direction R_{-e_1} in O(n). But $\mathbf{1}$ is definitely orientation preserving and R_{-e_1} is orientation reversing; hence a contradiction. This shows that Pin(n) has precisely two connected components. □

Recall that the identity component of a Lie group G is a subgroup of G. In particular, the identity component of Pin(n) is one of its subgroup, and is called the spin group Spin(n). This is not the only picture of the spin group.

Theorem 3.31. *The following statements are equivalent*

1. *Spin*(n) *is the identity component of Pin*(n).

2. *Spin*$(n) = Cl^+(\mathbb{R}^n) \cap Pin(n)$.

Proof. Suppose that $x \in \text{Spin}(n)$, then x is in the component of the identity. Furthermore since x is also in the pin group, we can write $x = x_1 x_2 \cdots x_p$, where $x_i \in \mathbb{S}^{n-1} \subseteq Cl(\mathbb{R}^n)$ and p is even. Note that for each i we can write $x_i = \sum_{j=1}^n \alpha_i^j e_j$ such that $\sum_j (\alpha_i^j)^2 = 1$. Then $x = \prod_{i=1}^p (\sum_{j=1}^n \alpha_i^j e_j)$. Distribute out this product, we obtain a sum of the form

$$x = \sum_{0 \leq r \leq n, 2|r} \alpha_{j_1 j_2 \cdots j_r} e_{j_1} e_{j_2} \cdots e_{j_r}, \tag{3.102}$$

where $\alpha_{j_1 j_2 \cdots j_r} \in \mathbb{R}$. By definition $x \in Cl^+(\mathbb{R}^n)$. Thus we have shown that $\text{Spin}(n) \subseteq Cl^+(\mathbb{R}^n) \cap \text{Pin}(n)$. Conversely, suppose x is an arbitrary element of $Cl^+(\mathbb{R}^n) \cap \text{Pin}(n)$. then we can write x as (112), where $\alpha_{j_1 j_2 \cdots j_r} \in \mathbb{R}$ and $\sum \alpha^2_{j_1 j_2 \cdots j_r} = 1$. Note that there is a path $\gamma_{j_1} : [0,1] \to Cl(\mathbb{R}^n)$ where $\gamma_{j_1}(0) = e_{j_1}$ and $\gamma_{j_1}(1) = \alpha_{j_1 \cdots j_r} e_{j_1}$, and for every $k \geq 2$, there is a path $\gamma_{j_k} : [0,1] \to Cl(\mathbb{R}^n)$ that connects e_{j_k} to $(-1)^{k+1} e_1$. Then consider

$$\gamma(t) := \sum_{0 \leq r \leq n, 2|r} \alpha_{j_1 \cdots j_r} \gamma_{j_1}(t) \cdots \gamma_{j_r}(t).$$

By construction, γ defines a continuous path $[0,1] \to \text{Pin}(n)$ with $\gamma(0) = x$ and $\gamma(1) = \sum \alpha^2_{j_1 j_2 \cdots j_r} = 1$. This shows that x is in the identity component of the pin group. Therefore, we obtain the equality $\text{Spin}(n) = Cl^+(\mathbb{R}^n) \cap \text{Pin}(n)$. □

Besides the two descriptions of the spin group, the above theorem also tells us that elements of $\text{Spin}(n)$ are products of even numbers of vectors of unit length in $\mathbb{R}^n$, so $\rho|_{\text{Spin}(n)}$ is a homomorphism onto $\text{SO}(n)$. Furthermore, $\text{Ker}\, \rho|_{\text{Spin}(n)} \subseteq \text{Ker}\, \rho = \mathbb{Z}/2$ but both -1 and 1 are elements of $\text{Spin}(n)$ that get sent to the identity matrix. Therefore, $\text{Ker}\, \rho|_{\text{Spin}(n)} = \mathbb{Z}/2$. As a result, we obtain

Proposition 3.32. *There is a short exact sequence*

$$1 \longrightarrow \mathbb{Z}/2 \hookrightarrow Spin(n) \xrightarrow{\rho} SO(n) \longrightarrow 1. \square$$

Corollary 3.33. *Spin(n) is a double cover for SO(n); hence it is a compact Lie group. For $n \geq 3$, Spin(n) is simply connected, thus the spin group is the universal cover of the special orthogonal group.*

Proof. Consider the exact homotopy sequence of fibration

$$\pi_1(\mathbb{Z}/2) \to \pi_1(\mathrm{Spin}(n)) \to \pi_1(\mathrm{SO}(n)) \xrightarrow{\partial} \pi_0(\mathbb{Z}/2) \to \pi_0(\mathrm{Spin}(n)) \to \pi_0(\mathrm{SO}(n)) \to 1,$$

where recall that $\pi_1(\mathbb{Z}/2)$, $\pi_0(\mathrm{SO}(n))$ and $\pi_0(\mathrm{Spin}(n))$ are trivial, while $\pi_1(\mathrm{SO}(n))$ and $\pi_0(\mathbb{Z}/2)$ are $\mathbb{Z}/2$ if $n \geq 3$. Now note that for $n \geq 2$, $\pi_0(\mathbb{Z}/2) \to \pi_0(\mathrm{Spin}(n))$ is trivial because both -1 and 1 are elements of the spin group. Then by exactness, ∂ is surjective, which implies that it is also an isomorphism. Hence, the map $\pi_1(\mathrm{Spin}(n)) \to \pi_1(\mathrm{SO}(n))$ is trivial. On the other hand, $\pi_1(\mathbb{Z}/2) \to \pi_1(\mathrm{Spin}(n))$ is trivial so that the kernel of $\pi_1(\mathrm{Spin}(n)) \to \pi_1(\mathrm{SO}(n))$ is also trivial. Therefore, $\pi_1(\mathrm{Spin}(n))$ is no other than 1. $\square$

Example 3.7. It is well-known that $\mathbb{S}^3$ is a double cover of SO(3) that is simply connected. Hence, it is also the universal cover. Then by the above corollary, $\mathrm{Spin}(3) \cong \mathbb{S}^3$, which means that $\mathbb{S}^3$ is also a Lie group.

Recall that a Clifford algebra $Cl(\mathbb{R}^n)$ over $\mathbb{R}^n$ equipped with its usual inner product is defined as $\mathcal{T}(\mathbb{R}^n)/\{v \otimes v + ||v||^2 \cdot 1\}$. If we were to quotient out the tensor algebra by the two-sided $\{v \otimes v - ||v||^2 \cdot 1\}$, we still obtain a Clifford algebra over $\mathbb{R}^n$ according to the universal property. Such algebra is denoted $Cl'(\mathbb{R}^n)$. Similarly when we replace $\mathbb{R}^n$ by $\mathbb{C}^n$, and the usual inner product of $\mathbb{R}^n$ by the standard complex symmetric bilinear form for $\mathbb{C}^n$ given by $(z, w) \mapsto \sum_{i=1}^n z^i w^i$, then the Clifford algebra over $\mathbb{C}^n$ is defined by $\mathcal{T}(\mathbb{C}^n)/\{z \otimes z + (\sum_{i=1}^n (z^i)^2) \cdot 1\}$. Note that on $\mathbb{C}^n$ every complex quadratic form is equivalent to $\sum_i (z^i)^2$ so that without ambiguity, we can denote the complex Clifford algebra $Cl^c(\mathbb{C}^n)$. We shall state without proofs the following results relating $Cl^c(\mathbb{C}^n)$ with $Cl(\mathbb{R}^n)$ and $Cl'(\mathbb{R}^n)$.

Proposition 3.34. *The complex Clifford algebra $Cl^c(\mathbb{C}^n)$ is the complexification of the real Clifford algebra*

$$Cl^c(\mathbb{C}^n) = Cl(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C} = Cl'(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}.$$

Proposition 3.35. *The following graded real algebras are isomorphic*

$$Cl(\mathbb{R}^{n+2}) = Cl'(\mathbb{R}^n) \otimes_{\mathbb{R}} Cl(\mathbb{R}^2), \quad Cl'(\mathbb{R}^{n+2}) = Cl(\mathbb{R}^n) \otimes_{\mathbb{R}} Cl'(\mathbb{R}^2).$$

Lemma 3.36. *There is an isomorphism $Cl^c(\mathbb{C}^2) = M(2, \mathbb{C}) = End(\mathbb{C}^2)$.*

Corollary 3.37. *There is an isomorphism $Cl^c(\mathbb{C}^{n+2}) = Cl^c(\mathbb{C}^n) \otimes_{\mathbb{C}} End(\mathbb{C}^2)$.*

Proof. By the complexification of the real Clifford algebra, we have
$Cl^c(\mathbb{C}^{n+2}) = Cl(\mathbb{R}^{n+2}) \otimes_{\mathbb{R}} \mathbb{C} = (Cl'(\mathbb{R}^n) \otimes_{\mathbb{R}} Cl(\mathbb{R}^2)) \otimes_{\mathbb{R}} \mathbb{C} =$
$(Cl'(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (Cl(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}) = Cl^c(\mathbb{C}^n) \otimes_{\mathbb{C}} Cl^c(\mathbb{C}^2)$; and by the above lemma, this is exactly isomorphic to $Cl^c(\mathbb{C}^n) \otimes_{\mathbb{C}} \mathrm{End}\,(\mathbb{C}^2)$. □

In fact there is an explicit isomorphism between $Cl^c(\mathbb{C}^{n+2}) \to Cl^c(\mathbb{C}^n) \otimes_{\mathbb{C}} \mathrm{End}\,(\mathbb{C}^2)$ and it is defined in the following way: Let $e_1, \cdots e_{n+2}$ be the generating elements of $Cl^c(\mathbb{C}^{n+2})$ and correspondingly, $e_1^*, \cdots e_n^*$ be those for $Cl^c(\mathbb{C}^n)$. Denote

$$g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

the generating basis of the algebra $M(2, \mathbb{C}) = \mathrm{End}\,(\mathbb{C}^2)$. Then
$Cl^c(\mathbb{C}^{n+2}) \cong Cl^c(\mathbb{C}^n) \otimes \mathrm{End}\,(\mathbb{C}^2)$ by the map

$$e_1 \mapsto 1 \otimes g_1, \quad e_2 \mapsto 1 \otimes g_2, \quad e_j \mapsto (ie_{j-2}^*) \otimes g_1 g_2, \ 3 \leq j \leq n+2.$$

Applying the result of Corollary 3.37 recursively, we immediately obtain

Corollary 3.38.

1. *If* $n = 2k$, *then* $Cl^c(\mathbb{C}^n) \cong \bigotimes_{j=1}^k M(2,\mathbb{C}) \cong End \bigotimes_{j=1}^k \mathbb{C}^2 \cong End(\mathbb{C}^{2^k})$.

2. *If* $n = 2k+1$, *then*
   $Cl^c(\mathbb{C}^n) \cong \bigotimes_{j=1}^k M(2,\mathbb{C}) \oplus \bigotimes_{j=1}^k M(2,\mathbb{C}) \cong End(\mathbb{C}^{2^k}) \oplus End(\mathbb{C}^{2^k})$.

The complex finite dimensional vector space $\mathbb{C}^{2^k}$ is called the *the complex* $n-$*spinors*, and is denoted Δ_n for $n = 2k$ or $2k+1$. With this notations, depending on the parity of n, we have $Cl^c(\mathbb{C}^n)$ isomorphic to either $\mathrm{End}\,(\Delta_n)$ or $\mathrm{End}\,(\Delta_n \oplus \Delta_n)$. Furthermore, the above isomorphisms can be described explicitly so that we would represent the complex Clifford algebra $Cl^c(\mathbb{C}^n)$ by the algebra of endomorphism on Δ_n. When n is even, the representation is the just the isomorphism $Cl^c(\mathbb{C}^n) \to \mathrm{End}\,(\Delta_n)$; on the other hand, when n is odd, the representation is the composition $Cl^c(\mathbb{C}^n) \to \mathrm{End}\,(\Delta_n) \oplus \mathrm{End}\,(\Delta_n) \to \mathrm{End}\,(\Delta_n)$, where the first map is the isomorphism and the latter map is the projection. Regardless, the representation $Cl^c(\mathbb{C}^n) \to \mathrm{End}\,(\Delta_n)$ is denoted κ_n and is called *the spin representation of the complex Clifford algebra.*

In particular, when $n = 2k$ [7], we would like to precisely write down a formula for κ_n. Note that $M(2,\mathbb{C})$ as a vector space over $\mathbb{C}$ has the following basis $\{I, g_1, g_2, T\}$, where g_1 and g_2 are defined as above and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Let $e_1, \cdots, e_n$ be basis elements that generates $Cl^c(\mathbb{C}^n)$. Then the formula for κ_n with respect to this choice of bases is given as follow: For each j,
$e_j \mapsto I \otimes \cdots \otimes I \otimes g_{\alpha(j)} \otimes T \otimes \cdots \otimes T$, where $\alpha(j)$ is 1 if j is odd and 2 otherwise.

[7]When n is odd, it is a similar treatment

Furthermore, the spin representation κ_n induces an $\mathbb{R}-$linear map $\kappa_n|_{\mathbb{R}^n} : \mathbb{R}^n \subset Cl(\mathbb{R}^n) \subset Cl^c(\mathbb{C}^n) \to \text{End}\,(\Delta_n)$. We claim that for each vector v in $\mathbb{R}^n$, $\kappa_n(v)^2\xi = -||v||^2\,\xi$, where ξ is any element of Δ_n. Indeed without loss of generality, assume $\xi = \xi^1 \otimes \cdots \otimes \xi^k$; write $v = v^1 e_1 + \cdots + v^n e_n$, where $v^j \in \mathbb{R}$. Note that $Tg_{\alpha(j)} + g_{\alpha(j)}T = 0$ and $T^2 = I$, then

$$\kappa_n(v)^2\xi = \sum_{j=1}^{n}\sum_{r=1}^{n} v^j v^r \xi^1 \otimes \cdots \otimes g_{\alpha(r)}\xi^r \otimes T\xi^{r+1} \otimes \cdots \otimes Tg_{\alpha(j)}\xi^j \otimes T^2\xi^{j+1} \otimes \cdots \otimes T^2\xi^k.$$

The right hand side of the above equality simplifies to be $\sum_{j=1}^{n}(v^j)^2\xi^1 \otimes \cdots \otimes g^2_{\alpha(j)}\xi^j \otimes T^2\xi^{j+1} \otimes \cdots \otimes T^2\xi^k =$ $\sum_{j=1}^{n}(v^j)^2\xi^1 \otimes \cdots \otimes (-\xi^j) \otimes \xi^{j+1} \otimes \cdots \otimes \xi^k = -||v||^2\xi$, which is what we need. We summarize the above discussion with the following proposition

Proposition 3.39. *The complex $n-$spinors Δ_n is a Clifford module with respect to the real Clifford algebra $Cl(\mathbb{R}^n)$. The action $\kappa_n(v)\xi$ of $\mathbb{R}^n$ on Δ_n is called the left Clifford multiplication of a vector and a spinor.* $\square$

Note that by definition, $\text{Spin}(n)$ is a a group of invertible elements in the real Clifford algebra. Hence the restriction of the spin representation of $Cl(\mathbb{R}^n)$ to the spin group $\kappa := \kappa_n|_{\text{Spin}(n)} : \text{Spin}(n) \to \text{GL}(\Delta_n)$ is called the *spin representation* of $\text{Spin}(n)$. κ is a faithful representation, i.e, every distinct element of $\text{Spin}(n)$ is mapped to distinct element of $\text{GL}(\Delta_n)$. Furthermore, one can show that there is a Hermitian inner product on Δ_n such that κ is a unitary representation, and the left Clifford multiplication by a vector on the spinors is skew-adjoint. To do this, first let $(\,,)^*$ be an arbitrary positive definite scalar product on Δ_n. Since $\text{Spin}(n)$ is a compact Lie group, we can define μ to be the Haar measure on it. Now for each fixed pair of spinors φ and ψ, the map $h^{\varphi\psi} : g \mapsto (\kappa(g)\varphi, \kappa(g)\psi)^*$ is a continuous function from $\text{Spin}(n) \to \mathbb{C}$; thus it is continuous, and in turns, measurable with respect to μ. Denote

$$(\varphi, \psi) := \int_{\mathrm{Spin}(n)} h^{\varphi\psi}(g)\, d\mu(g) = \int_{\mathrm{Spin}(n)} (\kappa(g)\varphi, \kappa(g)\psi)^*\, d\mu(g). \tag{3.103}$$

Lemma 3.40. $(\, ,)$ *is a well-defined Hermitian inner product on* Δ_n*.*

Proof: It is not hard to see that $(\, ,)$ is bilinear. What remains for us to show that $(\, ,)$ is actually well-defined, i.e, it is independent of the choice of $g \in \mathrm{Spin}(n)$. Let s be another element of $\mathrm{Spin}(n)$ other than g. With respect to the Haar measure, we immediately yield

$$\int_{\mathrm{Spin}(n)} h^{\varphi\psi}(g)\, d\mu(g) = \int_{\mathrm{Spin}(n)} h^{\varphi\psi}((sg^{-1})g)\, d\mu(sg^{-1}g) = \int_{\mathrm{Spin}(n)} h^{\varphi\psi}(s)\, d\mu(s). \ \square$$

Now we claim that with respect to the Hermitian inner product defined in (113), κ is a unitary representation. Indeed, let $s \in \mathrm{Spin}(n)$, consider

$$(\kappa(s)\varphi, \kappa(s)\psi) = \int_{\mathrm{Spin}(n)} h^{\kappa(s)\varphi\kappa(s)\psi}(g)\, d\mu(g) = \int_{\mathrm{Spin}(n)} h^{\varphi\psi}(sg)\, d\mu(sg) = (\varphi, \psi),$$

which shows that $\kappa(s)^*\kappa(s) = \kappa(s)\kappa(s)^* = \mathrm{Id}_{\Delta_n}$ with respect to $(\, ,)$. This observation has the following implication, since $\kappa_n(v)^2 = -||v|^2\mathrm{Id}_{\Delta_n}$ for each $v \in \mathbb{R}^n$ and without loss of generality assume $v \neq 0$ and $||v|| = 1$, $\kappa_n(v)^* = -\kappa_n(v)$. As a result

Lemma 3.41. *With respect to* $(\, ,)$ *defined in* (3.103)*, we have*
$(\kappa_n(v)\varphi, \psi) + (\varphi, \kappa_n(v)\psi) = 0$ *for each* $v \in \mathbb{R}^n$ *and* $\varphi, \psi \in \Delta_n$*.* $\square$

Finally we can write the spin representation as $\kappa : \mathrm{Spin}(n) \to \mathrm{U}(\Delta_n)$. For each $g \in \mathrm{Spin}(n)$, since $\kappa(g)^{-1} = \kappa(g)^*$, $|\det \kappa(g)| = 1$. This statement can be made even stronger.

Proposition 3.42. *If* $\kappa : Spin(n) \to U(\Delta_n)$ *is the spin representation, then*

$$det\, \kappa(g) = 1$$

for every $g \in Spin(n)$ so that κ is the representation of the spin group into the the special unitary group $SU(\Delta_n)$ of the space of spinors.

Proof. Consider the group homomorphism that happens to be also continuous with respect to the smooth structure of $\mathrm{Spin}(n)$ and $\mathbb{S}^1$, $f : \mathrm{Spin}(n) \to \mathbb{S}^1$ given by $f(g) := \det \kappa(g)$. Since $\mathrm{Spin}(n)$ is simply connected and $\mathbb{R}$ is the universal cover of $\mathbb{R}$, then there exists a continuous lift $F : \mathrm{Spin}(n) \to \mathbb{R}$ such that $f(g) = \exp(2\pi i F(g))$. Note that F is a group homomorphism and $\mathrm{Spin}(n)$ is compact, so $F(\mathrm{Spin}(n)) \subset \mathbb{R}$ is a subgroup of $\mathbb{R}$ that is bounded in an interval. This is only possible when $F(\mathrm{Spin}(n))$ is the trivial group, in other words, $F \equiv 0$. Therefore, $\det \kappa(g) = f(g) = \exp(0) = 1$. □

3.7 Spin structures and spin manifolds

Let (M, g) be a connected compact $n-$dimensional Riemannian manifold. Denote $\mathrm{SO}_n := O(M, n)$ the principal $\mathrm{SO}(n)-$bundle of ordered orthonormal frames over M.

Definition 3.14. (Spin structure) A spin structure on M is a principal $\mathrm{Spin}(n)-$bundle $(\widetilde{\mathrm{SP}}_n, \varrho)$ over M where $\varrho : \widetilde{\mathrm{SP}}_n \to \mathrm{SO}_n$ is a double cover map and ϱ restricted to each fiber is the double cover $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ such that the following diagram is commutative

$$\begin{array}{ccccc}
\widetilde{\mathrm{SP}}_n \times \mathrm{Spin}(n) & \longrightarrow & \widetilde{\mathrm{SP}}_n & \xrightarrow{\tilde{\pi}} & M \\
\downarrow{\scriptstyle \varrho\times\rho} & & \downarrow{\scriptstyle \varrho} & & \| \\
\mathrm{SO}_n \times \mathrm{SO}(n) & \longrightarrow & \mathrm{SO}_n & \xrightarrow{\pi} & M
\end{array}$$

Here the horizontal maps are the actions of the groups on the bundles, and the surjective smooth invariant maps from the bundles onto the underlying manifolds respectively. If M possesses a spin structure, M is called a *spin manifold.*

Definition 3.15. Two spin structures $(\widetilde{\mathrm{SP}}^1_n, \varrho_1)$ and $(\widetilde{\mathrm{SP}}^2_n, \varrho_2)$ are *spin equivalent* if and only if there exists a $\mathrm{Spin}(n)-$equivariant $f : \widetilde{\mathrm{SP}}^1_n \to \widetilde{\mathrm{SP}}^2_n$ such that the following diagram commutes

$$\begin{array}{ccc} \widetilde{\mathrm{SP}}^1_n & \xrightarrow{f} & \widetilde{\mathrm{SP}}^2_n \\ \downarrow \varrho_1 & & \downarrow \varrho_2 \\ \mathrm{SO}_n & = & \mathrm{SO}_n \end{array}$$

Note that the above definition gives us an equivalent relation of spin structures on M. Thus, all the spin structures on M are divided into equivalence classes. One should be cautious to realize that though the language used here insists upon the relation defined on M, it really is not the usual equivalence of bundle over the underlying manifold but rather over the principal bundle SO_n. Certainly, there are two spin structures over M that are not spin equivalent even if the corresponding principal $\mathrm{Spin}(n)-$bundles are equivalent over M. For example, $\mathbb{RP}^2$ and the trivial $\mathrm{SO}(n)-$bundle $\mathbb{RP}^2 \times \mathrm{SO}(n)$ satisfy the assertion. We shall see the reason why this is so later in details.

Suppose that $(\widetilde{\mathrm{SP}}_n, \varrho)$ is some fixed spin structure on M. Consider $\mathrm{SO}_{n,p}$ a fiber of SO_n. Because of the identification $\mathrm{SO}_{n,p} \cong \mathrm{SO}(n)$, we have $\pi_1(\mathrm{SO}_{n,p}) = \mathbb{Z}/2$. Let $\iota : \mathrm{SO}_{n,p} \hookrightarrow \mathrm{SO}_n$ be the inclusion map and $\alpha \in \pi_1(\mathrm{SO}_{n,p})$ be it's non-trivial element. Denote $\alpha_\# := \iota_\#(\alpha) \in \pi_1(\mathrm{SO}_n)$. By some general theory of covering spaces, since $\varrho : \widetilde{\mathrm{SP}}_n \to \mathrm{SO}_n$ is a double cover, $H(\widetilde{\mathrm{SP}}_n, \varrho) = \varrho_\#(\pi_1(\widetilde{\mathrm{SP}}_n)) \subset \pi_1(\mathrm{SO}_n)$ is a subgroup of index 2.

Lemma 3.43. *$\alpha_\#$ is not an element of $H(\widetilde{SP}_n, \varrho)$.*

Proof. Suppose otherwise that $\alpha_\# \in H(\widetilde{\mathrm{SP}}_n, \varrho)$, that means $\iota_\#(\pi_1(\mathrm{SO}_{n,p})) \subseteq \varrho_\#(\pi_1(\widetilde{\mathrm{SP}}_n))$. At the same time, $\mathrm{SO}_{n,p} \cong \mathrm{SO}(n)$ is locally-connected and path connected; thus there exists a unique lift $f : \mathrm{SO}_{n,p} \to \widetilde{\mathrm{SP}}_n$ of ι such that $\varrho \circ f = \iota$. In particular, $f(\mathrm{SO}_{n,p})$ lies in some fiber of $\widetilde{\mathrm{SP}}_n$ identified with $\mathrm{Spin}(n)$. Hence we can view $f : \mathrm{SO}(n) \to \mathrm{Spin}(n)$ where $\rho \circ f = \mathrm{Id}_{\mathrm{SO}(n)}$, which implies that $\rho_\# \circ f_\# = \mathrm{Id}_{\pi_1(\mathrm{SO}(n))}$. However for $n \geq 3$, $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$ and $\pi_1(\mathrm{Spin}(n)) = 1$, this leads to a contradiction. □

Proposition 3.44. *The equivalence classes of spin structures on M are in bijective correspondence with collection of subgroups $H \subseteq \pi_1(SO_n)$ of index 2 that do not contain $\alpha_\#$.*

Consider the short exact sequences of spaces $0 \to \mathrm{SO}_{n,p} \hookrightarrow \mathrm{SO}_n \to M \to 0$ which induces the following exact homotopy sequence of fibration

$$\cdots \longrightarrow \pi_2(M) \xrightarrow{\partial} \pi_1(\mathrm{SO}_{n,p}) = \mathbb{Z}/2 \xrightarrow{\iota_\#} \pi_1(\mathrm{SO}_n) \xrightarrow{\pi_\#} \pi_1(M) \to 1. \tag{3.104}$$

A subgroup $H \subseteq \pi_1(\mathrm{SO}_n)$ of index 2 is normal. Thus H is normal in $\pi_1(\mathrm{SO}_n)$ so that we have a non-trivial homomorphism $f_H : \pi_1(\mathrm{SO}_n) \to \pi_1(\mathrm{SO}_n)/H = \mathbb{Z}/2$ and vice versa. On the other hand, the condition $\alpha_\# \notin H$ is equivalent to the fact that the composition of maps

$$\pi_1(\mathrm{SO}_{n,p}) = \mathbb{Z}/2 \xrightarrow{\iota_\#} \pi_1(\mathrm{SO}_n) \xrightarrow{f_H} \pi_1(\mathrm{SO}_n)/H = \mathbb{Z}/2 = \pi_1(\mathrm{SO}_{n,p})$$

$f_H \circ \iota_\# = \mathrm{Id}_{\mathbb{Z}/2}$. Thus the collection of subgroups $H \subseteq \pi_1(\mathrm{SO}_n)$ of index 2 that do not contain $\alpha_\#$ is in one-to-one and onto correspondence with the collection of homomorphisms $f : \pi_1(\mathrm{SO}_n) \to \pi_1(\mathrm{SO}_{n,p}) = \mathbb{Z}/2$ such that $f \circ \iota_\# = \mathrm{Id}_{\mathbb{Z}/2}$. Therefore, we can rephrase Proposition 3.44 in the following way

Corollary 3.45. *The equivalence classes of spin structures on M are in bijective correspondence with the collection of homomorphisms $f : \pi_1(SO_n) \to \mathbb{Z}/2$ where $f \circ \iota_\# = Id_{\mathbb{Z}/2}$.*

Yet there is another more succinct way to describe the relationship between the equivalence classes of spin structures on M and the algebraic topology of SO_n. First, let's recall some algebra facts about the universal property of the abelianization of a group G. In general, G^{ab} is the abelianization of G if and only if it is an abelian group equipped with a homomorphism $\phi : G \to G^{ab}$ that is universal among all abelian groups equipped with homomorphisms from G. As a result, it is not hard to see that $G^{ab} \cong G/[G,G]$.

Another application of the universal property of $G/[G,G]$ is as follow: Suppose $\varphi \in \mathrm{Hom}(G, \mathbb{Z}/2)$. By the universal property, there exists uniquely a homomorphism $f \in \mathrm{Hom}(G/[G,G]; \mathbb{Z}/2)$ such that $f \circ \phi = \varphi$. Conversely, a homomorphism from

$G/[G,G] \to \mathbb{Z}/2$ gives rise to a homomorphism from $G \to \mathbb{Z}/2$ by composition. Hence it is not hard to see that $\mathrm{Hom}(G/[G,G];\mathbb{Z}/2) = \mathrm{Hom}(G;\mathbb{Z}/2)$. Replace $G := \pi_1(\mathrm{SO}_n)$, we obtain

$$H^1(\mathrm{SO}_n;\mathbb{Z}/2) := \mathrm{Hom}(H_1(\mathrm{SO}_n);\mathbb{Z}/2) \tag{3.105}$$

$$:= \mathrm{Hom}(\pi_1(\mathrm{SO}_n)/[\pi_1(\mathrm{SO}_n),\pi_1(\mathrm{SO}_n)];\mathbb{Z}/2) = \mathrm{Hom}(\pi_1(\mathrm{SO}_n);\mathbb{Z}/2). \tag{3.106}$$

This means that a homomorphism $f : \pi_1(\mathrm{SO}_n) \to \mathbb{Z}/2$ gives rise to an element which we also denote f belonging to the first cohomology group $H^1(\mathrm{SO}_n;\mathbb{Z}/2)$. At the same time, the inclusion $\iota : \mathrm{SO}_{n,p} \hookrightarrow \mathrm{SO}_n$ induces the homomorphism
$\iota^* : H^1(\mathrm{SO}_n;\mathbb{Z}/2) \to H^1(\mathrm{SO}_{n,p};\mathbb{Z}/2) = \mathrm{Hom}(\pi_1(\mathrm{SO}_{n,p});\mathbb{Z}/2) = \mathrm{Hom}(\mathbb{Z}/2;\mathbb{Z}/2) = \mathbb{Z}/2.$
Hence, the condition $f \circ \iota_\# = \mathrm{Id}_{\mathbb{Z}/2}$ is equivalent to the condition $\iota^*(f)$ is not the trivial map from $\mathbb{Z}/2 \to \mathbb{Z}/2$.

Consider the long exact sequence of cohomology groups

$$0 \longrightarrow H^1(M;\mathbb{Z}/2) \xrightarrow{\pi^*} H^1(\mathrm{SO}_n;\mathbb{Z}/2) \xrightarrow{\iota^*} H^1(\mathrm{SO}_{n,p};\mathbb{Z}/2) = \mathbb{Z}/2 \xrightarrow{\partial} H^2(M;\mathbb{Z}/2) \longrightarrow \cdots .$$

We have $\iota^*(f)$ is non-trivial in $H^1(\mathrm{SO}_{n,p};\mathbb{Z}/2) = \mathbb{Z}/2$ if and only if the non-trivial element $1 \in \mathbb{Z}/2$ belongs to to the image of ι^*. By exactness, $\partial(1) = 0$. The element $w_2(M) := \partial(1) \in H^2(M;\mathbb{Z}/2)$ is called the *second Stiefel-Whitney class* of M. The discussion helps us summarize the necessary and sufficient topological condition for a connected Riemannian manifold to posses spin structures.

Theorem 3.46. *The equivalence classes of spin structures on M are in bijectively correspondence with the elements of $f \in H^1(SO_n;\mathbb{Z}/2)$ such that $\iota^*(f) \neq 0$ in $\mathbb{Z}/2$. Moreover, M is a spin manifold if and only if the second Stiefel-Whitney class $w_2(M) = 0$. When the second Stiefel-Whitney class of M vanishes, the spin structures on M are completely determined by the first cohomology group $H^1(M;\mathbb{Z}/2)$.* □

Corollary 3.47. *Suppose M is simply connected. M is a spin manifold if and only if $\pi_1(SO_n) = \mathbb{Z}/2$ and in this case, M has a unique spin structure up-to spin equivalence.*

Proof. Since $\pi_1(M) = 1$, $H^1(M; \mathbb{Z}/2) = 0$. ι^* is now an injective homomorphism from $H^1(\mathrm{SO}_n; \mathbb{Z}/2) = \mathrm{Hom}(\mathbb{Z}/2; \mathbb{Z}/2) = \mathbb{Z}/2 \to \mathbb{Z}/2$; hence it is also surjective. Thus 1 belongs to the image of ι^* so that $w_2(M) = 0$. Therefore, M is spin. In this case, obviously there is one and only one element in $H^1(\mathrm{SO}_n; \mathbb{Z}/2)$ gets mapped to 1 by ι^*. As a result, M possesses a unique spin structure up-to spin equivalence.

Conversely suppose M is spin, then $w_2(M) = 0$. Therefore, ι^* is surjective. Earlier observation tells us ι^* is injective so then it is an isomorphism, which means that $H^1(\mathrm{SO}_n; \mathbb{Z}/2) = \mathbb{Z}/2$. But the assertion is only possible when $\pi_1(\mathrm{SO}_n) = \mathbb{Z}/2$. □

Corollary 3.48. *If M is $2-$connected, i.e, $\pi_2(M) = 1$ and simply connected, then M is a spin manifold and it only has one spin structure up-to spin equivalence.*

Proof. Since both π_1 and π_2 of M are trivial, (3.104) gives us an isomorphism $\pi_1(\mathrm{SO}_n) \cong \pi_1(\mathrm{SO}_{n,p}) = \mathbb{Z}/2$. By Corollary 3.47, M is spin and its spin structure is unique up-to equivalence. □

Example 3.8. Spin manifolds are not rare. It is well-known that $\mathbb{S}^n$ is $(n-1)-$connected. In particular, $\mathbb{S}^3$ is $2-$connected and also simply connected. Hence $\mathbb{S}^3$ is spin and there is only one spin structure up-to spin equivalence on it.

Example 3.9. Consider the complex projective plane $\mathbb{CP}^n$. Note that $\pi_1(\mathbb{CP}^n) = 1$ so wether or not $\mathbb{CP}^n$ is spin depends on when $\pi_1(\mathrm{SO}_{2n})$ is $\mathbb{Z}/2$. As it turns out, $\pi_1(\mathrm{SO}_{2n})$ is $\mathbb{Z}/2$ when $n \equiv 1 \mod 2$ and trivial otherwise (cite reference here). Therefore, $\mathbb{CP}^{2k+1}$ is spin, while $\mathbb{CP}^{2k}$ is not.

From now on when dealing with spin manifold M of dimension n, we shall fix a spin structure on it denoted by $\widetilde{\mathrm{SP}}_n$. The associated vector bundle $\Sigma_n := \widetilde{\mathrm{SP}}_n \times_\kappa \Delta_n$ with respect to the spin representation κ is called a *spinor bundle* over M. Note that Σ_n is a complex vector bundle over M of rank $2^{\frac{n}{2}}$. Since each fibers of Σ_n is identified with the complex $n-$spinors, and Δ_n is equipped with a Hermitian inner product defined previously, the spinor bundle also has a Hermitian metric that is smoothly varying across its fibers. As principal $\text{Spin}(n)-$bundle, $\widetilde{\text{SP}}_n$ is always equipped with a principal connection. From what we have seen before, a choice of principal connection induces a connection on the associated vector bundle via parallel translation. Our goal here is to construct a connection on Σ_n that is compatible with the metric and with the Levi-Civita connection ∇^{LC}. Consider the connection $1-$form ω on SO_n produced by ∇^{LC}, $\omega : T\text{SO}_n \to \mathfrak{so}(n)$. We first observe the following

Lemma 3.49. *The Lie algebra $\mathfrak{spin}(n)$ coincides with the linear subspace $span_{\mathbb{R}}\{e_ie_j : 1 \leq i < j \leq n.\} \subset Cl(\mathbb{R}^n)$ so that $dim_{\mathbb{R}}\, \mathfrak{spin}(n) = n(n-1)/2$. Furthermore, the differential $d_1\rho : \mathfrak{spin}(n) \to \mathfrak{so}(n)$ is an isomorphism where $d_1\rho(e_ie_j) = 2(E_{ij} - E_{ji})$. Therefore, $\mathfrak{spin}(n)$ can be identified with $\mathfrak{so}(n)$.*

Proof. cite reference □

Thus ω lifts uniquely to a $\mathfrak{spin}(n)-$valued $1-$form, $\widetilde{\omega} : T\widetilde{\text{SP}}_n \to \mathfrak{spin}(n)$ given by $\widetilde{\omega} = (d_1\rho)^{-1} \circ \omega \circ d\varrho$. We claim that $\widetilde{\omega}$ satisfies Proposition 3.20. Indeed let $u \in \mathfrak{spin}(n)$, X_u is the killing field on $\widetilde{\text{SP}}_n$, and $p \in \widetilde{\text{SP}}_n$ arbitrarily. Note that

$$d\varrho(X_u(p)) = \frac{d}{dt}\varrho(p \cdot \exp(tu))\bigg|_{t=0} = \frac{d}{dt}(\varrho(p) \cdot \rho(\exp(tu)))\bigg|_{t=0} \tag{3.107}$$

$$= \frac{d}{dt}(\varrho(p) \cdot \exp(td_1\rho(u)))\bigg|_{t=0} = X_{d_1\rho(u)}(\varrho(p)), \tag{3.108}$$

which is the killing field associated with $d_1\rho(u)$ on SO_n. Thus $\widetilde{\omega}(X_u) = (d_1\rho)^{-1} \circ \omega(X_{d_1\rho(u)}) = (d_1\rho)^{-1} \circ d_1\rho(u) = u$. For the second criterion, let X by an arbitrary vector field on $\widetilde{\text{SP}}_n$ and fix a point $p \in \widetilde{\text{SP}}_n$. For any $g \in \text{Spin}(n)$, note that $\text{Ad}(\rho(g)^{-1}) = (d_1\rho) \circ \text{Ad}(g^{-1}) \circ (d_1\rho)^{-1}$ and combining with the repetitive use of chain-rule, we have

$$(R_g)^*\widetilde{\omega}_p(X(p)) = \widetilde{\omega}_{p\cdot g}(d_pR_g\,X(p)) = (d_1\rho)^{-1}\circ\omega_{\varrho(p)\cdot\rho(g)}(d_{p\cdot g}\varrho\cdot d_pR_g\,X(p)) \tag{3.109}$$

$$= (d_1\rho)^{-1}\circ\omega_{\varrho(p)\cdot\rho(g)}(d_p(\varrho\circ R_g)\,X(p)) \tag{3.110}$$

$$= (d_1\rho)^{-1}\circ\omega_{\varrho(p)\cdot\rho(g)}(d_p(R_{\rho(g)}\circ\varrho)\,X(p)) \tag{3.111}$$

$$= (d_1\rho)^{-1}\circ\omega_{\varrho(p)\cdot\rho(g)}(d_{\varrho(p)}R_{\rho(g)}\cdot d_p\varrho\,X(p)) \tag{3.112}$$

$$= (d_1\rho)^{-1}\circ(R_{\rho(g)})^*\omega_{\varrho(p)}(d_p\varrho\,X(p)) \tag{3.113}$$

$$= (d_1\rho)^{-1}\circ\mathrm{Ad}(\rho(g)^{-1})\omega_{\varrho(p)\cdot\rho(g)}(d_{p\cdot g}\varrho\,X(p\cdot g)) \tag{3.114}$$

$$= (d_1\rho)^{-1}\circ(d_1\rho)\circ\mathrm{Ad}(g^{-1})\circ(d_1\rho)^{-1}\circ\omega_{\varrho(p\cdot g)}\cdot d_{p\cdot g}\varrho\,X(p\cdot g) \tag{3.115}$$

$$= \mathrm{Ad}(g^{-1})\widetilde{\omega}_{p\cdot g}(X(p\cdot g)). \tag{3.116}$$

We conclude that $\widetilde{\omega}$ defines a connection 1−form on $\widetilde{\mathrm{SP}}_n$. Now following the construction for (99), we obtain a connection $\nabla^{\Sigma_n}:\mathfrak{X}(M)\otimes\Gamma(\Sigma_n)\to\Gamma(\Sigma_n)$. We would like to show the following two important properties of ∇^{Σ_n}

Proposition 3.50. *The connection $\nabla^{\Sigma_n}:\Gamma(\Sigma_n)\to\Gamma(T^*M\otimes\Sigma_n)$ satisfies Definition 3.4, i.e,*

1. $(\nabla^{\Sigma_n}_X\xi,\psi)+(\xi,\nabla^{\Sigma_n}_X\psi) = X\,(\xi,\psi)$ *for each* $X\in\mathfrak{X}(M)$ *and* $\xi,\psi\in\Gamma(\Sigma_n)$.

2. $\nabla^{\Sigma_n}_X(\kappa_n(Y)\psi) = \kappa_n(\nabla^{LC}_X Y)\psi+\kappa_n(Y)\nabla^{\Sigma_n}_X\psi$ *for each* $X,Y\in\mathfrak{X}(M)$ *and* $\psi\in\Gamma(\Sigma_n)$.

If the above is true, then combine with Lemma 3.41 we yield the main result of this subsection

Theorem 3.51. *The spinor bundle Σ_n is a Clifford bundle over the spin manifold M.* □

We conclude the subsection by providing proof of Proposition 3.50. Suppose $n=2k$, [8] Let $U\subseteq M$ be an open subset such that we have a local section associated canonically with

[8] Again, similar treatment for $n=2k+1$.

a local trivialization of SO_n, $s : U \to \pi^{-1}(U) \subset \mathrm{SO}_n$. Since M is spin, there exists a unique lift $\widetilde{s} : U \to \widetilde{\pi}^{-1}(U) \subset \widetilde{\mathrm{SP}}_n$ such that the following diagram commutes

$$\begin{array}{ccc} U \subseteq M & \xrightarrow{\ \widetilde{s}\ } & \widetilde{\pi}^{-1}(U) \subset \widetilde{\mathrm{SP}}_n \\ \| & & \downarrow \varrho \\ U \subseteq M & \xrightarrow{\ s\ } & \pi^{-1}(U) \subset \mathrm{SO}_n \end{array}$$

Then $\widetilde{s}$ can be considered a local section associated canonically with a local trivialization of $\widetilde{\mathrm{SP}}_n$ on $U \subseteq M$. If we denote x^{ψ}, $\mathrm{x}^{\xi} \in \Gamma(\widetilde{\mathrm{SP}}_n, \Delta_n)^{\mathrm{Spin}(n)}$ be the corresponding maps to sections ψ, ξ respectively in the sense of Lemma 3.23, then on U one can write $\xi = (\widetilde{s}, \mathrm{x}^{\xi} \circ \widetilde{s})$, $\psi = (\widetilde{s}, \mathrm{x}^{\psi} \circ \widetilde{s})$. Note that $\widetilde{\pi} \circ \widetilde{s}$ is the identity on U; thus $d\widetilde{\pi} \circ d\widetilde{s}$ is the identity of TU. Therefore, $d\widetilde{s}(X) = \widetilde{X}$ at least on U, where $\widetilde{X}$ is the horizontal lift of X to $\mathfrak{X}(\widetilde{\mathrm{SP}}_n)$.

Proof of Proposition 3.50.1: It suffices to prove that metric compatibility condition is true on U. We have $\nabla^{\Sigma_n}_X \xi = (\widetilde{s}, d\mathrm{x}^{\xi} \circ d\widetilde{s}\,(X))$. Hence,

$$(\nabla^{\Sigma_n}_X \xi, \psi) = (d\mathrm{x}^{\xi} \circ d\widetilde{s}\,(X), \mathrm{x}^{\psi} \circ \widetilde{s}). \tag{3.117}$$

Similarly, we also have

$$(\xi, \nabla^{\Sigma_n}_X \psi) = (\mathrm{x}^{\xi} \circ \widetilde{s}, d\mathrm{x}^{\psi} \circ d\widetilde{s}\,(X)). \tag{3.118}$$

On the other hand, let $\{\sigma_1, \cdots \sigma_{2^k}\}$ be an ordered orthonormal basis with respect to the Hermitian inner product $(\ ,\ )$ on Δ_n so that we can write

$$\mathrm{x}^{\xi} = \sum_{i=1}^{2^k} \mathrm{x}^{\xi i} \sigma_i, \quad \mathrm{x}^{\psi} = \sum_{j=1}^{2^k} \mathrm{x}^{\psi j} \sigma_j,$$

where $\mathrm{x}^{\xi i}$'s and $\mathrm{x}^{\psi j}$'s belong to $\Gamma(\widetilde{\mathrm{SP}}_n, \mathbb{C})$ appropriately. Then it is not hard to see that $(\mathrm{x}^{\xi} \circ \widetilde{s}, \mathrm{x}^{\psi} \circ \widetilde{s}) = \sum_i (\mathrm{x}^{\xi i} \circ \widetilde{s})(\mathrm{x}^{\psi i} \circ \widetilde{s})$. As a result,

$$X(\xi, \psi) = X(\mathrm{x}^{\xi} \circ \widetilde{s}, \mathrm{x}^{\psi} \circ \widetilde{s}) = X \sum_{i=1}^{2^k} (\mathrm{x}^{\xi i} \circ \widetilde{s})(\mathrm{x}^{\psi i} \circ \widetilde{s}) \tag{3.119}$$

$$= \sum_{i=1}^{2^k} (d\mathrm{x}^{\xi i} \circ d\widetilde{s}(X))(\mathrm{x}^{\psi i} \circ \widetilde{s}) + \sum_{i=1}^{2^k} (\mathrm{x}^{\xi i} \circ \widetilde{s})(d\mathrm{x}^{\psi i} \circ d\widetilde{s}(X)) \tag{3.120}$$

(3.117), (3.118) and (3.120) combined gives us the equality. □

Proof of 3.50.1: Again, it suffices to show that the Levi-Civita compatibility condition is true on U. Let $\{\partial_1, \cdots, \partial_n\}$ be an oriented local synchronuous orthonormal frame field at m on U. It is enough to show that

$$\nabla^{\Sigma_n}_{\partial_i}(\kappa_n(Y)\psi) = \kappa_n(\nabla^{LC}_{\partial_i} Y)\psi + \kappa_n(Y)\nabla^{\Sigma_n}_{\partial_i}\psi. \tag{3.121}$$

We shall work from the left hand side of (3.121). Suppose $Y = \sum_j Y^j \partial_j$, $Y^j \in \Gamma(M)$. Then at m, using Leibniz rule for the connection ∇^{Σ_n} and chain rule we obtain

$$\mathrm{LHS} = \sum_{j=1}^{n} \nabla^{\Sigma_n}_{\partial_i}(Y^j \kappa_n(\partial_j)\psi) = \sum_{j=1}^{n} Y^j \nabla^{\Sigma_n}_{\partial_i}(\kappa_n(\partial_j)\psi) + \sum_{j=1}^{n} (\partial_i Y^j)\kappa_n(\partial_j)\psi \tag{3.122}$$

$$= \sum_{j=1}^{n} Y^j (\widetilde{s}, d\kappa_n(\partial_j)\, d(\mathrm{x}^{\psi} \circ \widetilde{s})(\partial_i)) + \sum_{j=1}^{n} (\partial_i Y^j)(\widetilde{s}, \kappa_n(\partial_j)(\mathrm{x}^{\psi} \circ \widetilde{s})) \tag{3.123}$$

$$= \sum_{j=1}^{n} Y^j (\widetilde{s}, \kappa_n(\partial_j)\, d(\mathrm{x}^{\psi} \circ \widetilde{s})(\partial_i)) + \sum_{j=1}^{n} (\widetilde{s}, \kappa_n((\partial_i Y^j)\partial_j)(\mathrm{x}^{\psi} \circ \widetilde{s})) \tag{3.124}$$

$$= \sum_{j=1}^{n} (\widetilde{s}, \kappa_n(Y^j \partial_j)\, d(\mathrm{x}^{\psi} \circ \widetilde{s})(\partial_i)) + \sum_{j=1}^{n} (\widetilde{s}, \kappa_n((\partial_i Y^j)\partial_j)(\mathrm{x}^{\psi} \circ \widetilde{s})) \tag{3.125}$$

$$= \left(\widetilde{s}, \kappa_n\left(\sum_{j=1}^{n} Y^j \partial_j\right) d(\mathrm{x}^{\psi} \circ \widetilde{s})(\partial_i)\right) + \left(\widetilde{s}, \kappa_n\left(\sum_{j=1}^{n} (\partial_i Y^j)\partial_j\right)(\mathrm{x}^{\psi} \circ \widetilde{s})\right) \tag{3.126}$$

$$= (\widetilde{s}, \kappa_n(Y)\, d(\mathrm{x}^{\psi} \circ \widetilde{s})(\partial_i)) + (\widetilde{s}, \kappa_n(\nabla^{LC}_{\partial_i} Y)(\mathrm{x}^{\psi} \circ \widetilde{s})) \tag{3.127}$$

$$= \kappa_n(Y)\nabla^{\Sigma_n}_{\partial_i}\psi + \kappa_n(\nabla^{LC}_{\partial_i}Y)\psi = \text{RHS}. \quad \square \tag{3.128}$$

From now on when there is no ambiguity and to avoid notation clutter, we shall write $\kappa_n(X)\psi := X \cdot \psi$ for $X \in \mathfrak{X}(M)$ and $\psi \in \Gamma(\Sigma_n)$. Since Σ_n is a Clifford bundle over the spin manifold M, it possesses a Dirac operator $D : \Gamma(\Sigma_n) \to \Gamma(\Sigma_n)$. When $n = 2k$ [9], the spin representation decomposes into two irreducible representations
$\kappa = \kappa^{\pm} : \text{Spin}(n) \to \text{SU}(\Delta^+_{2k} \oplus \Delta^-_{2k})$ [9]. Thus when the spin manifold M is even dimensional, the spinor bundle Σ_n decomposes correspondingly into $\Sigma^+_n \oplus \Sigma^-_n$, i.e, Σ_n is $\mathbb{Z}/2-$graded; and it is not hard to see that D is an odd operator.

Let E be any $U(m)$ vector bundle over M of rank m. We always have a canonical scalar product that varies smoothly across the fibers of E. In the presence of this auxiliary bundle, one can define another Clifford bundle structure over M, $\Sigma_n \otimes E \to M$. We shall often refer to this bundle as *twisted spinor bundle*. To see why this is true, we shall divide the problem into 3 parts.

Part 1. Since both Σ_n and E have canonical scalar products, the bundle tensor gives a natural scalar product. In particular, if $\psi_j \otimes \phi_j \in C^\infty(\mathfrak{s} \otimes E)$ are twisted spinors $(j = 1, 2)$, then

$$\langle \psi_1 \otimes \phi_1, \psi_2 \otimes \phi_2 \rangle := \langle \psi_1, \psi_2 \rangle_{\mathfrak{s}} \langle \phi_1, \phi_2 \rangle_E.$$

By construction, the above scalar product varies smoothly across the fibers of $\Sigma_n \otimes E$.

Part 2. The left Clifford multiplication by the covectors to sections of $\Sigma_n \otimes E$ is borrowed from κ. Specifically, we consider the homomorphism $\kappa \otimes \mathbf{1}_E : T^*X \to \text{End}(\Sigma_n \otimes E)$, or equivalently,

$$\kappa := \rho \otimes \mathbf{1}_E : T^*X \otimes \Sigma_n \otimes E \to \Sigma_n \otimes E$$

so that for every $v \in \Gamma(T^*X)$ and twisted spinor $\psi \otimes \phi$, we have

[9]When $n = 2k+1$, there is no decomposition in the spin representation. In fact, $\kappa : \text{Spin}(n) \to \text{SU}(\Delta_{2k+1})$ is irreducible.

$\kappa(v)(\psi \otimes \phi) = (\kappa(v)\psi) \otimes \phi$. Direct calculations still give $\rho(v)^2 = -g(v,v)\mathbf{1}_{\mathfrak{s}\otimes E}$. Furthermore to check the skew-adjoint-ness of ρ with the scalar product defined in Part 1, we take two arbitrary twisted spinors $\psi_j \otimes \phi_j$, and compute

$$\langle \rho(v)\psi_1 \otimes \phi_1, \psi_2 \otimes \phi_2 \rangle = \langle \rho(v)\psi_1, \psi_2 \rangle_{\mathfrak{s}} \langle \phi_1, \phi_2 \rangle_E = -\langle \psi_1, \rho(v)\psi_2 \rangle_{\mathfrak{s}} \langle \phi_1, \phi_2 \rangle_E. \tag{3.129}$$

The right hand side of (3.129) is exactly $-\langle \psi_1 \otimes \phi_1, \rho(v)\psi_2 \otimes \phi_2 \rangle$.

Part 3. We need a Clifford bundle connection of $\Sigma \otimes E$. Constructing one is not difficult. Let ∇_A be any unitary connection of E and consider $\nabla_A := \nabla^{\Sigma_n} \otimes \mathbf{1}_E + \mathbf{1}_{\Sigma_n} \otimes \nabla_A$. We shall show that ∇_A is a compatible connection with respect to the two structures defined in the two previous parts. For the compatibility with the scalar product, let $\psi_j \otimes \phi_j$ be any twisted spinors, we have

$$\langle \nabla_A(\psi_1 \otimes \phi_1), \psi_2 \otimes \phi_2 \rangle = \langle \nabla^{\Sigma_n}\psi_1 \otimes \phi_1 + \psi_1 \otimes \nabla_A \phi_1, \psi_2 \otimes \phi_2 \rangle \tag{3.130}$$

$$= \langle \nabla^{\Sigma_n}\psi_1, \psi_2 \rangle_{\Sigma_n} \langle \phi_1, \phi_2 \rangle_E + \langle \psi_1, \psi_2 \rangle_{\Sigma_n} \langle \nabla_A \phi_1, \phi_2 \rangle_E \tag{3.131}$$

$$= \mathrm{d}\langle \psi_1, \psi_2 \rangle_{\Sigma_n} \langle \phi_1, \phi_2 \rangle_E + \langle \psi_1, \psi_2 \rangle_{\Sigma_n} \mathrm{d}\langle \phi_1, \phi_2 \rangle_E. \tag{3.132}$$

The last equation is exactly $\mathrm{d}\langle \psi_1 \otimes \phi_1, \psi_2 \otimes \phi_2 \rangle$, where d is the usual exterior derivative on X. Finally to verify the compatibility with the Levi–Civita connection, let v, w be vector fields on X and $\psi \otimes \phi$ be a twisted spinor. Then

$$\nabla_v(\kappa(w)\psi \otimes \phi) = \nabla_v^{\Sigma_n}(\kappa(w)\psi) \otimes \phi + \kappa(w)\psi \otimes \nabla_{Av}\phi \tag{3.133}$$

$$= \rho(\nabla_v^{LC} w)\psi \otimes \phi + \rho(w)(\nabla_v^{\Sigma_n}\psi \otimes \phi + \psi \otimes \nabla_{Av}\phi). \tag{3.134}$$

The last equality simplified gives $\kappa(\nabla_v^{LC} w)\psi \otimes \phi + \kappa(w)\nabla_{Av}(\psi \otimes \phi)$.

Because of the Clifford bundle structure of $\Sigma_n \otimes E$, there is also a globally defined Dirac operator D_A associated to the twisted spinor bundle. And just as the untwisted case,

D_A is defined by the composition

$$\Gamma(\Sigma_n \otimes E) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes \Sigma_n \otimes E) \xrightarrow{\kappa} \Gamma(\Sigma_n \otimes E).$$

We often refer to D_A as the twisted Dirac operator. When n is even, the $\mathbb{Z}/2-$grading of Σ_n also makes the twisted spinor bundle $\mathbb{Z}/2-$graded, $\Sigma_n \otimes E = \Sigma_n^+ \otimes E \oplus \Sigma_n^- \otimes E$. And D_A also respects the splitting, $D_A^{\pm} : \Gamma(\Sigma_n^{\pm} \otimes E) \to \Gamma(\Sigma_n^{\mp} \otimes E)$. Sections of the bundle $\Sigma_n^{\pm} \otimes E$ will be denoted by *positive (negative) twisted spinors*. Note that the rank of $\Sigma_n^{\pm} \otimes E$ is exactly $2^{n/2-1}$. If E is a line bundle, $\Sigma_n \otimes E$ is the *spin*^{c} bundle over the spin manifold M.

4 Analysis of Dirac operator

The majority content of this section is based on Chapter 3 of [17], Chapter 5 of [21], Chapter 1 of [11].

4.1 General properties

Throughout this section, we assume M is an $n-$dimensional compact Riemannian spin manifold without boundary. Let Σ_n be a fixed spinor bundle over M, and with it we have the spinor Dirac operator D. We view D as a differential operator from $\Gamma(\Sigma_n) \to \Gamma(\Sigma_n)$. One can impose different Hermitian inner products on the linear space $\Gamma(\Sigma_n)$, the following are the two we have encountered

A. $(\, , \,)$ is a local inner product that is smoothly varying across the fibers of Σ_n induced from the natural Hermitian inner product on the complex $n-$spinors Δ_n.

B. The global L^2-inner product defined by $\langle \, , \, \rangle_{L^2} = \int_M (\, , \,) \, \mathrm{vol}$.

Since Σ_n is a Clifford bundle, D has the Weitzenbock formula $D^2 = \nabla^{\Sigma_n *} \nabla^{\Sigma_n} + \mathbf{K}^{\Sigma_n}$ and thus it enjoys all of the L^2-properties we have proved previously, namely $\mathbf{K}^{\Sigma_n}$, D^2 and D

are L^2-self-adjoint. Note that the global L^2-inner product gives a norm on $\Gamma(\Sigma_n)$ defined by $||\psi||_{L^2}^2 = \int_M (\psi, \psi)\,\mathrm{vol}$, which makes $\Gamma(\Sigma_n)$ a normed linear space over $\mathbb{C}$. The completion of $\Gamma(\Sigma_n)$ with respect to $||\cdot||_{L^2}$ denotes the Banach space $L^2(\Sigma_n)$ equipped now with the induced norm $||\cdot||_2$. It is not hard to see that $||\cdot||_2$ satisfies the parallelogram law. Therefore, $L^2(\Sigma_n)$ is also a Hilbert space equipped with a Hermitian inner product satisfying $\langle\psi, \psi\rangle_2 = ||\psi||_2^2$. One of the questions we are interested in is to what extent D remains self-adjoint when $\Gamma(\Sigma_n)$ is replaced with $L^2(\Sigma_n)$. To answer this, we need some general theory about elliptic operators. Our setting is the following, suppose E is a certain vector bundle over M equipped with a metric and a connection ∇ that is both compatible with the Levi-Civita connection and the metric on E. Like before, we view $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ as a first order covariant derivative in the direction of a vector field X on M. Apply ∇ k times in k directions to obtain a $k^{th}-$order covariant derivative $\nabla^k\psi$, which is a smooth section of the bundle $(\otimes_{j=1}^k T^*M) \otimes E$ for $\psi \in \Gamma(E)$. Using the multi-index notation, we define

Definition 4.1. A *differential operator* $\mathcal{P}$ *on* M *of order* k is linear map from $\Gamma(E) \to \Gamma(E)$ such that for every $m \in M$ and a local trivialization $U \times \mathbb{F}^r = E|_U$, $\mathcal{P}$ can be written locally as

$$\mathcal{P} = \sum_{|\alpha| \le k} P_\alpha \nabla^{|\alpha|}_{\otimes_{j=1}^{\alpha_1}\partial_1 \otimes \cdots \otimes_{j=1}^{\alpha_n}\partial_n}.$$

Here $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ and P_α is a certain $r \times r-$matrix whose entries are smooth $\mathbb{F}-$valued functions defined on U. In other words, $P_\alpha \in \mathrm{End}(E)$. And we do require that $P_\alpha \neq 0$ for some α.

Definition 4.2. Let $\omega \in \Omega^1(M) = \Gamma(T^*M)$ and $\mathcal{P}$ be a differential operator as in Definition 4.1. On a trivialization U of E given in Definition 4.1, locally $\omega = \sum_j \omega^j dx_j$.

The *total symbol* of $\mathcal{P}$ with respect to ω is a bundle map $\sigma_{\mathcal{P}}(\omega) \in \text{End}(E)$ given by

$$\sigma_{\mathcal{P}}(\omega) = \sum_{|\alpha| \leq k} i^{|\alpha|} \omega^{\alpha} P_{\alpha} : E \to E, \quad e \mapsto \sigma_{\mathcal{P}}(\omega)(e) = \sum_{|\alpha| \leq k} i^{|\alpha|} \omega^{\alpha} P_{\alpha}\, e.$$

Here $\omega^{\alpha} = (\omega^1)^{\alpha_1} \cdots (\omega^n)^{\alpha_n}$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$. The *principal symbol* of $\mathcal{P}$ is the total symbol restricted to the highest order, i.e,

$$\sigma_{\mathcal{P}}^k(\omega) := \sigma_{\mathcal{P}}(\omega)|_{|\alpha|=k} = i^k \sum_{|\alpha|=k} \omega^{\alpha} P_{\alpha} : E \to E, \quad e \mapsto \sigma_{\mathcal{P}}^k(\omega)(e) = i^k \sum_{|\alpha|=k} \omega^{\alpha} P_{\alpha}\, e.$$

We then can view symbols of linear differential operators as polynomials in T^*M and out-puts are vector-bundle valued. The scalar multiplication by $i^{|\alpha|}$ is by convention so that if $\mathcal{P}$ is L^2-adjoint, so is its principal symbols. One should note that local formulas of $\sigma_{\mathcal{P}}$ and $\sigma_{\mathcal{P}}^k$ are independent of the choices of local trivialization and coordinate charts.

Definition 4.3. Let $\mathcal{P} : \Gamma(E) \to \Gamma(E)$ be a differential operator of order k. We say $\mathcal{P}$ is *elliptic* if and only if for each non-trivial covector field ω ($1-$form) on M and at each $m \in M$, $\sigma_{\mathcal{P}}^k(\omega)_m : E_m \to E_m$ is an isomorphism. In other words, $\mathcal{P}$ is elliptic if and only if $\sigma_{\mathcal{P}}^k : \Omega^1(M) \to \text{GL}(E)$.

In our context, the spinor Dirac operator D is an honest first order differential operator. The local formula of D has already been established, here we assume that $\{\partial_1, \cdots, \partial_n\}$ is an oriented synchronous orthonormal frame field around $m \in M$ and $(x_1, \cdots, x_n)$ is the corresponding normal coordinate. Suppose ω is an arbitrary non-trivial covector field on M so that we can write $\omega = \sum_{j=1}^n \omega^j dx_j$. Thus the principal symbol of D, which coincides with the total symbol, is

$$\sigma_D(\omega) = \sigma_D^1(\omega) = i \sum_{j=1}^{n} \omega^j \kappa_n(\partial_j) = i \sum_{j=1}^{n} \kappa_n(\omega^j \partial_j) = i\kappa_n \left(\sum_{j=1}^{n} \omega^j \partial_j \right) = i\kappa_n(\omega^{\#}), \qquad (4.1)$$

where $\omega^{\#}$ is the corresponding vector field of ω via the canonical identification

$T^*M \cong TM$, $\langle \omega^\#, X\rangle = \omega(X)$ for any vector field X on M. In short, $\sigma^1_D(\omega) = i\omega^\# \cdot$, the operation $\cdot$ signifies the left Clifford multiplication by vector fields. Note that $\sigma^1_D(\omega)\sigma^1_D(\omega) = i^2\omega^\# \cdot \omega^\# \cdot = ||\omega^\#||^2 \mathrm{Id}_{\Sigma_n}$. This shows exactly that D is an elliptic operator. Summarizing the discussion, we obtain

Proposition 4.1. *The spinor Dirac operator $D : \Gamma(\Sigma_n) \to \Gamma(\Sigma_n)$ is an elliptic first order differential operator. The principal symbol $\sigma_D(\omega)$ is an L^2-self-adjoint operator.* □

Remark 6. Similar calculations would show that any generalized Dirac operator is a first order elliptic operator. In particular, the Hodge-Dirac operator $d + d^*$ is elliptic.

To say anything more about elliptic operators, we need the language of Sobolev spaces and various related things. Note that the metrics on M and Σ_n induce a natural metric on $(\otimes^k_{j=1} T^*M) \otimes \Sigma_n$. Then on $\Gamma(\Sigma_n)$, we can define the *basic Sobolev $k-$inner product* as follow, for each $\psi, \xi \in \Gamma(\Sigma_n)$,

$$\langle \psi, \xi\rangle_{L^2_k} := \int_M (\psi, \xi) + (\nabla^{\Sigma_n}\psi, \nabla^{\Sigma_n}\xi) + \cdots + ((\nabla^{\Sigma_n})^k\psi, (\nabla^{\Sigma_n})^k\xi)\,\mathrm{vol}.$$

Thus the *basic Sobolev $k-$norm* on $\Gamma(\Sigma_n)$ is

$$||\psi||^2_{L^2_k} = \int_M \sum_{j=0}^{k} ((\nabla^{\Sigma_n})^j\psi, (\nabla^{\Sigma_n})^j\psi)\,\mathrm{vol}.$$

Again, $\Gamma(\Sigma_n)$ is a a normed linear space over $\mathbb{C}$ with respect to $||\cdot||_{L^2_k}$. The completion of $\Gamma(\Sigma_n)$ with respect to the basic Sobolev $k-$norm gives us a Banach space $L^2_k(\Sigma_n)$, where the induced norm is now just called the *Sobolev $k-$norm* $||\cdot||_{2,k}$. We call $L^2_k(\Sigma_n)$ the *Sobolev space.* And since $||\cdot||_{2,k}$ satisfies the parallelogram laws, $L^2_k(\Sigma_n)$ is also a Hilbert space equipped with an Hermitian inner-product $\langle\ ,\ \rangle_{2,k}$ satisfying $\langle\psi,\psi\rangle_{2,k} = ||\psi||^2_{2,k}$. When $k = 0$, it is not hard to see that $L^2_0(\Sigma_n)$ is isometric to $L^2(\Sigma_n)$; we shall use them interchangeably from time to time. Furthermore if $k \le k'$, then $||\psi||_{2,k} \le ||\psi||_{2,k'}$. This means that $H^{2,k'}(\Sigma_n)$ sits inside $L^2_k(\Sigma_n)$ naturally.

Similarly, one can also define the C^k-norm on $\Gamma(\Sigma_n)$. First, for each $\psi \in \Gamma(\Sigma_n)$, $||\psi||^2_{C^0} := \sup_M(\psi, \psi)$. Then for each k non-negative integer,

$$||\psi||^2_{C^k} := \max_{0\leq j\leq k}\{||\psi||^2_{C^0}, ||\nabla^{\Sigma_n}\psi||^2_{C^0}, \cdots, ||(\nabla^{\Sigma_n})^k\psi||^2_{C^0}\}.$$

Observe that,

$$||\psi||^2_{2,k} = \sum_{j=0}^{k}\int_M ((\nabla^{\Sigma_n})^j\psi, (\nabla^{\Sigma_n})^j\psi)\,\mathrm{vol} \leq \sum_{j=0}^{k}\int_M ||(\nabla^{\Sigma_n})^j\psi||^2_{C^0}\,\mathrm{vol} \tag{4.2}$$

$$\leq \sum_{j=0}^{k}\int_M ||\psi||^2_{C^k}\,\mathrm{vol} = \sum_{j=0}^{k}||\psi||^2_{C^k}\mathrm{vol}(M) = \mathrm{vol}(M)\cdot(k+1)\cdot||\psi||^2_{C^k}. \tag{4.3}$$

Therefore, $||\psi||_{2,k} \leq \mathrm{vol}(M)^{1/2}\cdot(k+1)^{1/2}\cdot||\psi||_{C^k}$. There is a sort of converse to the inequality (4.3), which is the *Sobolev Embedding theorem.* The theorem and its proof will be discussed in the next subsection when the treatment of Sobolev spaces on sections of vector bundles is more fleshed out.

The goal is this subsection is to state the two following results:

Theorem 4.2. *If M is a compact Riemannian spin manifold without boundary of dimension n and Σ_n is a fixed spinor bundle over M, then the associated spinor Dirac operator D is an unbounded operator with respect to the Hilbert space $L^2(\Sigma_n)$. Furthermore, D is closable with respect to $L^2(\Sigma_n)$. In the case M is complete, D is essentially self-adjoint.*

Theorem 4.3. *Let M be a compact Riemannian spin manifold without boundary of dimension n and Spec(D) be the spectrum of the spinor Dirac operator associated with a fixed spinor bundle Σ_n. Then the following is true:*

1. *Spec(D) is a closed in $\mathbb{R}$ containing an unbounded discrete sequence of eigenvalues.*

2. *Each eigenspace E_λ of D is finite-dimensional and consists of smooth sections, i.e, $E_\lambda \subseteq \Gamma(\Sigma_n)$.*

3. *There is a Hilbert space direct sum decomposition*

$$L^2(\Sigma_n) = \bigoplus_{\lambda \in Spec(D)} E_\lambda.$$

4. *$Spec(D)$ is unbounded on both sides of $\mathbb{R}$ and if $n \not\equiv 3 \mod 4$, then it is symmetric about the origin.*

Remark 7. The same statements in Theorem 4.2 and Theorem 4.3 (except for Theorem 4.3.4) can be made for any first order formally self-adjoint elliptic operator. In particular, they hold for generalized Dirac operators.

4.2 Sobolev spaces

In this subsection, we give a more thorough discussion of Sobolev spaces beyond the basic definitions introduced previously. We work in almost generality, assume the setting of previous subsection and insist that the metric on E is a Hermitian one. Similar to the case of spinor bundle Σ_n, the notions of $L^2(E)$ and $L^2_k(E)$ and C^k-norm on $\Gamma(E)$ are well-defined. Furthermore, we also have $||\psi||_{2,k} \leq ||\psi||_{2,l}$ for $k \leq l$. Hence there is a natural inclusion

$$L^2_l(E) \hookrightarrow L^2_k(E).$$

Equipping and completing $\Gamma(E)$ with an equivalent norm $||\cdot||'_{2,k}$ does not change the inclusion above. Indeed, suppose for each k there are postive constants C_k and c_k such that $c_k||\psi||_{2,k} \leq ||\psi||'_{2,k} \leq C_k||\psi||_{2,k}$ for any $\psi \in \Gamma(E)$; and observe

$$||\psi||'_{2,k} \leq C_k||\psi||_{2,k} \leq C_k||\psi||_{2,l} = (C_k/c_l)c_l||\psi||_{2,l} \leq (C_k/c_l)||\psi||'_{2,l}.$$

It is rather difficult to obtain an in-equivalent Sobolev norm on $\Gamma(E)$. In fact, the Sobolev norm depends on a choice of metric of M, Hermitian metric of E, and compatible connection ∇ on E. If we change one of these "varying factors", say, a compatible connection, yet we still have

Proposition 4.4. *Suppose* $||\cdot||'_{2,k\geq 1}$ *is a Sobolev* $k-$*norm for a different compatible connection* ∇'. $||\cdot||'_{2,k}$ *is equivalent to* $||\cdot||_{2,k}$.

Proof. It is sufficient to show the equivalence in the case of basic Sobolev norm. Since ∇' is another connection on E, there is an $\mathcal{A} \in \Gamma(\mathrm{End}(E))$ such that $\nabla' = \nabla + \mathcal{A}$. Note that $m \mapsto ||\mathcal{A}(m)||$ is a smooth function on compact M, thus it achieves a supremum; which we will also denote by $||\mathcal{A}||$. Then for any $\psi \in \Gamma(E)$, we have $||\mathcal{A}\,\psi||_{L^2} \leq ||\mathcal{A}|| \cdot ||\psi||_{L^2}$. We proceed with induction.

In the case $k = 1$, by the Cauchy-Schwarz inequality,

$$\langle \nabla'\psi, \nabla'\psi\rangle_{L^2} = ||\nabla\psi||^2_{L^2} + 2\Re\langle\nabla\psi, \mathcal{A}\psi\rangle_{L^2} + ||\mathcal{A}\psi||^2_{L^2} \tag{4.4}$$

$$\leq ||\nabla\psi||^2_{L^2} + 2||\mathcal{A}||\,||\nabla\psi||_{L^2}||\psi||_{L^2} + ||\mathcal{A}||^2||\psi||^2_{L^2} \tag{4.5}$$

$$\leq ||\nabla\psi||^2_{L^2} + ||\mathcal{A}||(||\psi||^2_{L^2} + ||\nabla\psi||^2_{L^2}) + ||\mathcal{A}||^2||\psi||^2_{L^2} \tag{4.6}$$

$$= ||\nabla\psi||^2_{L^2} + ||\mathcal{A}|| \cdot ||\psi||^2_{2,1} + ||\mathcal{A}||^2||\psi||^2_{L^2}. \tag{4.7}$$

Thus we obtain,

$$||\psi||'^2_{2,1} = ||\psi||^2_{L^2} + ||\nabla'\psi||^2_{L^2} \leq ||\psi||^2 + ||\nabla\psi||^2_{L^2} + ||\mathcal{A}|| \cdot ||\psi||^2_{2,1} + ||\mathcal{A}||^2||\psi||^2_{L^2} \tag{4.8}$$

$$\leq ||\psi||^2_{2,1} + ||\mathcal{A}|| \cdot ||\psi||^2_{2,1} + ||\mathcal{A}||^2||\psi||^2_{2,1} \tag{4.9}$$

$$= (1 + ||\mathcal{A}|| + ||\mathcal{A}||^2) \cdot ||\psi||^2_{2,1}. \tag{4.10}$$

(4.4) – (4.10) can be replicated when we swap ∇' with ∇. As a result,

$$(1 + ||\mathcal{A}|| + ||\mathcal{A}||^2)^{-1/2} \cdot ||\psi||_{2,1} \leq ||\psi|'_{2,1} \leq (1 + ||\mathcal{A}|| + ||\mathcal{A}||^2)^{1/2} \cdot ||\psi||_{2,1}.$$

Assume the two Sobolev norms are equivalent up-to k, that is, for each $1 \leq j \leq k$ there are

C_j and c_j positive and finite such that $c_j||\psi||_{L^2_j} \leq ||\psi||'_{L^2_j} \leq C_j||\psi||_{L^2_j}$. Consider

$$||\psi||'^2_{2,k+1} = \sum_{j=0}^{k+1} ||\nabla'^j \psi||^2_{L^2} = ||\psi||'^2_{L^2_k} + ||\nabla'^{k+1}\psi||^2_{L^2}.$$

Now by Cauchy-Schwarz inequality and the induction hypothesis, we have

$$||\nabla'^{k+1}\psi||^2_{L^2} = ||\nabla'^k \nabla\psi + \nabla'^k \mathcal{A}\psi||^2_{L^2} \leq 2||\nabla'^k \nabla\psi||^2_{L^2} + 2||\nabla'^k \mathcal{A}\psi||^2_{L^2} \tag{4.11}$$

$$\leq 2||\nabla\psi||'^2_{L^2_k} + 2||\mathcal{A}\psi||'^2_{L^2_k} \leq 2C_k^2||\nabla\psi||^2_{L^2_k} + 2C_k^2||\mathcal{A}\psi||^2_{L^2_k} \tag{4.12}$$

$$\leq 2C_k^2||\nabla\psi||^2_{L^2_k} + 2C_k^2 C||\psi||^2_{L^2_k}. \tag{4.13}$$

Therefore, we obtain

$$||\psi||'^2_{2,k+1} \leq C_k^2||\psi||^2_{L^2_k} + 2C_k^2||\nabla\psi||^2_{L^2_k} + 2CC_k^2||\psi||^2_{H^k} \tag{4.14}$$

$$\leq C_k^2||\psi||^2_{2,k+1} + 2C_k^2||\psi||^2_{2,k+1} + 2CC_k^2||\psi||^2_{2,k+1} = (3C_k^2 + 2CC_k^2)||\psi||^2_{k+1}. \tag{4.15}$$

Similarly, we can also show $||\psi||_{2,k+1} \lesssim ||\psi||'_{2,k+1}$. So the equivalency is also true at $k+1$. And our proof by induction is complete. □

In the proof above, at (4.12) we assume the following result, i.e, if $T : E \to E$ is a bundle map, then it extends to a bounded operator from $L^2_k(E) \to L^2_k(E)$. The proof of such assertion is independent of the proof Proposition 4.4, so nothing should be logically inconsistent. Furthermore, the mentioned assertion is part of an important property about the extension of differential operator of finite order to a bounded operator between Sobolev spaces. We shall give a detailed discussion of this in the form of the following proposition.

Proposition 4.5.

1. *The inclusion* $L^2_l(E) \hookrightarrow L^2_k(E)$ *is a bounded operator for* $k \leq l$.

2. $\nabla : L^2_k(E) \to L^2_{k-1}(E)$ *is bounded.*

3. *If $T : E \to E$ is a bundle map, then T extends to a bounded operator from $L^2_k(E) \to L^2_k(E)$ for any k.*

4. *If $\mathcal{P} : \Gamma(E) \to \Gamma(E)$ is a differential operator of order k, then $\mathcal{P}$ extends to a bounded operator from $L^2_l(E) \to L^2_{l-k}(E)$.*

Proof. If $\psi \in L^2_l(E)$, then by definition $||\psi||_{2,k} \leq ||\psi||_{2,l} < \infty$. On the other hand, if $\psi \in L^2_k(E)$ and without loss of generality $\psi \in \Gamma(E)$, then

$$||\nabla\psi||^2_{L^2_{k-1}} = ||\nabla\psi||^2_{L^2} + ||\nabla^2\psi||^2_{L^2} + \cdots + ||\nabla^k\psi||^2_{L^2} \leq ||\psi||^2_{L^2} + ||\nabla\psi||^2_{L^2} + \cdots + ||\nabla^k\psi||^2_{L^2},$$

which implies that $||\nabla\psi||_{L^2_{k-1}} \leq ||\psi||_{L^2_k}$. These two observations immediately imply (4.5.1) and (4.5.2). For (4.5.3), we need to show that there exists an C positive and finite such that $||T\psi||_{2,k} \leq C||\psi||_{2,k}$. We proceed with induction on k, and simultaneously we would like to show that each $k \geq 1$, $||[T, \nabla^k]\psi||_2 = ||[T, \nabla^k]\psi||_{2,0} \leq C'||\psi||_{2,k-1}$. The case $k = 0$ is when $L^2_0(E)$ isometric to $L^2(E)$, same argument as the beginning of the proof of Proposition 4.4 yields the result. Now assume that for each $1 \leq j \leq k$, there are positive and finite C_j and C'_j such that $||T\psi||_{2,j} \leq C_j||\psi||_{2,j}$ and $||[T, \nabla^j]\psi||_2 \leq C'_j||\psi||_{2,j-1}$. Note that

$$[T, \nabla^{k+1}] = [T, \nabla]\nabla^k + \nabla[T, \nabla]\nabla^{k-1} + \nabla^2[T, \nabla]\nabla^{k-2} + \cdots + \nabla^k[T, \nabla]. \tag{4.16}$$

$[T, \nabla] : L^2_0(E) = L^2(E) \to L^2(E) = L^2_0(E)$ is a bounded operator, in particular when restricted to E, $[T, \nabla]$ is a bundle map. Thus by the induction hypothesis, $[T, \nabla] : L^2_j(E) \to L^2_j(E)$ is a bounded operator for every $0 \leq j \leq k$. As a result from (4.16) and (4.5.2), $[T, \nabla^{k+1}] : L^2_k(E) \to L^2_0(E) = L^2(E)$ is also bounded. Since $L^2_{k+1}(E)$ naturally sits inside $L^2_k(E)$, for each $\psi \in L^2_{k+1}(E)$ and by triangle inequality we have

$$||[\nabla^{k+1}T\psi||_2 \leq ||T\nabla^{k+1}\psi||_2 + ||[T, \nabla^{k+1}]\psi||_2 \tag{4.17}$$

$$\leq ||T|| \cdot ||\nabla^{k+1}\psi||_2 + C'_{k+1}||\psi||_{2,k} \leq \max\{||T||, C'_{k+1}\}||\psi||_{2,k+1}. \tag{4.18}$$

Thus with the induction hypothesis, we obtain

$$||T\psi||_{2,k+1} = ||T\psi||_{2,k} + ||\nabla^{k+1}T\psi||_2 \tag{4.19}$$

$$\leq C_k||\psi||_{2,k} + \max\{||T||, C'_{k+1}\}||\psi||_{2,k+1} \leq C_{k+1}||\psi||_{2,k+1}. \tag{4.20}$$

This completes the induction proof of (4.5.3). Lastly, (4.5.4) is an immediate consequence of (4.5.2) and (4.5.3). □

We now take a closer look of the inclusion $L^2_1(E) \hookrightarrow L^2_0(E) = L^2(E)$. Let $\mathcal{B}$ be the closed unit ball in $L^2_1(E)$. We have $\mathcal{B}$ sits naturally in $L^2(E)$ and consider the closure of $\mathcal{B}$ in $L^2(E)$, denoted by $\overline{\mathcal{B}}^{L^2}$. Obviously, $\mathcal{B} \subseteq \overline{\mathcal{B}}^{L^2}$. Furthermore, we claim that the reverse inclusion is also true. Indeed, let $\psi \in \overline{\mathcal{B}}^{L^2} = \mathcal{B} \cup \partial\mathcal{B}^{L^2}$. If $\psi \in B$, there is nothing to show. However, if $\psi \in \partial\mathcal{B}^{L^2}$, then there is a sequence $\{\psi_j\}$ in $\mathcal{B}$ such that $||\psi_j - \psi||_2 \to 0$ as j tends to infinity. Pick a coordinate patch on M so that we have local frame field $\{\partial_1, \cdots, \partial_n\}$. As usual, denote $g_{rs} := \langle \partial_r, \partial_s \rangle$. Consider

$$||\nabla(\psi_j - \psi)||_2^2 = \int_M \left(\sum_{r=1}^n dx_r \otimes \nabla_{\partial_r}(\psi_j - \psi), \sum_{s=1}^n dx_s \otimes \nabla_{\partial_s}(\psi_j - \psi) \right) \text{vol} \tag{4.21}$$

$$= \int_M \sum_{r,s=1}^n g_{rs}(\nabla_{\partial_r}(\psi_j - \psi), \nabla_{\partial_s}(\psi_j - \psi)) \,\text{vol} \tag{4.22}$$

$$= 2\sum_{r<s} \int_M g_{rs}\Re(\nabla_{\partial_r}(\psi_j - \psi), \nabla_{\partial_s}(\psi_j - \psi)) \,\text{vol} + \sum_{r=1}^n \int_M g_{rr}||\nabla_{\partial_r}(\psi_j - \psi)||^2 \,\text{vol}. \tag{4.23}$$

By compactness of M and the Cauchy-Schwarz inequality, we can estimate further

$$||\nabla(\psi_j - \psi)||_2^2 \leq C \left(\sum_{r<s} \int_M 2||\nabla_{\partial_r}(\psi_j - \psi)|| \cdot ||\nabla_{\partial_s}(\psi_j - \psi)|| \,\text{vol} + \sum_{r=1}^n \int_M ||\nabla_{\partial_r}(\psi_j - \psi)||^2 \,\text{vol} \right) \tag{4.24}$$

$$\leq C \left(\sum_{r<s} \int_M (||\nabla_{\partial_r}(\psi_j - \psi)||^2 + ||\nabla_{\partial_s}(\psi_j - \psi)||^2) \operatorname{vol} + \sum_{r=1}^{n} \int_M ||\nabla_{\partial_r}(\psi_j - \psi)||^2 \operatorname{vol} \right) \tag{4.25}$$

$$\leq C' \sum_{r=1}^{n} \int_M ||\nabla_{\partial_r}(\psi_j - \psi)||^2 \operatorname{vol} = C' \sum_{r=1}^{n} ||\nabla_{\partial_r}(\psi_j - \psi)||_2^2. \tag{4.26}$$

Now in general if $\varphi \in L^2(E)$, we can write $\nabla_{\partial_r}\varphi = \partial_r\varphi + \Gamma_r(\varphi)$, where the first term is understood as taking derivative in the direction of ∂_r of each component function of φ and Γ_r is a bundle map from $E \to E$ constructed from the Christoffel symbols of ∇. Then $||\partial_r\varphi + \Gamma_r(\varphi)||_2 \leq ||\partial_r\varphi||_2 + ||\Gamma_r(\varphi)||_2$. By L^2-theory of compact spaces, ∂_r is a bounded operator from $L^2(E) \to L^2(E)$ and by Proposition 4.5.3, Γ_r is also bounded from $L^2(E) \to L^2(E)$. Thus ∇_{∂_r} is bounded on $L^2(E)$. Therefore combining with (166), we arrive at $||\nabla(\psi_j - \psi)||_2^2 \leq C''||\psi_j - \psi||_2^2$. And as $j \to \infty$, we obtain $||\nabla(\psi_j - \psi)||_2 \to 0$. Since $||\psi_j - \psi||_{2,1}^2 = ||\psi_j - \psi||_2^2 + ||\nabla(\psi_j - \psi)||_2^2$, it is also true that $\psi_j \to \psi$ in $L_1^2(E)$. By triangle inequality, $||\psi||_{2,1} \leq ||\psi_j - \psi||_{2,1} + ||\psi_j||_{2,1} \leq ||\psi_j - \psi||_{2,1} + 1$. Let $j \to \infty$, we yield $||\psi||_{2,1} \leq 1$. We summarize the discussion above in the following lemma

Lemma 4.6. *If $\mathcal{B}$ is the closed unit ball in $L_1^2(E)$, then $\mathcal{B} = \overline{\mathcal{B}}^{L^2}$.* $\square$

Let $\{U_\alpha\}$ be a finite open cover of M that trivializes E and $\{\rho_\alpha\}$ be the partition of unity subordinated to $\{U_\alpha\}$. If ψ is a smooth section of E that also belongs to $L_1^2(E)$, $\rho_\alpha\psi$ is considered to be a $\mathbb{C}^r-$valued smooth function with compact support on U_α. Thus if we would like show a certain property of ψ, it is enough to show that such property holds for every $\mathbb{C}^r-$valued smooth function $\rho_\alpha\psi$ with compact support in U_α. Furthermore when choosing coordinate charts to live on $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, we reduce the study of $L_1^2(E)$ to the study of smooth functions in $L_1^2(\mathbb{T}^n, \mathbb{C}^r)$ with compact supports.

Integration of a vector-valued smooth function is integrating component-wise with respect to a basis of a (Hermitian) inner product vector space. It is not hard to show that such definition of integration is well-defined regardless of choices of basis. With this, we can define the *Fourier transform of vector-valued smooth functions* by Fourier transforming each of their component functions. In particular for our context, suppose $\{\sigma_1, \cdots, \sigma_r\}$ is an orthonormal basis with respect to $\mathbb{C}^r$, any smooth function ψ in $L_1^2(\mathbb{T}^n, \mathbb{C}^r)$ with compact support can be written as

$$\psi(x) = \sum_{j=1}^{r} \psi^j(x)\sigma_j,$$

where $\psi^j \in \Gamma(\mathbb{T}^n, \mathbb{C})$ with compact support. For each $\nu \in \mathbb{Z}^n$, we define $\widehat{\psi}(\nu)$ as following

$$\widehat{\psi}(\nu) := \sum_{j=1}^{r} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \psi^j(x) e^{-i\nu\cdot x}\, dx \right) \sigma_j = \sum_{j=1}^{r} \widehat{\psi^j}(\nu)\sigma_j. \tag{4.27}$$

As a result, $||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r} = \sum_{j=1}^r |\widehat{\psi^j}(\nu)|^2$, which implies that

$$\sum_{\nu\in\mathbb{Z}^n} ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r} = \sum_{\nu\in\mathbb{Z}^n} \sum_{j=1}^{r} |\widehat{\psi^j}(\nu)|^2. \tag{4.28}$$

Since $L_1^2(\mathbb{T}^n, \mathbb{C}^r)$ sits naturally in $L_0^2(\mathbb{T}^n, \mathbb{C}^r) = L^2(\mathbb{T}^n, \mathbb{C}^r)$, $\psi \in L^2(\mathbb{T}^n, \mathbb{C}^r)$. By basic theory of Hilbert spaces and (4.28), we have

$$||\psi||_2^2 = \int_{\mathbb{T}^n} ||\psi(x)||^2_{\mathbb{C}^r}\, dx = \int_{\mathbb{T}^n} \sum_{j=1}^{r} |\psi^j(x)|^2\, dx = \sum_{j=1}^{r} \int_{\mathbb{T}^n} |\psi^j(x)|^2\, dx \tag{4.29}$$

$$= \sum_{j=1}^{r} (2\pi)^n \sum_{\nu\in\mathbb{Z}^n} |\widehat{\psi^j}(\nu)|^2 = (2\pi)^n \sum_{\nu\in\mathbb{Z}^n} ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r}. \tag{4.30}$$

The covariant derivative ∇ on $\Gamma(\mathbb{T}^n, \mathbb{C}^r)$ can be taken as directional derivative of each component functions. Thus we can write

$$\nabla\psi(x) = \sum_{j=1}^{r} \left(\sum_{s=1}^{n} dx_s \otimes (\partial_s \psi^j) \right)\Bigg|_x \sigma_j. \tag{4.31}$$

Therefore by Hilbert spaces theory, similarly we obtain

$$||\nabla\psi||_2^2 = \int_{\mathbb{T}^n} ||\nabla\psi(x)||^2_{\mathbb{C}^r}\,dx = \int_{\mathbb{T}^n} \sum_{j=1}^r \left|\sum_{s=1}^n dx_s \otimes (\partial_s\psi^j)\right|_x^2 dx \tag{4.32}$$

$$= \sum_{j=1}^r \int_{\mathbb{T}^n} \left\langle \sum_{s=1}^n dx_s \otimes (\partial_s\psi^j), \sum_{k=1}^n dx_k \otimes (\partial_k\psi^j) \right\rangle\Big|_x dx \tag{4.33}$$

$$= \sum_{j=1}^r \int_{\mathbb{T}^n} \sum_{s,k=1}^n g_{sk}(x)(\partial_s\psi^j(x))(\overline{\partial_k\psi^j(x)})\,dx \tag{4.34}$$

$$= \sum_{j=1}^r \int_{\mathbb{T}^n} \sum_{s=1}^n |\partial_s\psi^j(x)|^2\,dx = \sum_{j=1}^r\sum_{s=1}^n \int_{\mathbb{T}^n} |\partial_s\psi^j(x)|^2\,dx \tag{4.35}$$

$$= \sum_{j=1}^r\sum_{s=1}^n (2\pi)^n \sum_{\nu\in\mathbb{Z}^n} |\widehat{\partial_s\psi^j}(\nu)|^2 = \sum_{j=1}^r\sum_{s=1}^n (2\pi)^n \sum_{\nu\in\mathbb{Z}^n} |-i\nu_s\widehat{\psi^j}(\nu)|^2 \tag{4.36}$$

$$= (2\pi)^n \sum_{j=1}^r \sum_{\nu\in\mathbb{Z}^n} |\nu|^2|\widehat{\psi^j}(\nu)|^2 = (2\pi)^n \sum_{\nu\in\mathbb{Z}^n} |\nu|^2 \sum_{j=1}^r |\widehat{\psi^j}(\nu)|^2 = (2\pi)^n \sum_{\nu\in\mathbb{Z}^n} |\nu|^2 ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r}. \tag{4.37}$$

(4.30) and (4.37) combined yields for us a relation between the Sobolev 1−norm of ψ on $L^2_1(\mathbb{T}^n, \mathbb{C}^r)$ to Fourier coefficients $\widehat{\psi}(\nu)$'s of ψ

$$||\psi||^2_{2,1} = (2\pi)^n \sum_{\nu\in\mathbb{Z}^n} (1+|\nu|^2)||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r}. \tag{4.38}$$

Proposition 4.7. (Rellich) *The bounded inclusion $L^2_1(E) \hookrightarrow L^2(E)$ is compact.*

Proof. By the discussion above, we reduce the problem to showing the inclusion of $L^2_1(\mathbb{T}^n, \mathbb{C}^r)$ into $L^2(\mathbb{T}^n, \mathbb{C}^r)$ is compact. Denote $\mathcal{B}$ the closed unit ball in $L^2_1(\mathbb{T}^n, \mathbb{C}^r)$. By Lemma 4.6, we have $\mathcal{B} = \overline{\mathcal{B}}^{L^2(\mathbb{T}^n,\mathbb{C}^r)}$. Thus it is sufficient to show that for any arbitrary sequence $\{\psi_k\}$ of $\Gamma(\mathbb{T}^n, \mathbb{C}^r)$ such that $||\psi_k||_{2,1} \le 1$ and ψ_k's have compact supports on $\mathbb{T}^n$, there is convergent subsequence. We claim that there is a subsequence $\{\psi_{k_j}\}$ that is Cauchy in $L^2(\mathbb{T}^n, \mathbb{C}^r)$. And because $L^2(\mathbb{T}^n, \mathbb{C}^r)$ is complete; if such subsequence exits, it has to converge in $L^2(\mathbb{T}^n, \mathbb{C}^r)$.

Let $\nu, \eta \in \mathbb{Z}^n$, for any k we have $\widehat{\psi_k}(\nu) = \sum_{j=1}^{r} \widehat{\psi_k^j}(\nu)\sigma_j$, where ψ_k^j is smooth $\mathbb{C}^r-$valued function with compact supports on $\mathbb{T}^n$. By triangle inequality and Holder inequality, consider

$$||\widehat{\psi_k}(\nu) - \widehat{\psi_k}(\eta)||_{\mathbb{C}^r}^2 = \frac{1}{(2\pi)^{2n}} \sum_{j=1}^{r} \left| \int_{\mathbb{T}^n} \psi_k^j(x)(e^{-i\nu\cdot x} - e^{-i\eta\cdot x})\, dx \right|^2 \tag{4.39}$$

$$\leq \frac{1}{(2\pi)^{2n}} \sum_{j=1}^{r} \left(\int_{\mathbb{T}^n} |\psi_k^j(x)| \cdot |e^{-i\nu\cdot x} - e^{-i\eta\cdot x}|\, dx \right)^2 \tag{4.40}$$

$$\leq \frac{1}{(2\pi)^{2n}} \sum_{j=1}^{r} \left(\int_{\mathbb{T}^n} |\psi_k^j(x)|^2\, dx \right) \left(\int_{\mathbb{T}^n} |e^{-i\nu\cdot x} - e^{-i\eta\cdot x}|^2\, dx \right) \tag{4.41}$$

$$= \frac{1}{(2\pi)^{2n}} ||\psi_k||_2^2 \int_{\mathbb{T}^n} |e^{-i\nu\cdot x} - e^{-i\eta\cdot x}|^2\, dx \tag{4.42}$$

$$\leq \frac{1}{(2\pi)^{2n}} ||\psi_k||_{2,1}^2 \int_{\mathbb{T}^n} |e^{-i\nu\cdot x} - e^{-i\eta\cdot x}|^2\, dx \tag{4.43}$$

$$\leq \frac{1}{(2\pi)^{2n}} \int_{\mathbb{T}^n} |e^{-i\nu\cdot x} - e^{-i\eta\cdot x}|^2\, dx. \tag{4.44}$$

Note that $e^{-i\nu\cdot x} - e^{-i\eta\cdot x} = e^{-i\eta\cdot x}(e^{-i(\nu-\eta)\cdot x} - 1)$. As a result,

$$|e^{-i\nu\cdot x} - e^{-i\eta\cdot x}|^2 = |e^{-i(\nu-\eta)\cdot x} - 1|^2 = 2 - 2\cos((\eta - \nu)\cdot x). \tag{4.45}$$

By the inequality $1 - t^2/2 \leq \cos t$ for all $t \in \mathbb{R}$ and the Cauchy-Schwarz inequality, we estimate $(4, 45)$ from above

$$|e^{-i(\nu-\eta)\cdot x} - 1|^2 \leq ((\eta - \nu)\cdot x)^2 \leq |\eta - \nu|^2 |x|^2. \tag{4.46}$$

Therefore, the above estimate of (4.44) becomes

$$||\widehat{\psi_k}(\nu) - \widehat{\psi_k}(\eta)||_{\mathbb{C}^r}^2 \leq (2\pi)^{-2n} |\nu - \eta|^2 \int_{\mathbb{T}^n} |x|^2\, dx \leq C(2\pi)^{-2n} |\nu - \eta|^2, \tag{4.47}$$

for some positive and finite C. Let $\epsilon > 0$ arbitrarily, then whenever $|\nu - \eta| < C^{-1/2}(2\pi)^n \epsilon$.

Then (4.47) implies $||\widehat{\psi_k}(\nu) - \widehat{\psi_k}(\eta)||_{\mathbb{C}^r} < \sqrt{C}(2\pi)^{-n}C^{-1/2}(2\pi)^n\epsilon = \epsilon$. In particular, we have shown that $\{\widehat{\psi_k}\}$ is uniformly equicontinuous in $C(\mathbb{Z}^n, \mathbb{C}^r)$. Furthermore for any k and $\nu \in \mathbb{Z}^n$, by triangle inequality and Holder inequality, we have

$$||\widehat{\psi_k}(\nu)||^2_{\mathbb{C}^r} \leq \frac{1}{(2\pi)^n}\sum_{j=1}^{r}\int_{\mathbb{T}^n}|\psi_k^j(x)|^2\,dx = \frac{1}{(2\pi)^n}||\psi_k||_2^2 \leq \frac{1}{(2\pi)^n}||\psi_k||^2_{2,1} \leq (2\pi)^{-n}. \quad (4.48)$$

Thus $\{\widehat{\psi_k}\}$ is also uniformly bounded. By the Arzela-Ascoli theorem, there is a subsequence $\{\widehat{\psi_{k_j}}\}$ that is uniformly Cauchy on compact subsets of $\mathbb{Z}^n$.

For $N > 0$, by (4.29) we have

$$||\psi_{j_k} - \psi_{j_l}||_2^2 = (2\pi)^n\sum_{|\nu|>N}||\widehat{\psi_{j_k}}(\nu) - \widehat{\psi_{j_l}}(\nu)||^2_{\mathbb{C}^r} + (2\pi)^n\sum_{|\nu|\leq N}||\widehat{\psi_{j_k}}(\nu) - \widehat{\psi_{j_l}}(\nu)||^2_{\mathbb{C}^r}. \quad (4.49)$$

Now the first summation of (4.49), using (4.38), we estimate from above

$$(2\pi)^n\sum_{|\nu|>N}||\widehat{\psi_{j_k}}(\nu) - \widehat{\psi_{j_l}}(\nu)||^2_{\mathbb{C}^r} \leq (2\pi)^n\sum_{|\nu|>N}\frac{1+|\nu|^2}{1+N^2}||\widehat{\psi_{j_k}}(\nu) - \widehat{\psi_{j_l}}(\nu)||^2_{\mathbb{C}^r} \quad (4.50)$$

$$\leq \frac{||\psi_{j_k} - \psi_{j_l}||^2_{2,1}}{1+N^2} \leq \frac{4}{1+N^2}. \quad (4.51)$$

For the second summation of (4.49), let $\epsilon > 0$, using the uniformly Cauchy subsequence $\{\widehat{\psi_{j_k}}\}$, for k and l large enough (depending on N and ϵ), we have

$$(2\pi)^n\sum_{|\nu|\leq N}||\widehat{\psi_{j_k}}(\nu) - \widehat{\psi_{j_l}}(\nu)||^2_{\mathbb{C}^r} \leq (2\pi)^n\sum_{|\nu|\leq N}\frac{\epsilon^2}{2C'_N(2\pi)^n} = \frac{\epsilon^2}{2}, \quad (4.52)$$

where $C'_N = \sum_{|\nu|\leq N} 1$. To finish the argument, we simply choose N in (4.51) large enough (depending on ϵ) so that $4/(1+N^2) < \epsilon^2/2$. Therefore, for k and l large enough (depending only on ϵ), we have $||\psi_{j_k} - \psi_{j_l}||_2 < \epsilon$. This shows that $\{\psi_{j_k}\}$ is Cauchy in $L^2(\mathbb{T}^n, \mathbb{C}^r)$. With our beginning observation, the proof is complete. $\square$

(4.38) can be generalized to other Sobolev $k-$norms. In particular using induction, one can prove that for $\psi \in L^2_k(\mathbb{T}^n, \mathbb{C}^r)$ and ψ is smooth with compact support,

$$||\psi||^2_{2,k} = (2\pi)^n \sum_{\nu \in \mathbb{Z}^n} (1 + |\nu|^2 + |\nu|^4 + \cdots + |\nu|^{2k}) ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r}. \tag{4.53}$$

Thus using similar argument, the result can be extended to the following statement

Proposition 4.8. *For $l < k$, the inclusion $L^2_k(E) \hookrightarrow L^2_l(E)$ is compact.* □

Theorem 4.9. (Sobolev Embedding theorem) *If k and l are natural numbers such that $l > \frac{n}{2} + k$, then there is a positive finite constant c (depending on n, l, and k) such that*

$$||\psi||_{C^k} \leq c||\psi||_{2,l}, \text{ for } \psi \in \Gamma(E).$$

Thus the inclusion $L^2_l(E) \hookrightarrow C^k(E)$ is a bounded operator.

Proof. Again we reduce the problem to showing $||\psi||_{C^k} \leq c(l,k,n)||\psi||_{2,l}$ for $\psi \in \Gamma(\mathbb{T}^n, \mathbb{C}^r)$ with compact support. Note that by the Binomial theorem, for each $\nu \in \mathbb{Z}^n$ and positive integer l we have $(1 + |\nu|^2)^l = \sum_{j=0}^{l} \binom{l}{j} |\nu|^{2j}$. Let $c_1 := \min\{\binom{l}{j} : 0 \leq j \leq l\}$ and $c_2 := \max\{\binom{l}{j} : 0 \leq j \leq l\}$. Thus we obtain,

$$c_1(1 + |\nu|^2 + |\nu|^4 + \cdots + |\nu|^{2l}) \leq (1 + |\nu|^2)^l \leq c_2(1 + |\nu|^2 + |\nu|^4 + \cdots + |\nu|^{2l}).$$

This shows that the two following Sobolev $l-$norms are equivalent

$$(2\pi)^n \sum_{\nu \in \mathbb{Z}^n} (1 + |\nu|^2 + \cdots + |\nu|^{2l}) ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r}, \quad (2\pi)^n \sum_{\nu \in \mathbb{Z}^n} (1 + |\nu|^2)^l ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r}. \tag{4.54}$$

The later one will be used in this proof. We claim the following

$$||\nabla^k \psi(x)||^2_{\mathbb{C}^r} \leq \frac{C_{k,n}}{(2\pi)^n} \left(\sum_{\nu \in \mathbb{Z}^n} (1 + |\nu|^2)^{-l+k} \right) ||\psi||^2_{2,l}, \tag{4.55}$$

where $C_{k,n} := \sum_{|\alpha|=k} 1$. Note that $\nabla^k : \Gamma(\mathbb{T}^n, \mathbb{C}^r) \to \Gamma((\otimes_{j=1}^k T^*\mathbb{T}^n) \otimes \mathbb{C}^r)$, so for each $\psi \in \Gamma(\mathbb{T}^n, \mathbb{C}^r)$ with compact support, in coordinate, we can write

$$\nabla^k \psi = \sum_{|\alpha|=k} \otimes_{s=1}^n (dx_s)^{\otimes \alpha_s} \otimes \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}} \psi := \sum_{|\alpha|=k} \otimes_{s=1}^n (dx_s)^{\otimes \alpha_s} \otimes \partial^{|\alpha|} \psi. \tag{4.56}$$

As a result, the Fourier inversion formula, triangle inequality, and the Cauchy-Schwarz inequality yield us the above estimate

$$||\nabla^k \psi(x)||^2_{\mathbb{C}^r} = \left\langle \sum_{|\alpha|=k} \otimes_{s=1}^n (dx_s)^{\otimes \alpha_s} \otimes \partial^{|\alpha|}\psi, \sum_{|\beta|=k} \otimes_{j=1}^n (dx_j)^{\otimes \alpha_j} \otimes \partial^{|\beta|}\psi \right\rangle \Bigg|_x \tag{4.57}$$

$$= \sum_{|\alpha|=k} ||\partial^{|\alpha|}\psi(x)||^2_{\mathbb{C}^r} = \frac{1}{(2\pi)^{2n}} \sum_{|\alpha|=k} \left|\left| \sum_{\nu \in \mathbb{Z}^n} e^{i\nu \cdot x} \widehat{\partial^{|\alpha|}\psi}(\nu) \right|\right|^2_{\mathbb{C}^r} \tag{4.58}$$

$$\leq \frac{1}{(2\pi)^{2n}} \sum_{|\alpha|=k} \left(\sum_{\nu \in \mathbb{Z}^n} ||\widehat{\partial^{|\alpha|}\psi}(\nu)||_{\mathbb{C}^r} \right)^2 \tag{4.59}$$

$$= \frac{1}{(2\pi)^{2n}} \sum_{|\alpha|=k} \left(\sum_{\nu \in \mathbb{Z}^n} (1+|\nu|^2)^{\frac{-l+k}{2}} (1+|\nu|^2)^{\frac{l-k}{2}} ||\widehat{\partial^{|\alpha|}\psi}(\nu)||_{\mathbb{C}^r} \right)^2 \tag{4.60}$$

$$\leq \frac{1}{(2\pi)^{2n}} \left(\sum_{\nu \in \mathbb{Z}^n} (1+|\nu|^2)^{-l+k} \right) \sum_{|\alpha|=k} \sum_{\nu \in \mathbb{Z}^n} (1+|\nu|^2)^{l-k} \nu_1^{2\alpha_1} \cdots \nu_n^{2\alpha_n} ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r}. \tag{4.61}$$

Now observe the following, by a silly estimate we have

$$\sum_{|\alpha|=k} \sum_{\nu \in \mathbb{Z}^n} (1+|\nu|^2)^{l-k} \nu_1^{2\alpha_1} \cdots \nu_n^{2\alpha_n} ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r} \leq \sum_{\nu \in \mathbb{Z}^n} (1+|\nu|^2)^{l-k} ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r} \sum_{|\alpha|=k} |\nu|^{2k} \tag{4.62}$$

$$= \sum_{\nu \in \mathbb{Z}^n} (1+|\nu|^2)^{l-k} ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r} |\nu|^{2k} C_{k,n} \leq C_{k,n} \sum_{\nu \in \mathbb{Z}^n} (1+|\nu|^2)^k (1+|\nu|^2)^{l-k} ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r} \tag{4.63}$$

$$= C_{k,n} \sum_{\nu \in \mathbb{Z}^n} (1+|\nu|^2)^l ||\widehat{\psi}(\nu)||^2_{\mathbb{C}^r} = C_{k,n} ||\psi||^2_{2,l}. \tag{4.64}$$

Thus (4.60) becomes (4.54) as desired. If $l > \frac{n}{2} + k$, it is also true that $l > \frac{n}{2} + k - j$ for each $j = 0, \cdots, k$. Apply (4.54) for each j, we have

$$||\nabla^{k-j}\psi(x)||^2_{C^0} = \sup_{x\in\mathbb{T}^n} ||\nabla^{k-j}\psi(x)||^2_{\mathbb{C}^r} \le \frac{C_{k-j,n}}{(2\pi)^{2n}} \left(\sum_{\nu\in\mathbb{Z}^n} (1+|\nu|^2)^{-l+k-j}\right) ||\psi||^2_{2,l}. \qquad (4.65)$$

Denote $c(l,k,n) = \max_{0\le j\le k}\left\{(2\pi)^{-n}\sqrt{C_{k-j,n}}\left(\sum_{\nu\in\mathbb{Z}^n}(1+|\nu|^2)^{-l+k-j}\right)^{1/2}\right\}$. Such a constant makes sense and is a finite positive number because $\sum_{\nu\in\mathbb{Z}^n}(1+|\nu|^2)^{-l+k-j} < \infty$ by the $p-$series test ($p = 2(l-k+j) \ge 2$). Therefore, $||\psi||_{C^k} \le c(l,k,n)||\psi||_{2,l}$. □

We end this subsection by summarizing various results in the context of spinor bundle Σ_n and its associated spinor Dirac operator D over M.

Theorem 4.10. *Let M be an $n-$dimensional compact Riemannian spin manifold without boundary and Σ_n be a fixed spinor bundle over M equipped with the associated spinor Dirac operator D. Then the following are true*

1. *The inclusion $L^2_l(\Sigma_n) \hookrightarrow L^2_k(\Sigma_n)$ is a bounded operator for $k < l$.*

2. $\nabla^{\Sigma_n} : L^2_k(\Sigma_n) \to L^2_{k-1}(\Sigma_n)$ *is bounded.* □

3. *If $T : \Sigma_n \to \Sigma_n$ is a bundle map, then T extends to a bounded operator from $L^2_k(\Sigma_n) \to L^2_k(\Sigma_n)$ for any k.* □

4. *D extends to a bounded operator from $L^2_l(\Sigma_n) \to L^2_{l-1}(\Sigma_n)$.* □

5. **(Rellich)** *For $l < k$, the inclusion $L^2_k(\Sigma_n) \hookrightarrow L^2_l(\Sigma_n)$ is compact.* □

6. **(Sobolev embedding theorem)** *If l and k are natural numbers such that $l > \frac{n}{2} + k$, then there is a positive finite constant $c(l,k,n)$ such that $||\psi||_{C^k} \le c(l,k,n)||\psi||_{2,l}$ for each spinor $\psi \in \Gamma(\Sigma_n)$. Consequently, the inclusion $L^2_l(\Sigma_n) \hookrightarrow C^k(\Sigma_n)$ is a bounded operator between Banach spaces.* □

7. *As a consequence of* 4.10.5, 4.10.6, *and* (4.3), *for* l *and* k *are natural numbers where* $l > \frac{n}{2} + k$, *any sequence* $\{\psi_n\}$ *in* $L^2_l(\Sigma_n)$ *that is uniformly bounded in the Sobolev* $l-$*norm has a subsequence that converges in* $C^k(\Sigma_n)$. □

Remark 8. Statements in Theorem 4.10 can also be made for any first order formally self-adjoint elliptic operator.

5 Global Kuranishi model

In this section, we describe various analytical background that goes into the proof of our main Theorem 1.8. The majority content of this section can be found in details at Appendix B1, B2 of [23]. We decide to include it this thesis for the sake of self-containment.

The *Kuranishi model* is an extension of the implicit function theorem which turns the local analysis of the zero set of a Fredholm map near a singular point into a finite dimensional eigenvalue problem. Such model was also used by Furuta in his proof of the 10/8-Theorem. We will describe the model in a more abstract setting.

Let X and Y be infinite dimensional Banach spaces and $f : X \to Y$ be a smooth map. For every $x \in X$, we denote $d_x f : X \to Y$ by its linearization of f at x, and $B(x, R)$ by the open ball centered at x of radius R, whereas $\overline{B}(x, R)$ denotes the closed ball.

Lemma 5.1. *Let* X *be a Banach space and* $\psi : X \to X$ *be a continuously differential map such that* $\psi(0) = 0$ *and* $\|1 - d_x\psi\| \leq \gamma$ *for all* $x \in X$ *with* $\|x\| < R$, $\gamma < 1$ *is some constant. Then the restriction of* ψ *to* $B(0, R)$ *is injective,* $\psi(B(0, R))$ *is an open set, and* $\psi^{-1} : \psi(B(0, R)) \to B(0, R)$ *is continuously differentiable with* $d_y\psi^{-1} = [d_{\psi^{-1}(y)}\psi]^{-1}$. *Moreover,* $B(0, R(1 - \gamma)) \subset \psi(B(0, R)) \subset B(0, R(1 + \gamma))$.

Proof. First, we show the containment. Let $f = 1 - \psi$. Since $f(0) = 0$ and $\|d_x f\| \leq \gamma$ for any $\|x\| < R$, f is a contraction on $B(0, R)$. This means that $\|f(x_1) - f(x_2)\| \leq \gamma \|x_1 - x_2\|$ for any $x_1, x_2 \in B(0, R)$. By triangle inequality, for any

$x_1, x_2 \in B(0, R)$ we obtain

$$\|\psi(x_1) - \psi(x_2)\| = \|x_1 - f(x_1) - x_2 + f(x_2)\| \leq \|x_1 - x_2\| + \|f(x_1) - f(x_2)\| \tag{5.1}$$

$$\leq (1+\gamma)\|x_1 - x_2\| \leq (1+\gamma)R. \tag{5.2}$$

Similarly,

$$\|\psi(x_1) - \psi(x_2)\| \geq \|x_1 - x_2\| - \|f(x_1) - f(x_2)\| \geq (1-\gamma)\|x_1 - x_2\|. \tag{5.3}$$

(5.2) shows that $\psi(B(0,R)) \subset B(0, R(1+\gamma))$, while (5.3) shows that ψ restricted to $B(0,R)$ is injective. Next we show that $B(0, R(1-\gamma)) \subset \psi(B(0,R))$. Let $y \in B(0, R(1-\gamma))$ and consider the map $g(x) = f(x) + y$. Since f is a contraction on $B(0,R)$, g is also a contraction on $\overline{B}(0, R-\epsilon)$ whenever ϵ is small. Hence, it has a unique fixed point for $\|y\| = (R-\epsilon)(1-\gamma)$ and $\|x\| \leq R - \epsilon$. So, we have

$$g(x) = x, \qquad \text{which implies that} \qquad \psi(x) = y.$$

Let $\epsilon \to 0$, and we indeed obtain $B(0, R(1-\gamma)) \subset \psi(B(0,R))$. Similar argument and we would also have that $\psi(B(0,R))$ is open and $\psi^{-1} : \psi(B(0,R)) \to B(0,R)$ is continuous.

Finally, we prove that ψ^{-1} is continuously differentiable. Let $x_0 \in B(0,R)$ and $y_0 \in \psi(B(0,R))$ such that $\psi(x_0) = y_0$, we need to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that for any $\|y - y_0\| < \delta$, we have

$$\left\|\psi^{-1}(y) - \psi^{-1}(y_0) - [d_{x_0}\psi]^{-1}(y - y_0)\right\| \leq \epsilon \|y - y_0\|.$$

Note that $[d_{x_0}\psi]^{-1}$ should be well-defined. This is because $\|1 - d_{x_0}\psi\| \leq \gamma < 1$, so that

$$[d_{x_0}\psi]^{-1} = \sum_{k=0}^{\infty}(1 - d_{x_0}\psi)^k.$$

Furthermore, $\|[d_{x_0}\psi]^{-1}\| \leq 1/(1-\gamma)$. Since ψ is continuously differentiable, for any $\epsilon > 0$, there is a $R > \delta > 0$ such that if $x \in B(0,R)$ and $\|x - x_0\| < \delta$, then

$$\|\psi(x) - \psi(x_0) - d_{x_0}\psi(x - x_0)\| \leq \epsilon(1-\gamma)^2 \|x - x_0\| .$$

By (5.3), if $\|y - y_0\| < (1-\gamma)\delta$, then we have $\|x - x_0\| < \delta$. So

$$\begin{aligned}
\left\|\psi^{-1}(y) - \psi^{-1}(y_0) - [d_{x_0}\psi]^{-1}(y - y_0)\right\| &= \left\|[d_{x_0}\psi]^{-1}(y - y_0 - d_{x_0}\psi(x - x_0)\right\| && (5.4)\\
&\leq \frac{1}{1-\gamma} \|\psi(x) - \psi(x_0) - d_{x_0}\psi(x - x_0)\| && (5.5)\\
&\leq \epsilon(1-\gamma) \|x - x_0\| && (5.6)\\
&\leq \epsilon \|\psi(x) - \psi(x_0)\| = \epsilon \|y - y_0\| . && (5.7)
\end{aligned}$$

(5.7) is true because of (5.3) again. □

Theorem 5.2. (Inverse function theorem) *Let $f : X \to Y$ be continuously differentiable. Suppose that $D = d_{x_0}f : X \to Y$ has a bounded inverse. Choose $c > 0$ and $\delta > 0$ such that*

$$\left\|D^{-1}\right\| \leq c, \qquad \|d_x f - D\| \leq \frac{1}{2c}$$

for $\|x - x_0\| < \delta$. Then the following is true

1. *f restricted to $B(x_0, \delta)$ is injective and $f(B(x_0,\delta))$ is open in Y containing $B(f(x_0), \delta/2c)$.*
2. *The inverse map $f^{-1} : B(f(x_0), \delta/2c) \to B(x_0, \delta)$ is continuously differentiable.*
3. *If $x_1, x_2 \in B(x_0, \delta)$ then $\|x_1 - x_2\| \leq 2c \|f(x_1) - f(x_2)\|$.*

Proof. Direct application of Lemma 5.1 where without loss of generality we assume $x_0 = 0$ and $f(0) = 0$. □

Given a bounded linear operator $D : X \to Y$ between Banach spaces, a *pseudo-inverse* of D is a bounded linear operator $T : Y \to X$ which satisfies $TDT = T$ and $DTD = D$. Note that it is a fact that every Fredholm operator has a pseudo-inverse [reference].

Theorem 5.3. (Kuranishi) *Let X and Y be Banach spaces and $f : X \to Y$ be a smooth map such that $f(0) = 0$. Suppose that $D = d_0 f$ has a pseudo-inverse $T : Y \to X$. Let $Y_0 = ker\, T$. Then there exist an open neighborhood U of 0 in X, a local diffeomorphism $g : U \to g(U) \subset X$, and a smooth map $f_0 : U \to Y_0$ such that*

$$f(g(x)) = Dx + f_0(x)$$

for any $x \in U$ and

$$g(0) = 0, \quad d_0 g = 1, \quad f_0(0) = 0, \quad d_0 f(0) = 0.$$

Moreover, if f is equivariant with respect to the action of some compact lie group G on X and Y then the maps g and f_0 can be chosen to be equivariant.

Proof. We define $\psi : X \to X$ given by $\psi(x) = x + T(f(x) - Dx)$. Immediately, we see that $\psi(0) = 0$ and $d_0\psi = 0$. By Theorem 5.2, there is an open neighborhood U of 0 in X has an inverse. Let $g = \psi^{-1}$ and $f_0 = (1 - DT) \circ f \circ g$. Then we have

$$D\psi(x) = Dx + DT(f(x) - Dx) = DTf(x).$$

This means that on U, $D = DT \circ f \circ g$. Therefore,

$$f(g(x)) = Dx - DTf(g(x)) + f(g(x)) = Dx + (1 - DT)f(g(x)) = Dx + f_0(x).$$

If G acts on X and Y and f is $G-$equivariant, then D is also equivariant and its pseudo-inverse can also chosen to be $G-$equivariant. Hence, the construction of g and f_0

above are also going to be equivariant with respect to the $G-$action. $\square$

We now consider the abstract set-up of the global Kuranishi model. Let $P_n : X \to X$ and $Q_n : Y \to Y$ be sequences of projections where $DP_n = Q_n D$, $TQ_n = P_n T$, $im\, Q_n \subset im\, D$, $ker\, D \subset ker\, P_n$ and $\lim_n P_n x = 0$, $\lim_n Q_n y = 0$ for all $x \in X$, $y \in Y$. We defined ψ_n, g_n, f_n as in the proof of Theorem 5.3 where $P = TD$ is replaced with P_n and $Q = DT$ is replaced with Q_n:

$$\psi_n(x) = x + TQ_n(f(x) - Dx), \quad g_n = \psi_n^{-1}, \quad f_n = (1 - Q_n) \circ (D + (f - D) \circ \psi_n^{-1}).$$

Similar calculations as in the proof of Theorem 5.3, we have $f \circ g_n = Q_n D + f_n$. Note then the zeros of f on $g_n(U)$ is exactly the image of of the zeroes of the restriction of $f_n : ker\, P_n \to ker\, Q_n$. These domains will limit to the entire X as $n \to \infty$ if $d_x f - D$ is a uniform family of compact operators.

Lemma 5.4. *Let X, Y, Z be Banach spaces and $Q_n : Y \to Z$ be a sequence of bounded linear operators such that*

$$\lim_{n\to\infty} Q_n y = 0$$

for all $y \in Y$. Moreover, let $\{K_\alpha\}_{\alpha \in A}$ by a collection of bounded linear operators $K_\alpha : X \to Y$, indexed by a set A, such that the set

$$B = \{K_\alpha x : \alpha \in A, x \in X, \|x\| \leq 1\} \subset Y$$

has compact closure. Then

$$\lim_{n\to\infty} \sup_\alpha \|Q_n K_\alpha\| = 0.$$

Proof. First note that there is a c such that $\|Q_n\| \leq c$ for all n. Let $\epsilon > 0$ and since B has compact closure, we may cover it by finitely number of balls centered at $y_1, \cdots, y_N$ of radius $\epsilon/2c$. We have that there is an n_0 such that for all $j = 1, \cdots, N$ and $n \geq n_0$,

$\|Q_n y_j\| \leq \epsilon/2$. Also, for an α and $x \in X$ where $\|x\| \leq 1$, $K_\alpha x \in B(y_j, \epsilon/2c)$. Then

$$\|Q_n K_\alpha x\| \leq \|Q_n\| \cdot \|K_\alpha x - y_j\| + \|Q_n y_j\| \leq c \cdot \frac{\epsilon}{2c} + \frac{\epsilon}{2} = \epsilon.$$

The above estimate is true as long as $n \geq n_0$. Thus, we obtain the result. □

6 Rarita-Schwinger-Seiberg-Witten equations

6.1 Rarita-Schwinger operator

A Rarita-Schwinger operator can be defined in general setting. Let (X^{2n}, g) be a closed Riemannian even dimensional manifold for now. Recall that a Clifford bundle $S \to X$ is a complex vector bundle that is equipped with a Hermitian inner product, a compatible connection ∇, and a Clifford multiplication $\rho : TX \otimes S \to S$. As usual, since the manifold is even dimensional, there is an orthogonal decomposition of $S = S^+ \oplus S^-$ such that ρ exchanges the chirality.

Consider $ker\, \rho$, which is a parallel bundle of $TX \otimes S$ over X. With respect to the chirality of S, we also have an orthogonal decomposition $ker\, \rho = (ker\, \rho)^+ \oplus (ker\, \rho)^-$. Let $\iota : S^\mp \to T^*X \otimes S^\pm = TX \otimes S^\pm$ be the embedding defined $\iota(\varphi)(X) = -\frac{1}{2n}\rho(X)\varphi$. Thus, we have another orthogonal decomposition $S^\pm \otimes TX = \iota(S^\mp) \oplus (ker\, \rho)^\pm$.

On the bundle S, there are two natural first order differential operators: The Dirac operator $D : \Gamma(S) \to \Gamma(S)$ defined by $D = \rho \circ \nabla$, and the twistor operator $P : \Gamma(S) \to \Gamma(ker\, \rho)$ defined by $\pi \circ \nabla$ where $\pi : TX \otimes S \to ker\, \rho$ is the orthogonal projection. Taking into account of the chirality, we note that $D^\pm : \Gamma(S^\pm) \to \Gamma(S^\mp)$, whereas $P^\pm : \Gamma(S^\pm) \to (ker\, \rho)^\pm$.

Next we consider the twisted Dirac operator $\mathcal{D}^\pm : \Gamma(S^\pm \otimes TX) \to \Gamma(S^\mp \otimes TX)$. Follow the computations in [25], with respect to the decomposition $S^\pm \otimes TX = \iota(S^\mp) \oplus (ker\, \rho)^\pm$,

$\mathcal{D}^{\pm}$ takes the following matrix form

$$\mathcal{D}^{\pm} = \begin{pmatrix} \frac{1-n}{n}\iota \circ D^{\pm} \circ \iota^{-1} & 2\iota \circ P^{+*} \\ \frac{1}{n}P^{+} \circ \iota^{-1} & Q^{\pm} \end{pmatrix}.$$

Definition 6.1. The Rarita-Schwinger operator associated to the Clifford bundle $S \to X$ is defined by $Q^{\pm} := \pi \circ \mathcal{D}^{\pm}|_{\Gamma(ker\,\rho)^{\pm}}$.

The above construction carries out similarly when (X, g) is a closed simply connected smooth spin 4−manifold. In this case, a Clifford bundle is given by a *spin*^{c} bundle $\mathfrak{s}_{1/2} \otimes L \to X$ where $\mathfrak{s}_{1/2}$ is the associated vector bundle to the only principal $Spin(4)$−bundle $P_{Spin(4)} \to X$ and L is any complex line bundle. Notation-wise, the operators defined previously all come with extra subscript: $D^{\pm}_A, \mathcal{D}^{\pm}_A, Q^{\pm}_A$ with A being a unitary connection on L. Borrowing the language from Physics, we call sections of $\mathfrak{s}^{\pm}_{1/2} \otimes L$ positive (negative) twisted 1/2−spinors and sections of $(ker\,\rho)^{\pm} := \mathfrak{s}^{\pm}_{3/2} \otimes L$ positive (negative) twisted 3/2−spinors.

Rarita-Schwinger operator Q_A is a first order elliptic operator, thus it has a well-defined index.

Proposition 6.1. *Let X be a closed simply connected smooth spin 4−manifold and Q_A be the Rarita-Schwinger operator associated to the twisted 3/2−spinor bundle $\mathfrak{s}_{3/2} \otimes L \to X$ where A is a unitary connection of L. Then the index of Q_A is given by*

$$index_{\mathbb{C}}\, Q_A = \frac{19}{8}\,\sigma(X) + \frac{5}{2}\,c_1(L)^2.$$

Proof. Apply the equivariant version of Atiyah-Singer index theorem for first order elliptic operator (Theorem 13.13 of [17]), we have

$$index_{\mathbb{C}}\, Q_A = \left(\frac{ch(\mathfrak{s}^{+}_{3/2} \otimes L) - ch(\mathfrak{s}^{-}_{3/2} \otimes L)}{\chi(TX)}\, \hat{\mathcal{A}}(TX)^2 \right) [X], \tag{6.1}$$

where $\chi(TX)$ and $\hat{A}(TX)$ are the usual Euler class and the $\hat{A}-$class of TX, ch is the Chern character. On the other hand, using the orthogonal decomposition of $\mathfrak{s}^{\pm}_{1/2} \otimes L \otimes TX$, we obtain

$$ch(\mathfrak{s}^{\pm}_{1/2} \otimes L)\, ch(TX) = ch(\mathfrak{s}^{\pm} \otimes L \otimes TX) = ch(\mathfrak{s}^{\pm}_{3/2} \otimes L) + ch(\mathfrak{s}^{\mp}_{1/2} \otimes L). \tag{6.2}$$

Subtract the two version of (6.2) from each other to get

$$ch(\mathfrak{s}^{+}_{3/2} \otimes L) - ch(\mathfrak{s}^{-}_{3/2} \otimes L) = (ch(\mathfrak{s}^{+}_{1/2} \otimes L) - ch(\mathfrak{s}^{-}_{1/2} \otimes L))(ch(TX) + 1). \tag{6.3}$$

Substitute (6.3) into (6.1), recall that $\hat{\mathcal{A}}(TX) = (ch(\mathfrak{s}^{+}_{1/2} - ch(\mathfrak{s}^{-}_{1/2}))\, \hat{\mathcal{A}}(TX)^2/\chi(TX)$ to get

$$index_{\mathbb{C}}\, Q_A = ch(TX \otimes L)\hat{\mathcal{A}}(TX)[X] + ch(L)\hat{\mathcal{A}}(TX)[X] = index_{\mathbb{C}}\, \mathcal{D}_A + \left(-\frac{1}{8}\sigma(X) + \frac{1}{2}c_1(L)^2\right) \tag{6.4}$$

Let's calculate $ch(TX \otimes L) = ch(T_{\mathbb{C}}X \otimes L)$. Recall (Chapter 2.7 of [18]):

Fact 1. $ch(T_{\mathbb{C}}X \otimes L) = ch(\mathfrak{s}^{+}_{1/2} \otimes \mathfrak{s}^{-}_{1/2} \otimes L) = ch(\mathfrak{s}^{+}_{1/2})ch(\mathfrak{s}^{-}_{1/2})ch(L)$.

Note that $ch(\mathfrak{s}^{+}_{1/2})ch(\mathfrak{s}^{-}_{1/2}) = 4 - c_2(\mathfrak{s}^{+}_{1/2}) - 2c_2(\mathfrak{s}^{-}_{1/2})$, and $ch(L) = 1 + c_1(L) + c_1(L)^2/2$. Thus, $ch(T_{\mathbb{C}} \otimes L) = 4 + 2c_1(L)^2 - 2c_2(\mathfrak{s}^{+}_{1/2}) - 2c_2(\mathfrak{s}^{-}_{1/2}) + 4c_1(L)$. Combine with the fact that on 4−manifolds, $\hat{\mathcal{A}}(TX) = 1 - p_1(TX)/24$, where p_1 is the first Pontryagin class, we have (picking out top forms)

$$\hat{\mathcal{A}}(TX)ch(T_{\mathbb{C}}X \otimes L) = 2c_1(L)^2 - 2c_2(\mathfrak{s}^{+}_{1/2}) - 2c_2(\mathfrak{s}^{-}_{1/2}) - \frac{p_1(TX)}{6}. \tag{6.5}$$

Recall (Chapter 2.7 of [18]):

Fact 2. $c_2(\mathfrak{s}^{\pm}_{1/2})[X] = -3\,\sigma(X)/4 \mp \chi(X)/2$.

Since $p_1(TX)/3[X] = \sigma(X)$, use the above formulas, we rewrite (2.5) as

$$\hat{\mathcal{A}}(TX)ch(T_{\mathbb{C}}X \otimes L)[X] = 2c_1(L)^2[X] - 2\left(-\frac{3}{4}\,\sigma(X) - \frac{\chi(X)}{2}\right) + \tag{6.6}$$

$$- 2\left(-\frac{3}{4}\sigma(X) + \frac{\chi(X)}{2}\right) - \frac{\sigma(X)}{2} = 2c_1(L)^2[X] + \frac{5}{2}\sigma(X). \tag{6.7}$$

Therefore, (6.4) becomes $index_{\mathbb{C}}\, Q_A = 19\,\sigma(X)/8 + 5c_1(L)^2/2$. □

Remark 9. When $L = \underline{\mathbb{C}}$ and A is the trivial connection, from Proposition 2.2 we have $index_{\mathbb{C}}\, Q = 19\,\sigma(X)/8$.

6.2 Gauge theoretic equations for the Rarita-Schwinger operator

From here onward, X is always denoted by a closed simply connected smooth spin $4-$manifold. The Clifford multiplication ρ is an isometry $i\Lambda^+T^*X \to i\mathfrak{su}(\mathfrak{s}^+_{1/2})$. Let μ be a quadratic map defined by

$\mu : \mathfrak{s}^+_{1/2} \otimes L \otimes TX = \mathfrak{s}^+_{1/2} \otimes L \otimes T^*X = \mathrm{Hom}(TX, \mathfrak{s}^+_{1/2} \otimes L) \to \mathfrak{g}_L \otimes \mathfrak{su}(\mathfrak{s}^+_{1/2}) = i\mathfrak{su}(\mathfrak{s}^+_{1/2})$,

$$\mu(\psi) = \psi\psi^* - \frac{1}{2}\mathrm{tr}(\psi\psi^*)1_{\mathfrak{s}^+_{1/2}\otimes L}.$$

Besides looking for twisted Rarita-Schwinger fields, we impose a curvature condition for A. What follow are referred as the Rarita-Schwinger-Seiberg-Witten equations (RS-SW),

$$Q^+_A\psi = 0, \;\; F^+_A = \rho^{-1}(\mu(\psi)). \tag{6.8}$$

If A_0 is a fixed referenced unitary connection of L, then every other connection $A = A_0 + a$, where $a \in i\Omega^1(X)$. (2.8) can also be re-written as

$$Q^+_{A_0}\psi + \pi^-(a \cdot \psi) = 0, \;\; d^+a + F^+_{A_0} = \rho^{-1}(\mu(\psi)). \tag{6.9}$$

We define $\mathcal{C} = \Gamma(\mathfrak{s}^+_{3/2} \otimes L) \oplus i\Omega^1(X)$ to be the configuration space of the RS-SW equations. $\mathcal{R} = \Gamma(\mathfrak{s}^-_{3/2} \otimes L) \oplus i\Omega^+(X)$ is denoted by its range space. The gauge group $\mathcal{G}$ acts on $\mathcal{C}$ by pulling-back the connections and left multiplying by conjugation on the twisted 3/2−spinors. $\mathcal{G}$ also acts on $i\Omega^+(X)$ trivially. Not much different from the standard Seiberg-Witten theory, the following lemma tells us that solutions to (6.9) are preserved under the gauge group action.

Lemma 6.2. *If $(\psi, A_0 + a)$ is a solution to (6.9), then $h \cdot (\psi, A_0 + a)$ is also a solution for any $h \in \mathcal{G} = Maps\,(X, U(1))$.*

Proof. Since X is simply connected, h has global logarithm, i.e, there is a smooth real-valued function u such that $h = e^{iu}$. Then the action of h on a configuration can be re-described as

$$h \cdot (\psi, A_0 + a) = (e^{-iu}\psi, A_0 + a + idu).$$

We easily see that $d^+(a + idu) + F^+_{A_0} - \rho^{-1}(\mu(e^{-iu}\psi)) = d^+a + F^+_{A_0} - \rho^{-1}(\mu(\psi))$. At the same time,

$$Q^+_{A_0}(e^{-iu}\psi) + \pi^-(a \cdot e^{-iu}\psi + idu \cdot e^{-iu}\psi) = e^{-iu}Q^+_{A_0}\psi - ie^{-iu}\pi^-(du \cdot \psi) + \tag{6.10}$$

$$+ e^{-iu}\pi^-(a \cdot \psi) + ie^{-iu}\pi^-(du \cdot \psi) = e^{-iu}(Q^+_{A_0}\psi + \pi^-(a \cdot \psi)). \tag{6.11}$$

Therefore, obviously if $(\psi, A_0 + a)$ solves (6.9), then so does $(e^{-iu}\psi, A_0 + a + idu)$. □

RS-SW equations are in an abelian gauge theory. And just as the Seiberg-Witten equations, there is a gauge-fixing condition for the RS-SW equations. Recall that with respect to a referenced unitary connection A_0, A is in the Coulomb gauge if $d^*(A - A_0) = 0$.

Lemma 6.3. *Let $(\psi, A_0 + a)$ be any solution to (2.9). Then its orbit by the action of $\mathcal{G}$ can be represented uniquely up to a constant by another solution where the connection part is in the Coulomb gauge.*

Proof. Suppose $h = e^{iu}$ is an element of $\mathcal{G}$ such that $A_0 + a + idu$ is in the Coulomb gauge. That means to show the existence of a connection that is in the Coulomb gauge, we have to solve $d^*(a+idu)=0$. Note that since u is a smooth function, $d^*u=0$. Therefore, solving for u in $d^*(a+idu)=0$ is equivalent to solving for u in

$$d^*du + dd^*u = \Delta u = d^*(-ia) \tag{6.12}$$

But by Hodge's decomposition theorem, $\Omega^0(X) = ker\,\Delta \oplus d^*\Omega^1(X)$, which means that a solution u for (6.12) always exists up to a constant. □

From Lemma 6.2 and Lemma 6.3, we see that after the gauge group action and the gauge fixing condition, RS-SW becomes a non-linear system of elliptic differential equations

$$Q^+_{A_0}\psi + \pi^-(a\cdot\psi) = 0,\ \ d^*a = 0,\ \ d^+a + F^+_{A_0} = \rho^{-1}(\mu(\psi)). \tag{6.13}$$

When $L = \underline{\mathbb{C}}$ and A_0 is chosen to be the trivial connection, equations (6.13) become

$$Q^+\psi + \pi^-(a\cdot\psi) = 0,\ \ d^*a = 0,\ \ d^+a = \rho^{-1}(\mu(\psi)). \tag{6.14}$$

The configuration space $\mathcal{C}$ and the range space $\mathcal{R}$ of (2.14) become $\Gamma(\mathfrak{s}^+_{3/2}) \oplus i\Omega^1(X)$ and $\Gamma(\mathfrak{s}^-_{3/2}) \oplus i\Omega^0(X) \oplus i\Omega^+(X)$. Denote $\mathcal{F} = \mathfrak{D} \oplus \mathfrak{Q} : \mathcal{C} \to \mathcal{R}$ by the functional of (2.14), where $\mathfrak{D} = Q^+ \oplus (d^+ \oplus d^*)$ and $\mathfrak{Q}(\psi, a) = (\pi^-(a\cdot\psi), -\rho^{-1}(\mu(\psi)), 0)$. Note that $\mathcal{M}_g = \mathcal{F}^{-1}(0)$. For the remaining of the thesis, we focus only on this functional $\mathcal{F}$.

6.3 Pin(2)-equivariance of the RSSW functional

Let V be a real 4−dimensional vector space. V can be given a quaternionic structure by considering each vector v in V as the following 2×2−complex matrix

$$v = \begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix} := \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$$

This turns V into an algebra where the multiplication is defined by multiplication of these complex matrices. Furthermore, the usual Euclidean inner product on V agrees with the Frobenius inner product on V equipped with the quaternionic structure. We note that if v is not trivial, then $v = |v|^2 \cdot (v/|v|^2)$. And since $|v|^2 = \det v$, we can also think of V as real scalar multiplication of $SU(2)$.

Let $W = W^+ \oplus W^-$, where $W^\pm$ each is a copy of $\mathbb{C}^2$. We define an $\mathbb{R}-$linear map $\rho : V \to \mathrm{End}(W)$ by

$$\rho(v) = \begin{pmatrix} 0 & -\overline{v}^t \\ v & 0 \end{pmatrix}.$$

Let $\{e_1, e_2, e_3, e_4\}$ be any orthonormal basis of V. Then $\rho(e_i)\rho(e_j) + \rho(e_j)\rho(e_i) = -2\delta_{ij}$. So ρ is a Clifford multiplication. Furthermore, ρ exchanges the chirality of W. In particular if $\psi \in W^+$, then $\rho(v)\psi = v\,\psi$; where we view $v\,\psi$ as an element of W^-. Also note that $W^\pm$ corresponds to $\pm 1-$eigenspaces of the map $-\rho(e_1)\rho(e_2)\rho(e_3)\rho(e_4)$.

On $\mathbb{C}^2$, it is also naturally endowed with a quaternionic structure. Let $\psi \in \mathbb{C}^2$, and write $\psi = (\psi_1\ \psi_2)^t$. We can view ψ as a quaternion number in two different ways, either $\psi = \psi_1 + \psi_2 j$ or

$$\psi = \begin{pmatrix} \psi_1 & -\overline{\psi_2} \\ \psi_2 & \overline{\psi_1} \end{pmatrix}.$$

Now consider the expression $\rho(v)\psi$, where $v \in V$ and $\psi \in W^+$. As observed above, we have

$$\rho(v)\psi = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} z\psi_1 - \overline{w}\psi_2 \\ w\psi_1 + \overline{z}\psi_2 \end{pmatrix} \in W^-.$$

The above vector then can be identified with the matrix

$$\begin{pmatrix} z\psi_1 - \overline{w}\psi_2 & -\overline{w\psi_1} - z\overline{\psi_2} \\ w\psi_1 + \overline{z}\psi_2 & \overline{z\psi_1} - w\overline{\psi_2} \end{pmatrix} = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \begin{pmatrix} \psi_1 & -\overline{\psi_2} \\ \psi_2 & \overline{\psi_1} \end{pmatrix},$$

which is exactly the multiplication of v and ψ in the quaternion numbers. Hence, the representation (Clifford multiplication) is exactly the multiplication in the quaternions.

Consider the group $Spin(4) = SU(2) \times SU(2)$. Let $(p_-, p_+) \in \text{Spin}(4)$ arbitrarily. We define the following group homomorphism $Spin(4) \to GL(V)$ given by

$$(p_-, p_+) \mapsto (v \mapsto p_- v p_+^{-1}).$$

The map $v \mapsto p_- v p_+^{-1}$ is norm preserving; thus the above homomorphism is actually $Spin(4) \to SO(V)$. Either way, this is a representation of $Spin(4)$ on V. There is another representation of $Spin(4)$ onto W defined by

$$(p_-, p_+) \mapsto ((\psi, \phi) \mapsto (p_- \phi, p_+ \psi)).$$

This is the spinor representation of the spin group. When $Spin(4)$ only acts on W^+, the above action agrees with the Clifford multiplication ρ restricted to only unit vectors in V (as multiplication in the quaternions).

The group $Pin(2)$ is defined to be the disjoint union of two unit circles, $Pin(2) = S^1 \sqcup j \cdot S^1$. But we can also view $S^1 \sqcup j \cdot S^1 \subset SU(2)$. Let $p_0 \in Pin(2)$ and $\psi, \phi \in W^{\pm}$, respectively. There is another action on $W^{\pm}$ by $Pin(2)$ given by $\psi \mapsto \psi p_0^{-1}$ and $\phi \mapsto \phi p_0^{-1}$. We would like to emphasize again that every action we have listed so far is defined as multiplication in the quaternions. Combine and sum up, we obtain the following $Spin(4) \times Pin(2)-$modules:

- $V := {}_-\mathbb{H}_+$ is the module where the action of (p_-, p_+, p_0) is given by $p_- v p_+^{-1}$.
- $W^+ := {}_+\mathbb{H}$ is the module where the action is given by $p_+ \psi p_0^{-1}$.
- $W^- := {}_-\mathbb{H}$ is the module where action is defined by $p_- \phi p_0^{-1}$.
- $\tilde{\mathbb{R}}$ is the $Pin(2)-$module where the actions are $e^{i\theta} \cdot x = x$ and $j \cdot x = -x$. ${}_+\mathbb{H}_+$ is the

module where the action is given by $p_+ v p_+^{-1}$ so that $\tilde{\mathbb{R}} \oplus \Lambda^+ V$ is identified with ${}_+\mathbb{H}_+$.

On our manifold X, with respect to its only $Spin(4)-$bundle $P_{Spin(4)} \to X$, the above actions make the bundles $T^*X = TX, \mathfrak{s}^{\pm}_{1/2}$ equivariant with respect to $Spin(4) \times Pin(2)$. Since $Pin(2) \hookrightarrow Spin(4)$ by the diagonal map, these bundles are also $Pin(2)-$equivariant. Fiber-wise, they correspond to ${}_-\mathbb{H}_+$, ${}_+\mathbb{H}$, and ${}_-\mathbb{H}$, respectively. And fiber of $\underline{\mathbb{R}} \oplus \Lambda^+ T^*X$ corresponds to ${}_+\mathbb{H}_+$. We note that the Clifford multiplication $\rho : TX \otimes \mathfrak{s}^{\pm}_{1/2} \to \mathfrak{s}^{\mp}_{1/2}$ is induced exactly by the Clifford multiplication ${}_-\mathbb{H}_+ \times {}_\pm\mathbb{H} \to {}_\mp\mathbb{H}$ defined previously; and the later one is definitely $Pin(2)-$equivariant. Thus, $\rho : TX \otimes \mathfrak{s}^{+}_{1/2} \to \mathfrak{s}^{-}_{1/2}$ is $Pin(2)-$equivariant.

Lemma 6.4. *The bundles $\mathfrak{s}^{\pm}_{3/2}$ are $Spin(4) \times Pin(2)-$equivariant.*

Proof. We just need to check for positive $3/2-$spinors. Let $\psi = \sum e_\alpha \otimes \psi_\alpha \in \Gamma(\mathfrak{s}^{+}_{1/2})$, where $\{e_\alpha\}$ is a local orthonormal basis on X with respect to g. Introduce the action of (p_-, p_+, p_0), we obtain $\rho((p_-, p_+, p_0) \cdot \psi) = \rho\left(\sum p_- e_a p_+^{-1} \otimes p_+ \psi_a p_0^{-1}\right) = p_- \rho(\psi) p_0^{-1} = 0$. Here we use the equivariant property of ρ. Therefore, if ψ is a $3/2-$spinor, then so is $(p_-, p_+, p_0) \cdot \psi$. Note that this means that the $3/2-$spinor bundles are also $Pin(2)-$equivariance. □

With abuse of notation, recall that we also denote the Clifford multiplication $TX \otimes \mathfrak{s}^{+}_{1/2} \otimes TX \to \mathfrak{s}^{-}_{1/2} \otimes TX$ by ρ.

Lemma 6.5. *For $a \in \Omega^1(X)$ and $\psi \in \Gamma(\mathfrak{s}^{+}_{3/2})$, $(a, \psi) \mapsto a \cdot \psi$ is $Spin(4) \times Pin(2)-$equivariant.*

Proof. Write $\psi = \sum e_\alpha \otimes \psi_\alpha$. Then $\rho(a)\psi = \sum e_\alpha \otimes \rho(a)\psi_\alpha$. Introduce the (p_-, p_+, p_0) in two ways. First, we have $(p_-, p_+, p_0) \cdot \rho(a)\psi = \sum p_- e_\alpha p_+^{-1} \otimes p_- a \psi_\alpha p_0^{-1}$. On the other hand, $\rho(p_- a p_+^{-1}) \sum p_- e_\alpha p_+^{-1} \otimes p_+ \psi_\alpha p_0^{-1} = \sum p_- e_\alpha p_+^{-1} \otimes p_- a \psi_\alpha p_0^{-1}$. The two expressions are exactly the same; hence we have the equivariant property of the quadratic map. Again, this also implies $Pin(2)-$equivariant. □

Recall that the orthogonal projection $\pi^{-}:\Gamma(\mathfrak{s}^{-}_{1/2}\otimes TX)\to\Gamma(\mathfrak{s}^{-}_{3/2})$ can be explicitly given by $\pi^{-}(\phi)=\psi-\iota\circ\rho(\phi)=\phi+\frac{1}{4}\sum_{\alpha}e_{\alpha}\otimes e_{\alpha}\rho(\phi)$.

Lemma 6.6. *π^{-} is $Pin(2)-$equivariant.*

Proof. Let $\phi=\sum e_{\alpha}\otimes\phi_{\alpha}$. We only need to check for the action of j. First, we note

$$\pi^{-}(j\cdot\phi)=j\cdot\phi-\iota\circ\rho(j\cdot\phi)=j\cdot\phi-\iota\circ\rho\left(\sum je_{\alpha}j^{-1}\otimes j\phi_{\alpha}j^{-1}\right) \tag{6.15}$$
$$=j\cdot\phi-\iota(j\rho(\phi)j^{-1})=j\cdot\phi+\frac{1}{4}\sum_{\beta}e_{\beta}\otimes e_{\beta}j\rho(\phi)j^{-1} \tag{6.16}$$
$$=j\cdot\phi+\frac{1}{4}\sum_{e_{\beta}\neq j}e_{\beta}\otimes e_{\beta}j\rho(\phi)j^{-1}-\frac{1}{4}j\otimes\rho(\phi)j^{-1}. \tag{6.17}$$

On the other hand, we have

$$j\cdot\pi^{-}(\phi)=j\cdot(\phi-\iota\circ\rho(\phi))=j\cdot\phi+\frac{j}{4}\sum_{\beta}e_{\beta}\otimes e_{\beta}\rho(\phi) \tag{6.18}$$
$$=j\cdot\phi+\frac{1}{4}\sum_{\beta}je_{\beta}j^{-1}\otimes je_{\beta}\rho(\phi)j^{-1} \tag{6.19}$$
$$=j\cdot\phi+\frac{1}{4}\sum_{e_{\beta}\neq j}e_{\beta}\otimes e_{\beta}j\rho(\phi)j^{-1}-\frac{1}{4}j\otimes\rho(\phi)j^{-1}. \tag{6.20}$$

As a result, we have $\pi^{-}(j\cdot\phi)=j\cdot\pi^{-}(\phi)$. ☐

Let $\psi\in\mathfrak{s}^{+}_{3/2}$. Consider the action of (p_{-},p_{+},p_{0}) on ψ to obtain $p_{+}\psi p_{0}^{-1}$. Passing through quadratic map μ, we have

$$p_{+}\psi p_{0}^{-1}p_{0}\psi^{*}p_{+}^{-1}-\frac{1}{2}\mathrm{tr}(p_{+}\psi p_{0}^{-1}p_{0}\psi^{*}p_{+}^{-1})1=p_{+}\psi\psi^{*}p_{+}^{-1}-\frac{1}{2}\mathrm{tr}(\psi\psi^{*})p_{+}p_{+}^{-1}=p_{+}\mu(\psi)p_{+}^{-1}$$

This shows that

Lemma 6.7. *$\mu:\mathfrak{s}^{+}_{3/2}\to i\mathfrak{su}(\mathfrak{s}^{+}_{1/2})$ is equivariant with respect to Spin$(4)\times Pin(2)$; hence $Pin(2)-$equivariant.*

All aforementioned $Pin(2)$ actions have been unitary (or orthogonal). Therefore, the connections defined on the bundles are also $Pin(2)-$equivariant. In particular, $\mathcal{D}^+$ and $d+d^*$ are $Pin(2)-$equivariant. Combine with the previous lemmas, we obtain that the functional $\mathcal{F}:\mathcal{C}\to\mathcal{R}$ is a $Pin(2)-$equivariant map.

7 Finite dimensional approximation of the RSSW functional

7.1 Linearization of the RSSW functional

We consider the Sobolev completions of $\mathcal{C}_{L^2_k}$ and $\mathcal{R}_{L^2_{k-1}}$ by L^2_k and L^2_{k-1} norms, respectively. Naturally upon completion, they become Hilbert manifolds. As such, for the sake of convenience, we suppress their subscripts. There is a natural identification of the tangent spaces of the Hilbert manifolds $\mathcal{C}$ and $\mathcal{R}$ at a point (ψ,a). Consider a parametrization of a curve $\gamma_t:=(\psi_t,a_t)$ in $\mathcal{C}$ for $t\in(-\epsilon,\epsilon)$, where $\epsilon>0$ and $\psi_t=\psi+t\phi,\ \ a_t=a+tb$. Here ϕ is some other positive $3/2-$spinor and b is a purely imaginary valued $1-$form. Then a short calculation shows that the derivative of γ_t at $t=0$ gives us $\dot\gamma_0=(\phi,b)$.

Lemma 7.1. *$\mathcal{F}:\mathcal{C}\to\mathcal{R}$ is a smooth mapping between separable Hilbert manifolds and its differential at a point (ψ,a) is*

$$d_{(\psi,a)}\mathcal{F}(\phi,b)=\big(Q^+\phi+\pi^-(b\cdot\psi+a\cdot\phi),\ d^*b,\ d^+b-\rho^{-1}(\psi\phi^*+\phi\psi^*)_0\big)$$

Proof. Smoothness part comes from Sobolev regularity. Now let $\gamma_t=(\psi_t,a_t)$, $t\in(-\epsilon,\epsilon)$ such that $\gamma_0=(\psi,a)$ and $\dot\gamma_0=(\phi,b)$, where $\phi\in\Gamma(\mathfrak{s}^+_{3/2})$ and $b\in i\Omega^1(X)$. We have

$$\begin{aligned}d_{(\psi,a)}\mathcal{F}(\phi,b)&=\lim_{t\to0}\frac{\mathcal{F}(\gamma_t)-\mathcal{F}(\psi,a)}{t}\\&=\left(Q^+\phi+\lim_{t\to0}\frac{\pi^-(a_t\cdot\psi_t-a\cdot\psi)}{t},\ d^*b,\ d^+b-\lim_{t\to0}\frac{\rho^{-1}(\mu(\psi_t)-\mu(\psi))}{t}\right).\end{aligned}\tag{7.1}$$

Compute the first limit on the right hand side of (7.1), we have

$$\lim_{t\to 0}\frac{\pi^-(a_t\cdot\psi_t-a\cdot\psi)}{t}=\pi^-\left(\lim_{t\to 0}\frac{a_t\cdot\psi_t-a_t\cdot\psi}{t}+\lim_{t\to 0}\frac{a_t-a}{t}\cdot\psi\right)$$
$$=\pi^-(b\cdot\phi+a\cdot\psi). \tag{7.2}$$

For the other limit, we see that

$$\frac{(\psi_t\psi_t^*)_0-(\psi\psi^*)_0}{t}=\frac{\psi_t\psi_t^*-\psi\psi^*}{t}-\frac{1}{2}\mathrm{tr}\left(\frac{\psi_t\psi_t^*-\psi\psi^*}{t}\right)1$$
$$=\frac{\psi_t(\psi_t^*-\psi^*)}{t}+\frac{(\psi_t-\psi)\psi^*}{t}-\frac{1}{2}\mathrm{tr}\Big\{\frac{\psi_t(\psi_t^*-\psi^*)}{t}+\frac{(\psi_t-\psi)\psi^*}{t}\Big\}1. \tag{7.3}$$

Let $t\to 0$ and (7.3) approaches $(\psi\phi^*+\phi\psi^*)-\frac{1}{2}\mathrm{tr}(\psi\phi^*+\phi\psi^*)1$, which is exactly $(\psi\phi^*+\phi\psi^*)_0$. The above calculations give us exactly the formula for the differential of $\mathcal{F}$ at $(\psi,a)\in\mathcal{C}$. □

Lemma 7.2. *Suppose $k\geq 4, R_0>0$ and $\|(\psi,a)\|_{L^2_k}<R_0$. The linear operator $R_{(\psi,a)}:L^2_k(\mathfrak{s}^+_{3/2})\oplus L^2_k(iT^*X)\to L^2_{k-1}(\mathfrak{s}^-_{1/2}\otimes TX)$ given by $R_{(\psi,a)}(\phi,b)=b\cdot\psi+a\cdot\phi$ is a compact operator.*

Proof. Let $\{(\phi_j,b_j)\}_{j=1}^\infty$ be a uniformly bounded sequence in $L^2_k(\mathfrak{s}^+_{3/2})\oplus L^2_k(iT^*X)$. We need to show that there is a subsequence of $\{R_{\psi,a}(\phi_j,b_j)\}_{j=1}^\infty$ that converges in $L^2_{k-1}(\mathfrak{s}^-_{1/2}\otimes TX)$. By the Rellich lemma, there is a subsequence of $\{b_j\}_{j=1}^\infty$ that converges in L^2_{k-1}. Without loss of generality, we may denote such subsequence by $\{b_j\}_{j=1}^\infty$ again and suppose that $b_j\to b$ in $L^2_{k-1}(iT^*X)$. For an arbitrary $\epsilon>0$, let j be large enough such that $\|b_j-b\|_{L^2_{k-1}}<\epsilon/2\mathrm{const}R_0$. Then by the Sobolev multiplication theorem (Appendix IV of [6]), we have $\|(b_j-b)\cdot\psi\|_{L^2_{k-1}}\leq \mathrm{const}\,\|b_j-b\|_{L^2_{k-1}}R_0<\epsilon/2$. Similarly, for j also large enough, we also have $\|a\cdot(\phi_j-\phi)\|_{L^2_{k-1}}<\epsilon/2$. Therefore, $\{R_{(\psi,a)}(\phi_j,b_j)\}_{j=1}^\infty$ also converges in $L^2_{k-1}(\mathfrak{s}^-_{1/2}\otimes TX)$. It is not hard to see that $R_{(\psi,a)}$ is bounded. This shows that $R_{(\psi,a)}$ is indeed a compact operator. □

Lemma 7.3. *When $k \geq 4$ and $R_0 > 0$, the linear map $L_\psi : L^2_k(\mathfrak{s}^+_{3/2}) \to L^2_{k-1}(i\Lambda^+ T^*X)$ defined by $L_\psi(\phi) = \rho^{-1}(\psi\phi^* + \phi\psi^*)_0$ is a compact operator for any ψ such that $\|\psi\|_{L^2_k} < R_0$.*

Proof. It is sufficient to show that $\phi \mapsto \psi\phi^* + \phi\psi^*$ is compact. Suppose $\{\phi_j\}_{j=1}^\infty$ is a uniformly bounded sequence in $L^2_k(\mathfrak{s}^+_{3/2})$. The Rellich lemma tells us that there is a subsequence of $\{\phi_j\}_{j=1}^\infty$ that converges in $L^2_{k-1}(\mathfrak{s}^+_{3/2})$. For notational convenience, we denote such subsequence by $\{\phi_j\}_{j=1}^\infty$ again, and $\phi_j \to \phi$ in $L^2_{k-1}(\mathfrak{s}^+_{3/2})$. Now for any $\epsilon > 0$ and j is large enough, we have $\|\phi_j - \phi\|_{L^2_{k-1}} < \epsilon/2\text{const}R_0$. By the Sobolev multiplication theorem (Appendix IV of [6],

$$
\begin{aligned}
&\left\|\psi(\phi_j^* - \phi_j^*) + (\phi_j - \phi)\psi^*\right\|_{L^2_{k-1}} \leq \left\|\psi(\phi_j^* - \phi_j^*)\right\|_{L^2_{k-1}} + \|(\phi_j - \phi)\psi^*\|_{L^2_{k-1}} \leq \\
&\leq \text{const}R_0 \left\|\phi_j^* - \phi^*\right\|_{L^2_{k-1}} + \text{const}\,\|\phi_j - \phi\|_{L^2_{k-1}} R_0. \qquad (7.4)
\end{aligned}
$$

The last inequality of (7.4) is less than ϵ. As a result, L_ψ is compact as desired. □

Note that the formula in Lemma 7.1 is exactly $d_{(\psi,a)}\mathcal{F} = \mathcal{D} + d_{(\psi,a)}\mathcal{Q}$. Since orthogonal projection π^- is bounded and Lemma 7.2 tells us that $R_{(\psi,a)}$ is compact whenever $\|(\psi, a)\|_{L^2_k} < R_0$, $\pi^- \circ R_{(\psi,a)}$ is also compact. By Lemma 3.3, we immediately see that $d_{(\psi,a)}\mathcal{Q}$ is a compact operator whenever $\|(\psi, a)\|_{L^2_k} < R_0$. Obviously, the linearization of $\mathcal{F}$ and $\mathcal{Q}$ are also $Pin(2)-$equivariant. To summarize the main point of this subsection, we have the following proposition.

Proposition 7.4. *The operator $d_{(\psi,a)}\mathcal{Q} = d_{(\psi,a)}\mathcal{F} - \mathcal{D} : \mathcal{C} \to \mathcal{R}$ is compact and equivariant under the $Pin(2)-$action. Furthermore, the set*

$$
\bigcup_{(\psi,a)\in B(0,R_0)} d_{(\psi,a)}\mathcal{Q}(\overline{B(0,1)})
$$

has compact closure in $\mathcal{R}$ for any prefixed $R_0 > 0$.

Since $\mathcal{D}$ is elliptic, $\mathcal{D}^*\mathcal{D}$ and $\mathcal{D}\mathcal{D}^*$ share the same real discrete spectrum $\{\lambda_1, \lambda_2, \cdots\}$. Each eigenspace associated to an eigenvalue λ is finite dimensional, a consequence of ellipticity of $\mathcal{D}$. Now we denote $\mathcal{C}^{\lambda_n}$ by the (orthogonal) direct sum of all eigenspaces of $\mathcal{D}^*\mathcal{D}$ where the eigenvalues are strictly larger λ_n, and $\mathcal{C}_{\lambda_n}$ by the direct sum of all eigenspaces of $\mathcal{D}^*\mathcal{D}$ where the eigenvalues are lesser than or equal to λ_n. Similarly, define $\mathcal{R}^{\lambda_n}$ and $\mathcal{R}_{\lambda_n}$ for $\mathcal{D}\mathcal{D}^*$. Note that both $\mathcal{C}_{\lambda_n}$ and $\mathcal{R}_{\lambda_n}$ are finite dimensional, and

$$\mathcal{D} : \mathcal{C} = \mathcal{C}_{\lambda_n} \oplus \mathcal{C}^{\lambda_n} \to \mathcal{R}_{\lambda_n} \oplus \mathcal{R}^{\lambda_n} = \mathcal{R}$$

respects the orthogonal decomposition. Furthermore, $\mathcal{D}$ is "norm-preserving".

Lemma 7.5. *The map $\mathcal{D} : \mathcal{C}^{\lambda_n} \to \mathcal{R}^{\lambda_n}$ is an isomorphism between Hilbert spaces.*

Proof. We only need to show that $\mathcal{D}$ is bijective. First, let $v \in \mathcal{C}^{\lambda_n}$ such that $\mathcal{D}v = 0$. Without loss of generality, suppose that v is an eigenvector associated to an eigenvalue $\lambda > \lambda_n$. Then $\mathcal{D}^*\mathcal{D}v = \lambda\, v$. This immediately implies that $\lambda\, v = 0$. And since $\lambda \neq 0$, $v = 0$. So we have $\mathcal{D}$ injective. For surjectivity, we let $w \in \mathcal{R}^{\lambda_n}$ and without loss of generality assume that w is an eigenvector associated to an eigenvalue $\lambda > \lambda_n$. Then $\mathcal{D}\mathcal{D}^*w = \lambda\, w$. A quick calculation show that $(1/\lambda)\mathcal{D}^*w \in \mathcal{C}^{\lambda_n}$. And thus, we have $\mathcal{D}$ onto. □

Let $\pi^{\lambda_n} : \mathcal{R} \to \mathcal{R}^{\lambda_n}$ be the orthogonal projection. Lemma 3.5 tells us that the restricted map $\mathcal{D}$ is an isomorphism so there is a well-defined bounded inverse $\mathcal{D}^{-1} : \mathcal{R}^{\lambda_n} \to \mathcal{C}^{\lambda_n}$. To get a finite dimensional approximation for $\mathcal{F}$, we use the global Kuranishi model. The idea is straightforward. We first define the following map $\phi_n : \mathcal{C} \to \mathcal{C}$

$$\phi_n := 1_{\mathcal{C}} + \mathcal{D}^{-1}\pi^{\lambda_n}\mathcal{Q}$$

We would like to show that for a fixed radius $R > 0$ and large enough n, ϕ_n is injective with a well-defined differential inverse. Then the map $f_n := (1-\pi^{\lambda_n})\mathcal{F}\phi_n^{-1}$ from $\mathcal{C}_{\lambda_n} \to \mathcal{R}_{\lambda_n}$ is a finite dimensional approximation for $\mathcal{F}$. To prove this, we need to recall following technical lemmas.

Lemma 7.6. (Lemma 5.4) *Let X, Y, Z be Banach spaces and $Q_n : Y \to Z$ be a sequence of bounded linear operators such that*

$$\lim_{n\to\infty} Q_n y = 0$$

for all $y \in Y$. Moreover, let $\{K_\alpha\}_{\alpha\in A}$ by a collection of bounded linear operators $K_\alpha : X \to Y$, indexed by a set A, such that the set

$$B = \{K_\alpha x : \alpha \in A, x \in X, \|x\| \leq 1\} \subset Y$$

has compact closure. Then

$$\lim_{n\to\infty} \sup_\alpha \|Q_n K_\alpha\| = 0.$$

Proof. See section 5 □

Lemma 7.7. (Lemma 5.1) *Let X be a Banach space and $\psi : X \to X$ be a continuously differential map such that $\psi(0) = 0$ and $\|1 - d_x\psi\| \leq \gamma$ for all $x \in X$ with $\|x\| < R$, $\gamma < 1$ is some constant. Then the restriction of ψ to $B(0,R)$ is injective, $\psi(B(0,R))$ is an open set, and $\psi^{-1} : \psi(B(0,R)) \to B(0,R)$ is continuously differentiable with $d_y\psi^{-1} = [d_{\psi^{-1}(y)}\psi]^{-1}$. Moreover, $B(0, R(1-\gamma)) \subset \psi(B(0,R)) \subset B(0, R(1+\gamma))$.*

Proof. see section 5 □

Lemma 7.8. *Let $\phi_n v = u$. Then $\mathcal{F}v = 0$ if and only if $u \in \mathcal{C}_{\lambda_n}$ and $(1-\pi^{\lambda_n})\mathcal{F}v = 0$.*

Proof. Suppose that $\mathcal{F}v = \mathcal{D}v + \mathcal{Q}v = 0$, this is equivalent to $\mathcal{Q}v = -\mathcal{D}v$. Then from $\phi_n v = u$, we obtain $v - \mathcal{D}_1^{-1}\pi^{\lambda_n}\mathcal{D}v = u$. Note that $\mathcal{D}$ and π^{λ_n} commute. As a result, $v - \pi^{\lambda_n} v = u$. But this is equivalent to saying $u \in \mathcal{C}_{\lambda_n}$. Obviously, $(1-\pi^{\lambda_n})\mathcal{F}v = 0$.

Conversely if $u \in \mathcal{C}_{\lambda_n}$ and $(1-\pi^{\lambda_n})\mathcal{F}v = 0$, then $u = v + \mathcal{D}_1^{-1}\pi^{\lambda_n}(\mathcal{F}-\mathcal{D})v = v + \mathcal{D}^{-1}\pi^{\lambda_n}\mathcal{F}v - \pi^{\lambda_n}v$. Apply $\mathcal{D}$ to both sides, we obtain $\mathcal{D}u = \mathcal{D}v + \mathcal{F}v - \pi^{\lambda_n}\mathcal{D}v = (1-\pi^{\lambda_n})\mathcal{D}v + \mathcal{F}v$. Since $u \in \mathcal{C}_{\lambda_n}$, $\mathcal{D}u \in \mathcal{R}_{\lambda_n}$. And note that $(1-\pi^{\lambda_n})\mathcal{D}v$ is clearly also $\mathcal{R}_{\lambda_n}$. Hence $\mathcal{F}v \in \mathcal{R}_{\lambda_n} \cap \mathcal{R}^{\lambda_n}$. This is possible only if $\mathcal{F}v = 0$. □

Proposition 7.9. *Assume that $\mathcal{M}_g := \mathcal{F}^{-1}(0)$ is compact. Then there exists an $C > 0$ and n large enough such that $f_n : \mathcal{C}_{\lambda_n} \to \mathcal{R}_{\lambda_n}$ defined above has no zero as long as $u \in \mathcal{C}_{\lambda_n}$ and $\|u\|_{L^2_k} = C$.*

Proof. With $\mathcal{M}_g$ being compact, there is an $R > 0$ such that if $\|v\|_{L^2_k} \geq R$, we have $\mathcal{F}v \neq 0$. Note that $\phi_n(0) = 0$ and for any $v \in \mathcal{C}$, $d_v\phi_n = 1_\mathcal{C} + \mathcal{D}^{-1}\pi^{\lambda_n}d_v\mathcal{Q}$ so that $d_v\phi_n - 1_\mathcal{C} = \mathcal{D}^{-1}\pi^{\lambda_n}d_v\mathcal{Q}$. Since $\mathcal{D} : \mathcal{C}^{\lambda_n} \to \mathcal{R}^{\lambda_n}$ is an isomorphism, we have

$$\|d_v\phi_n - 1_\mathcal{C}\| = \left\|\pi^{\lambda_n}d_v\mathcal{Q}\right\|.$$

Note that $\lim_{n\to\infty}\pi^{\lambda_n}w = 0$ for all $w \in \mathcal{R}$ and Proposition 7.4 tells us that $\{d_v\mathcal{Q}\}_{v\in B(0,3R)}$ is a uniform family of compact operators. So, by Lemma 7.6, we have

$$\lim_{n\to\infty}\sup_{v\in B(0,3R)}\left\|\pi^{\lambda_n}d_v\mathcal{Q}\right\| = 0.$$

This implies that for an n large enough, say for all $n \geq n_0(R)$,

$$\|d_v\phi_n - 1_\mathcal{C}\| \leq \frac{1}{2} \text{ whenever } v \in B(0,3R).$$

Then by Lemma 7.7, ϕ_n restricted to $B(0,3R)$ is injective and has a well-defined continuously differentiable inverse. Furthermore, $\phi_n(B(0,3R))$ is open and

$$\phi_n(B(0,R)) \subset B(0,3R/2) \subset \phi_n(B(0,3R)) \subset B(0,9R/2).$$

Suppose that there is $u \in \partial B(0, 3R/2) \cap \mathcal{C}_{\lambda_n}$ such that $f_n u = 0$. Let $v = \phi_n^{-1} u$ or $u = \phi_n v$. By Lemma 7.8, $\mathcal{F} v = 0$. But since $\|v\|_{L^2_k} = \|\phi_n^{-1} u\|_{L^2_k} \geq R$, this leads to a contradiction. Therefore, for any $u \in \mathcal{C}_{\lambda_n}$ where $\|u\|_{L^2_k} = C$ and $C > 3R/2$ such that $B(0, C) \subset \phi_n(B(0, 3R))$, $f_n u \neq 0$. □

Remark 10. The finite dimensional approximation construction of $\mathcal{F}$ above is similar to Furuta's in his 10/8th-paper [10].

Remark 11. There is another way to obtain finite dimensional approximation for $\mathcal{F}$ and one still gets the same result in Proposition 7.9. Alternatively, we define $g_n := (1 - \pi^{\lambda_n})(\mathcal{D} + \mathcal{Q}\phi_n^{-1})$. With the same hypothesis in Proposition 3.9 and a specified ball $B(0, R)$ in $\mathcal{C}$ such that $\mathcal{F}$ is never zero outside such ball indicated in the beginning of the above proof, ϕ_n restricted to $B(0, 3R)$ is still injective and has a well-defined continuously differential inverse for large enough n. Then one note that

$$\mathcal{D}\phi_n - \mathcal{D} - \pi^{\lambda_n}\mathcal{F} + \pi^{\lambda_n}\mathcal{D} = \mathcal{D}\phi_n - \mathcal{D} - \pi^{\lambda_n}(\mathcal{D} + \mathcal{Q}) + \pi^{\lambda_n}\mathcal{D} \tag{7.5}$$

$$= \mathcal{D}\phi_n - \mathcal{D} - \pi^{\lambda_n}\mathcal{Q} = \mathcal{D}(1_{\mathcal{C}} + \mathcal{D}^{-1}\pi^{\lambda_n}\mathcal{Q}) - \mathcal{D} - \pi^{\lambda_n}\mathcal{Q} = 0. \tag{7.6}$$

Apply ϕ_n^{-1} on the right of the above equations, and we get exactly that $\mathcal{D} - \mathcal{D}\phi_n^{-1} - \pi^{\lambda_n}\mathcal{F}\phi_n^{-1} + \pi^{\lambda_n}\mathcal{D}\phi_n^{-1} = 0$. Therefore,

$$g_n = (1 - \pi^{\lambda_n})(\mathcal{D} + \mathcal{F}\phi_n^{-1} - \mathcal{D}\phi_n^{-1}) \tag{7.7}$$

$$= \mathcal{D} + \mathcal{F}\phi_n^{-1} - \mathcal{D}\phi_n^{-1} - \pi^{\lambda_n}\mathcal{D} - \pi^{\lambda_n}\mathcal{F}\phi_n^{-1} + \pi^{\lambda_n}\mathcal{D}\phi_n^{-1} \tag{7.8}$$

$$= \mathcal{F}\phi_n^{-1} - \pi^{\lambda_n}\mathcal{D}. \tag{7.9}$$

Hence, $\mathcal{F}\phi_n^{-1} = \pi^{\lambda_n}\mathcal{D} + g_n$. So if $u \in \mathcal{C}_{\lambda_n}$ and $\|u\|_{L^2_k} \geq 3R/2$, then $\|\phi_n^{-1}u\|_{L^2_k} \geq R$. This implies that $\mathcal{F}\phi_n^{-1}u \neq 0$. Because $\mathcal{D}u \in \mathcal{R}_{\lambda_n}$, so $\pi^{\lambda_n}\mathcal{D}u = 0$. As a result, $g_n u \neq 0$ also.

Remark 12. The hypothesis that there is a ball of a certain radius centered at 0 in $\mathcal{C}$ such that $\mathcal{F} \neq 0$ outside such ball is not needed to obtain a finite dimensional approximation for

$\mathcal{F}$. This condition only ensures that once we have a finite dimensional approximation f_n of $\mathcal{F}$, $f_n \neq 0$ also on a specified sphere whose existence depends on the aforementioned ball.

Note that the map $f_n : \mathcal{C}_{\lambda_n} \to \mathcal{R}_{\lambda_n}$ in Proposition 7.9 is also $Pin(2)-$equivariant by construction. Recall that we have shown $\mathcal{D}$ to be $Pin(2)-$equivariant. Thus, $\mathcal{C}_{\lambda_n}$ and $\mathcal{R}_{\lambda_n}$ are finite dimensional representations of $Pin(2)$. So, they can be written as

$$\mathcal{C}_{\lambda_n} = \mathbb{H}^t \oplus \mathbb{R}^s; \quad \mathcal{R}_{\lambda_n} = \mathbb{H}^r \oplus \mathbb{R}^q.$$

Since $\mathcal{D} : \mathcal{C}^{\lambda_n} \to \mathcal{R}^{\lambda_n}$ is an isomorphism by Lemma 7.5, $2\, index_{\mathbb{C}}\, Q^+ = 4(t-r)$ and $index\,(d^+ \oplus d^*) + 1 = s - q$. The $+1$ is needed because $\mathcal{R}$ does not contain constant functions. We know that $index\,(d^+ \oplus d^*) = -b_2^+(X) - 1$; and Proposition 2.2 tells us that $index\, Q^+ = 19\,\sigma(X)/8$. Hence, $t = r + 19\,\sigma(X)/16$ and $q = s + b_2^+(X)$, where $b_2^+(X) = \dim H^+(X)$. Let $k = 19\,\sigma(X)/16$ and $m = b_2^+(X)$. As a result,

$$\mathcal{C}_{\lambda_n} = \mathbb{H}^{r+k} \oplus \mathbb{R}^s; \quad \mathcal{R}_{\lambda_n} = \mathbb{H}^r \oplus \mathbb{R}^{s+m}.$$

Now the complexification of f_n would still be $Pin(2)-$equivariant. Denote the complexified f_n by itself again for convenience. We have a smooth equivariant map

$$f_n : V := \mathbb{H}^{2r+2k} \oplus \mathbb{C}^s := V_0 \oplus V_1 \to W_0 \oplus W_1 := \mathbb{H}^{2r} \oplus \mathbb{C}^{s+m} := W.$$

The above map also induces a smooth $Pin(2)-$equivariant map $f : \mathcal{B}V/\mathcal{S}V \to \mathcal{B}W/\mathcal{S}W$, where $\mathcal{B}$ and $\mathcal{S}$ denotes the closed unit ball and closed unit sphere of a vector space. In the next section, using equivariant $K-$theory, we will show that if such a map exists, then we must have $m \geq 2k + 1$.

8 Proof of Theorem 1.8

8.1 Equivariant K-theory

So far, we have seen that if the moduli space of solutions $\mathcal{M}_g$ is compact, then there exists a $Pin(2)-$equivariant map $f : \mathcal{B}V/\mathcal{S}V \to \mathcal{B}W/\mathcal{B}W$ between spheres, where V and W are $Pin(2)-$representations constructed in the previous section. If the following proposition holds, then we immediately obtain Theorem 1.8.

Proposition 8.1. *If there exists a $Pin(2)-$equivariant map $f : \mathcal{B}V/\mathcal{S}V \to \mathcal{B}W/\mathcal{S}W$, where $V = \mathbb{H}^{2r+2k} \oplus \mathbb{C}^s := V_0 \oplus V_1$ and $W = \mathbb{H}^{2r} \oplus \mathbb{C}^{s+m} := W_0 \oplus W_1$ are $Pin(2)-$representations, then either $k = 0$ or $m \geq 2k + 1$.*

Indeed, suppose that Proposition 4.1 is true for now, apply it to the setting where f is the induced map of a finite dimensional approximation for $\mathcal{F}$ where $\mathcal{F}^{-1}(0) = \mathcal{M}_g$ is compact. Since X is a closed simply connected smooth spin $4-$manifold whose intersection form is indefinite, obviously $b_2(X) \geq 2$. The inequality in Theorem 1.2 satisfies vacuously when $k = 0$. Otherwise, $m \geq 2k + 1$. Equivalently, $b_2^+(X) \geq 19\,\sigma(X)/8 + 1$. Scaling the inequality by 2, we have

$$b_2(X) + \sigma(X) = 2\,b_2^+(X) = 2m \geq 4k + 2 = \frac{19}{4}\,\sigma(X) + 2.$$

The above inequality implies that $b_2(X) \geq 15\,\sigma(X)/4 + 2$ as claimed.

Therefore, what remains for us to do in this section is to prove Proposition 8.1. To do that, we need to recall some facts about equivariant $K-$theory for $Pin(2)$. The results and definitions that are about to be listed exist in a variety of places in the literature; for example, see [23]. We summarize them here for the sake of self-containment.

Let M be any compact Hausdorff space and G is any compact Lie group acting on M. An $G-$equivariant complex vector bundle is a complex vector bundle $\pi : E \to M$ that carries a $G-$action such that π is equivariant. The group $K_G(M)$ is the Grothendieck group of the semigroup of equivalence classes of complex $G-$vector bundles over M. For two complex $G-$vector bundles $E \to M$ and $F \to M$, we write $E \ominus F$ as its equivalence class in $K_G(M)$. Note that $K_G(\star)$ is exactly the Grothendieck ring of the set containing classes of equivalent finite dimensional unitary representations of G, which is denoted by the representation ring $\mathcal{R}(G)$. The functor K_G is a homotopy invariant; and hence, any contractible space M has $K_G(M) \cong \mathcal{R}(G)$. The functor K_G is also contravariant. In the case where M has no $G-$action, $K_G(M) := K(M) \otimes \mathcal{R}(G)$.

If $N \subset M$ that is also compact and has a G action, the inclusion $\star \to M/N$ induces a group homomorphism $K_G(M/N) \to \mathcal{R}(G)$. The relative K_G-group $K_G(M,N)$ is defined to be the kernel of aforementioned group homomorphism. Elements of $K_G(M,N)$ are represented by $E \ominus_\varphi F$ such that $\varphi : E|_M \to E|_N$ is an isomorphism.

Let V be a finite dimensional unitary representation of G. Naturally, $\mathcal{B}V$ and $\mathcal{S}V$ are compact and have $G-$actions. So we set $M := \mathcal{B}V$ and $N := \mathcal{S}V$. Then, define the *equivariant Thom class* $\tau_V \in K_G(\mathcal{B}V, \mathcal{S}V)$ by

$$\tau_V := \Lambda^{0,\text{even}} V^* \ominus_\varphi \Lambda^{0,\text{odd}} V^*.$$

The following theorem is an important and deep result that tells us $K_G(\mathcal{B}V, \mathcal{S}V)$ is a module over $\mathcal{R}(G)$, and as a module, it is generated by τ_V.

Theorem 8.2 (Bott). *Suppose V is a finite dimensional unitary representation of G. Then $K_G(\mathcal{B}V, \mathcal{S}V)$ is naturally isomorphic to $\mathcal{R}(G)$ via the homomorphism*

$$\mathcal{R}(G) \to K_G(\mathcal{B}V, \mathcal{S}V), \quad \rho \mapsto \rho \otimes \tau_V.$$

The above theorem lets us define a notion called K_G-theoretic degree. Let V and W are finite dimensional unitary representations of G. Suppose we have a smooth equivariant map $f : \mathcal{B}V/\mathcal{S}V \to \mathcal{B}W/\mathcal{S}W$ so that one has an induced map

$f^* : K_G(\mathcal{B}W/\mathcal{S}W) \to K_G(\mathcal{B}V/\mathcal{S}V)$. Since $f^*\tau_W \in K_G(\mathcal{B}V, \mathcal{S}V)$, by Theorem 4.1, there must be a unique element $a_f \in \mathcal{R}(G)$ such that $f^*\tau_W = a_f \otimes \tau_V$. This element $a_f \in \mathcal{R}(G)$ associated to a smooth map f between spheres is the K_G-theoretic degree of f. Note that a_f could be formal difference between two unitary representations. However, in the case where G acts trivially on V and W, a_f is something more familiar:

Lemma 8.3. *Suppose the group $G_0 = S^1$ acts trivially on the finite dimensional Hermitian vector spaces V and W. Let $f : \mathcal{B}V/\mathcal{S}V \to \mathcal{B}W/\mathcal{S}W$ be smooth and equivariant. Then the induced map*

$$f^* : K_{G_0}(\mathcal{B}W/\mathcal{S}W) \to K_{G_0}(\mathcal{B}V/\mathcal{S}V)$$

satsfies $f^*(\tau_W) = deg(f)\,\tau_V$, where $deg(f)$ is the usual degree of a smooth map. Consequently, if V and W have different dimensions, $f^*\tau_W = 0$.

Proof. See reference [23]. □

Beside Theorem 8.2 and Lemma 8.3, we also need to know exactly what the representation ring of $Pin(2)$ looks like before proving Proposition 8.1.

Lemma 8.4. *The representation ring of $Pin(2)$ is naturally isomorphic to the quotient ring over $\mathbb{Z}$ (also an $\mathbb{Z}-$module)*

$$\mathcal{R}(Pin(2)) \cong \frac{\mathbb{Z}[d,h]}{\langle d^2 - 1, dh - h\rangle}.$$

In particular, d is associated with the unitary representation of $Pin(2)$ over $\mathbb{C}$ where $j \mapsto -1$ and $e^{it} \mapsto 1$; and h is associated with the usual representation of the group on $\mathbb{H}$. In particular, every $1-$dimensional unitary representation is either associated with 1 or d; and every $2-$dimensional unitary representation is of the form

$$j \mapsto \begin{pmatrix} 0 & (-1)^n \\ 1 & 0 \end{pmatrix}, \quad e^{it} \mapsto \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

Proof. See reference [23]. $\square$

Lemma 8.5. *Suppose $m \geq 1$. Let $f : \mathcal{B}V/\mathcal{S}V \to \mathcal{B}W/\mathcal{S}W$ be the smooth $Pin(2)-$map where V and W are defined as in Proposition 8.1. Let a_f be the K_G-theoretic degree of f. Then $tr\,(a_f(e^{it})) = 0$ for all t. Note that if a_f happens to a formal difference between two representations of $Pin(2)$, then we take $tr\,(a_f(e^{it}))$ to be the difference between the traces of the two representations evaluated at e^{it}.*

Proof. We first set up some notations. Let $G_0 = S^1$, which is a topological cyclic subgroup of $G := Pin(2)$. Note that G_0 acts trivially on both V_0 and W_0. Also denote $f_0 : \mathcal{B}V_0/\mathcal{S}V_0 \to \mathcal{B}W_0/\mathcal{S}W_0$ by the natural restriction of f. So, f_0 is obviously smooth and G_0-equivariant. For any finite dimensional unitary representation U of G, we also define a version of equivariant Thom class λ_U in $K_G(\mathcal{B}U) \cong \mathcal{R}(G)$ by denoting $\lambda_U = \Lambda^{0,\mathrm{even}}U^* \ominus \Lambda^{0,\mathrm{odd}}U^*$. Correspondingly, λ_U^0 is denoted by the restriction of λ_U to G_0.

Next, consider the following commutative diagram

$$\begin{array}{ccc}
\tau_W \in K_G(\mathcal{B}W, \mathcal{S}W) & \xrightarrow{f^*} & K_G(\mathcal{B}V, \mathcal{S}V) \ni a_f \otimes \tau_V \\
\downarrow & & \downarrow \\
\tau_W^0 \in K_{G_0}(\mathcal{B}W, \mathcal{S}W) & \xrightarrow{f^*} & K_{G_0}(\mathcal{B}V, \mathcal{S}V) \ni a_f^0 \otimes \tau_V^0 \\
\downarrow & & \downarrow \\
\lambda_{W_1}^0 \otimes \tau_{W_0}^0 \in K_{G_0}(\mathcal{B}W_0, \mathcal{S}W_0) & \xrightarrow{f_0^*} & K_{G_0}(\mathcal{B}V_0, \mathcal{S}V_0) \ni \lambda_{V_1}^0 \otimes a_f^0 \otimes \tau_{V_0}^0
\end{array}$$

Here the top two vertical maps are just restrictions. On the other hand, since f_0^* is a $\mathcal{R}(G_0)-$linear map and by Lemma 8.3, we have $f_0^*(\lambda_{W_1}^0 \otimes \tau_{W_0}^0) = \lambda_{W_1}^0 \otimes \deg(f_0)\, \tau_{V_0}^0$. Therefore, $\lambda_{V_1}^0 \otimes a_f^0 = \deg(f_0)\, \lambda_{W_1}^0$. But because $m \geq 1$, W_0 and V_0 have different dimensions; so, $\lambda_{V_1}^0 \otimes a_f^0 = 0$. Taking the traces of these two elements $\lambda_{V_1}^0$ and a_f^0 evaluated at e^{it}, we have

$$\mathrm{tr}\,(\lambda_{V_1}^0(e^{it}))\,\mathrm{tr}\,(a_f^0(e^{it})) = 0.$$

Since $\Lambda^{0,\mathrm{even}}V_1^*$ and $\Lambda^{0,\mathrm{odd}}V_1^*$ are inequivalent unitary representations of G, $\mathrm{tr}\,(\lambda_{V_1}^0(\cdot))$

cannot be zero. Hence, $\operatorname{tr}(a_f^0(e^{it})) = 0$ as claimed for any t. □

Lemma 8.6. *For $d, h \in \mathcal{R}(Pin(2))$ in Lemma 8.4, we have $\lambda_d = 1 - d$ and $\lambda_h = 2 - h$. Furthermore as elements in $\mathcal{R}(Pin(2))$, $(1-d)^\alpha = 2^{\alpha-1}(1-d)$ and $(2-h)^\beta(1-d) = 2^\beta(1-d)$ for any integers $\alpha \geq 2, \beta \geq 1$.*

Proof. d is associated to the unitary representation of $Pin(2)$ on $\mathbb{C}$. Note that $\Lambda^{0,0}\mathbb{C}$ is the base field which corresponds to the trivial representation. And $\Lambda^{0,1}\mathbb{C}$ (which is generated by dz) is canonically isomorphic to $\mathbb{C}$; so, it corresponds to d. As a result, $\lambda_d = \Lambda^{0,0}\mathbb{C} - \Lambda^{0,1}\mathbb{C} = 1 - d$.

We identify $\mathbb{H} = \mathbb{C}^2$ as previously. Since $\Lambda^{0,0}\mathbb{C}^2$ is the base field which also corresponds to the trivial representation and $\Lambda^{0,2}\mathbb{C}^2$ (which is generated by $d\bar{z}_1 \wedge d\bar{z}_2$) is isomorphic to $\mathbb{C}$, $\Lambda^{0,0}\mathbb{C}^2 \oplus \Lambda^{0,2}\mathbb{C}^2 = 1 + 1 = 2$ in $\mathcal{R}(Pin(2))$. On the other hand, $\Lambda^{0,1}\mathbb{C}^2$ (which is generated by $d\bar{z}_1$, $d\bar{z}_2$) is isomorphic to $\mathbb{C}^2 = \mathbb{H}$. Hence, in $\mathcal{R}(Pin(2))$, $\Lambda^{0,1}\mathbb{C}^2 = h$. Therefore, $\lambda_h = 2 - h$.

Now we prove the two identities by induction. Recall that in $\mathcal{R}(Pin(2))$, $(1-d)^2 = 1 - 2d + d^2 = 2(1-d)$—this is our base case. Suppose that the $(1-d)^\alpha = 2^{\alpha-1}(1-d)$ for all α up to $n \geq 2$. Then one sees that

$$(1-d)^{n+1} = (1-d)\,2^{n-1}(1-d) = 2^{n-1}(1-d)^2 = 2^n(1-d).$$

This completes the induction. Similarly, since $dh - h = 0$ in the representation ring, we have $(2-h)(1-d) = 2 - 2d - h + dh = 2(1-d)$—this is the base case for $\beta = 1$. Assume that the identity holds for all β up to $n \geq 1$. Then we see that

$$(2-h)^{n+1}(1-d) = (2-h)\,2^n(1-d) = 2^n\,2(1-d) = 2^{n+1}(1-d).$$

The induction is concluded. We have our result as claimed. □

8.2 Proof of Proposition 8.1

With V, W, f set up as in the end of previous subeciton, and all the notations set up in the proof of Lemma 8.5, since we have a smooth equivariant map $\mathcal{B}W \to \mathcal{B}W/\mathcal{S}W$, this induces a homomorphism that is also $\mathcal{R}(G)-$linear $K_B(\mathcal{B}W, \mathcal{S}V) \to K_G(\mathcal{B}W)$ taking $\tau_W \mapsto \lambda_W$. Similarly, for V. Note that both $K_G(\mathcal{B}W)$ and $K_G(\mathcal{B}V)$ are naturally isomorphic to $\mathcal{R}(G)$. So, we have the following commutative diagram

$$\begin{array}{ccc}
\tau_W \in K_G(\mathcal{B}W, \mathcal{S}W) & \xrightarrow{f^*} & K_G(\mathcal{B}V, \mathcal{S}V) \ni a_f \otimes \tau_V \\
\downarrow & & \downarrow \\
\lambda_W \in K_G(\mathcal{B}W) & \xrightarrow{f^*} & K_G(\mathcal{B}V) \ni a_f \otimes \lambda_V \\
\downarrow & & \downarrow \\
\mathcal{R}(G) & \xrightarrow{\cong} & \mathcal{R}(G)
\end{array}$$

Thus, $\lambda_W = a_f \lambda_V$. Now because λ splits direct sum of representations into product of their λ's in the representation ring, we have

$$\lambda_V = (2-h)^{2r+2k}(1-d)^s, \quad \lambda_W = (2-h)^{2r}(1-d)^{s+m}.$$

By Lemma 8.6, we see that in $\mathcal{R}(G)$,

$$\lambda_V = (2-h)^{2r+2k}\, 2^{s-1}(1-d) = 2^{s-1}\, 2^{2r+2k}(1-d) = 2^{2r+2k+s-1}(1-d). \tag{8.1}$$

Similarly,

$$\lambda_W = (2-h)^{2r}\, 2^{m-1}(1-d) = 2^{m-1}\, 2^{2r}(1-d) = 2^{2r+s+m-1}(1-d). \tag{8.2}$$

From (8.1) and (8.2), we obtain

$$2^{2r+s+m-1}(1-d) = 2^{2r+2k+s-1}\, a_f(1-d). \tag{8.3}$$

Since $\operatorname{tr}((1-d)(j)) = \operatorname{tr}(1(j)) - \operatorname{tr}(d(j)) = 1-(-1) = 2$, when apply traces evaluated at j to (8.3), we have

$$2^{2r+s+m} = 2^{2r+2k+s} \operatorname{tr}(a_f(j)) \tag{8.4}$$

Now as element in $\mathcal{R}(G)$, by Lemma 8.4, we can write a_f as

$$a_f = \alpha_0\, 1 + \alpha_1\, d + \alpha_\ell\, h^\ell + \cdots + \alpha_2\, h.$$

Here $\alpha_0, \ldots, \alpha_\ell$ are all integers. When evaluating the trace of $a_f(j)$, we have

$$\operatorname{tr}(a_f(j)) = \alpha_0 - \alpha_1 + \alpha_\ell \operatorname{tr}(h^\ell(j)) + \cdots + \alpha_2 \operatorname{tr}(h(j)).$$

By Lemma 8.4 again, $\operatorname{tr}(h(j)) = 0$ and thus the trace of any power of h evaluated at j is also zero, it turns out that $\operatorname{tr}(a_f(j)) = \alpha_0 - \alpha_1 \in \mathbb{Z}$. Combine with (8.4), it implies that $m \geq 2k$. So, $\operatorname{tr}(a_f(j)) \geq 1$. Thus either $k = 0$ or for $m \geq 1$, $\operatorname{tr}(a_f(j))$ cannot be 1. Assume otherwise, evaluating the trace of a_f at i gives us

$$\operatorname{tr}(a_f(i)) = \alpha_0 + \alpha_1 + \alpha_\ell \operatorname{tr}(h^\ell(i)) + \cdots + \alpha_2 \operatorname{tr}(h(i)).$$

By Lemma 8.4, the trace of any power of h evaluated at i is exactly zero; and by Lemma 8.5, $\operatorname{tr}(a_f(i)) = 0$. As a result, $\alpha_0 + \alpha_1 = 0$. But it is impossible to find integer numbers α_0, α_1 that satisfy $\alpha_0 + \alpha_1 = 0$ and $\alpha_0 - \alpha_1 = 1$ simultaneously. Therefore, $\operatorname{tr}(a_f(j)) \geq 2$. Going back to (8.4), we obtain $m \geq 2k+1$ as claimed.

9 Proof of Theorem 1.10

9.1 The elliptic complex of the RSSW equations

Recall that the RSSW equations for a closed simply connected spin 4−manifold X are given by

$$Q^+\psi + \pi^-(a\cdot\psi) = 0, \;\; d^+a = \rho^{-1}(\mu(\psi))$$

Let $h = e^{iu} \in \mathcal{G}$ where u is a real-valued function, and $(\psi, a) \in \mathcal{C}$ be a point in the configuration space. Note that

$$h \cdot (\psi, a) = (e^{-iu}\psi, a + idu)$$

If h is a stabilizer of (ψ, a), then $du = 0$. Thus, u is constant. From $e^{-iu}\psi = \psi$, we have two possibilities: either $\psi \equiv 0$ or $u = 0$ when ψ is not identically 0. The obervation motivates the follow definition.

Definition 9.1. A configuration $(\psi, a) \in \mathcal{C}$ is *irreducible* if ψ is not identically 0 so that its stabilizer in $\mathcal{G}$ is exactly 1. (ψ, a) is said to be reducible if $\psi \equiv 0$, equivalently its stabilizer in $\mathcal{G}$ is $U(1)$. We denote $\mathcal{C}^*$ by the space of irreducible configurations of the RSSW equations. Consequently, $\mathcal{G}$ acts on $\mathcal{C}^*$ freely.

Definition 9.2. We define the functional of the RSSW equations
$\mathcal{F} : L^2_k((T^*X \otimes i\mathbb{R}) \oplus \mathfrak{s}^+_{3/2}) \to L^2_{k-1}((\Lambda^+T^*X \otimes i\mathbb{R}) \oplus \mathfrak{s}^-_{3/2})$ to be

$$\mathcal{F}(a, \psi) = (d^+a - \rho^{-1}(\mu(\psi)), Q^+\psi + \pi^-(a\cdot\psi)).$$

At a configuration (a, ψ), the multiplication map $M_{(a,\psi)} : L^2_{k+1}(\mathcal{G}) \to L^2_k((T^*X \otimes i\mathbb{R}) \oplus \mathfrak{s}^+_{3/2})$ is given by

$$M_{(a,\psi)}(e^{iu}) = (a + idu, e^{-iu}\psi).$$

A similar calculations in Lemma 7.1 shows that the linearizations of $\mathcal{F}$ at (a,ψ) and of $M_{(a,\psi)}$ at the identity are

$$d_{(a,\psi)}\mathcal{F}(b,\phi) = (d^+b - \rho^{-1}(d_\psi\mu(\phi)), Q^+\phi + \pi^-(b\cdot\psi + a\cdot\phi)) \tag{9.1}$$

$$dM_{(a,\psi)}(iw) = (idw, -iw\psi). \tag{9.2}$$

Putting them together, we have the following sequence of maps

$$\mathcal{E}(a,\psi) : 0 \to L^2_{k+1}(X, i\mathbb{R}) \to L^2_k((T^*X\otimes i\mathbb{R})\oplus\mathfrak{s}^+_{3/2}) \to L^2_{k-1}((\Lambda^+T^*X\otimes i\mathbb{R})\oplus\mathfrak{s}^-_{3/2}) \to 0, \tag{9.3}$$

where the second map is given by (9.2) and the third map is given by (9.1). We claim that $\mathcal{E}(a,\psi)$ is an elliptic complex for each (a,ψ) that is a solution to the RSSW equations, and we also going to calculate its Euler characteristic.

Proposition 9.1. *At any solution (a,ψ) to the RSSW equations, $\mathcal{E}(a,\psi)$ is an elliptic complex.Furthermore, its Euler characteristic is given by $1 + b_2^+(X) - \frac{19}{4}\sigma(X)$.*

Proof. First we show that $\mathcal{E}(a,\psi)$ is a complex when (a,ψ) is a solution to the RSSW equations. Indeed, at $iw \in L^2_{k+1}(X, i\mathbb{R})$ we have

$$d_{(a,\psi)}\mathcal{F}\circ dM_{(a,\psi)}(iw) = (id^+dw - \rho^{-1}(d_\psi\mu(-iw\psi)), Q^+(-iw\psi) + \pi^-(idw\cdot\psi + a\cdot(-iw\psi))).$$

Note that

1. $id^+dw = iP^+d^2w = 0$.

2. $d_\psi\mu(-iw\psi) = \psi(-iw\psi)^* + (-iw\psi)\psi^* - \frac{1}{2}tr\,(\psi(-iw\psi)^* + (-iw\psi)\psi^*)1 = 0$.

3. Leibniz rule tells us that

$$Q^+(-iw\psi) = -iQ^+(w\psi) = -i\pi^-\mathcal{D}^+(w\psi) = -i\pi^-(w\mathcal{D}^+\psi + \rho(dw)\psi) \tag{9.4}$$

$$= -wQ^+\psi - \pi^-(\rho(idw)\psi) \tag{9.5}$$

4. (9.5) let us rewrite the second component of $d_{(a,\psi)}\mathcal{F} \circ dM_{(a,\psi)}(iw)$ as

$$Q^+(-iw\psi) + \pi^-(idw \cdot \psi + a \cdot (-iw\psi)) = -iwQ^+\psi - \pi^-(\rho(idw)\psi) \tag{9.6}$$

$$+ \pi^-(\rho(idw)\psi) - iw\pi^-(\rho(a)\psi) \tag{9.7}$$

$$= -iw(Q^+\psi + \pi^-(\rho(a)\psi)) = 0. \tag{9.8}$$

As a result, $d_{(a,\psi)}\mathcal{F} \circ dM_{(a,\psi)}(iw) = 0$, which means that $\mathcal{E}(a,\psi)$ is a complex as claimed.

Next we show $\mathcal{E}(a,\psi)$ is elliptic. To do this, we deform the complex as follows: For $t \in [0,1]$, denote $\mathcal{E}(a,\psi)_t$ by

$$\mathcal{E}(a,\psi)_t : 0 \to L^2_{k+1}(X, i\mathbb{R}) \to L^2_k((T^*X \otimes i\mathbb{R}) \oplus \mathfrak{s}^+_{3/2}) \to L^2_{k-1}((\Lambda^+T^*X \otimes i\mathbb{R}) \oplus \mathfrak{s}^-_{3/2}) \to 0, \tag{9.9}$$

where the second map is given by:

$$d_{(a,\psi)}\mathcal{F}_t(b,\phi) = (d^+b - t\rho^{-1}(d_\psi\mu(\phi)), Q^+\phi + \pi^-(tb \cdot \psi + a \cdot \phi))$$

and the third map is given by:

$$dM_{(a,\psi)t}(iw) = (idw, -tiw\psi)$$

Similar calculations show that $\mathcal{E}(a,\psi)_t$ is also a complex for any $t \in [0,1]$ and (a,ψ) solves the RSSW equations. Note that $\mathcal{E}(a,\psi)_1$ is the the original complex (9.3). While at $t = 0$, we have the second map of (9.9) is given by $(d, 0)$ and the third map is given by

$(d^+, Q^+ \cdot + \pi^-(a\cdot))$. In fact, $\mathcal{E}(a,\psi)_0 = \mathcal{A} \oplus \mathcal{B}$ where

$$\mathcal{A} : 0 \longrightarrow L^2_{k+1}(X, i\mathbb{R}) \xrightarrow{d} L^2_k(T^*X \otimes i\mathbb{R}) \xrightarrow{d^+} L^2_{k-1}(\Lambda^+ T^*X \otimes i\mathbb{R}) \longrightarrow 0,$$

and

$$\mathcal{B} : 0 \longrightarrow 0 \longrightarrow L^2_k(\mathfrak{s}^+_{3/2}) \xrightarrow{Q^+ \cdot + \pi^-(a\cdot)} L^2_{k-1}(\mathfrak{s}^-_{3/2}) \longrightarrow 0.$$

Since the map $\phi \mapsto Q^+\phi + \pi^-(a \cdot \phi)$ is a compact perturbation of the Fredholm map Q^+, it is also Fredholm. Furthermore, its index is also the *index* Q^+. Hence, $\mathcal{B}$ is an elliptic complex. Recall that $\mathcal{A}$ is the elliptic complex introduced in subsection 2.4. As a result, $\mathcal{E}(a,\psi)_0$ is elliptic. In turn, the original complex (9.3) is also elliptic.

Denote $\chi(\mathcal{E})$ by the (real) Euler characteristic of an elliptic complex $\mathcal{E}$. Euler characteristic is preserved under continuous deformation of elliptic complex. By Proposition 6.1 and the discussion at the end of subsection 2.4, we have

$$\chi(\mathcal{E}(a,\psi)) = \chi(\mathcal{E}(a,\psi)_0) = \chi(A) + \chi(B) = -\mathit{index}\,(d^+ + d^*) - \mathit{index}\, Q^+ \tag{9.10}$$

$$= 1 + b_2^+(X) - \frac{19}{4}\sigma(X) \tag{9.11}$$

as claimed. □

Let $\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}$ and $\mathcal{M}^*$ be a collection of $[a,\psi] \in \mathcal{B}^*$ that are irreducible solution to the RSSW equations. Suppose $[a,\psi] \in \mathcal{M}^*$. Then the zeroth cohomology $\mathcal{H}^0_{(a,\psi)}$ of $\mathcal{E}(a,\psi)$ is trivial, while the first cohomology $\mathcal{H}^1_{(a,\psi)}$ is a finite dimensional linear subspace of $T_{[a,\psi]}\mathcal{B}^*$. This is called the *Zariski tangent space* of the moduli space $\mathcal{M}$. The second cohomology $\mathcal{H}^2_{(a,\psi)}$ of $\mathcal{E}(a,\psi)$ is called the *obstruction space.* $\mathcal{M}$ is called *smooth* at an irreducible solution of the obstruction space is trivial. If this is the case, by the implicit function theorem, a neighborhood of $[a,\psi] \in \mathcal{M}$ is a smooth finite dimensional submanifold of the banach manifold $\mathcal{B}^*$ where the tangent space at $[a,\psi]$ is the Zariski tangent space. Note that

$$\chi(\mathcal{E}(a,\psi)) = dim\,\mathcal{H}^0_{(a,\psi)} - dim\,\mathcal{H}^1_{(a,\psi)} + dim\,\mathcal{H}^2_{(a,\psi)} = -dim\,\mathcal{H}^1_{(a,\psi)}$$

Thus, we arrive at the following corollary

Corollary 9.2. *If $[a,\psi] \in \mathcal{M}^*$, then the dimension of the Zariski tangent space to $\mathcal{M}$ at $[a,\psi]$ is*

$$d = \frac{19}{4}\sigma(X) - b_2^+(X) - 1.$$

In other words if $\mathcal{M}$ is smooth at $[a,\psi]$, then around $[a,\psi]$, $\mathcal{M}$ is smooth manifold of dimension d.

9.2 Parametrized transversality

Let $F : M \to N$ be a smooth Fredholm map between Banach manifolds, this just means that dF is always a Fredholm operator at each point in M. Sard-Smale theorem [24] would tell us that the pre-image of a regular value of F would be a finite dimensional manifold whose dimension is $index\, dF$. Moreover, given an arbitrary point $y \in N$, even if $F^{-1}(y)$ is not a submanifold of finite dimension in X, Sard-Smale theorem still allows us to choose another y' that is in some neighborhood of y such that $F^{-1}(y')$ is indeed a submanifold. However, in practice, it is often not the case that we have such degree of freedom. When there are actions of Lie group G on both X and Y, typically we want y to be a fixed point of the group action so that $F^{-1}(y)$ is also $G-$equivariant. This further restricts the choices of y and Sard-Smale theorem cannot be applied directly. In situation like this, we can adjust our map F by a certain appropriate perturbation: Construct $\tilde{F} : M \times W \to N$ that satisfies the following

1. For any $(x,w) \in \tilde{F}^{-1}(y)$, $d_{(x,w)}\tilde{F}$ is surjective so that y is a regular value of $\tilde{F}$.

2. This means that $\tilde{F}^{-1}(y) \subset M \times W$ as a Banach submanifold.

3. Consider $\pi : M \times W \to N$ a projection onto the second factor. Then the restriction of π to $\tilde{F}^{-1}(y)$ is a Fredholm map. This means that genereically, $\pi^{-1}(w) := F_w^{-1}(y)$ is finite dimensional manifold of dimension equals to $index\, d\pi$.

This method of obtaining transversality is used for ASD equations where the perturbation W is the Banach manifold of conformal classes of metric on X. While in the case Seiberg-Witten equations, W is $i\Omega^+(X)$. For our RSSW equations being in a similar set-up with the Seiberg-Witten equations but with Rarita-Schwinger operator, we also use $i\Omega^+(X)$ as our perturbation.

Definition 9.3. We define the following map $\tilde{\mathcal{F}} : \mathcal{C}_{L^2_k} \times L^2_{k-1}(\Lambda^+ T^*X \otimes i\mathbb{R}) \to \mathcal{R}_{L^2_{k-1}}$ to be

$$\tilde{\mathcal{F}}(a, \psi, \omega) = (d^+ a - \mu(\psi) - \omega, Q^+\psi + \pi^-(a \cdot \psi)).$$

Note that $\tilde{\mathcal{F}}|_{\mathcal{C}\times\{0\}} = \mathcal{F}$ that is defined in Definition 9.2. Furthermore, $R^\pm : \psi \mapsto Q^\pm\psi + \pi^\mp(a \cdot \psi)$ has an analytic continuation property because it is also a twisted Rarita-Schwinger operator.

Lemma 9.3. *Suppose V, W, H are Hilbert spaces. Let $T : V \oplus H \to W \oplus H$ be given by $T(v, h) = (Rv, Qv - h)$ where $R : V \to W$ and $Q : V \to H$ are linear so that T is also a linear map. Then R is surjective if and only if T is also surjective.*

Proof. Denote $\pi_1 : W \oplus H \to W$ and $\pi_2 : W \oplus H \to H$, respectively, by the projections onto the appropriate subspaces of the direct sum. Note that $T = (\pi_1 \circ T) \oplus (\pi_2 \circ T)$. Then if we can show that $\pi_i \circ T$ is surjective for $i = 1, 2$, then T is also surjective.

Let $\iota_2 : H \to V \oplus H$ be the inclusion map given by $h \mapsto (0, h)$. Since $\pi_2 \circ T \circ \iota_2(h) = -h$–which is clearly surjective, $\pi_2 \circ T$ is surjective. Similarly, let $\iota_1 : V \to V \oplus H$ be the inclusion $v \mapsto (v, 0)$. We note that $\pi_1 \circ T \circ \iota_1 = R$ and R is surjective. Therefore, $\pi_1 \circ T$ is also surjective. □

Lemma 9.4. *Suppose that for all $(a,\psi,\omega) \in \tilde{\mathfrak{F}}^{-1}(0)$ where $\psi \neq 0$, $\mathcal{H}^2_{(a,\psi)} = 0$. Then $d\tilde{\mathfrak{F}}$ is surjective at any $(a,\psi,\omega) \in \tilde{\mathfrak{F}}^{-1}(0)$ where $\psi \neq 0$.*

Proof. Similar calculations to the proof of Lemma 7.1, note that the differential of $\tilde{\mathfrak{F}}$ at any point (a,ψ,ω) is given by

$$d_{(a,\psi,\omega)}\tilde{\mathfrak{F}}(b,\phi,\eta) = (d^+b - \rho^{-1}(d_\psi\mu(\phi)) - \rho^{-1}(\eta), Q^+\phi + \pi^-(b\cdot\psi + a\cdot\phi)).$$

By Lemma 9.3, it suffices to show that the map
$G : (b,\phi) \mapsto Q^+\phi + \pi^-(b\cdot\psi + a\cdot\phi) = R^+\phi + \pi^-(b\cdot\psi)$ is surjective. But since $\mathcal{H}^2_{(a,\psi)} = 0$, $d_{(a,\psi)}\mathfrak{F}$ is surjective. By Lemma 9.3 again, $d_{(a,\psi)}\mathfrak{F}$ is surjective if and only if its second component which is the same as G is surjective. And we our result as claimed. □

Let $\pi : \mathcal{C}_{L^2_k} \times L^2_{k-1}(\Lambda^+T^*X \otimes i\mathbb{R}) \to L^2_{k-1}(\Lambda^+T^*X \otimes i\mathbb{R})$ be the projection onto the second component. From Lemma 9.4, we see that 0 is a regular value of $\tilde{\mathfrak{F}}$, which means $\tilde{\mathfrak{F}}^{-1}(0)$ is a submanifold $\mathcal{N}$ of $\mathcal{C}_{L^2_k} \times L^2_{k-1}(\Lambda^+T^*X \otimes i\mathbb{R})$.

Lemma 9.5. *Suppose that for all $(a,\psi,\omega) \in \tilde{\mathfrak{F}}^{-1}(0)$ where $\psi \neq 0$, $\mathcal{H}^2_{(a,\psi)} = 0$. Consider $\pi : \mathcal{N} \to L^2_{k-1}(\Lambda^+T^*X \otimes i\mathbb{R})$. Then $d\pi$ is always a Fredholm operator restricted to the solutions where $\psi \neq 0$. Furthermore, $\text{index}_{\mathbb{R}}\, d\pi = 19\sigma(X)/4 - b_2^+(X) - 1$.*

Proof. Recall the functional $\mathcal{F}$ defined at the end of subection 6.2. First, we claim that

$$T_{(a,\psi,\omega)}\mathcal{N} = \{(b,\phi,\eta) : d_{(a,\psi)}\mathcal{F}(b,\phi) = (0,0,\eta)\}. \tag{9.12}$$

Consider the following complex associated to the functional $\tilde{\mathfrak{F}}$

$$\tilde{\mathfrak{E}} : 0 \longrightarrow L^2_{k+1}(X, i\mathbb{R}) \to \mathcal{C}_{L^2_k} \oplus L^2_k(\Lambda^+T^*X \otimes i\mathbb{R})) \to \mathcal{R}_{L^2_{k-1}} \longrightarrow 0,$$

where the second map is the differential at the identity of the map
$\tilde{M}(a,\psi,\omega)(e^{iu}) = (a + idu, e^{-iu}\psi, \omega)$, and the third map is the differential at (a,ψ,ω) of $\tilde{\mathfrak{F}}$.

Similar argument as in the proof of Proposition 9.1, $\tilde{\mathfrak{E}}$ is an elliptic complex. And we see that at each irreducible solution of $\tilde{\mathfrak{F}}$, its Zarisky tangent space is the first cohomology of $\tilde{\mathfrak{E}}$. Consequently,

$$T_{(a,\psi,\omega)}\mathcal{N} = \frac{ker\, d_{(a,\psi,\omega)}\tilde{\mathfrak{F}}}{im\, d\tilde{M}_{(a,\psi,\omega)}}.$$

Note that if (b, ϕ, η) is in the kernel of $d_{(a,\psi,\omega)}\tilde{\mathfrak{F}}$, then

$$d^+ b - \rho^{-1}(d_\psi \mu(\phi)) = \eta, \qquad Q^+\phi + \pi^-(b \cdot \psi + a \cdot \phi) = 0.$$

Now for (b, ϕ, η) to be in the orthogonal complement of $d\tilde{M}_{(a,\psi,\omega)}$, it is necessary and sufficient that $b \in im\, d^\perp$. Immediately, this implies that $d^* b = 0$. As a result, (9.12) is true as claimed.

Next we show that $d\pi$ restricted to $T_{(a,\psi,\omega)}\mathcal{N}$ is always a Fredholm operator. Note that from (9.12), we already have $ker\, d_{(a,\psi,\omega)}\pi = ker\, d_{(a,\psi)}\mathcal{F}$. On the other, it is not hard to see that

$$im\, d_{(a,\psi,\omega)}\pi = \{\eta \in L^2_{k-1}(\Lambda^+ T^* X \otimes i\mathbb{R}) : (0, 0, \eta) = d_{(a,\psi)}\mathcal{F}(b, \phi)\} \tag{9.13}$$

$$= im\, d_{(a,\psi)}\mathcal{F} \cap (0 \oplus 0 \oplus L^2_{k-1}(\Lambda^+ T^* X \otimes i\mathbb{R})). \tag{9.14}$$

Hence, $d_{(a,\psi,\omega)}\pi$ has closed image of finite codimension. This implies that $d\pi$ is Fredholm. To show that its real Fredholm index is $19\sigma(X)/4 - b_2^+(X) - 1$, we claim that $dim\, coker\, d\pi = dim\, coker\, d\mathcal{F}$. Equivalently, it suffices to show the following

$$(\Gamma(\mathfrak{s}^-_{3/2}) \oplus \Omega^0 \oplus \{0\}) \cap im\, d\mathcal{F}^\perp = \{0\}. \tag{9.15}$$

Indeed, assume (9.15) holds for now, then we have

$$\mathcal{R} = [(\Gamma(\mathfrak{s}^-_{3/2}) \oplus \Omega^0 \oplus \{0\}) \cap im\, d\mathcal{F}^\perp]^\perp = (0 \oplus 0 \oplus \Gamma(\Lambda^+ T^* X \otimes i\mathbb{R})) + im\, d\mathcal{F}.$$

Combine with (9.14), we also have

$$\mathcal{R} = \frac{0 \oplus 0 \oplus \Gamma(\Lambda^+ T^*X \otimes i\mathbb{R})}{im\, d\pi} \oplus im\, d\mathcal{F}.$$

But $\mathcal{R} = im\, d\mathcal{F}^{\perp} \oplus im\, d\mathcal{F}$. Therefore,

$$\frac{L^2_{k-1}(\Lambda^+ T^*X \otimes i\mathbb{R})}{im\, d\pi} \cong im\, d\mathcal{F}^{\perp}$$

so that $dim\, coker\, d\pi = dim\, coker\, d\mathcal{F}$ holds. From calculations in the previous sections, we have $index\, d\pi = index\, d\mathcal{F} = 19\sigma(X)/4 - b_2^+(X) - 1$ as desired.

We wrap it up by proving (9.15). Suppose $(\sigma, u, 0) \in (\Gamma(\mathfrak{s}^-_{3/2}) \oplus \Omega^0 \oplus \{0\}) \cap im\, d\mathcal{F}^{\perp}$ and $(\sigma, u, 0) \neq 0$. Then $R^-\sigma = 0$. Furthermore, for any (b, ϕ), we have

$$0 = \langle (\sigma, u, 0), (R^+\phi + \pi^-(b \cdot \psi), d^*b, d^+b - \rho^{-1}(d_\psi \mu(\phi)) \rangle \tag{9.16}$$

$$= \langle \sigma, R^+\phi \rangle + \langle \sigma, \pi^-(b \cdot \psi) \rangle + lau, d^*b \rangle = \langle R^-\sigma, \phi \rangle + \langle \sigma, b \cdot \psi \rangle + \langle u, d^*b \rangle \tag{9.17}$$

So, $\langle \sigma, b \cdot \psi \rangle = -\langle u, d^*b \rangle$ for all purely imaginary $1-$form b. In particular, if $d^*b = 0$, then $\langle \sigma, b \cdot \psi \rangle = 0$. Define $b \in i\Omega^1$ such that $\langle b, a \rangle = \langle \sigma, a \cdot \psi \rangle$. Since $R^+\psi = 0$ and $R^-\sigma = 0$, the divergence theorem tells us that

$$d^*b = \langle \mathcal{D}^+\psi, \sigma \rangle - \langle \psi, \mathcal{D}^-\sigma \rangle = \langle R^+\psi, \sigma \rangle - \langle \psi, R^-\sigma \rangle = 0.$$

Let $a := b$ and we arrive at $\langle b, b \rangle = \langle \sigma, b \cdot \psi \rangle = 0$, which means that $b = 0$. Since $\psi \neq 0$, $\sigma = 0$ on some open set. But σ solve R^-. So by analytic continuation, $\sigma = 0$ on all of X. Since $\langle u, d^*b \rangle = 0$ for any b, we also have $u = 0$ on all of X. This is a contradiction. □

Lemma 9.6. *When $b_2^+(X) \geq 1$, there is a perturbation by $\omega \in i\Omega^+(X)$ such that there is no reducible solutions to the perturbed RSSW equations.*

Proof. Recall that (a, ψ) is a reducible solution to the RSSW equations when $\psi = 0$. As a result, $d^+a = \omega$. Since $b_2^+(X) \geq 1$, there is a non-trivla harmonic class $[\omega] \in \mathcal{H}^+$. Take this to be our perturbation of the RSSW equations, and there would be no imaginary $1-$form a such that $d^+a = \omega$. Consequently, there would not be any reducible solution to such a pertrubation of the RSSW equations. $\square$

Theorem 9.7. (Theorem 1.10) *Suppose X is a simply connected smooth spin $4-$manifold such that $b_2^+(X) \geq 1$. Furthermore assume that for every (a, ψ) an solution to the RSSW equations, $\mathcal{H}^2_{(a,\psi)} = 0$. Then there is a generic self-dual $2-$form ω on X such that the following holds. The gauge equivalence classes of pairs $[a, \psi]$ that solves the perturbed RSSW equations:*

$$Q^+\psi + \pi^-(a \cdot \psi) = 0, \qquad d^+a = \rho^{-1}(\mu(\psi)) + \omega$$

forms a smooth manifold of dimension

$$d = \frac{19}{4}\sigma(X) - b_2^+(X) - 1.$$

Proof. By Lemma 9.5 and the Sard-Smale theorem, for a generic $[\omega] \in \mathcal{H}^+$, $\pi^{-1}(0)$ is finite dimensional manifold whose dimension is exactly

$$d = \frac{19}{4}\sigma(X) - b_2^+(X) - 1.$$

By Lemma 9.6, all solutions to the perturbation by ω of the RSSW equations are irreducible. Thus, the gauge group acts freely. Therefore, there is no singularity in the moduli space and that

$$\{[a, \psi] : Q^+\psi + \pi^-(a \cdot \psi) = 0, \quad d^+a = \rho^{-1}(\mu(\psi)) + \omega\}$$

is a smooth finite dimensional manifold of dimension d. $\square$

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