Diederich-Fornæss Index on Boundaries Containing Crescents

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Diederich-Fornæss Index on Boundaries Containing Crescents

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Abstract

The worm domain developed by Diederich and Fornæss is a classic example of a bounded pseudoconvex domains that fails to satisfy global regularity of the Bergman Projection, due to the set of weakly pseudoconvex points that form an annulus in its boundary. We instead examine a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^2$ whose set of weakly pseudoconvex points form a crescent in its boundary. In 2019, Harrington had shown that these types of domains satisfy global regularity of the Bergman Projection based on the existence of good vector fields. In this thesis we study the Regularized Diederich-Fornæss index of these domains, another sufficient condition for global regularity of the Bergman Projection.

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1 Introduction

We begin by introducing a fundamental question in the study of several complex variables: when is the Bergman Projection globally regular on a domain? Given some smooth bounded domain $\Omega \subset \mathbb{C}^n, n \geq 2$, the Bergman Projection, denoted by P, takes some $f \in L^2(\Omega)$ and projects it into the Bergman space, i.e., the space of holomorphic L^2 functions on Ω. When $Pf \in C^\infty(\overline{\Omega})$ for every $f \in C^\infty(\overline{\Omega})$, we say that P is globally regular on Ω and that Ω satisfies condition R (see Def. 6.3.7 in [5] and [2]). Global regularity of the Bergman Projection is closely tied to the regularity of the $\overline{\partial}$-Neumann operator (see Thm. 5.5 in [15]) which is critical to solving the inhomogenous Cauchy-Riemann equation in several complex variables (see [5] for more background). We specify several complex variables as the study in one complex variable is fundamentally different. For example, in one complex variable we have the Riemann mapping theorem which states: any simply connected domain in $\mathbb{C}$ not equal to $\mathbb{C}$ is biholomorphic to the unit disk. This means the function theory in one complex variable does not depend on the geometry of the domain in question so much as the topology, and we can reduce the study of simply connected domains to the studying of the unit disk. However there is no such equivalent theorem in several complex variables, with a classic counterexample being that the polydisc is not biholomorphic to the unit ball in $\mathbb{C}^n$ (Thm. 1.7.1 of [5].) All of this is to say, we must examine the geometry of the regions in higher dimensions to solve the Cauchy-Riemann equation. Condition R gives us a useful tool in classifying domains of higher dimension. Theorem 6.6.13 in [5] (first developed in [2]) tells us that given two smooth bounded pseudoconvex domains D_1 and D_2 in $\mathbb{C}^n, n \geq 2$, and a biholomorphism f from D_1 to D_2, if both domains satisfy condition R then f extends smoothly to the boundary. While not as encompassing as the Riemann mapping theorem, this does allow us to classify similar domains that satisfy condition R.

Because of its uses, it is of great interest to identify sufficient conditions for Ω to satisfy condition R. One such condition is the existence of what we will call good vector fields on

$\partial\Omega$ developed by Boas and Straube and refined in [4].

Definition 1.1. Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain with smooth boundary and let ρ be a defining function of Ω. We say that Ω admits a family of good vector fields if there exists a constant $C > 0$ such that for every $\varepsilon > 0$ there exists a $(1, 0)$ vector field X_ε with smooth coefficients on some neighborhood U_ε of the set of weakly pseudoconvex points $K \subset \partial\Omega$ satisfying

1. $C^{-1} < |X_\varepsilon \rho| < C$ on U_ε,
2. $|\arg X_\varepsilon \rho| < \varepsilon$ on U_ε, and
3. $|\partial\rho([X_\varepsilon, \partial/\partial\bar{z}_j])| < \varepsilon$ on U_ε for $j = 1$ and $j = 2$.

This condition is concerned with the geometry of our domain, specifically on the weakly pseudoconvex points of the boundary as the strictly pseudoconvex points are known to not be a hindrance to global regularity. We gain from this family of good vector fields that on the weakly pseudoconvex points, the commutators from property 3 are well behaved. That is, the commutators between the vector field X_ε and $\overline{\partial}$ are small, and thus we can obtain good Sobolev estimates necessary for global regularity.

The existence of these good vector fields has also been linked directly to an object known as D'Angelo's 1-form α [8]. We will use the notation adopted from equation (5.85) in [15] to give a precise definition for such a form α. Given a defining function ρ for Ω this one form can be written as;

$$\alpha = \frac{1}{4|\partial\rho|^2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial\bar{z}_j \partial z_k} \frac{\partial\rho}{\partial z_j} dz_k + \frac{\partial^2 \rho}{\partial z_j \partial\bar{z}_k} \frac{\partial\rho}{\partial\bar{z}_j} d\bar{z}_k. \tag{1}$$

Boas and Straube mention in [3] that when restricted to the tangent space of the set of weakly pseudoconvex points of $\partial\Omega$, α is a closed form. Thus when this set is a submanifold, α forms a cohomology class of the set of weakly pseudoconvex points of $\partial\Omega$ that does not depend on the specific defining function used in the construction of α. When this cohomology class is also exact on the set of weakly pseudoconvex points, then we obtain the existence of good vector fields [3].

The second condition we will examine in this thesis is related to the Diederich-Fornæss index developed by Diederich and Fornæss [10] and formally defined by Kohn in [13]. One may look to [12] or Remark iii after Corollary 5.11 in [15] for more history on the Diederich-Fornæss index. We will be looking at the Regularized Diederich-Fornæss index of 1, adopting the terminology of Dall'ara and Mongodi in [7]. For a smooth, bounded, pseudoconvex domain Ω, the Diederich-Fornæss Index is the supremum over all exponents $0 < \eta < 1$ such that there exists some smooth defining function ρ_η for Ω and $-(-\rho_\eta)^\eta$ is plurisubharmonic on Ω. When the Diederich-Fornæss index is equal to 1 and

$$\liminf_{\eta \to 1^-} \sqrt{1-\eta} \sup_{\partial\Omega} |\nabla \varphi_\eta| = 0,$$

where $\rho_\eta = e^{-\varphi_\eta}\delta$ and δ is the signed distance function on Ω, we say that Ω has Regularized Diederich-Fornæss index of 1. It has been shown that if Ω has Regularized Diederich-Fornæss index of 1, then Ω satisfies condition R [13],[11].

We have already discussed how pseudoconvexity (a geometric property) plays a role in defining global regularity. For a time, it was thought to be true that global regularity was satisfied on all bounded pseudoconvex domains. However, Diederich and Fornæss [9] developed a series of bounded pseudoconvex domains now referred to as worm domains that do not satisfy condition R, as shown by Christ [6]. This can also be related back to the 1-form α. The set of weakly pseudoconvex points of the boundary of a worm domain forms an annulus, and when α is restricted to this annulus one can show that it is not exact [3]. Barret has shown that Sobolev estimates and thus the regularity of the $\overline{\partial}$-Neumann operator fails on these worm domains as well [1]. Liu has explicitly calculated the Diederich-Fornæss index of these worm domains in [14], which has salvaged some information from these otherwise poorly behaved domains.

While worm domains are a strong counterexample due to the annulus of weakly pseudoconvex points, for this thesis we will instead look at bounded pseudoconvex domains with a crescent in the boundary. That is, a domain whose set of weakly pseudoconvex points is homotopic to the annulus yet has simply connected interior. We will formally define these types of domains later in Sections 2. Harrington has shown that on these types of domains we achieve good vector fields and thus global regularity [12]. Harrington also shows that these types of domains have Diederich-Fornæss index of 1, yet the gradient estimate required for the regularized Diederich-Fornæss index is not shown to also be satisfied. This gives us a partial relationship between the good vector fields and the regularized Diederich-Fornæss index on these domains with crescents in the boundary. The goal of this thesis is to find a stronger relationship between these two sufficient conditions for condition R. In Sections 3 and 4 we construct an explicit function that is sufficient to show that our domains of interest have Diederich-Fornæss index of 1 as well as provide an estimate related to the gradient. As far as we know, such an explicit example has not been computed outside of a specific case of a plurisubharmonic defining function as discussed in Remark 6.3 of [11].

Theorem 1.1 (Main Theorem)**.** *Let $\Omega \subset \mathbb{C}^2$ be a smooth, bounded, pseudoconvex domain that contains a crescent K_γ in its boundary, and $\partial\Omega$ is strictly pseudoconvex on $\partial\Omega \backslash K_\gamma$. Then for every $0 < \eta < 1$ there exists some smooth, bounded, real-valued function φ_η and a neighborhood $\tilde{U}$ of $\partial\Omega$ such that $-(-\rho)^\eta$ is strictly plurisubharmonic on $\tilde{U} \cap \Omega$ where $\rho = \delta e^{-\varphi_\eta}$ and δ is the signed distance function on Ω. Furthermore,*

$$\sup_{K_\gamma} |\nabla \varphi_\eta| \leq O\left(\frac{1-\eta}{\eta} e^{\frac{4\pi D^2 \eta}{1-\eta}}\right).$$

We can also obtain an immediate corollary by following the method of Harrington [12] (based on the method of Diederich and Fornæss [9]) that allows us to find an extension of ρ to all of Ω.

Corollary 1.2. *Let $\Omega \subset \mathbb{C}^2$ be a smooth, bounded, pseudoconvex domain that contains a crescent K_γ in its boundary, and $\partial\Omega$ is strictly pseudoconvex on $\partial\Omega \backslash K_\gamma$. Then for every $0 < \eta < 1$, there exists a smooth defining function ρ_η for Ω such that $-(-\rho_\eta)^\eta$ is strictly plurisubharmonic on Ω.*

In Section 5 we construct a specific crescent region K_a and define a group of automorphisms of said crescent, as well as some useful properties of those automorphisms. In Section 5 we take the constructed crescent and ask the question: does such a function as we attempted to construct in Sections 3 and 4 exist that also satisfies the required gradient estimate for the Regularized Diederich-Fornæss index? We use a method similar to that used by Diederich and Fornæss in their study of worm domains to find such a function. By averaging a candidate function over a family of automorphisms of the crescent region, we are able to develop a representative function as well as identify some important properties.

Theorem 1.3. *Let $\Omega \subset \mathbb{C}^2$ be a bounded, smooth domain containing a crescent region K_a in its boundary such that $\partial\Omega$ is strictly pseudoconvex on $\partial\Omega \backslash K_a$. Let φ be a smooth, real valued, subharmonic function on K_a such that if we define $\rho = \delta e^{-\varphi}$ then ρ is a defining function for Ω and $-(-\rho)^\eta$ is plurisubharmonic on Ω for some $0 < \eta < 1$ and there exists some constant E such that $\sup_{K_a} |\nabla\varphi| < E$. Then for every $\varepsilon > 0, r > 0$ there exists a real valued, subharmonic function $\varphi_\varepsilon \in C^2(\partial\Omega)$ satisfying the following conditions;*

1. *There exists a defining function $\rho_\varepsilon = \delta e^{-\varphi_\varepsilon}$ for Ω where $-(-\rho_\varepsilon)^\eta$ is plurisubharmonic on Ω.*
2. $|\nabla\varphi_\varepsilon| < \left(E + \frac{a+1}{1-a}\right)\left(1 + \frac{2}{r(1-a)}\right)^2 + \frac{a+1}{1-a}$ *on* $K_a \backslash B_{(1,0)}(r)$.
3. *There exists some negative valued convex function $\psi : \mathbb{R} \to \mathbb{R}$ such that*

$$\sup_{K_a \backslash B_{(1,0)}(r)} \left| \varphi_\varepsilon(z) - \left[\frac{\eta - 1}{\eta} \log\left(-\psi\left(\frac{|z|^2 - 1}{|z-1|^2}\right)\right) - 2\mathrm{Im}\log(z)\right]\right| < \varepsilon.$$

We will end the thesis with our final conclusions and a discussion of future work.

2 Definitions and Notation

First, we give a general definition of a crescent region contained in the boundary of some domain in $\mathbb{C}^2$ which we will call K_γ. Also as a clarification of notation, we define an open ball in $\mathbb{C}$ as follows:

$$B_c(r) = \{z \in \mathbb{C} : |z - c| < r\}.$$

Definition 2.1. Let $0 < \gamma < 1$, and $6^{\frac{-\gamma}{4(1-\gamma)}} < R < 2^{\frac{-\gamma}{2(1-\gamma)}}$. Let J_1 and J_2 be Jordan regions in $\mathbb{C}$ with smooth boundary such that;

- $J_1 \subset J_2$
- $\partial J_1 \cap \partial J_2 = 1$
- $0 \in J_1$
- $\{z_1 : |z_1| = 1, z_1 \neq 1\} \subset J_2 \backslash \overline{J_1}$
- $\overline{J_2 \backslash J_1} \cap B_1(R) \subset \{z_1 : |\mathrm{Im} z_1| \geq |\mathrm{Re} z_1 - 1|^\gamma\}$

Define S_γ by

$$S_\gamma = \overline{J_2 \backslash J_1}.$$

Define a crescent region $K_\gamma \subset \mathbb{C}^2$ by

$$K_\gamma = \{(z_1, z_2) \in \mathbb{C} : z_1 \in S_\gamma, z_2 = 0\}.$$

We define the diameter D of K_γ by $D = \sup_{p_1, p_2 \in K_\gamma} |p_1 - p_2|$.

Remark 1. By construction of $K_\gamma, D > 2$.

Example 2.1. Given $\gamma > \frac{1}{2}, r_1 > 1, \frac{1}{2} < r_2 < 1$ and $S_\gamma = \{z \in \mathbb{C} : z \in \overline{B_{1-r_1}(r_1)} \backslash B_{1-r_2}(r_2)\}$, $K_\gamma = \{z \in \mathbb{C}^2 : z_1 \in S_\gamma, z_2 = 0\}$ is a crescent region.

By Proposition 1.2 of [12], if $K_\gamma \subset \partial\Omega$ for some domain $\Omega \subset \mathbb{C}^2$ and $\partial\Omega$ is strictly pseudoconvex on $\partial\Omega \backslash K_\gamma$, then Ω admits a family of good vector fields and thus satisfies condition R. To show that such a domain Ω exists, we will construct a defining function following the setup used in Lemma 7.1 of [12] which is based on the construction of the worm domains found in [9].

Example 2.2. Given a smooth function $\mu : \mathbb{C} \to \mathbb{R}, A > 0, m > 1, B > \left(\frac{mA^{m-1}+3}{m-1}\right)^{1/(m-1)}$ such that

1. $\mu(0) > A$
2. $\liminf_{|z|\to\infty} \mu(z) > 0$
3. $\mu(e^{i\theta}) \le 0$ for all $\theta \in \mathbb{R}$
4. $(\mu(z))^m \frac{\partial^2\mu}{\partial z \partial \bar{z}}(z) + \left|\frac{\partial\mu}{\partial z}(z)\right|^2 > 0$ and $\left|\frac{\partial\mu}{\partial z}(z)\right| > \frac{(\mu(z))^m}{|z|}$ whenever $0 < \mu(z) < A$
5. $\{z \in \mathbb{C} : \mu(z) < A\}$ is bounded

and $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\chi(t) = \begin{cases} 0 & t \le 0 \\ \exp(-1/t^{m-1}) & t > 0, \end{cases}$$

define the domain $\Omega \subset \mathbb{C}^2$ by the defining function $\rho : \mathbb{C}^2 \to \mathbb{R}$ where

$$\rho(z_1, z_2) = |z_2 + \exp(i \log |z_1|^2)|^2 - 1 + \exp(A^{1-m}B^{m-1})\chi(B^{-1}\mu(z_1)).$$

Ω contains a set of weakly pseudoconvex points K in its boundary where $K = \{(z_1, z_2) \in \mathbb{C}^2 : \mu(z_1) \le 0, z_2 = 0\}$.

If we want the above K to look like a crescent, given $0 < s < 1$ we can define μ by

$$\mu(z) = (|z|^2 - 1)^2 - s^2|z - 1|^4.$$

In this case, $K_\gamma = \{z \in \mathbb{C}^2 : z_1 \in \overline{B_{\frac{s}{1+s}}(\frac{1}{1+s})} \backslash B_{\frac{-s}{1-s}}(\frac{1}{1-s}), z_2 = 0\}$.

Now that we have a defining function for such domains, we can consider D'Angelo's 1-form α and a (1,1)-form β used in [12] on K_γ. We first note that on $K_\gamma, z_2 = 0$ and χ as defined above vanishes along with all of its derivatives. Given the defining function ρ from Example 2.2,

$$|\partial\rho|^2|_{K_\gamma} = \frac{1}{4}\left|\exp(-i\log|z_1|^2)\right|^2 = \frac{1}{4}.$$

So for the 1-form α on K_γ,

$$\alpha = \sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial\bar{z}_j\partial z_k}\frac{\partial\rho}{\partial z_j}dz_k + \frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}\frac{\partial\rho}{\partial\bar{z}_j}d\bar{z}_k$$
$$= \frac{i}{z_1}dz_1 + \exp(-i\log|z_1|^2)dz_2 - \frac{i}{\bar{z}_1}d\bar{z}_1 + \exp(i\log|z_1|^2)d\bar{z}_2.$$

For our purposes, it will be useful to look at the projection of α onto its (1,0) and (0,1)-components, which we will denote by $\pi_{1,0}\alpha$ and $\pi_{0,1}\alpha$ respectively. So,

$$\pi_{1,0}\alpha = \frac{1}{2}\left(\frac{i}{z_1}dz_1 + \exp(-i\log|z_1|^2)dz_2\right)$$

and

$$\pi_{0,1}\alpha = \frac{1}{2}\left(\frac{-i}{\bar{z}_1}d\bar{z}_1 + \exp(i\log|z_1|^2)d\bar{z}_2\right).$$

We will also define the form β by

$$\beta = i\sum_{j,k=1}^{2} \partial\left(\frac{\partial\rho}{\partial z_j}\right)\left(I_{j,k} - 4\frac{\partial\rho}{\partial\bar{z}_j}\frac{\partial\rho}{\partial z_k}\right) \wedge \overline{\partial}\left(\frac{\partial\rho}{\partial\bar{z}_k}\right),$$

where $I_{j,k}$ are the entries of the identity matrix. This form plays an important role in [12] for finding good vector fields on Ω. For our purposes, we will restrict β to the region K_γ

where $z_2 = 0$. So using our defining function ρ gives us;

$$\beta = \frac{i}{4|z_1|^2} dz_2 \wedge d\bar{z}_2.$$

We can now construct a new operator which will be important for the examination of the Diederich-Fornæss index of Ω.

Definition 2.2. Given a real valued function $f \in C^2(S_\gamma \backslash \{1\})$ and $0 < \eta < 1$, define the operator P_η acting on f as follows;

$$P_\eta f = i\partial\overline{\partial} f + 2\beta - i\frac{\eta}{1-\eta}(\partial f - 2\pi_{1,0}\alpha) \wedge (\overline{\partial} f - 2\pi_{0,1}\alpha). \tag{2}$$

Harrington has shown that when there exists a smooth, bounded, real-valued f such that $P_\eta f > 0$ on S_γ for every $0 < \eta < 1$, then the Diederich-Fornæss index of a domain Ω whose set of weakly pseudoconvex points form a crescent K_γ in its boundary is 1 [12]. To clarify what it means for a (1,1)-form to be positive, we will start by examining a (1,1)-form on $\mathbb{C}$. Given some function g, $ig(z)dz \wedge d\bar{z}$ is written with standard orientation in $\mathbb{C}$. Given $z = x + iy$, if we expand the term $idz \wedge d\bar{z}$, we see

$$idz \wedge d\bar{z} = i\,(dx + idy) \wedge (dx - idy) = i\,(-idx \wedge dy + idy \wedge dx) = 2dx \wedge dy.$$

Since $dx \wedge dy$ is the standard orientation in two real variables, we say $2g(z)dx \wedge dy > 0$ if and only if $g(z)$ is real valued and $g(z) > 0$.

In $\mathbb{C}^2$, a (1,1)-form would look like

$$ig_1(z)dz_1 \wedge d\bar{z}_1 + ig_2(z)dz_1 \wedge d\bar{z}_2 + i\overline{g_2(z)}dz_2 \wedge d\bar{z}_1 + ig_3(z)dz_2 \wedge d\bar{z}_2.$$

So the functions $g_1, g_2, \overline{g_2}$ and g_3 form the entries of a Hermitian matrix. We say the above form is positive if and only if the Hermitian matrix has positive eigenvalues. That is, if

$g_1(z) > 0, g_3(z) > 0$, and $g_1(z)g_3(z) - |g_2(z)|^2 > 0$.

3 Result on One-Dimensional Crescent Region

While [12] shows that for every $0 < \eta < 1$ there exists some φ_η such that $P_\eta \varphi_\eta > 0$ on S_γ, we will create such a function while also examining an upper bound on its gradient. First, we note that when we restrict to this submanifold, we obtain $\beta \equiv 0$ and $\alpha = \frac{i}{z}dz - \frac{i}{\bar{z}}d\bar{z}$.

Lemma 3.1. *Given S_γ as defined in Definition 2.1, for every $0 < \eta < 1$, there exists a constant $E_\eta > 0$ and a smooth real valued function φ_η plurisubharmonic on S_γ such that*

$$i\partial\overline{\partial}\varphi_\eta - i\frac{\eta}{1-\eta}\left(\partial\varphi_\eta - \frac{i}{z}dz\right) \wedge \left(\overline{\partial}\varphi_\eta + \frac{i}{\bar{z}}d\bar{z}\right) > 0 \tag{3}$$

on S_γ and

$$\sup_{z \in S_\gamma} |\nabla \varphi_\eta(z)| < E_\eta. \tag{4}$$

Furthermore, $E_\eta = O\left(\frac{1-\eta}{\eta}e^{\frac{4\pi D^2 \eta}{1-\eta}}\right)$.

Proof. Let $0 < \eta < 1$. Choose ε such that

$$0 < \varepsilon < \min\left\{\frac{4\pi}{D^2}, \frac{1-\eta}{D^2\eta}\right\}, \tag{5}$$

and M such that

$$1 < M < \frac{1}{2}R^{\frac{-2(1-\gamma)}{\gamma}}. \tag{6}$$

Now, choose some A such that

$$\frac{1}{R^2} < A < \frac{M}{R^2}. \tag{7}$$

Now, given such an A, we have $AR^2 - 1 > 0$. Also, given that $\varepsilon < \frac{4\pi}{D^2}$, and $R < 1 < D$, we also have that $\varepsilon < \frac{4\pi}{R^2}$, and so

$$1 < e^{\frac{4\pi}{\varepsilon} - R^2}. \tag{8}$$

So, there exists some constant B such that

$$0 < B < \frac{AR^2 - 1}{e^{\frac{4\pi}{\varepsilon} - R^2} - 1}. \tag{9}$$

Now, starting with

$$B < \frac{AR^2 - 1}{e^{\frac{4\pi}{\varepsilon} - R^2} - 1},$$

we can see

$$B(e^{\frac{4\pi}{\varepsilon} - R^2} - 1) < AR^2 - 1,$$

and by dividing both sides by B and then adding 1 to both sides we have,

$$e^{\frac{4\pi}{\varepsilon} - R^2} < \frac{AR^2}{B} + 1 - \frac{1}{B}.$$

Taking the log of both sides shows that

$$\frac{4\pi}{\varepsilon} - R^2 < \log\left(\frac{AR^2}{B} + 1 - \frac{1}{B}\right),$$

or equivalently

$$\frac{4\pi}{\varepsilon} - R^2 < \log\big(AR^2 + B - 1\big) - \log(B).$$

This can be rearranged to show

$$\log(B) < R^2 + \log(AR^2 + B - 1) - \frac{4\pi}{\varepsilon}.$$

Lastly, multiplying both sides by ε gives us

$$\varepsilon \log(B) < \varepsilon R^2 + \varepsilon \log(AR^2 + B - 1) - 4\pi.$$

So, choose a C such that

$$\varepsilon \log(B) < C < \varepsilon R^2 + \varepsilon \log(AR^2 + B - 1) - 4\pi. \tag{10}$$

Define a new function λ on S_γ by

$$\lambda(z) = \frac{2M}{R^2}(\operatorname{Re} z - 1)^2.$$

When $z \in S_\gamma \cap \overline{B_1(R)}$, $|\operatorname{Re} z - 1| \leq |\operatorname{Im} z|^{\frac{1}{\gamma}}$ by definition of S_γ. Also, since we are inside $B_1(R)$, $|\operatorname{Im} z|^{\frac{1}{\gamma}} \leq R^{\frac{1}{\gamma}}$. Given the upper bound from (6), we can see that

$$\sqrt{2M} < R^{\frac{-(1-\gamma)}{\gamma}}.$$

By reciprocating both sides, we have

$$\frac{1}{\sqrt{2M}} > R^{\frac{1-\gamma}{\gamma}},$$

and by multiplying both sides by R, we get

$$\frac{R}{\sqrt{2M}} > R^{\frac{1}{\gamma}}.$$

Now, we have that on $K_\gamma \cap \overline{B_1(R)}$, $|\operatorname{Re} z - 1| < \frac{R}{\sqrt{2M}}$, and so

$$0 < \lambda(z) < 1.$$

Now define

$$\varphi_1(z) = i\log(z) - i\log(\bar{z}) + \varepsilon|z-1|^2 \tag{11}$$

on $S_\gamma \backslash \{1\}$ with the branch cut $\{z : \operatorname{Im} z = 0, \operatorname{Re} z \geq 0\}$. More precisely,

$$\varphi_1(z) = -2\arg(z) + \varepsilon|z-1|^2 \tag{12}$$

with $\arg(z) \in (0, 2\pi)$. Now, computing some derivatives of φ_1, we can see

$$\begin{aligned} \overline{\partial}\varphi_1 &= \left(\frac{-i}{\bar{z}} + \varepsilon(z-1)\right) d\bar{z}, \\ \partial\varphi_1 &= \left(\frac{i}{z} + \varepsilon(\bar{z}-1)\right) dz, \end{aligned}$$

and

$$i\partial\overline{\partial}\varphi_1 = i\varepsilon dz \wedge d\bar{z}.$$

Since $\varepsilon > 0, \varphi_1$ is plurisubharmonic on $S_\gamma \backslash \{1\}$. We now want to check that φ_1 satisfies (3). Computing the left hand side of (3) for φ_1, we get:

$$\begin{aligned} & i\partial\overline{\partial}\varphi_1 - i\frac{\eta}{1-\eta}\left(\partial\varphi_1 - \frac{i}{z}dz\right) \wedge \left(\overline{\partial}\varphi_1 + \frac{i}{\bar{z}}d\bar{z}\right) \\ &= i\varepsilon dz \wedge d\bar{z} - i\frac{\eta}{1-\eta}\left(\frac{i}{z} + \varepsilon(\bar{z}-1) - \frac{i}{z}\right) dz \wedge \left(\frac{-i}{\bar{z}} + \varepsilon(z-1) + \frac{i}{\bar{z}}\right) d\bar{z} \\ &= i\varepsilon dz \wedge d\bar{z} - i\frac{\eta}{1-\eta}\left(\varepsilon(\bar{z}-1)dz\right) \wedge \left(\varepsilon(z-1)d\bar{z}\right) \\ &= i\varepsilon dz \wedge d\bar{z} - i\frac{\eta}{1-\eta}\varepsilon^2|z-1|^2 dz \wedge d\bar{z} \\ &= i\varepsilon\left(1 - \frac{\eta}{1-\eta}\varepsilon|z-1|^2\right) dz \wedge d\bar{z}. \end{aligned}$$

Now, using (5), we can see that

$$i\varepsilon\left(1 - \frac{\eta}{1-\eta}\varepsilon|z-1|^2\right) dz \wedge d\bar{z} > i\varepsilon\left(1 - \frac{|z-1|^2}{D^2}\right) dz \wedge d\bar{z} > 0.$$

We would also like to find an upper bound on $\sup_{S_\gamma \backslash \{1\}} |\nabla\varphi_1(z)|$. By the triangle inequality,

$$|\nabla\varphi_1| \leq |\nabla(i\log(z) - i\log(\bar{z}))| + \varepsilon|\nabla|z-1|^2|.$$

Now, computing the gradients, we have

$$|\nabla(i\log(z) - i\log(\bar{z}))| = \frac{2}{|z|}$$

and

$$|\nabla|z-1|^2| = 2|z-1|.$$

And so, we can see

$$|\nabla\varphi_1| \leq \frac{2}{|z|} + 2\varepsilon|z-1|.$$

Since $0 \notin S_\gamma$, and S_γ is closed, there exists some $0 < \zeta < \infty$ such that

$$\frac{2}{|z|} < \zeta.$$

Also, we have by definition of D and (5) that

$$2\varepsilon|z-1| < 2\frac{4\pi}{D} < 8\pi.$$

This gives us;

$$\sup_{S_\gamma\setminus\{1\}} |\nabla\varphi_1| < \zeta + 8\pi.$$

Let $E_{\eta,1} = \zeta + 8\pi$.

Now define

$$\varphi_2(z) = -\varepsilon\log(A|z-1|^2 + B - \lambda(z)) - 2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right) + C \tag{13}$$

on $S_\gamma \cap B_1(R)$. To show φ_2 is a real valued function, we must show that

$$B > \lambda(z) - A|z-1|^2$$

on $S_\gamma \cap \overline{B_1(R)}$. Define Ψ on $S_\gamma \cap \overline{B_1(R)}$ by

$$\Psi(z) = \frac{2M}{R^2}(\operatorname{Re} z - 1)^2 - A|z-1|^2.$$

We wish to maximize Ψ over $S_\gamma \cap \overline{B_1(R)}$. Since $S_\gamma \cap \overline{B_1(R)} \subset \{z : |\operatorname{Im} z| \geq |\operatorname{Re} z - 1|^\gamma\} \cap \overline{B_1(R)} := \Omega_\gamma$, it suffices to maximize Ψ over Ω_γ. Also, given the symmetry of Ω_γ and Ψ, we will find the maximum over

$$\Omega_+ = \Omega_\gamma \cap \{z : \operatorname{Im} z \geq 0\}.$$

Taking the Laplacian of Ψ;

$$\frac{\partial^2}{\partial z \partial \bar{z}} \Psi(z) = \frac{M}{R^2} - A > 0,$$

so Ψ is subharmonic. By the maximum value principle for subharmonic functions, Ψ must obtain its maximum on the boundary of Ω_+. Restricting Ψ to $\Omega_R = \Omega_+ \cap \{z : |z-1| = R\}$, we can let $z = 1 + Re^{i\theta}, 0 < \theta < \pi$. Then,

$$\Psi(z)|_{\Omega_R} = 2M\cos^2(\theta) - AR^2.$$

Since this restriction is convex in θ its maximum occurs at the endpoints of Ω_R, i.e., $\{z : (\operatorname{Re} z - 1) = \operatorname{Im} z^{\frac{1}{\gamma}}$ or $-(\operatorname{Re} z - 1) = \operatorname{Im} z^{\frac{1}{\gamma}}$, and $|z-1| = R\}$. Now, restricting Ψ to the curves $\Omega_1 = \Omega_+ \cap \{z : \operatorname{Re} z - 1 = \operatorname{Im} z^{\frac{1}{\gamma}}\}$ and $\Omega_2 = \Omega_+ \cap \{z : -(\operatorname{Re} z - 1) = \operatorname{Im} z^{\frac{1}{\gamma}}\}$ both give us

$$\Psi(z)|_{\Omega_1} = \Psi(z)|_{\Omega_2} = \frac{2M}{R^2}\operatorname{Im} z^{\frac{2}{\gamma}} - A(\operatorname{Im} z^{\frac{2}{\gamma}} + \operatorname{Im} z^2).$$

Letting $\operatorname{Im} z = y$, we will rewrite this restriction as

$$\psi(y) = \left(\frac{2M}{R^2} - A\right) y^{\frac{2}{\gamma}} - Ay^2.$$

To check for critical points, we take the derivative in y to get

$$\psi'(y) = \left(\frac{2M}{R^2} - A\right) \frac{2}{\gamma} y^{\frac{2-\gamma}{\gamma}} - 2Ay,$$

and by setting $\psi' = 0$, we have

$$\left(\frac{2M}{R^2} - A\right) \frac{2}{\gamma} y^{\frac{2-\gamma}{\gamma}} = 2Ay.$$

One solution is $y = 0$. Also, dividing both sides by $2y$ gives

$$\left(\frac{2M}{R^2} - A\right) \frac{1}{\gamma} y^{\frac{2-2\gamma}{\gamma}} = A.$$

Isolating the y term on the left hand side gives

$$y^{\frac{2-2\gamma}{\gamma}} = \gamma A \left(\frac{2M}{R^2} - A\right)^{-1},$$

and then, by taking the appropriate root of both sides, we have

$$y_0 = \left(\gamma A \left(\frac{2M}{R^2} - A\right)^{-1}\right)^{\frac{\gamma}{2-2\gamma}}.$$

Now to check for concavity at y_0 and 0,

$$\psi''(y) = \left(\frac{2M}{R^2} - A\right) \frac{2}{\gamma} \frac{2-\gamma}{\gamma} y^{\frac{2-2\gamma}{\gamma}} - 2A.$$

So evaluating at 0,

$$\psi''(0) = -2A < 0,$$

and so 0 is a local maximum for ψ with $\psi(0) = 0$. Also evaluation at y_0 gets

$$\begin{aligned}
\psi''(y_0) &= \left(\frac{2M}{R^2} - A\right)\frac{2}{\gamma}\frac{2-\gamma}{\gamma}\left(\left(\gamma A\left(\frac{2M}{R^2} - A\right)^{-1}\right)^{\frac{\gamma}{2-2\gamma}}\right)^{\frac{2-2\gamma}{\gamma}} - 2A \\
&= \left(\frac{2M}{R^2} - A\right)\frac{2}{\gamma}\frac{2-\gamma}{\gamma}\gamma A\left(\frac{2M}{R^2} - A\right)^{-1} - 2A \\
&= \frac{2-\gamma}{\gamma}2A - 2A \\
&= 2A\left(\frac{2-\gamma}{\gamma} - 1\right) \\
&= 2A\left(\frac{2-2\gamma}{\gamma}\right) \\
&= 4A\left(\frac{1-\gamma}{\gamma}\right) > 0.
\end{aligned}$$

So, y_0 is a local minimum for ψ, meaning the maximum of ψ occurs on the endpoints of its domain, being either $y = 0$ or $0 < y_1 < R$ such that $y_1^2 + y_1^{\frac{2}{\gamma}} = R^2$. If $0 < y_0 < y_1$ then we can use the fact that ψ is an increasing function for $y > y_0$, and the fact that on our domain $y < R$ to give us $\psi(y_1) < \psi(R)$. Evaluating $\psi(R)$:

$$\begin{aligned}
\psi(R) &= \left(\frac{2M}{R^2} - A\right)R^{\frac{2}{\gamma}} - AR^2 \\
&= \frac{2M}{R^2}R^{\frac{2}{\gamma}} - AR^{\frac{2}{\gamma}} - AR^2 \\
&= 2MR^{\frac{2-2\gamma}{\gamma}} - A(R^{\frac{2}{\gamma}} + R^2).
\end{aligned}$$

If however $y_0 > y_1$, we can use the fact that ψ is a decreasing function for $0 < y < y_1$ to give us that the maximum of ψ is 0. Since y_1 happens to also be the local maximum we found in the first restriction, $\max\{0, \psi(R)\} > \Psi(z)|_{\Omega_R}$ for all $z \in \Omega_R$. Now we have

$$\begin{aligned}\sup_{\Omega_+} \Psi &< \max\{0, \psi(R)\} \\ &= \max\{0, 2MR^{\frac{2-2\gamma}{\gamma}} - A(R^{\frac{2}{\gamma}} + R^2)\}.\end{aligned}$$

Now, recalling (6),

$$M < \frac{1}{2}R^{\frac{2\gamma-2}{\gamma}},$$

so we have

$$M < \frac{1}{2}R^{\frac{2\gamma-2}{\gamma}} + \frac{1}{2}.$$

Multiplying both sides by $2R^{\frac{2}{\gamma}}$ gives us

$$2MR^{\frac{2}{\gamma}} < R^2 + R^{\frac{2}{\gamma}}.$$

Now we divide both sides by $R^2(R^2 + R^{\frac{2}{\gamma}})$ to give us

$$\frac{2MR^{\frac{2}{\gamma}}}{R^2(R^2 + R^{\frac{2}{\gamma}})} < \frac{1}{R^2}.$$

Since $\frac{1}{R^2} < A$, we then know

$$\frac{2MR^{\frac{2}{\gamma}}}{R^2(R^2 + R^{\frac{2}{\gamma}})} < A,$$

and we can multiply both sides by $R^2 + R^{\frac{2}{\gamma}}$ to see that

$$2MR^{\frac{2-2\gamma}{\gamma}} < A(R^2 + R^{\frac{2}{\gamma}}).$$

Thus, we know that the supremum of Ψ on our domain is 0. By (9), φ_2 is a real valued function. Now, computing some derivatives of φ_2;

$$\begin{aligned}
\overline{\partial}\varphi_2 &= \frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\overline{\partial}(A|z-1|^2 + B - \lambda(z)) - \frac{i}{\bar{z}}d\bar{z} \\
&= \frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\left(A(z-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)\right)d\bar{z} - \frac{i}{\bar{z}}d\bar{z}, \\
\partial\varphi_2 &= \frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\partial(A|z-1|^2 + B - \lambda(z)) + \frac{i}{z}dz \\
&= \frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\left(A(\bar{z}-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)\right)dz + \frac{i}{z}dz,
\end{aligned}$$

and

$$\begin{aligned}
i\partial\overline{\partial}\varphi_2 &= \Big[i\partial\left(\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\right)\left(A(z-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)\right) \\
&\quad + i\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\partial\left(A(z-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)\right) - i\partial\frac{i}{\bar{z}}\Big]d\bar{z} \\
&= \Big[i\frac{-\varepsilon(-A(\bar{z}-1) + \frac{2M}{R^2}(\operatorname{Re} z - 1))(A(z-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1))}{(-A|z-1|^2 - B + \lambda(z))^2} \\
&\quad + i\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\left(A - \frac{M}{R^2}\right)\Big]dz \wedge d\bar{z} \\
&= \Big[i\frac{\varepsilon(-A(\bar{z}-1) + \frac{2M}{R^2}(\operatorname{Re} z - 1))(-A(z-1) + \frac{2M}{R^2}(\operatorname{Re} z - 1))}{(-A|z-1|^2 - B + \lambda(z))^2} \\
&\quad + i\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\left(A - \frac{M}{R^2}\right)\Big]dz \wedge d\bar{z} \\
&= \Big[i\frac{\varepsilon(A^2|z-1|^2 + \frac{2M}{R^2}\lambda(z) - A\frac{2M}{R^2}(\operatorname{Re} z - 1)(\bar{z} - 1 + z - 1))}{(-A|z-1|^2 - B + \lambda(z))^2} \\
&\quad + i\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\left(A - \frac{M}{R^2}\right)\Big]dz \wedge d\bar{z} \\
&= \Big[i\frac{\varepsilon(A^2|z-1|^2 + \frac{2M}{R^2}\lambda(z) - 2A\lambda(z))}{(-A|z-1|^2 - B + \lambda(z))^2} \\
&\quad + i\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\left(A - \frac{M}{R^2}\right)\Big]dz \wedge d\bar{z} \\
&= i\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\left(\frac{A^2|z-1|^2 + 2\lambda(z)(\frac{M}{R^2} - A)}{-A|z-1|^2 - B + \lambda(z)} + A - \frac{M}{R^2}\right)dz \wedge d\bar{z}.
\end{aligned}$$

Given (9), we have

$$\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)} < 0,$$

and by (7) as well as (9) we have

$$\frac{A^2|z-1|^2 + 2\lambda(z)\big(\frac{M}{R^2} - A\big)}{-A|z-1|^2 - B + \lambda(z)} + A - \frac{M}{R^2} < \frac{A^2|z-1|^2 + 2\lambda(z)\big(\frac{M}{R^2} - A\big)}{-A|z-1|^2 - B + \lambda(z)} < 0,$$

and therefore the product is positive. So we have φ_2 is plurisubharmonic on $S_\gamma \cap B(1, R)$. Also,

$$\begin{aligned}
& i\partial\overline{\partial}\varphi_2 - i\frac{\eta}{1-\eta}\left(\partial\varphi_2 - \frac{i}{z}dz\right) \wedge \left(\overline{\partial}\varphi_2 + \frac{i}{\bar{z}}d\bar{z}\right) \\
&= i\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\left(\frac{A^2|z-1|^2 + 2\lambda(z)\left(\frac{M}{R^2} - A\right)}{-A|z-1|^2 - B + \lambda(z)} + \left(A - \frac{M}{R^2}\right)\right) dz \wedge d\bar{z} \\
&\quad - i\frac{\eta}{1-\eta}\frac{\varepsilon\left(A(\bar{z}-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)\right)}{-A|z-1|^2 - B + \lambda(z)}\frac{\varepsilon\left(A(z-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)\right)}{-A|z-1|^2 - B + \lambda(z)} dz \wedge d\bar{z} \\
&= i\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\left(\frac{A^2|z-1|^2 + 2\lambda(z)\left(\frac{M}{R^2} - A\right)}{-A|z-1|^2 - B + \lambda(z)} + \left(A - \frac{M}{R^2}\right)\right) dz \wedge d\bar{z} \\
&\quad - i\frac{\varepsilon\eta}{1-\eta}\left(\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\right)\frac{A^2|z-1|^2 + 2\lambda(z)\left(\frac{M}{R^2} - A\right)}{-A|z-1|^2 - B + \lambda(z)} dz \wedge d\bar{z} \\
&= i\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)}\Bigg(\frac{A^2|z-1|^2 + 2\lambda(z)\left(\frac{M}{R^2} - A\right)}{-A|z-1|^2 - B + \lambda(z)} \\
&\quad - \frac{\varepsilon\eta}{1-\eta}\frac{A^2|z-1|^2 + 2\lambda(z)\left(\frac{M}{R^2} - A\right)}{-A|z-1|^2 - B + \lambda(z)} + \left(A - \frac{M}{R^2}\right)\Bigg) dz \wedge d\bar{z} \\
&= i\frac{-\varepsilon}{A|z-1|^2 + B - \lambda(z)}\Bigg(\left(1 - \frac{\varepsilon\eta}{1-\eta}\right)\frac{A^2|z-1|^2 + 2\lambda(z)\left(\frac{M}{R^2} - A\right)}{-A|z-1|^2 - B + \lambda(z)} \\
&\quad + A - \frac{M}{R^2}\Bigg) dz \wedge d\bar{z}.
\end{aligned}$$

Similar to before, we have

$$\frac{\varepsilon}{-A|z-1|^2 - B + \lambda(z)} < 0,$$

and by (5) we have that

$$\left(1 - \frac{\varepsilon\eta}{1-\eta}\right) > 0.$$

Now, using (7), we see

$$\left(1 - \frac{\varepsilon\eta}{1-\eta}\right) \frac{A^2|z-1|^2 + 2\lambda(z)\left(\frac{M}{R^2} - A\right)}{-A|z-1|^2 - B + \lambda(z)} + \left(A - \frac{M}{R^2}\right) < \left(1 - \frac{\varepsilon\eta}{1-\eta}\right) \frac{A^2|z-1|^2 + 2\lambda(z)\left(\frac{M}{R^2} - A\right)}{-A|z-1|^2 - B + \lambda(z)} < 0,$$

and so the above product is a positive form.

We must also find some upper bound on $\sup_{S_\gamma \cap B(1,R)} |\nabla \varphi_2(z)|$. By the triangle inequality, we can see

$$|\nabla \varphi_2| \leq \varepsilon \left|\nabla \log(A|z-1|^2 + B - \lambda)\right| + 2 \left|\nabla \arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)\right|.$$

Computing gradients on the right hand side,

$$\left|\nabla \arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)\right| = \frac{1}{|z|}$$

and

$$\left|\nabla \log(A|z-1|^2 + B - \lambda)\right| = 2\left|\frac{A(\bar{z}-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)}{A|z-1|^2 + B - \frac{2M}{R^2}(\operatorname{Re} z - 1)^2}\right|.$$

So currently,

$$|\nabla \varphi_2| \leq 2\varepsilon \left|\frac{A(\bar{z}-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)}{A|z-1|^2 + B - \frac{2M}{R^2}(\operatorname{Re} z - 1)^2}\right| + \frac{2}{|z|}.$$

Using the same ζ as before,

$$|\nabla \varphi_2| \leq 2\varepsilon \left|\frac{A(\bar{z}-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)}{A|z-1|^2 + B - \frac{2M}{R^2}(\operatorname{Re} z - 1)^2}\right| + \zeta.$$

Now, we need to try and maximize

$$w = \left| \frac{A(\bar{z}-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)}{A|z-1|^2 + B - \frac{2M}{R^2}(\operatorname{Re} z - 1)^2} \right|$$

over $S_\gamma \cap B(1, R)$. This is equivalent to maximizing

$$w^2 = \left| \frac{A(\bar{z}-1) - \frac{2M}{R^2}(\operatorname{Re} z - 1)}{A|z-1|^2 + B - \frac{2M}{R^2}(\operatorname{Re} z - 1)^2} \right|^2,$$

which can be rewritten as

$$w^2 = \frac{\left(A - \frac{2M}{R^2}\right)^2 (\operatorname{Re} z - 1)^2 + A^2 \operatorname{Im} z^2}{\left(\left(A - \frac{2M}{R^2}\right)(\operatorname{Re} z - 1)^2 + A \operatorname{Im} z^2 + B\right)^2}.$$

Define s, t, and c as follows:

$$s = (\operatorname{Re} z - 1)^2,$$
$$t = \operatorname{Im} z^2,$$
$$c = A - \frac{2M}{R^2}.$$

Now we define a function $f(s,t)$ by

$$f(s,t) = \frac{c^2 s + A^2 t}{(cs + At + B)^2}. \tag{14}$$

We now wish to maximize f over the region bounded between the curves $t = R^2 - s$, $s = 0$, and $t = s^\gamma$. Since $\gamma < 1$, we can also enlarge our domain to be contained within the lines $t = R^2 - s$, $s = 0$, and $s = t$. It is worth noting that by construction, the denominator of f is never equal to 0. We begin by finding the Laplacian of f, so we first find the second partial derivatives in t and s. First with respect to t,

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{A^2(cs+At+B)^2-(c^2s+A^2t)2(cs+At+B)A}{(cs+At+B)^4}\\ &= \frac{A^2(cs+At+B)-2A(c^2s+A^2t)}{(cs+At+B)^3}\\ &= \frac{(A^2c-2Ac^2)s-A^3t+A^2B}{(cs+At+B)^3},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial t^2} &= \frac{-A^3(cs+At+B)^3-((A^2c-2Ac^2)s-A^3t+A^2B)3A(cs+At+B)^2}{(cs+At+B)^6}\\ &= \frac{-A^3(cs+At+B)-3A((A^2c-2Ac^2)s-A^3t+A^2B)}{(cs+At+B)^4}\\ &= \frac{(6A^2c^2-4A^3c)s+2A^4t-4A^3B}{(cs+At+B)^4}.\end{aligned}$$

Then taking the partial derivatives with respect to s,

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{c^2(cs+At+B)^2-(c^2s+A^2t)2(cs+At+B)c}{(cs+At+B)^4}\\ &= \frac{c^2(cs+At+B)-2c(c^2s+A^2t)}{(cs+At+B)^3}\\ &= \frac{(c^2A-2cA^2)t-c^3s+c^2B}{(cs+At+B)^3},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial s^2} &= \frac{-c^3(cs+At+B)^3-((c^2A-2cA^2)t-c^3s+c^2B)3c(cs+At+B)^2}{(cs+At+B)^6}\\ &= \frac{-c^3(cs+At+B)-3c((c^2A-2cA^2)t-c^3s+c^2B)}{(cs+At+B)^4}\\ &= \frac{(6A^2c^2-4Ac^3)t+2c^4s-4Bc^3}{(cs+At+B)^4}.\end{aligned}$$

So, the Laplacian is:

$$\Delta f = \frac{(6A^2c^2 - 4A^3c)s + 2A^4t - 4A^3B + (6A^2c^2 - 4Ac^3)t + 2c^4s - 4Bc^3}{(cs + At + B)^4}.$$

We note that based on the sign of A, B, c, s and t, if we show that $A^3 + c^3 < 0$, then the Laplacian is positive. Since $-c = 2\frac{M}{R^2} - A > A > 1, -c^3 > A^3$. This means f is subharmonic so its supremum must occur on the boundary of the region.

Restricting to the line $s = t$, we want to find critical points for

$$f(s, s) = \frac{(A^2 + c^2)s}{((A + c)s + B)^2}.$$

Taking the derivative and setting it equal to zero, we see

$$\frac{\partial}{\partial s} f(s, s) = \frac{B(A^2 + c^2) - s(A^2 + c^2)(A + c)}{((A + c)s + B)^3} = 0 \iff s = \frac{B}{A + c}.$$

However, since $\frac{B}{A+c} < 0$, this falls outside of our boundary.

Next, restricting to the line $s + t = R^2$, we wish to find any critical points for

$$f(R^2 - t, t) = \frac{(A^2 - c^2)t + c^2R^2}{((A - c)t + B + cR^2)^2}.$$

Similar to before,

$$\frac{\partial}{\partial t} f(R^2 - t^2, t^2) = \frac{-t(A - c)(A^2 - c^2) + (B + cR^2)(A^2 - c^2) - 2c^2R^2(A - c)}{((A - c)t + B + cR^2)^3} = 0$$
$$\iff t = \frac{B(A + c) + cR^2(A - c)}{(A + c)(A - c)}.$$

To check if this value of t lies on the boundary of our domain, we wish to check if

$$0 < \frac{B(A + c) + cR^2(A - c)}{(A + c)(A - c)} < \frac{1}{2}R^2.$$

Given the sign of A, B, and c, we can see the left hand inequality is satisfied. However, by multiplying everything by $(A+c) < 0$, we get

$$\frac{B(A+c)}{A-c} + cR^2 > \frac{1}{2}R^2(A+c),$$

and by subtracting the cR^2 term and simplifying, we get

$$\frac{B(A+c)}{A-c} > \frac{1}{2}R^2(A-c).$$

By checking the sign of both sides we come to a contradiction since $A-c > 0$ and $A+c < 0$.

Restricting to the final line of $s = 0$, we wish to find critical points for

$$f(0,t) = \frac{A^2 t}{(At+B)^2}.$$

We find

$$\frac{\partial}{\partial t} f(0,t) = \frac{A^2 B - A^3 t}{(At+B)^3} = 0 \iff t = \frac{B}{A}.$$

If $\frac{B}{A} > R^2$, then the three corners of our region are candidates for the maximum point. Plugging these points into our equation,

$$f(0,0) = 0,$$
$$f(0,R^2) = \frac{A^2 R^2}{(AR^2+B)^2},$$

and

$$f\left(\frac{R^2}{2}, \frac{R^2}{2}\right) = \frac{A^2R^2 - 2AM + 2\frac{M^2}{R^2}}{(AR^2 - M + B)^2}.$$

Now, recalling (7), since $A < \frac{M}{R^2}$, we have that $2\frac{M^2}{R^2} - 2AM > 0$. So, since the numerator of $f\left(\frac{R^2}{2}, \frac{R^2}{2}\right)$ is greater than the numerator of $f(0, R^2)$, and similarly it has a smaller denominator, $f\left(\frac{R^2}{2}, \frac{R^2}{2}\right) > f(0,R^2) > f(0,0)$. However, if $\frac{B}{A} < R^2$, then $f(0, \frac{B}{A}) = \frac{A}{4B}$ is also a candidate for a maximum point. So, we have the two cases:

$$\sup f(s,t) = \begin{cases} \frac{A^2R^2 - 2AM + \frac{2M^2}{R^2}}{(AR^2 - M + B)^2}, \frac{B}{A} > R^2 \\ \max\left(\frac{A^2R^2 - 2AM + \frac{2M^2}{R^2}}{(AR^2 - M + B)^2}, \frac{A}{4B}\right), \frac{B}{A} < R^2 \end{cases}.$$

So, we now have

$$|\nabla \varphi_2| \leq 2\varepsilon(\sup f(s,t))^{\frac{1}{2}} + \zeta.$$

Let $E_{\eta,2} = 2\varepsilon(\sup f(s,t))^{\frac{1}{2}} + \zeta$.

Now, taking a look at how $i\log(z) - i\log(\bar{z})$ and $-2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$ compare, one can see that when $\operatorname{Re} z > 0, \operatorname{Im} z > 0$,

$$i\log(z) - i\log(\bar{z}) = -2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right).$$

However when $\operatorname{Re} z > 0, \operatorname{Im} z < 0$,

$$i\log(z) - i\log(\bar{z}) = -2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right) - 4\pi.$$

Now, we will show that near $|z-1| = R, \varphi_1 > \varphi_2$ and near $z = 1, \varphi_2 > \varphi_1$. By (10),

$$\varphi_2(1) = -\varepsilon\log(B) + C > 0.$$

Choose some $\mu_1 > 0$ so that

$$\varphi_2(1) > \mu_1 > 0.$$

By the continuity of φ_2, there is a $\delta_1 > 0$ such that $\varphi_2 > \mu_1$ on $S_\gamma \cap B(1,\delta_1)$. Now, by (12), we can see that $\varphi_1 < \varepsilon|z-1|^2$. We can choose

$$\delta_1 < \left(\frac{\mu_1}{\varepsilon}\right)^{\frac{1}{2}},$$

so that we can have

$$\varphi_1(z) < \varepsilon|z-1|^2 \leq \varepsilon\delta_1^2 < \varphi_2(1),$$

on $S_\gamma \cap B(1,\delta_1)\backslash\{1\}$. Now, letting $|z-1| = R$,

$$\varphi_2(z) = -\varepsilon\log(AR^2 + B - \lambda) - 2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right) + C.$$

Taking the sup of λ, we have

$$\varphi_2(z) < -\varepsilon\log(AR^2 + B - 1) - 2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right) + C.$$

Also when $|z-1| = R$,

$$\varphi_1(z) = i\log(z) - i\log(\bar{z}) + \varepsilon R^2.$$

So $\varphi_2 < \varphi_1$ if,

$$C - \varepsilon\log(AR^2 + B - 1) - 2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right) < i\log(z) - i\log(\bar{z}) + \varepsilon R^2.$$

Given the above relationship between $i\log(z) - i\log(\bar{z})$ and $-2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$, this will always hold so long as

$$-\varepsilon\log(AR^2 + B - 1) + C < \varepsilon R^2 - 4\pi,$$

which is true given (10). Since $\varphi_1 > \varphi_2$ when $|z-1| = R$, by continuity, there is some δ_2 such that $\varphi_1 > \varphi_2$ on $S_\gamma \cap B_1(R)\backslash\overline{B_1(R-\delta_2)}$. Let $\delta = \min\{\delta_1, \delta_2\}$, which gets us the following:

$$\varphi_1 < \varphi_2 \text{ on } S_\gamma \cap B_1(\delta)\backslash\{1\},$$
$$\varphi_1 > \varphi_2 \text{ on } S_\gamma \cap B_1(R)\backslash\overline{B_1(R-\delta)}.$$

We now wish to find the following values,

$$\xi_1 = \inf_{\{z:|z-1|=\delta\}} \varphi_2 - \varphi_1$$
$$\xi_2 = \inf_{\{z:|z-1|=R-\delta\}} \varphi_1 - \varphi_2.$$

Looking first at ξ_1,

$$\varphi_2 - \varphi_1|_{\{z:|z-1|=\delta\}} = -\varepsilon \log\left(A\delta^2 + B - \frac{2M}{R^2}(\operatorname{Re} z - 1)^2\right) - 2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right) + C$$
$$- i\log(z) + i\log(\bar{z}) - \varepsilon\delta^2$$

Given the relationship between $-2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$ and $i\log(z) - i\log(\bar{z})$,

$$\varphi_2 - \varphi_1|_{\{z:|z-1|=\delta\}} \leq -\varepsilon \log\left(A\delta^2 + B - \frac{2M}{R^2}(\operatorname{Re} z - 1)^2\right) + C - \varepsilon\delta^2$$

To attain the infimum, we need to maximize $A\delta^2 + B - \frac{2M}{R^2}(\operatorname{Re} z - 1)^2$, so on our restricted domain, we attain the infimum when $\operatorname{Re} z = 1$, getting us

$$\xi_1 = -\varepsilon\log(A\delta^2 + B) + C - \varepsilon\delta^2.$$

For ξ_2,

$$\varphi_1 - \varphi_2|_{\{z:|z-1|=R-\delta\}} = i\log(z) - i\log(\bar{z}) + \varepsilon(R-\delta)^2$$
$$+ \varepsilon\log\left(A(R-\delta)^2 + B - \frac{2M}{R^2}(\operatorname{Re} z - 1)^2\right) + 2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right) - C.$$

Again from the relation between $-2\arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$ and $i\log(z) - i\log(\bar{z})$,

$$\varphi_1 - \varphi_2|_{\{z:|z-1|=R-\delta\}} \leq \varepsilon(R-\delta)^2 + \varepsilon\log\left(A(R-\delta)^2 + B - \frac{2M}{R^2}(\operatorname{Re} z - 1)^2\right) - C - 4\pi.$$

To attain the infimum, we need to maximize $\frac{2M}{R^2}(\operatorname{Re} z - 1)^2 = \lambda$. Since the sup of λ is 1, we can see

$$\xi_2 = \varepsilon(R-\delta)^2 + \varepsilon \log(A(R-\delta) + B - 1) - C - 4\pi.$$

Choose the constant ξ such that,

$$0 < \xi < \min\{\xi_1, \xi_2\}.$$

Now, let $\chi \in C^\infty(\mathbb{R})$ be a convex function such that $\chi(x) = |x|$ when $|x| \geq 1$. Now, define φ_η such that

$$\varphi_\eta = \begin{cases} \varphi_1 \text{ on } B_1(R) \backslash B_1(R - \frac{\delta}{2}) \\ \varphi_2 \text{ on } B_1(\frac{\delta}{2}) \\ \frac{1}{2}(\varphi_1 + \varphi_2) + \frac{1}{2}\xi\chi(\frac{1}{\xi}(\varphi_1 - \varphi_2)) \text{ on } B_1(R - \frac{\delta}{2}) \backslash B_1(\frac{\delta}{2}) \end{cases}.$$

By the construction of ξ, $\varphi_\eta = \varphi_1$ on a neighborhood of $\{z : |z-1| = R - \delta\}$, and similarly $\varphi_\eta = \varphi_2$ on a neighborhood of $\{z : |z-1| = \delta\}$. Thus, φ_η is smooth on $\overline{S_\gamma}$. The convexity of χ implies that $|\chi'| \leq 1$, so we define,

$$t = \frac{1}{2}\left(1 + \chi'\left(\frac{1}{\xi}(\varphi_1 - \varphi_2)\right)\right).$$

Note that by construction, $0 \leq t \leq 1$. We will also compute ∂t:

$$\partial t = \frac{1}{2\xi}\chi''\left(\frac{1}{\xi}(\varphi_1 - \varphi_2)\right)(\partial\varphi_1 - \partial\varphi_2).$$

Now, we wish to compute $i\partial\overline{\partial}\varphi_\eta$. Starting with $\overline{\partial}\varphi_\eta$,

$$\begin{aligned}
\overline{\partial}\varphi_\eta &= \frac{1}{2}(\overline{\partial}\varphi_1 + \overline{\partial}\varphi_2) + \frac{1}{2}\xi\chi'\left(\frac{1}{\xi}(\varphi_1 - \varphi_2)\right)\frac{1}{\xi}(\overline{\partial}\varphi_1 - \overline{\partial}\varphi_2) \\
&= \frac{1}{2}(\overline{\partial}\varphi_1 + \overline{\partial}\varphi_2) + \frac{1}{2}\chi'\left(\frac{1}{\xi}(\varphi_1 - \varphi_2)\right)(\overline{\partial}\varphi_1 - \overline{\partial}\varphi_2) \\
&= \frac{1}{2}\left(1 + \chi'\left(\frac{1}{\xi}(\varphi_1 - \varphi_2)\right)\right)\overline{\partial}\varphi_1 + \frac{1}{2}\left(1 - \chi'\left(\frac{1}{\xi}(\varphi_1 - \varphi_2)\right)\right)\overline{\partial}\varphi_2 \\
&= t\overline{\partial}\varphi_1 + (1-t)\overline{\partial}\varphi_2.
\end{aligned}$$

Now, taking $\partial\overline{\partial}\varphi_\eta$,

$$\begin{aligned}
\partial\overline{\partial}\varphi_\eta &= \partial t \wedge \overline{\partial}\varphi_1 + t\partial\overline{\partial}\varphi_1 - \partial t \wedge \overline{\partial}\varphi_2 + (1-t)\partial\overline{\partial}\varphi_2 \\
&= \partial t \wedge (\overline{\partial}\varphi_1 - \overline{\partial}\varphi_2) + t\partial\overline{\partial}\varphi_1 + (1-t)\partial\overline{\partial}\varphi_2 \\
&= \frac{1}{2\xi}\chi''\left(\frac{1}{\xi}(\varphi_1 - \varphi_2)\right)(\partial\varphi_1 - \partial\varphi_2) \wedge (\overline{\partial}\varphi_1 - \overline{\partial}\varphi_2) \\
&\qquad + t\partial\overline{\partial}\varphi_1 + (1-t)\partial\overline{\partial}\varphi_2.
\end{aligned}$$

Given the convexity of χ, we have that $\chi'' \geq 0$, and so the multiplication of i gives us that

$$\frac{1}{2\xi}\chi''\left(\frac{1}{\xi}(\varphi_1 - \varphi_2)\right) i(\partial\varphi_1 - \partial\varphi_2) \wedge (\overline{\partial}\varphi_1 - \overline{\partial}\varphi_2) \geq 0.$$

Thus,

$$i\partial\overline{\partial}\varphi_\eta \geq it\partial\overline{\partial}\varphi_1 + i(1-t)\partial\overline{\partial}\varphi_2.$$

By applying our computations of (3) for φ_1 and φ_2, we have

$$\begin{aligned}
i\partial\overline{\partial}\varphi_\eta > it\frac{\eta}{1-\eta}\left(\partial\varphi_1 - \frac{i}{z}dz\right) \wedge \left(\overline{\partial}\varphi_1 + \frac{i}{\bar{z}}d\bar{z}\right) \\
+ i(1-t)\frac{\eta}{1-\eta}\left(\partial\varphi_2 - \frac{i}{z}dz\right) \wedge \left(\overline{\partial}\varphi_2 + \frac{i}{\bar{z}}d\bar{z}\right).
\end{aligned}$$

Letting $\theta_1 = \partial\varphi_1 - \frac{i}{z}dz$ and $\theta_2 = \partial\varphi_2 - \frac{i}{z}dz$, we simplify this to

$$i\partial\overline{\partial}\varphi_\eta > i\frac{\eta}{1-\eta}(t\theta_1 \wedge \overline{\theta_1} + (1-t)\theta_2 \wedge \overline{\theta_2}).$$

Now, we wish to show that for $(1,0)$ forms θ_1 and θ_2, we have

$$i(t\theta_1 \wedge \overline{\theta_1} + (1-t)\theta_2 \wedge \overline{\theta_2}) \geq i(t\theta_1 + (1-t)\theta_2) \wedge (t\overline{\theta_1} + (1-t)\overline{\theta_2}).$$

First, by expanding the right hand side we get,

$$\begin{aligned} &i(t\theta_1 + (1-t)\theta_2) \wedge (t\overline{\theta_1} + (1-t)\overline{\theta_2}) \\ &\qquad = i(t^2\theta_1 \wedge \overline{\theta_1} + t(1-t)(\theta_1 \wedge \overline{\theta_2} + \theta_2 \wedge \overline{\theta_1}) + (1-t)^2\theta_2 \wedge \overline{\theta_2}). \end{aligned}$$

By subtracting the above from $i(t\theta_1 \wedge \overline{\theta_1} + (1-t)\theta_2 \wedge \overline{\theta_2})$, we have the following:

$$\begin{aligned} &i(t-t^2)\theta_1 \wedge \overline{\theta_1} + ((1-t) - (1-t)^2)\theta_2 \wedge \overline{\theta_2} + t(1-t)(-\theta_1 \wedge \overline{\theta_2} - \theta_2 \wedge \overline{\theta_1})) \\ &= it(1-t)(\theta_1 \wedge \overline{\theta_1} + \theta_2 \wedge \overline{\theta_2} - \theta_1 \wedge \overline{\theta_2} - \theta_2 \wedge \overline{\theta_1}) \\ &= it(1-t)(\theta_1 \wedge (\overline{\theta_1} - \overline{\theta_2}) - \theta_2 \wedge (\overline{\theta_1} - \overline{\theta_2})) \\ &= it(1-t)((\theta_1 - \theta_2) \wedge (\overline{\theta_1} - \overline{\theta_2})) \geq 0. \end{aligned}$$

Thus, we get that

$$i\partial\overline{\partial}\varphi_\eta > i\frac{\eta}{1-\eta}(t\theta_1 + (1-t)\theta_2) \wedge (t\overline{\theta_1} + (1-t)\overline{\theta_2}).$$

By substituting for θ_1 and θ_2 and then simplifying, we now see

$$i\partial\overline{\partial}\varphi_\eta > i\frac{\eta}{1-\eta}\left(\partial\varphi_\eta - \frac{i}{z}dz\right) \wedge \left(\overline{\partial}\varphi_\eta + \frac{i}{\overline{z}}d\overline{z}\right).$$

We extend φ_η to the rest of S_γ by defining

$$\varphi_\eta = \varphi_1 \text{ on } S_\gamma \backslash B_1(R).$$

So (3) holds for φ_η on all of S_γ. Now we must estimate the gradient of φ_η. By use of the triangle inequality,

$$|\nabla \varphi_\eta| \leq \frac{1}{2}(|\nabla \varphi_1| + |\nabla \varphi_2|) + \frac{1}{2}\left|\chi'\left(\frac{1}{\xi}(\varphi_1 - \varphi_2)\right)\right| |\nabla \varphi_1 - \nabla \varphi_2|.$$

Again by the convexity of χ, we get

$$|\nabla \varphi_\eta| \leq \frac{1}{2}(|\nabla \varphi_1| + |\nabla \varphi_2|) + \frac{1}{2}|\nabla \varphi_1 - \nabla \varphi_2|,$$

and applying the triangle inequality one last time yields,

$$|\nabla \varphi_\eta| \leq |\nabla \varphi_1| + |\nabla \varphi_2| \leq E_{\eta,1} + E_{\eta,2}.$$

Letting $E_\eta = E_{\eta,1} + E_{\eta,2}$, we have shown φ_η satisfies (4).

When we let η be close to 1, that is, for $1 > \eta > \frac{\log(2 - \frac{1}{AR^2}) + R^2}{\log(2 - \frac{1}{AR^2}) + R^2 + 4\pi D^2}$, we can find that

$$0 < \varepsilon < \frac{1 - \eta}{D^2 \eta} < \frac{4\pi}{\log(2 - \frac{1}{AR^2}) + R^2}.$$

When this is true, we can also see that

$$\frac{B}{A} < \frac{R^2 - \frac{1}{A}}{\exp(\frac{4\pi}{\varepsilon} - R^2) - 1} < R^2.$$

However, we can show in the case of $\frac{B}{A} < R^2$, $\frac{A}{4B} > \frac{A^2R^2 - 2AM + \frac{2M^2}{R^2}}{(AR^2 - M + B)^2}$ for sufficiently small B.

This is equivalent to showing

$$A(AR^2 - M + B)^2 > \left(A^2R^2 - 2AM + \frac{2M^2}{R^2}\right)4B.$$

Rearranging this inequality shows that it is sufficient to find when the following is true;

$$0 > -AB^2 + \left(2A^2R^2 - 6AM + 8\frac{M^2}{R^2}\right)B - A(AR^2 - M)^2.$$

As this is a quadratic polynomial in B with negative leading coefficient, the minimal solution to the quadratic formula will serve as an upper bound on B such that the polynomial is negative. Thus we would like to know when;

$$B < \frac{A^2R^2 + 4\frac{M^2}{R^2} - 3AM}{A} - \frac{2\sqrt{-A^3R^2M + 4A^2M^2 - 6A\frac{M^3}{R^2} + 4\frac{M^4}{R^4}}}{A}. \tag{15}$$

To confirm this is a real solution, since $6^{\frac{-\gamma}{4(1-\gamma)}} < R$, one can check that

$$\left(\frac{3}{2}\right)^{\frac{1}{2}} > \frac{1}{2}R^{\frac{-2(1-\gamma)}{\gamma}} > M.$$

This can be rearranged to give us

$$\left(\frac{2}{3}\right)^{\frac{1}{2}} M < 1.$$

Dividing both sides by R^2 gives us

$$\left(\frac{2}{3}\right)^{\frac{1}{2}} \frac{M}{R^2} < \frac{1}{R^2} < A.$$

This can again be rearranged to show

$$3A^2R^2 > 2\frac{M^2}{R^2},$$

and we can add $A^2R^2 + 4\frac{M^2}{R^2}$ to see

$$4A^2R^2 + 4\frac{M^2}{R^2} > A^2R^2 + 6\frac{M^2}{R^2}.$$

Since $AR < \frac{M}{R}$, we can multiply the left-hand side by $\frac{M^2}{R^2}$ and the right hand side by $AR\frac{M}{R} = AM$ to get

$$4A^2M^2 + 4\frac{M^4}{R^4} > A^3R^2M + 6A\frac{M^3}{R^2}.$$

This tells us that (15) is a real value for B, and we will also show it is in fact positive. Since $(AR - \frac{M}{R})^2 > 0$, we can see that

$$A^2R^2 + \frac{M^2}{R^2} > 2AM.$$

We multiply both sides of this by A^2R^2 to yield

$$A^4R^4 + A^2M^2 > 2A^3R^2M.$$

Next, we add $16A^2M^2 + 16\frac{M^4}{R^4} - 6A^3R^2M - 24A\frac{M^3}{R^2}$ to both sides to get

$$A^4R^4 - 6A^3R^2M + 17A^2M^2 - 24A\frac{M^3}{R^2} > 4\left(4A^2M^2 + 4\frac{M^4}{R^4} - A^3R^2M - 6A\frac{M^3}{R^2}\right).$$

The left hand side can be factored to yield

$$\left(A^2R^2 + 4\frac{M^2}{R^2} - 3AM\right)^2 > 4\left(4A^2M^2 + 4\frac{M^4}{R^4} - A^3R^2M - 6A\frac{M^3}{R^2}\right),$$

and so (15) is a positive upper bound for B. So for

$$1 > \eta > \frac{\log\left(\frac{2A^2R^2 - A + 4\frac{M^2}{R^2} - 3AM - 2(4A^2M^2 + 4\frac{M^4}{R^4} - 6A\frac{M^3}{R^2} - A^3R^2M)^{\frac{1}{2}}}{A^2R^2 + 4\frac{M^2}{R^2} - 3AM - 2(4A^2M^2 + 4\frac{M^4}{R^4} - 6A\frac{M^3}{R^2} - A^3R^2M)^{\frac{1}{2}}}\right) + R^2}{\log\left(\frac{2A^2R^2 - A + 4\frac{M^2}{R^2} - 3AM - 2(4A^2M^2 + 4\frac{M^4}{R^4} - 6A\frac{M^3}{R^2} - A^3R^2M)^{\frac{1}{2}}}{A^2R^2 + 4\frac{M^2}{R^2} - 3AM - 2(4A^2M^2 + 4\frac{M^4}{R^4} - 6A\frac{M^3}{R^2} - A^3R^2M)^{\frac{1}{2}}}\right) + R^2 + 4\pi D^2},$$

we can rearrange this inequality to find

$$\varepsilon < \frac{1-\eta}{D^2\eta} < \frac{4\pi}{\log\left(\frac{2A^2R^2 - A + 4\frac{M^2}{R^2} - 3AM - 2(4A^2M^2 + 4\frac{M^4}{R^4} - 6A\frac{M^3}{R^2} - A^3R^2M)^{\frac{1}{2}}}{A^2R^2 + 4\frac{M^2}{R^2} - 3AM - 2(4A^2M^2 + 4\frac{M^4}{R^4} - 6A\frac{M^3}{R^2} - A^3R^2M)^{\frac{1}{2}}}\right) + R^2}.$$

This new upper bound on ε can then be used to find that B is in fact bounded above by (15). So when η is sufficiently close to 1,

$$E_{\eta,2} = \varepsilon\left(\frac{A}{B}\right)^{\frac{1}{2}} + \zeta.$$

Given the upper bound on B, we can see that

$$E_{\eta,2} > \varepsilon\left(\frac{A(\exp(\frac{4\pi}{\varepsilon} - R^2) - 1)}{AR^2 - 1}\right)^{\frac{1}{2}} + \zeta.$$

If $\eta > \frac{1}{4\pi+1}$, then $\frac{1-\eta}{D^2\eta} < \frac{4\pi}{D^2}$, so we can let $\varepsilon = \frac{1-\eta}{2D^2\eta}$. Substituting this into our inequality gives us,

$$E_{\eta,2} > \frac{1-\eta}{2D^2\eta}\left(\frac{A(\exp(\frac{8\pi D^2\eta}{1-\eta} - R^2) - 1)}{AR^2 - 1}\right)^{\frac{1}{2}} + \zeta.$$

Since A, R, and D are not dependent on η, we can combine many of these terms to some constant c_1 independent of η. So,

$$E_{\eta,2} > c_1\frac{1-\eta}{\eta}\exp\left(\frac{4\pi D^2\eta}{1-\eta}\right) + \zeta.$$

Adding $E_{\eta,1} = \zeta + 8\pi$, we get

$$E_{\eta,1} + E_{\eta,2} > c_1\frac{1-\eta}{\eta}\exp\left(\frac{4\pi D^2\eta}{1-\eta}\right) + 2\zeta + 8\pi$$

Thus, as $\eta \to 1^-$,

$$E_\eta = E_{\eta,1} + E_{\eta,2} = O\left(\frac{1-\eta}{\eta}e^{\frac{4\pi D^2\eta}{1-\eta}}\right).$$

□

Remark 2. If we consider this type of behaviour in comparison to the gradient estimate needed for the Regularized Diederich-Fornæss index, we would see that

$$\liminf_{\eta \to 1^-} \frac{(1-\eta)^{\frac{3}{2}}}{\eta} \exp\left(\frac{4\pi D^2 \eta}{1-\eta}\right) \neq 0,$$

and this limit is in fact unbounded.

This means either more work is required to find a sharper estimate on the gradient of the current function, or further alterations must be done to create a function satisfying the required gradient estimate. Either way, we move to the next section where we extend our function to be defined on the entirety of Ω in several steps.

4 Extension to a Neighborhood of the Two-Dimensional Crescent Region

Now we prove that such a function defined on K_γ exists by reintroducing the dz_2 and $d\bar{z}_2$ components of our operator.

Lemma 4.1. *Given K_γ as defined above, for every $0 < \eta < 1$, there exists a constant $E'_\eta > 0$ and a smooth function φ'_η plurisubharmonic on K_γ such that*

$$i\partial\overline{\partial}\varphi'_\eta + 2\beta - i\frac{\eta}{1-\eta}(\partial\varphi'_\eta - 2\pi_{1,0}\alpha) \wedge (\overline{\partial}\varphi'_\eta - 2\pi_{0,1}\alpha) > 0, \tag{16}$$

on K_γ and

$$\sup_{z \in K_\gamma} |\nabla\varphi'_\eta(z)| < E'_\eta. \tag{17}$$

Proof. Given φ_η from Lemma 3.1, define φ'_η by

$$\varphi'_\eta(z) = \varphi_\eta(z_1) + s|z_2|^2, \tag{18}$$

where $s > 0$ is a constant to be chosen later. Our goal is to isolate the $dz_1 \wedge d\bar{z}_1$, $dz_2 \wedge d\bar{z}_2$, and $dz_1 \wedge d\bar{z}_2$ coefficients, so that

$$\begin{aligned} &i\partial\overline{\partial}\varphi'_\eta + 2\beta - i\frac{\eta}{1-\eta}(\partial\varphi'_\eta - 2\pi_{1,0}\alpha) \wedge (\overline{\partial}\varphi'_\eta - 2\pi_{0,1}\alpha) \\ &\qquad = i\mathfrak{A}dz_1 \wedge d\bar{z}_1 + i\mathfrak{B}dz_2 \wedge d\bar{z}_2 + i\mathfrak{C}dz_1 \wedge d\bar{z}_2 + i\overline{\mathfrak{C}}dz_2 \wedge d\bar{z}_1 \\ &\qquad\qquad = i\mathfrak{A}\left(dz_1 + \frac{\overline{\mathfrak{C}}}{\mathfrak{A}}dz_2\right) \wedge \left(d\bar{z}_1 + \frac{\mathfrak{C}}{\mathfrak{A}}d\bar{z}_2\right) + i\left(\mathfrak{B} - \frac{|\mathfrak{C}|^2}{\mathfrak{A}}\right)dz_2 \wedge d\bar{z}_2, \end{aligned}$$

and then show that $\mathfrak{A} > 0, \mathfrak{B} > 0$, and $\mathfrak{A}\mathfrak{B} - |\mathfrak{C}|^2 > 0$. First, looking at derivatives of φ'_η, restricted to K_γ

$$\begin{aligned} \partial\varphi'_\eta &= \frac{\partial\varphi_\eta}{\partial z_1}dz_1, \\ \overline{\partial}\varphi'_\eta &= \frac{\partial\varphi_\eta}{\partial\bar{z}_1}d\bar{z}_1, \end{aligned}$$

and

$$\partial\overline{\partial}\varphi'_\eta = \frac{\partial^2\varphi_\eta}{\partial z_1\partial\bar{z}_1}dz_1 \wedge d\bar{z}_1 + s dz_2 \wedge d\bar{z}_2.$$

We now note that $i\partial\overline{\partial}\varphi'_\eta > 0$ so our function is plurisubharmonic. Substituting the derivatives, α, and β into (16) yields:

$$\begin{aligned} &i\partial\overline{\partial}\varphi'_\eta + 2\beta - i\frac{\eta}{1-\eta}(\partial\varphi'_\eta - 2\pi_{1,0}\alpha) \wedge (\overline{\partial}\varphi'_\eta - 2\pi_{0,1}\alpha) = \\ &\qquad i\left(\frac{\partial^2\varphi_\eta}{\partial z_1\partial\bar{z}_1}dz_1 \wedge d\bar{z}_1 + s dz_2 \wedge d\bar{z}_2\right) + \frac{i}{2|z_1|^2}dz_2 \wedge d\bar{z}_2 \\ &\quad - i\frac{\eta}{1-\eta}\left(\left(\frac{\partial\varphi_\eta}{\partial z_1} - \frac{i}{z_1}\right)dz_1 - e^{-i\log|z_1|^2}dz_2\right) \wedge \left(\left(\frac{\partial\varphi_\eta}{\partial\bar{z}_1} + \frac{i}{\bar{z}_1}\right)d\bar{z}_1 - e^{i\log|z_1|^2}d\bar{z}_2\right). \end{aligned}$$

Regrouping terms gives us,

$$\begin{aligned}
i\partial\overline{\partial}\varphi'_\eta + 2\beta - i\frac{\eta}{1-\eta}(\partial\varphi'_\eta - 2\pi_{1,0}\alpha)\wedge(\overline{\partial}\varphi'_\eta - 2\pi_{0,1}\alpha) = \\
i\left(\frac{\partial^2\varphi_\eta}{\partial z_1\partial\bar{z}_1} - \frac{\eta}{1-\eta}\left(\frac{\partial\varphi_\eta}{\partial z_1} - \frac{i}{z_1}\right)\left(\frac{\partial\varphi_\eta}{\partial\bar{z}_1} + \frac{i}{\bar{z}_1}\right)\right) dz_1\wedge d\bar{z}_1 \\
+ i\left(s + \frac{1}{2|z_1|^2} - \frac{\eta}{1-\eta}\right) dz_2\wedge d\bar{z}_2 + i\frac{\eta}{1-\eta}\left(e^{i\log|z_1|^2}\left(\frac{\partial\varphi_\eta}{\partial z_1} - \frac{i}{z_1}\right)\right) dz_1\wedge d\bar{z}_2 \\
+ i\frac{\eta}{1-\eta}\left(e^{-i\log|z_1|^2}\left(\frac{\partial\varphi_\eta}{\partial\bar{z}_1} + \frac{i}{\bar{z}_1}\right)\right) dz_2\wedge d\bar{z}_1.
\end{aligned}$$

Now, we know that $\mathfrak{A} > 0$ from Lemma 3.1, and $\mathfrak{B} > 0$ given s sufficiently large. Now, to check that $\mathfrak{A}\mathfrak{B} - |\mathfrak{C}|^2 > 0$,

$$\begin{aligned}
\mathfrak{A}\mathfrak{B} - |\mathfrak{C}|^2 &= \left(\frac{\partial^2\varphi_\eta}{\partial z_1\partial\bar{z}_1} - \frac{\eta}{1-\eta}\left(\frac{\partial\varphi_\eta}{\partial z_1} - \frac{i}{z_1}\right)\left(\frac{\partial\varphi_\eta}{\partial\bar{z}_1} + \frac{i}{\bar{z}_1}\right)\right)\left(s + \frac{1}{2|z_1|^2} - \frac{\eta}{1-\eta}\right) \\
&\quad - \frac{\eta^2}{(1-\eta)^2}\left(\frac{\partial\varphi_\eta}{\partial z_1} - \frac{i}{z_1}\right)\left(\frac{\partial\varphi_\eta}{\partial\bar{z}_1} + \frac{i}{\bar{z}_1}\right) \\
&= \frac{\partial^2\varphi_\eta}{\partial z_1\partial\bar{z}_1}\left(s + \frac{1}{2|z_1|^2} - \frac{\eta}{1-\eta}\right) - \frac{\eta}{1-\eta}\left(s + \frac{1}{2|z_1|^2}\right)\left(\frac{\partial\varphi_\eta}{\partial z_1} - \frac{i}{z_1}\right)\left(\frac{\partial\varphi_\eta}{\partial\bar{z}_1} + \frac{i}{\bar{z}_1}\right) \\
&= \left(s + \frac{1}{2|z_1|^2}\right)\left(\frac{\partial^2\varphi_\eta}{\partial z_1\partial\bar{z}_1} - \frac{\eta}{1-\eta}\left(\frac{\partial\varphi_\eta}{\partial z_1} - \frac{i}{z_1}\right)\left(\frac{\partial\varphi_\eta}{\partial\bar{z}_1} + \frac{i}{\bar{z}_1}\right)\right) - \frac{\eta}{1-\eta}\frac{\partial^2\varphi_2}{\partial z_1\partial\bar{z}_1} \\
&= \left(s + \frac{1}{2|z_1|^2}\right)\mathfrak{A} - \frac{\eta}{1-\eta}\frac{\partial^2\varphi_\eta}{\partial z_1\partial\bar{z}_1}.
\end{aligned}$$

Since φ_η is smooth on our bounded domain, we have $\sup_{z\in K_\gamma}\{\frac{\partial^2\varphi_\eta}{\partial z_1\partial\bar{z}_1}\} < \infty$. So, we have

$$\mathfrak{A}\mathfrak{B} - |\mathfrak{C}|^2 > \left(s + \frac{1}{2|z_1|}\right)\mathfrak{A} - \frac{\eta}{1-\eta}\sup_{K_\gamma}\frac{\partial^2\varphi_\eta}{\partial z_1\partial\bar{z}_1}.$$

Then, we have that $\inf_{z\in K_\gamma}\mathfrak{A} > 0$, so

$$\mathfrak{A}\mathfrak{B} - |\mathfrak{C}|^2 > s\inf_{K_\gamma}\mathfrak{A} - \frac{\eta}{1-\eta}\sup_{K_\gamma}\frac{\partial^2\varphi_\eta}{\partial z_1\partial\bar{z}_1}.$$

We can choose s large enough so that we have our desired result. Now, to estimate the gradient,

$$|\nabla\varphi'_\eta| = |\nabla\varphi_\eta + s\nabla|z_2|^2| = |\nabla\varphi_\eta + s|z_2||.$$

Restricting to K_γ where $z_2 = 0$, and using the result of Lemma 3.1,

$$|\nabla\varphi'_\eta| = |\nabla\varphi_\eta| \le E_\eta.$$

□

Since the results of [12] depend on a function satisfying our properties in a neighborhood of K_γ, we show very simply that our result above extends to such a neighborhood.

Lemma 4.2. *Given K_γ as defined above, for every $0 < \eta < 1$, there exists a constant $E_\eta > 0$ and a smooth function $\tilde\varphi_\eta$ plurisubharmonic on a neighborhood of K_γ such that*

$$i\partial\overline{\partial}\tilde\varphi_\eta + 2\beta - i\frac{\eta}{1-\eta}(\partial\tilde\varphi_\eta - 2\pi_{1,0}\alpha) \wedge (\overline{\partial}\tilde\varphi_\eta - 2\pi_{0,1}\alpha) > 0, \tag{19}$$

on a neighborhood of K_γ and

$$\sup_{z\in K_\gamma} |\nabla\tilde\varphi_\eta(z)| < E_\eta. \tag{20}$$

Proof. Let $\tilde\varphi_\eta = \varphi'_\eta$. By Lemma 3.1, the desired traits hold on K_γ, and the smoothness of the function implies there is some neighborhood satisfying the lemma. □

We now have to tools to prove the main theorem.

Proof of Theorem 1.1. Given $\tilde\varphi_\eta$ from Lemma 4.2, by Lemma 5.3 of [12] for every $0 < \eta < 1$ there exists a smooth, bounded, real-valued function φ on a neighborhood U of $\partial\Omega$ such that $P_\eta\varphi > 0$. Then by Lemma 5.4 of [12], there exists some neighborhood $\tilde U$ of $\partial\Omega$ such that $-(-\rho)^\eta = -e^{-\eta\varphi}(-\delta)^\eta$ is strictly plurisubharmonic on $\tilde U \cap \Omega$. Furthermore, Lemma 3.1 provides us with the order of magnitude for the upper bound on the gradient. □

We can use the work of [12] to find an extension of this defining function as well.

Proof of Corollary 1.2. Given φ from Theorem 1.1, for every $0 < \eta < 1$, the proof of Theorem 2.11 (1) in section 5 of [12] gives us a smooth defining function ρ such that $-(-\rho)^\eta$ is plurisubharmonic on Ω. □

5 Defining a Crescent and a Family of Automorphisms

For Section 6, we will narrow our focus to a specific class of crescent regions we will denote as K_a, as well as a family of automorphism H_s of K_a. While we acknowledge that by definition our crescent regions lie in $\mathbb{C}^2$, for the sake of notation we neglect to include the z_2 coordinates in the coming discussion as they are not relevant.

Definition 5.1. Let $a \in \mathbb{R}$ be a constant such that $0 < a < 1$. Define the circles K_1 and K_2 as follows;

$$K_1 = \left\{ z \in \mathbb{C} : \left| z - \frac{a}{a-1} \right| < \frac{1}{1-a} \right\},$$

$$K_2 = \left\{ z \in \mathbb{C} : \left| z - \frac{a}{a+1} \right| < \frac{1}{1+a} \right\}.$$

Now, define the region K_a to be as follows:

$$K_a = \overline{K_1 \backslash K_2}. \tag{21}$$

K_a satisfies the conditions in Definition 2.1 to be an S_γ for $\gamma > \frac{1}{2}$.

This type of domain is helpful to us as it is constructed by a group of circles that all intersect at a single point $z = 1$. This allows us to construct our family of automorphisms of K_a that fix $z = 1$ by mapping the crescent region to the unit disk, performing an automorphism of the disk, and then mapping back to K_a.

Definition 5.2. Let $t \in \mathbb{R}$ be a constant such that $-1 < t < 1$. Let D denote the unit disk in $\mathbb{C}$. Given K_a as defined above, we let $z \in K_a$, and $w \in D$. Now, define the maps F, G, and f_t as follows;

$$F : K_a \to D, F(z) = \frac{1 - \exp\left(\frac{i\pi}{2a}\frac{z+1}{z-1}\right)}{1 + \exp\left(\frac{i\pi}{2a}\frac{z+1}{z-1}\right)},$$

$$f_t : D \to D, f_t(w) = \frac{w + t}{tw + 1},$$

and

$$G : D \to K_a, G(w) = \frac{\log\left(\frac{1-w}{1+w}\right) + \frac{i\pi}{2a}}{\log\left(\frac{1-w}{1+w}\right) - \frac{i\pi}{2a}}.$$

The map F consists of the composition of several isomorphisms:

1. $\frac{z+1}{z-1}$ takes K_a to the region in $\mathbb{C}$ bounded by the lines $\operatorname{Re} z = a$ and $\operatorname{Re} z = -a$.

2. $\frac{i\pi}{2a}z$ rotates and stretches that region to be a strip of $\mathbb{C}$ bounded by the lines $\operatorname{Im} z = i\frac{\pi}{2}$ and $\operatorname{Im} z = -i\frac{\pi}{2}$.

3. $\exp(z)$ takes that band and maps it to the right half plane of $\mathbb{C}$.

4. $\frac{1-z}{1+z}$ then takes that half plane and maps it to D.

Thus, F is an isomorphism from K_a to D, and one can check that G is the inverse of F. Likewise, f_t is an automorphism of D that fixes 1 and -1. By taking the composition of these three maps, we get

$$G \circ f_t \circ F(z) = \frac{\log\left(\frac{1-t}{1+t}\right)(z-1) + \frac{i\pi z}{a}}{\log\left(\frac{1-t}{1+t}\right)(z-1) + \frac{i\pi}{a}}.$$

We know that $\log\left(\frac{1-t}{1+t}\right)$ is a bijection from $(-1, 1)$ into $\mathbb{R}$, so we now define a new parameter $s \in \mathbb{R}$ where $s = \log\left(\frac{1-t}{1+t}\right)$.

Now, define a new function H_s where

$$H_s : K_a \to K_a, H_s(z) = \frac{s(z-1) + \frac{i\pi}{a}z}{s(z-1) + \frac{i\pi}{a}}. \tag{22}$$

H_s is an automorphism of K_a that fixes $z = 1$.

Now that we have defined our family of automorphisms of K_a, we prove some useful properties of these holomorphisms that we will make use of in Section 6.

Lemma 5.1. *Let δ be a real number such that $\delta > 0$. If C_b is a circle contained in K_a that passes through the point $z = 1$, then C_b can be defined by $C_b = \{z \in K_a : z = b + (1-b)e^{i\theta}\}$ where $\frac{a}{a-1} < b < \frac{a}{a+1}$. H_s satisfies the following properties;*

1. *H_s maps any C_b to itself.*

2. *$H_s(H_t(z)) = H_{s+t}(z)$.*

3. *$\lim_{s\to\infty} H_s(z) = 1$ with $\mathrm{Im} H_s(z) > 0$ on $K_a \backslash B_1(\delta)$.*

4. *$\lim_{s\to -\infty} H_s(z) = 1$ with $\mathrm{Im} H_s(z) < 0$ on $K_a \backslash B_1(\delta)$.*

5. *$\left|\frac{\partial}{\partial z} H_s\right|$ is bounded above by some constant $E_{\delta,a} = \left(1 + \frac{2}{\delta(1-a)}\right)^2$ on $K_a \backslash B_1(\delta)$.*

Proof. We will start with property 1. Let $z = b + (1-b)e^{i\theta_1}$. We wish to show that

$$H_s(b + (1-b)e^{i\theta_1}) = b + (1-b)e^{i\theta_2}.$$

So,

$$\begin{aligned} H_s(b + (1-b)e^{i\theta_1}) &= \frac{s(b + (1-b)e^{i\theta_1} - 1) + \frac{i\pi}{a}(b + (1-b)e^{i\theta_1})}{s(b + (1-b)e^{i\theta_1} - 1) + \frac{i\pi}{a}} \\ &= \frac{s((b-1) + (1-b)e^{i\theta_1}) + \frac{i\pi}{a}(b + (1-b)e^{i\theta_1})}{s((b-1) + (1-b)e^{i\theta_1}) + \frac{i\pi}{a}}. \end{aligned}$$

Now, we will add and subtract a b, and combine the negative b with our current term after finding the common denominator.

$$H_s(b+(1-b)e^{i\theta_1}) = b - b + \frac{s((b-1)+(1-b)e^{i\theta_1}) + \frac{i\pi}{a}(b+(1-b)e^{i\theta_1})}{s((b-1)+(1-b)e^{i\theta_1}) + \frac{i\pi}{a}}$$
$$= b + \frac{s((b-1)+(1-b)e^{i\theta_1}) + \frac{i\pi}{a}(b+(1-b)e^{i\theta_1}) - sb((b-1)+(1-b)e^{i\theta_1}) - b\frac{i\pi}{a}}{s((b-1)+(1-b)e^{i\theta_1}) + \frac{i\pi}{a}}$$
$$= b + \frac{s(1-b)((b-1)+(1-b)e^{i\theta_1}) + \frac{i\pi}{a}(1-b)e^{i\theta_1}}{s((b-1)+(1-b)e^{i\theta_1}) + \frac{i\pi}{a}}.$$

After removing a factor of $(1-b)$ from the numerator, and a factor of $(1-b)$ from all $((b-1)+(1-b)e^{i\theta_1})$ terms we have;

$$H_s(b+(1-b)e^{i\theta_1}) = b + (1-b)\frac{s(1-b)(e^{i\theta_1}-1) + \frac{i\pi}{a}e^{i\theta_1}}{s(1-b)(e^{i\theta_1}-1) + \frac{i\pi}{a}}.$$

All that remains is to show that

$$\frac{s(1-b)(e^{i\theta_1}-1) + \frac{i\pi}{a}e^{i\theta_1}}{s(1-b)(e^{i\theta_1}-1) + \frac{i\pi}{a}} = e^{i\theta_2},$$

for some θ_2. It is sufficient to show

$$\left|\frac{s(1-b)(e^{i\theta_1}-1) + \frac{i\pi}{a}e^{i\theta_1}}{s(1-b)(e^{i\theta_1}-1) + \frac{i\pi}{a}}\right|^2 = 1.$$

Computing this modulus gives us;

$$\left|\frac{s(1-b)(e^{i\theta_1}-1)+\frac{i\pi}{a}e^{i\theta_1}}{s(1-b)(e^{i\theta_1}-1)+\frac{i\pi}{a}}\right|^2 = \frac{s(1-b)(e^{i\theta_1}-1)+\frac{i\pi}{a}e^{i\theta_1}}{s(1-b)(e^{i\theta_1}-1)+\frac{i\pi}{a}} * \frac{s(1-b)(e^{-i\theta_1}-1)-\frac{i\pi}{a}e^{-i\theta_1}}{s(1-b)(e^{-i\theta_1}-1)-\frac{i\pi}{a}}$$

$$= \frac{s^2(1-b)^2(2-2\cos(\theta_1)+\frac{i\pi}{a}e^{i\theta_1}s(1-b)(e^{-i\theta_1}-1)-\frac{i\pi}{a}e^{-i\theta_1}s(1-b)(e^{i\theta_1}-1)+\frac{\pi^2}{a^2}}{s^2(1-b)^2(2-2\cos(\theta_1)+\frac{i\pi}{a}s(1-b)(e^{-i\theta_1}-1)-\frac{i\pi}{a}s(1-b)(e^{i\theta_1}-1)+\frac{\pi^2}{a^2}}$$

$$= \frac{s^2(1-b)^2(2-2\cos(\theta_1)+\frac{i\pi}{a}s(1-b)(1-e^{i\theta_1})-\frac{i\pi}{a}s(1-b)(1-e^{-i\theta_1})+\frac{\pi^2}{a^2}}{s^2(1-b)^2(2-2\cos(\theta_1)+\frac{i\pi}{a}s(1-b)((e^{-i\theta_1}-1)-(e^{i\theta_1}-1))+\frac{\pi^2}{a^2}}$$

$$= \frac{s^2(1-b)^2(2-2\cos(\theta_1)+\frac{i\pi}{a}s(1-b)((1-e^{i\theta_1})-(1-e^{-i\theta_1}))+\frac{\pi^2}{a^2}}{s^2(1-b)^2(2-2\cos(\theta_1)+\frac{i\pi}{a}s(1-b)((e^{-i\theta_1}-1)-(e^{i\theta_1}-1))+\frac{\pi^2}{a^2}}$$

$$= \frac{s^2(1-b)^2(2-2\cos(\theta_1)+\frac{i\pi}{a}s(1-b)(e^{-i\theta_1}-e^{i\theta_1})+\frac{\pi^2}{a^2}}{s^2(1-b)^2(2-2\cos(\theta_1)+\frac{i\pi}{a}s(1-b)(e^{-i\theta_1}-e^{i\theta_1})+\frac{\pi^2}{a^2}} = 1.$$

For property 2,

$$\begin{aligned}
H_s(H_t(z)) &= \frac{s\left(\frac{t(z-1)+\frac{i\pi}{a}z}{t(z-1)+\frac{i\pi}{a}}-1\right)+\frac{i\pi}{a}\left(\frac{t(z-1)+\frac{i\pi}{a}z}{t(z-1)+\frac{i\pi}{a}}\right)}{s\left(\frac{t(z-1)+\frac{i\pi}{a}z}{t(z-1)+\frac{i\pi}{a}}-1\right)+\frac{i\pi}{a}} \\
&= \frac{s\left(\frac{\frac{i\pi}{a}z-\frac{i\pi}{a}}{t(z-1)+\frac{i\pi}{a}}\right)+\frac{i\pi}{a}\left(\frac{t(z-1)+\frac{i\pi}{a}z}{t(z-1)+\frac{i\pi}{a}}\right)}{s\left(\frac{\frac{i\pi}{a}z-\frac{i\pi}{a}}{t(z-1)+\frac{i\pi}{a}}\right)+\frac{i\pi}{a}} \\
&= \frac{s\left(\frac{z-1}{t(z-1)+\frac{i\pi}{a}}\right)+\left(\frac{t(z-1)+\frac{i\pi}{a}z}{t(z-1)+\frac{i\pi}{a}}\right)}{s\left(\frac{z-1}{t(z-1)+\frac{i\pi}{a}}\right)+1} \\
&= \frac{s\left(\frac{z-1}{t(z-1)+\frac{i\pi}{a}}\right)+\left(\frac{t(z-1)+\frac{i\pi}{a}z}{t(z-1)+\frac{i\pi}{a}}\right)}{s\left(\frac{z-1}{t(z-1)+\frac{i\pi}{a}}\right)+\frac{t(z-1)+\frac{i\pi}{a}}{t(z-1)+\frac{i\pi}{a}}} \\
&= \frac{s(z-1)+t(z-1)+\frac{i\pi}{a}z}{s(z-1)+t(z-1)+\frac{i\pi}{a}} \\
&= \frac{(s+t)(z-1)+\frac{i\pi}{a}z}{(s+t)(z-1)+\frac{i\pi}{a}} = H_{s+t}(z).
\end{aligned}$$

Now, let δ be some positive real number. For properties 3 and 4, we will rewrite H_s as follows;

$$H_s(z) = \frac{s(z-1)+\frac{i\pi}{a}z}{s(z-1)+\frac{i\pi}{a}} = 1 + \frac{\frac{i\pi}{a}(z-1)}{s(z-1)+\frac{i\pi}{a}}.$$

Then, looking at $|H_s(z)-1|^2$, we see;

$$|H_s(z)-1|^2 = \frac{\frac{\pi^2}{a^2}|z-1|^2}{|s(z-1)+\frac{i\pi}{a}|^2} = \frac{\frac{\pi^2}{a^2}|z-1|^2}{s^2|z-1|^2+\frac{\pi^2}{a^2}+\frac{2\pi s}{a}\mathrm{Im}\, z}.$$

If $|z-1| \geq \delta$, then the above converges uniformly to 0 for large values of s. Now, to inspect the imaginary part separately, we will start by rationalizing $H_s(z)-1$.

$$\begin{aligned} H_s(z)-1 &= \frac{\frac{i\pi}{a}(z-1)}{s(z-1)+\frac{i\pi}{a}} \\ &= \frac{\frac{i\pi}{a}(z-1)}{s(z-1)+\frac{i\pi}{a}} \frac{s(\bar{z}-1)-\frac{i\pi}{a}}{s(\bar{z}-1)-\frac{i\pi}{a}} \\ &= \frac{\frac{i\pi s}{a}|z-1|^2+\frac{\pi^2}{a^2}(z-1)}{s^2|z-1|^2+\frac{\pi^2}{a^2}+2\mathrm{Re}\left(\frac{-i\pi}{a}s(z-1)\right)} \\ &= \frac{\frac{\pi^2}{a^2}(\mathrm{Re}\, z-1)+i(\frac{\pi}{a}s|z-1|^2+\frac{\pi^2}{a^2}\mathrm{Im}\, z)}{s^2|z-1|^2+\frac{\pi^2}{a^2}+\frac{2\pi s}{a}\mathrm{Im}\, z}. \end{aligned}$$

So, we have now;

$$\mathrm{Im} H_s(z) = \frac{\frac{\pi}{a}s|z-1|^2+\frac{\pi^2}{a^2}\mathrm{Im}\, z}{s^2|z-1|^2+\frac{\pi^2}{a^2}+\frac{2\pi s}{a}\mathrm{Im}\, z}$$

Our goal is to show that for $|z-1| \geq \delta$ this term uniformly approaches 0 from above for large positive s, and uniformly approaches 0 from below for large negative s. Clearly, because of the degree of s, this term is approaching 0 for large s, so we need only check the sign. It is worth noting that by construction, the denominator is strictly positive so we only need to check the sign of the numerator.

For property 3, we start by supposing $s > 0$. If $\mathrm{Im}\, z > 0$, then clearly $\mathrm{Im} H_s(z) > 0$. Now suppose $\mathrm{Im}\, z < 0$. Since $|z-1| \geq \delta$ and on K_a, $-\mathrm{Im}\, z \leq \frac{1}{1-a}$. So we can see the above is positive so long as

$$s > \frac{\frac{\pi}{a}\frac{1}{1-a}}{\delta^2} \geq \frac{\frac{-\pi}{a}\mathrm{Im}\, z}{|z-1|^2} > 0.$$

So we have a uniform lower bound for s that gives us the convergence we desire. Similarly, for property 4, we suppose $s < 0$. If $\mathrm{Im}\, z < 0$, then $\mathrm{Im} H_s(z) < 0$. If $\mathrm{Im}\, z > 0$, then $\mathrm{Im} H_s(z) < 0$

if

$$s < \frac{\frac{-\pi}{a}\operatorname{Im} z}{|z-1|^2} < 0.$$

As before, $|z-1| \geq \delta$ and $-\operatorname{Im} z \geq \frac{-1}{1-a}$. So we have a similar uniform upper bound on s as follows;

$$s < \frac{\frac{\pi}{a}\frac{-1}{1-a}}{\delta^2} \leq \frac{\frac{-\pi}{a}\operatorname{Im} z}{|z-1|^2} < 0.$$

Lastly for property 5, let $E_{\delta,a} = \left(1 + \frac{2}{\delta(1-a)}\right)^2$. We wish to show that $\left|\frac{\partial H_s(z)}{\partial z}\right| \leq E_{\delta,a}$ on $K_a \backslash B_1(\delta)$. For sake of notation, let $h_s(z) = \left|\frac{\partial H_s(z)}{\partial z}\right|$. Then,

$$\begin{aligned} h_s(z) &= \left|\frac{(s+\frac{i\pi}{a})(s(z-1)+\frac{i\pi}{a}) - s(s(z-1)+\frac{i\pi}{a}z)}{(s(z-1)+\frac{i\pi}{a})^2}\right| \\ &= \left|\frac{\frac{-\pi^2}{a^2}}{(s(z-1)+\frac{i\pi}{a})^2}\right| \\ &= \frac{\frac{\pi^2}{a^2}}{(s(\operatorname{Re} z-1))^2 + (s\operatorname{Im} z + \frac{\pi}{a})^2}. \end{aligned}$$

It is worth noting that if $s = 0, h_s(z) = 1 < E_{\delta,a}$, so suppose $s \neq 0$. Without loss of generality, suppose $s > 0$. Setting up our inequality, we wish to show;

$$\frac{\frac{\pi^2}{a^2}}{(s(\operatorname{Re} z-1))^2 + (s\operatorname{Im} z + \frac{\pi}{a})^2} \leq E_{\delta,a}.$$

First, we have a positive denominator on the left hand side and so we can multiply both sides by that denominator and divide both sides by $E_{\delta,a}$ to get:

$$\frac{\pi^2}{a^2 E_{\delta,a}} \leq (s(\operatorname{Re} z-1))^2 + \left(s\operatorname{Im} z + \frac{\pi}{a}\right)^2.$$

Then we divide both sides by s^2 to get

$$\frac{\pi^2}{a^2 s^2 E_{\delta,a}} \leq (\operatorname{Re} z-1)^2 + \left(\operatorname{Im} z + \frac{\pi}{as}\right)^2.$$

Geometrically speaking, we can see that $h_s(z) \leq E_{\delta,a}$ so long as z lies outside of the circle centered at $1 - \frac{i\pi}{as}$ with radius of $\frac{\pi}{asE_{\delta,a}^{\frac{1}{2}}}$, or if that circle lies entirely within $B_1(\delta)$. We will define this circle as $K_{E_{\delta,a}}$. We will show that one of the following cases must be true:

1. $K_{E_{\delta,a}} \cap K_a = \emptyset$

2. $K_{E_{\delta,a}} \subset \overline{B_1(\delta)}$.

First, looking at $E_{\delta,a}$, we have the following set of equations;

$$\begin{aligned} E_{\delta,a} = \left(1 + \frac{2}{\delta(1-a)}\right)^2 &\implies \frac{1-a}{2} E_{\delta,a}^{\frac{1}{2}} = \frac{1-a}{2} + \frac{1}{\delta} \\ &\implies \frac{1-a}{2}(E_{\delta,a}^{\frac{1}{2}} - 1) = \frac{1}{\delta} \\ &\implies \frac{1-a}{2} \frac{(E_{\delta,a}^{\frac{1}{2}} + 1)(E_{\delta,a}^{\frac{1}{2}} - 1)}{E_{\delta,a}^{\frac{1}{2}}} = \frac{1}{\delta} \frac{E_{\delta,a}^{\frac{1}{2}} + 1}{E_{\delta,a}^{\frac{1}{2}}} \\ &\implies \frac{\pi(1-a)}{2a} \frac{E_{\delta,a} - 1}{E_{\delta,a}^{\frac{1}{2}}} = \frac{\pi}{a\delta} \frac{E_{\delta,a}^{\frac{1}{2}} + 1}{E_{\delta,a}^{\frac{1}{2}}}. \end{aligned}$$

So, given any $s > 0$, one of the following cases must be true;

1. $s < \frac{\pi(1-a)}{2a} \frac{E_{\delta,a}-1}{E_{\delta,a}^{\frac{1}{2}}}$

2. $s \geq \frac{\pi}{a\delta} \frac{E_{\delta,a}^{\frac{1}{2}}+1}{E_{\delta,a}^{\frac{1}{2}}}$.

If the first inequality is true, we can see the following by multiplying both sides by $\frac{2}{1-a}$;

$$\frac{2s}{1-a} < \frac{\pi}{a} \frac{E_{\delta,a} - 1}{E_{\delta,a}^{\frac{1}{2}}}.$$

Then after dividing both sides by s and rewriting $\frac{E_{\delta,a}-1}{E_{\delta,a}^{\frac{1}{2}}}$ as $E_{\delta,a}^{\frac{1}{2}} - \frac{1}{E_{\delta,a}^{\frac{1}{2}}}$, we have

$$\frac{2}{1-a} < \frac{\pi}{as} \left(E_{\delta,a}^{\frac{1}{2}} - \frac{1}{E_{\delta,a}^{\frac{1}{2}}} \right).$$

Now we multiply both sides of the inequality by $\frac{\pi}{asE_{\delta,a}^{\frac{1}{2}}}$ to get

$$\frac{2\pi}{as(1-a)E_{\delta,a}^{\frac{1}{2}}} < \frac{\pi^2}{a^2s^2}\left(1 - \frac{1}{E_{\delta,a}}\right).$$

We can distribute the right hand side and then add one of the resulting terms to see

$$\frac{2\pi}{as(1-a)E_{\delta,a}^{\frac{1}{2}}} + \frac{\pi^2}{a^2s^2E_{\delta,a}} < \frac{\pi^2}{a^2s^2}.$$

Now, we add $\frac{1}{(1-a)^2}$ to both sides to yield

$$\frac{2\pi}{as(1-a)E_{\delta,a}^{\frac{1}{2}}} + \frac{\pi^2}{a^2s^2E_{\delta,a}} + \frac{1}{(1-a)^2} < \frac{1}{(1-a)^2} + \frac{\pi^2}{a^2s^2}.$$

Now, the left hand side is a perfect square, and the right hand side can be rewritten to show the following;

$$\left(\frac{\pi}{asE_{\delta,a}^{\frac{1}{2}}} + \frac{1}{1-a}\right)^2 < \left(1 - \frac{a}{a-1}\right)^2 + \left(\frac{-\pi}{as} - 0\right)^2$$

Now, we can recognize that the left-hand side of this inequality represents the square of the sum of radii of circles K_2 and $K_{E_{\delta,a}}$, and the right-hand side represents the square of the distance between the centers of said circles. That is;

$$\left(\frac{\pi}{asE_{\delta,a}^{\frac{1}{2}}} + \frac{1}{1-a}\right)^2 < \left|\left(1 - i\frac{\pi}{as}\right) - \frac{a}{a-1}\right|^2,$$

which implies that

$$\frac{\pi}{asE_{\delta,a}^{\frac{1}{2}}} + \frac{1}{1-a} < \left|\left(1 - i\frac{\pi}{as}\right) - \frac{a}{a-1}\right|.$$

This tells us that $K_{E_{\delta,a}}$ must lie entirely outside of K_2 and thus case 1 is satisfied. If we look at when $s \geq \frac{\pi}{a\delta}\frac{E_{\delta,a}^{\frac{1}{2}}+1}{E_{\delta,a}^{\frac{1}{2}}}$, we can see the following by multiplying both sides by $\frac{\delta}{s}$;

$$\delta \geq \frac{\pi}{as}\frac{E_{\delta,a}^{\frac{1}{2}}+1}{E_{\delta,a}^{\frac{1}{2}}}.$$

We can separate the second term on the right-hand side to see

$$\delta \geq \frac{\pi}{as}\left(1+\frac{1}{E_{\delta,a}^{\frac{1}{2}}}\right),$$

and then distribute the right-hand side to get

$$\delta \geq \frac{\pi}{as}+\frac{\pi}{asE_{\delta,a}^{\frac{1}{2}}}.$$

Since the center of $K_{E_{\delta,a}}$ is $1-i\frac{\pi}{as}$, and its radius is $\frac{\pi}{asE_{\delta,a}^{\frac{1}{2}}}$, this tells us that the point inside $K_{E_{\delta,a}}$ furthest from $z=1$ is still contained in $\overline{B_1(\delta)}$, thus satisfying case 2. □

6 Representative Form

Now that we have constructed a family of automorphisms for our crescent region K_a that satisfy nice properties, including nice gradient estimates away from $z=1$, we will use those automorphisms to obtain a better understanding of our functions φ_η. Since these automorphisms are inherently connected to the geometry of our crescent region, we are able to see how this geometry changes our sample function into a more general representative form. Since we are again looking at a region defined in one complex variable, we note that the operator P_η takes the form like that in (3) where β vanishes.

Lemma 6.1. *Given a region K_a as defined in Definition 5.1 and a real number η with $0<\eta<1$, if there exists a real valued, smooth, subharmonic function φ_η on K_a satisfying the following conditions on K_a;*

1. $P_\eta \varphi_\eta(z) > 0$,

2. *There exists some finite real constant* E_η *such that* $|\nabla \varphi_\eta| < E_\eta$,

then for all real numbers $\varepsilon > 0, \delta > 0$ *there exists some real valued, subharmonic function* $\tilde{\varphi}_{\eta,\varepsilon} \in C^2(K_a)$ *satisfying the following conditions;*

1. $P_\eta \tilde{\varphi}_{\eta,\varepsilon}(z) > 0$ *on* K_a.

2. $|\nabla \tilde{\varphi}_{\eta,\varepsilon}| < \left(E_\eta + \frac{a+1}{1-a}\right)\left(1 + \frac{2}{\delta(1-a)}\right)^2 + \frac{a+1}{1-a}$ *on* $K_a \backslash B_1(\delta)$.

3. *There exists some negative valued convex function* $\psi : \mathbb{R} \to \mathbb{R}$ *such that*

$$\sup_{K_a \backslash B_1(\delta)} \left| \tilde{\varphi}_{\eta,\varepsilon}(z) - \left[\frac{\eta - 1}{\eta} \log\left(-\psi\left(\frac{|z|^2 - 1}{|z-1|^2} \right)\right) - 2\mathrm{Im}\log(z) \right] \right| < \varepsilon.$$

Proof. Let $0 < \eta < 1$, $\varepsilon > 0$, $\delta > 0$, and let φ_η be a real valued, smooth, subharmonic function on K_a satisfying properties 1 and 2.

We now construct the map Φ_η on K_a by,

$$\Phi_\eta(z) = -\exp\left(\frac{-\eta}{1-\eta}(\varphi_\eta(z) + 2\mathrm{Im}\log(z)) \right). \tag{23}$$

Note that $\Phi_\eta(z) < 0$. We find the Laplacian of Φ_η is as follows;

$$\frac{\partial \Phi_\eta(z)}{\partial \bar{z}} = \Phi_\eta(z) \frac{-\eta}{1-\eta} \left(\frac{\partial \varphi_\eta(z)}{\partial \bar{z}} + \frac{i}{\bar{z}} \right),$$

$$\begin{aligned}\frac{\partial^2 \Phi_\eta(z)}{\partial z \partial \bar{z}} &= \Phi_\eta(z) \left(\frac{-\eta}{1-\eta} \right)^2 \left(\frac{\partial \varphi_\eta(z)}{\partial z} - \frac{i}{z} \right) \left(\frac{\partial \varphi_\eta(z)}{\partial \bar{z}} + \frac{i}{\bar{z}} \right) + \Phi_\eta(z) \frac{-\eta}{1-\eta} \frac{\partial^2 \varphi_\eta(z)}{\partial z \partial \bar{z}} \\ &= \Phi_\eta(z) \frac{-\eta}{1-\eta} \left(\frac{\partial^2 \varphi_\eta(z)}{\partial z \partial \bar{z}} - \frac{\eta}{1-\eta} \left(\frac{\partial \varphi_\eta(z)}{\partial z} - \frac{i}{z} \right) \left(\frac{\partial \varphi_\eta(z)}{\partial \bar{z}} + \frac{i}{\bar{z}} \right) \right) \\ &= \frac{-\eta}{1-\eta} \Phi_\eta(z) P_\eta \varphi_\eta(z).\end{aligned}$$

This is strictly positive since $\Phi_\eta(z)\frac{-\eta}{1-\eta} > 0$, and condition 1 of φ_η. So we can see that;

$$\left|\frac{\partial \Phi_\eta(z)}{\partial z}\right| = |\Phi_\eta(z)| \left(\frac{\eta}{1-\eta}\right) \left|\frac{\partial \varphi_\eta(z)}{\partial z} - \frac{i}{z}\right|.$$

By the triangle inequality we get

$$\left|\frac{\partial \Phi_\eta(z)}{\partial z}\right| \leq |\Phi_\eta(z)| \left(\frac{\eta}{1-\eta}\right) \left(\left|\frac{\partial \varphi_\eta(z)}{\partial z}\right| + \left|\frac{1}{z}\right|\right),$$

and by using the definition of K_a and property 2 of φ_η we see

$$\left|\frac{\partial \Phi_\eta(z)}{\partial z}\right| \leq |\Phi_\eta(z)| \left(\frac{\eta}{1-\eta}\right) \left(E_\eta + \frac{a+1}{1-a}\right).$$

So we have the constant $E_{\eta,a} = \frac{\eta}{1-\eta}(E_\eta + \frac{a+1}{1-a})$ such that

$$\left|\frac{\partial \Phi_\eta(z)}{\partial z}\right| \leq |\Phi_\eta(z)| E_{\eta,a}. \tag{24}$$

Now, given $M \in \mathbb{N}$ and $H_s(z)$ as defined in (22), define the map $\tilde{\Phi}_{\eta,M}$ as

$$\tilde{\Phi}_{\eta,M}(z) = \fint_{-M}^{M} \Phi_\eta(H_s(z))ds.$$

Since M and s are independent of z, we see that

$$\frac{\partial \tilde{\Phi}_{\eta,M}(z)}{\partial \bar{z}} = \fint_{-M}^{M} \frac{\partial \Phi_\eta(H_s(z))}{\partial \overline{w}} \frac{\partial \overline{H_s(z)}}{\partial \bar{z}} ds,$$

due to H_s being holomorphic.

Then we can see;

$$\frac{\partial^2\tilde{\Phi}_{\eta,M}(z)}{\partial z\partial\bar{z}} = \fint_{-M}^{M}\frac{\partial}{\partial z}\left(\frac{\partial\Phi_\eta(H_s(z))}{\partial\overline{w}}\frac{\partial\overline{H_s(z)}}{\partial\bar{z}}\right)ds$$

$$= \fint_{-M}^{M}\left(\frac{\partial^2\Phi_\eta(H_s(z))}{\partial w\partial\overline{w}}\frac{\partial H_s(z)}{\partial z}+\frac{\partial^2\Phi_\eta(H_s(z))}{\partial\overline{w}^2}\frac{\partial\overline{H_s(z)}}{\partial z}\right)\frac{\partial\overline{H_s(z)}}{\partial\bar{z}}+\frac{\partial\Phi_\eta(H_s(z))}{\partial\overline{w}}\frac{\partial^2\overline{H_s(z)}}{\partial z\partial\bar{z}}ds$$

$$= \fint_{-M}^{M}\frac{\partial^2\Phi_\eta(H_s(z))}{\partial w\partial\overline{w}}\left|\frac{\partial H_s(z)}{\partial z}\right|^2 ds.$$

Since Φ_η is strictly subharmonic and H_s is non-constant for $s \neq 0$, we have

$$\frac{\partial^2\tilde{\Phi}_{\eta,M}(z)}{\partial z\partial\bar{z}} > 0.$$

Now, looking at $\left|\frac{\partial\tilde{\Phi}_{\eta,M}(z)}{\partial z}\right|$,

$$\left|\frac{\partial\tilde{\Phi}_{\eta,M}(z)}{\partial z}\right| = \left|\fint_{-M}^{M}\frac{\partial\Phi_\eta(H_s(z))}{\partial w}\frac{\partial H_s(z)}{\partial z}ds\right| \leq \fint_{-M}^{M}\left|\frac{\partial\Phi_\eta(H_s(z))}{\partial w}\right|\left|\frac{\partial H_s(z)}{\partial z}\right|ds.$$

Using (24), we have

$$\left|\frac{\partial\tilde{\Phi}_{\eta,M}(z)}{\partial z}\right| \leq E_{\eta,a}\fint_{-M}^{M}|\Phi_\eta(H_s(z))|\left|\frac{\partial H_s(z)}{\partial z}\right|ds.$$

Thanks to property 5 of Lemma 5.1, $\left|\frac{\partial H_s(z)}{\partial z}\right|$ is bounded above by $E_{\delta,a} = \left(1+\frac{2}{\delta(1-a)}\right)^2$ on $K_a\backslash B_1(\delta)$. We now have the current upper bound for $\left|\frac{\partial}{\partial z}\tilde{\Phi}_{\eta,M}(z)\right|$,

$$\left|\frac{\partial\tilde{\Phi}_{\eta,M}(z)}{\partial z}\right| \leq E_{\eta,a}E_{\delta,a}\fint_{-M}^{M}|\Phi_\eta(H_s(z))|ds.$$

Now, since $\Phi_\eta < 0$ and consequently $\tilde{\Phi}_{\eta,M} < 0$, we can see the following;

$$\begin{aligned}
\fint_{-M}^{M} |\Phi_\eta(H_s(z))| ds &= \fint_{-M}^{M} -\Phi_\eta(H_s(z)) ds \\
&= -\fint_{-M}^{M} \Phi_\eta(H_s(z)) ds \\
&= -\tilde{\Phi}_{\eta,M}(z) \\
&= |\tilde{\Phi}_{\eta,M}(z)|.
\end{aligned}$$

So we can rewrite the above inequality as;

$$\left| \frac{\partial \tilde{\Phi}_{\eta,M}(z)}{\partial z} \right| \leq |\tilde{\Phi}_{\eta,M}(z)| E_{\eta,a} E_{\delta,a}.$$

Now, we can rearrange the inequality to see

$$\left| \frac{\frac{\partial}{\partial z} \tilde{\Phi}_{\eta,M}(z)}{\tilde{\Phi}_{\eta,M}(z)} \right| = \left| \frac{\partial}{\partial z} \log(-\tilde{\Phi}_{\eta,M}(z)) \right| \leq E_{\eta,a} E_{\delta,a}. \tag{25}$$

Our goal now is to show that so long as $|z - 1| > \delta$, as M increases, $\tilde{\Phi}_{\eta,M}(z)$ converges uniformly to a function that is constant along circles C_b as described in Lemma 5.1. To do this, we will show that the derivative with regard to rotation along said circles approaches 0 as M increases without bound. Since we have shown that $H_t(z)$ represents a rotation along such circles, it suffices to show the following;

Given $z \in K_a$ such that $|z - 1| > \delta$,

$$\lim_{M \to \infty} \lim_{t \to 0} \frac{\tilde{\Phi}_{\eta,M}(H_t(z)) - \tilde{\Phi}_{\eta,M}(z)}{t} = 0.$$

Without loss of generality, assume $t > 0$. First, looking at $\tilde{\Phi}_{\eta,M}(H_t(z)) - \tilde{\Phi}_{\eta,M}(z)$,

$$\begin{aligned}\tilde{\Phi}_{\eta,M}(H_t(z)) - \tilde{\Phi}_{\eta,M}(z) &= \fint_{-M}^{M} \Phi_\eta(H_s(H_t(z)))ds - \fint_{-M}^{M} \Phi_\eta(H_s(z))ds \\ &= \frac{1}{2M}\left(\int_{-M}^{M} \Phi_\eta(H_{s+t}(z))ds - \int_{-M}^{M} \Phi_\eta(H_s(z))ds\right) \\ &= \frac{1}{2M}\left(\int_{-M+t}^{M+t} \Phi_\eta(H_s(z))ds - \int_{-M}^{M} \Phi_\eta(H_s(z))ds\right) \\ &= \frac{1}{2M}\left(\int_{M}^{M+t} \Phi_\eta(H_s(z))ds - \int_{-M}^{-M+t} \Phi_\eta(H_s(z))ds\right).\end{aligned}$$

Taking the limit of the difference quotient, we have;

$$\begin{aligned}\lim_{t\to 0} \frac{\tilde{\Phi}_{\eta,M}(H_t(z)) - \tilde{\Phi}_{\eta,M}(z)}{t} &= \lim_{t\to 0} \frac{1}{2M}\left(\fint_{M}^{M+t} \Phi_\eta(H_s(z))ds - \fint_{-M}^{-M+t} \Phi_\eta(H_s(z))ds\right) \\ &= \frac{\Phi_\eta(H_M(z)) - \Phi_\eta(H_{-M}(z))}{2M}.\end{aligned}$$

Using the definition of Φ_η, we have;

$$\begin{aligned}&\frac{\Phi_\eta(H_M(z)) - \Phi_\eta(H_{-M}(z))}{2M} \\ &\quad = \frac{\exp\left(\frac{-\eta}{1-\eta}(\varphi_\eta(H_{-M}(z)) + 2\mathrm{Im}\log(H_{-M}(z)))\right)}{2M} \\ &\qquad - \frac{\exp\left(\frac{-\eta}{1-\eta}(\varphi_\eta(H_{M}(z)) + 2\mathrm{Im}\log(H_{M}(z)))\right)}{2M}.\end{aligned}$$

Now, by properties 3 and 4 of Lemma 5.1, as $M \to \infty$, $H_M(z) \to 1$ with $\mathrm{Im}H_M(z) > 0$ and $H_{-M}(z) \to 1$ with $\mathrm{Im}H_{-M}(z) < 0$. So, taking the limit of the above, we see the numerator converges uniformly in M as follows;

$$\begin{aligned}\lim_{M\to\infty} \Phi_\eta(H_M(z)) - \Phi_\eta(H_{-M}(z)) &= \exp\left(\frac{-\eta}{1-\eta}(\varphi_\eta(1) + 4\pi)\right) - \exp\left(\frac{-\eta}{1-\eta}\varphi_\eta(1)\right) \\ &= \exp\left(\frac{-\eta}{1-\eta}\varphi_\eta(1)\right)\left(\exp\left(\frac{-\eta}{1-\eta}4\pi\right) - 1\right).\end{aligned}$$

We note that since $\frac{-\eta}{1-\eta}4\pi < 0, -1 < \exp\left(\frac{-\eta}{1-\eta}4\pi\right) - 1 < 0$. Also, since φ_η is continuous on $K_a, \varphi_\eta(1)$ has a strict lower bound, and thus the above equation has a strict lower bound. Thus, we have the uniform convergence of

$$\lim_{M\to\infty} \frac{\Phi_\eta(H_M(z)) - \Phi_\eta(H_{-M}(z))}{2M} = 0.$$

So, $\tilde{\Phi}_{\eta,M}$ is uniformly converging to a subharmonic function that is constant on these circles, and we will call this $\tilde{\Phi}_\eta$. That is, we have the function

$$\tilde{\Phi}_\eta(z) = \lim_{M\to\infty} \tilde{\Phi}_{\eta,M}(z).$$

We now wish to find a representation of this new function. First, we define a function $k : K_a \backslash \{1\} \to \mathbb{R}$ by;

$$k(z) = 1 + 2\mathrm{Re}\left(\frac{1}{z-1}\right).$$

Since this function is the real part of a holomorphic function, it is harmonic. For the purpose of the next step, we will rewrite $k(z)$.

$$\begin{aligned}
k(z) &= 1 + 2\mathrm{Re}\left(\frac{1}{z-1}\right) \\
&= 1 + \frac{1}{z-1} + \frac{1}{\bar{z}-1} \\
&= 1 + \frac{\bar{z}-1}{|z-1|^2} + \frac{z-1}{|z-1|^2} \\
&= \frac{|z-1|^2 + (z-1) + (\bar{z}-1) + 1 - 1}{|z-1|^2} \\
&= \frac{((z-1)+1)((\bar{z}-1)+1) - 1}{|z-1|^2} \\
&= \frac{|(z-1)+1|^2 - 1}{|z-1|^2} \\
&= \frac{|z|^2 - 1}{|z-1|^2}.
\end{aligned}$$

Now, using the representation for circles given in Lemma 5.1,

$$
\begin{aligned}
k(b+(1-b)e^{i\theta}) &= \frac{|b+(1-b)e^{i\theta}|^2-1}{|b+(1-b)e^{i\theta}-1|^2} \\
&= \frac{b^2+(1-b)^2+2b(1-b)\cos(\theta)-1}{(b-1)^2+(1-b)^2+2(b-1)(1-b)\cos(\theta)} \\
&= \frac{2b^2-2b-2b(b-1)\cos(\theta)}{2(b-1)^2-2(b-1)^2\cos(\theta)} \\
&= \frac{2b(b-1)(1-\cos(\theta))}{2(b-1)^2(1-\cos(\theta))} \\
&= \frac{b}{b-1}.
\end{aligned}
$$

So $k(z)$ is constant along each circle C_b. We now consider the family of circles $\{C_b\}_{b\in\left[\frac{a}{a-1},\frac{a}{a+1}\right]}$ to construct the next part of our representation.

We will let t be an element of the image of $k(z)$, i.e., $t=\frac{b}{b-1}$ for some $b\in\left[\frac{a}{a-1},\frac{a}{a+1}\right]$. Solving for b yields $b=\frac{t}{t-1}$. This means the circle $C_{\frac{t}{t-1}}\in\{C_b\}$, where $C_{\frac{t}{t-1}}=\{z\in\mathbb{C}|z=\frac{t}{t-1}+(1-\frac{t}{t-1})e^{i\theta},\theta\in[0,2\pi)\}$. Letting $\theta=\pi$ shows us that the point $z=\frac{t+1}{t-1}\in C_{\frac{t}{t-1}}$. Now, we consider some function $\psi:\mathbb{R}\to\mathbb{R}$ to be the function that takes an element t in the image of $k(z)$ and maps it to $\tilde{\Phi}_\eta(\frac{t+1}{t-1})$. Composing ψ with k then yields:

$$\psi(k(z))=\tilde{\Phi}_\eta(z).$$

Taking into account that k is harmonic, we can see that,

$$
\begin{aligned}
\frac{\partial^2\tilde{\Phi}_\eta(z)}{\partial z\partial\bar{z}} &= \psi''(k(z))\left|\frac{\partial k(z)}{\partial z}\right|^2+\psi'(k(z))\frac{\partial^2 k(z)}{\partial z\partial\bar{z}} \\
&= \psi''(k(z))\left|\frac{\partial k(z)}{\partial z}\right|^2.
\end{aligned}
$$

Since $\tilde{\Phi}_\eta$ is subharmonic, the above must be positive, and so it must hold that $\psi''(z)\geq 0$, which implies that ψ is a convex function.

Now, we will define a new sequence of functions $\tilde{\varphi}_{\eta,M}$ as,

$$\tilde{\varphi}_{\eta,M}(z) = \frac{\eta - 1}{\eta} \log(-\tilde{\Phi}_{\eta,M}(z)) - 2\mathrm{Im}\log(z).$$

Taking derivatives, we see;

$$\frac{\partial \tilde{\varphi}_{\eta,M}(z)}{\partial \bar{z}} = \frac{\eta - 1}{\eta} \frac{\frac{\partial}{\partial \bar{z}} \tilde{\Phi}_{\eta,M}(z)}{\tilde{\Phi}_{\eta,M}(z)} - \frac{i}{\bar{z}},$$

and so

$$\frac{\partial^2 \tilde{\varphi}_{\eta,M}(z)}{\partial z \partial \bar{z}} = \frac{\eta - 1}{\eta} \frac{\frac{\partial^2}{\partial z \partial \bar{z}} \tilde{\Phi}_{\eta,M}(z) \tilde{\Phi}_{\eta,M}(z) - \frac{\partial}{\partial \bar{z}} \tilde{\Phi}_{\eta,M}(z) \frac{\partial}{\partial z} \tilde{\Phi}_{\eta,M}(z)}{\tilde{\Phi}_{\eta,M}(z)^2}$$
$$= \frac{\eta - 1}{\eta} \frac{\frac{\partial^2}{\partial z \partial \bar{z}} \tilde{\Phi}_{\eta,M}(z) \tilde{\Phi}_{\eta,M}(z) - \left| \frac{\partial}{\partial z} \tilde{\Phi}_{\eta,M}(z) \right|^2}{\tilde{\Phi}_{\eta,M}(z)^2}.$$

Now, since $\tilde{\Phi}_{\eta,M}(z) < 0$ and $\tilde{\Phi}_{\eta,M}$ is strictly subharmonic, we can see that the numerator of the second term is negative. Since $\eta - 1 < 0$, we have a positive product. Thus, $\tilde{\varphi}_{\eta,M}(z)$ is strictly subharmonic. Finding an upper bound on its gradient, we have;

$$\left| \frac{\partial \tilde{\varphi}_{\eta,M}(z)}{\partial z} \right| = \left| \frac{\eta - 1}{\eta} \frac{\frac{\partial}{\partial z} \tilde{\Phi}_{\eta,M}(z)}{\tilde{\Phi}_{\eta,M}(z)} + \frac{i}{z} \right|$$
$$\leq \frac{1 - \eta}{\eta} E_{\eta,a} E_{\delta,a} + \frac{a+1}{1-a}.$$

By expanding $E_{\eta,a}$, we see

$$\left| \frac{\partial \tilde{\varphi}_{\eta,M}(z)}{\partial z} \right| \leq \frac{1 - \eta}{\eta} \left(\frac{\eta}{1 - \eta} \left(E_\eta + \frac{a+1}{1-a} \right) \right) E_{\delta,a} + \frac{a+1}{1-a}$$
$$= \left(E_\eta + \frac{a+1}{1-a} \right) E_{\delta,a} + \frac{a+1}{1-a}.$$

Combine all these terms into a new constant $E_{\eta,\delta,a}$, and we have an upper bound on the gradient that is independent of our choice of M. If we take the limit in M, this sequence will uniformly converge on $K_a \backslash B_1(\delta)$ as follows;

$$\begin{aligned}
\lim_{M\to\infty} \tilde{\varphi}_{\eta,M}(z) &= \lim_{M\to\infty} \frac{\eta-1}{\eta} \log(-\tilde{\Phi}_{\eta,M}(z)) - 2\mathrm{Im}\log(z) \\
&= \frac{\eta-1}{\eta} \log(-\tilde{\Phi}_{\eta}(z)) - 2\mathrm{Im}\log(z) \\
&= \frac{\eta-1}{\eta} \log(-\psi(k(z))) - 2\mathrm{Im}\log(z).
\end{aligned}$$

We will call this new function $\tilde{\varphi}_\eta$. That is;

$$\tilde{\varphi}_\eta(z) = \frac{\eta-1}{\eta} \log(-\psi(k(z))) - 2\mathrm{Im}\log(z).$$

Since this is a uniform convergence, we know that there exists some M_ε such that;

$$\sup_{K_a\backslash B_1(\delta)} |\tilde{\varphi}_{\eta,M_\varepsilon}(z) - \tilde{\varphi}_\eta(z)| < \varepsilon.$$

We will relabel this function $\tilde{\varphi}_{\eta,M_\varepsilon}$ as $\tilde{\varphi}_{\eta,\varepsilon}$, which is the function that satisfies our property 3. We now wish to show that $\tilde{\varphi}_{\eta,\varepsilon} \in C^2(K_a)$. Looking again at its construction, we have:

$$\begin{aligned}
\tilde{\varphi}_{\eta,\varepsilon}(z) &= \frac{\eta-1}{\eta} \log\left(\fint_{M_\varepsilon}^{M_\varepsilon} -\Phi_\eta(H_s(z))ds \right) - 2\mathrm{Im}\log(z) \\
&= \frac{\eta-1}{\eta} \log\left(\fint_{M_\varepsilon}^{M_\varepsilon} \exp\left(\frac{\eta}{\eta-1}(\varphi_\eta(H_s(z)) + 2\mathrm{Im}\log(H_s(z)) \right) ds \right) - 2\mathrm{Im}\log(z). \quad (26)
\end{aligned}$$

We note that while $\mathrm{Im}\log(z)$ is not continuous at $z = 1$, our branch cut gives us clearly defined limits as we approach from both the upper and lower half plane. One can also see that each composition likewise has well defined limits. So, a natural way of showing continuity at 1 is to let $z_{b,\theta} = b + (1-b)e^{i\theta}$ and check the limits as θ approach 0 and 2π. We do know that our branch cut gives us $\lim_{\theta\to 0} \mathrm{Im}\log(z_{b,\theta}) = \mathrm{Im}\log(z_{b,0}) = 0$, so we really only need to show that $\lim_{\theta\to 2\pi}\tilde{\varphi}_{\eta,\varepsilon}(z_{b,\theta}) = \tilde{\varphi}_{\eta,\varepsilon}(1)$. By evaluating at $z = 1$, we see;

$$\begin{aligned}
\tilde{\varphi}_{\eta,\varepsilon}(1) &= \frac{\eta-1}{\eta}\log(-\tilde{\Phi}_{\eta,M_\varepsilon}(1)) - 2\mathrm{Im}\log(1) \\
&= \frac{\eta-1}{\eta}\log\left(-\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(1))ds\right) - 0 \\
&= \frac{\eta-1}{\eta}\log\left(-\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(1)ds\right) \\
&= \frac{\eta-1}{\eta}\log(-\Phi_\eta(1)) \\
&= \frac{\eta-1}{\eta}\log\left(\exp\left(\frac{-\eta}{1-\eta}(\varphi_\eta(1) + 2\mathrm{Im}\log(1))\right)\right) \\
&= \varphi_\eta(1).
\end{aligned}$$

Then taking the limit as θ approaches 2π, we see;

$$\begin{aligned}
\lim_{\theta\to 2\pi}\tilde{\varphi}_{\eta,\varepsilon}(z_{b,\theta}) &= \lim_{\theta\to 2\pi}\frac{\eta-1}{\eta}\log(-\tilde{\Phi}_{\eta,M_\varepsilon}(z_{b,\theta})) - \lim_{\theta\to 2\pi}2\mathrm{Im}\log(z_{b,\theta}) \\
&= \frac{\eta-1}{\eta}\log(\lim_{\theta\to 2\pi}-\tilde{\Phi}_{\eta,M_\varepsilon}(z_{b,\theta})) - 4\pi \\
&= \frac{\eta-1}{\eta}\log\left(\fint_{M_\varepsilon}^{M_\varepsilon}\lim_{\theta\to 2\pi}\exp\left(\frac{\eta}{\eta-1}(\varphi_\eta(H_s(z_{b,\theta})) + 2\mathrm{Im}\log(H_s(z_{b,\theta})))\right)ds\right) - 4\pi \\
&= \frac{\eta-1}{\eta}\log\left(\fint_{M_\varepsilon}^{M_\varepsilon}\exp\left(\frac{\eta}{\eta-1}(\varphi_\eta(\lim_{\theta\to 2\pi}H_s(z_{b,\theta})) + \lim_{\theta\to 2\pi}2\mathrm{Im}\log(H_s(z_{b,\theta})))\right)ds\right) - 4\pi \\
&= \frac{\eta-1}{\eta}\log\left(\fint_{M_\varepsilon}^{M_\varepsilon}\exp\left(\frac{\eta}{\eta-1}(\varphi_\eta(1) + 4\pi)\right)ds\right) - 4\pi \\
&= \frac{\eta-1}{\eta}\log\left(\exp\left(\frac{\eta}{\eta-1}\varphi_\eta(1) + 4\pi\right)\right) - 4\pi \\
&= \varphi_\eta(1).
\end{aligned}$$

So our function is continuous on K_a. Checking first derivatives, we see

$$\begin{aligned}
\frac{\partial \tilde{\varphi}_{\eta,\varepsilon}(z)}{\partial z} &= \frac{\eta-1}{\eta}\frac{\frac{\partial}{\partial z}\tilde{\Phi}_{\eta,M_\varepsilon}(z)}{\tilde{\Phi}_{\eta,M_\varepsilon}(z)} + \frac{i}{z} \\
&= \frac{\eta-1}{\eta}\frac{\fint_{M_\varepsilon}^{M_\varepsilon} \Phi_\eta(H_s(z))\frac{\eta}{\eta-1}\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial z} - \frac{i}{H_s(z)}\right)\frac{\partial H_s(z)}{\partial z}ds}{\tilde{\Phi}_{\eta,M_\varepsilon}(z)} + \frac{i}{z} \\
&= \frac{\fint_{M_\varepsilon}^{M_\varepsilon} \Phi_\eta(H_s(z))\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial z} - \frac{i}{H_s(z)}\right)\frac{\frac{-\pi^2}{a^2}}{(s(z-1)+\frac{i\pi}{a})^2}ds}{\fint_{M_\varepsilon}^{M_\varepsilon} \Phi_\eta(H_s(z))ds} + \frac{i}{z}.
\end{aligned}$$

Then evaluating at $z = 1$ yields,

$$\begin{aligned}
\frac{\partial \tilde{\varphi}_{\eta,\varepsilon}(1)}{\partial z} &= \frac{\fint_{M_\varepsilon}^{M_\varepsilon} \Phi_\eta(H_s(1))\left(\frac{\partial\varphi_\eta(H_s(1))}{\partial z} - \frac{i}{H_s(1)}\right)\frac{\frac{-\pi^2}{a^2}}{(s(0)+\frac{i\pi}{a})^2}ds}{\fint_{M_\varepsilon}^{M_\varepsilon} \Phi_\eta(H_s(1))ds} + i \\
&= \frac{\fint_{M_\varepsilon}^{M_\varepsilon} \Phi_\eta(1)\left(\frac{\partial\varphi_\eta(1)}{\partial z} - i\right)ds}{\fint_{M_\varepsilon}^{M_\varepsilon} \Phi_\eta(1)ds} + i \\
&= \frac{\partial\varphi_\eta(1)}{\partial z}.
\end{aligned}$$

Similarly to before, we can see that evaluating the limits gives us the following;

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\partial \tilde{\varphi}_{\eta,\varepsilon}(z_{b,\theta})}{\partial z} &= \frac{\fint_{M_\varepsilon}^{M_\varepsilon} \lim_{\theta\to 0}\Phi_\eta(H_s(z_{b,\theta}))\left(\frac{\partial\varphi_\eta(1)}{\partial z} - i\right)\frac{\frac{-\pi^2}{a^2}}{(s(0)+\frac{i\pi}{a})^2}ds}{\fint_{M_\varepsilon}^{M_\varepsilon} \lim_{\theta\to 0}\Phi_\eta(H_s(z_{b,\theta}))ds} + i \\
&= \left(\frac{\partial\varphi_\eta(1)}{\partial z} - i\right)\frac{\fint_{M_\varepsilon}^{M_\varepsilon} \lim_{\theta\to 0}\Phi_\eta(H_s(z_{b,\theta}))ds}{\fint_{M_\varepsilon}^{M_\varepsilon} \lim_{\theta\to 0}\Phi_\eta(H_s(z_{b,\theta}))ds} + i \\
&= \frac{\partial\varphi_\eta(1)}{\partial z},
\end{aligned}$$

$$\lim_{\theta \to 2\pi} \frac{\partial \tilde{\varphi}_{\eta,\varepsilon}(z_{b,\theta})}{\partial z} = \frac{\fint_{M_\varepsilon}^{M_\varepsilon} \lim_{\theta \to 2\pi} \Phi_\eta(H_s(z_{b,\theta})) \left(\frac{\partial \varphi_\eta(1)}{\partial z} - i\right) \frac{\frac{-\pi^2}{a^2}}{(s(0)+\frac{i\pi}{a})^2} ds}{\fint_{M_\varepsilon}^{M_\varepsilon} \lim_{\theta \to 2\pi} \Phi_\eta(H_s(z_{b,\theta})) ds} + i$$
$$= \left(\frac{\partial \varphi_\eta(1)}{\partial z} - i\right) \frac{\fint_{M_\varepsilon}^{M_\varepsilon} \lim_{\theta \to 2\pi} \Phi_\eta(H_s(z_{b,\theta})) ds}{\fint_{M_\varepsilon}^{M_\varepsilon} \lim_{\theta \to 2\pi} \Phi_\eta(H_s(z_{b,\theta})) ds} + i$$
$$= \frac{\partial \varphi_\eta(1)}{\partial z}.$$

Since $\frac{\partial \tilde{\varphi}_{\eta,\varepsilon}}{\partial z}$ is continuous, and $\frac{\partial \tilde{\varphi}_{\eta,\varepsilon}}{\partial \bar{z}} = \overline{\frac{\partial \tilde{\varphi}_{\eta,\varepsilon}}{\partial z}}$, it follows that $\frac{\partial \tilde{\varphi}_{\eta,\varepsilon}}{\partial \bar{z}}$ is also continuous on K_a, which gives us $\tilde{\varphi}_{\eta,\varepsilon} \in C^1(K_a)$.

Now we look at the second derivatives, starting with $\frac{\partial^2}{\partial z^2}\tilde{\varphi}_{\eta,\varepsilon}$.

$$
\begin{aligned}
\frac{\partial^2 \tilde{\varphi}_{\eta,\varepsilon}(z)}{\partial z^2} &= \frac{\partial}{\partial z}\left(\frac{\eta-1}{\eta}\frac{\frac{\partial}{\partial z}\tilde{\Phi}_{\eta,M_\varepsilon}(z)}{\tilde{\Phi}_{\eta,M_\varepsilon}(z)} + \frac{i}{z}\right) \\
&= \frac{\eta-1}{\eta}\frac{\frac{\partial^2}{\partial z^2}\tilde{\Phi}_{\eta,M_\varepsilon}(z)\tilde{\Phi}_{\eta,M_\varepsilon}(z) - \frac{\partial}{\partial z}\tilde{\Phi}_{\eta,M_\varepsilon}(z)^2}{\tilde{\Phi}_{\eta,M_\varepsilon}(z)^2} - \frac{i}{z^2} \\
&= \frac{\eta-1}{\eta}\frac{\frac{\partial^2}{\partial z^2}\tilde{\Phi}_{\eta,M_\varepsilon}(z)\tilde{\Phi}_{\eta,M_\varepsilon}(z)}{\tilde{\Phi}_{\eta,M_\varepsilon}(z)^2} - \frac{\eta-1}{\eta}\left(\frac{\frac{\partial}{\partial z}\tilde{\Phi}_{\eta,M_\varepsilon}(z)}{\tilde{\Phi}_{\eta,M_\varepsilon}(z)}\right)^2 - \frac{i}{z^2} \\
&= \frac{\eta-1}{\eta}\frac{\frac{\partial^2}{\partial z^2}\tilde{\Phi}_{\eta,M_\varepsilon}(z)}{\tilde{\Phi}_{\eta,M_\varepsilon}(z)} - \frac{\eta-1}{\eta}\left(\frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))\frac{\eta}{\eta-1}\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial z} - \frac{i}{H_s(z)}\right)\frac{\frac{-\pi^2}{a^2}}{(s(z-1)+\frac{i\pi}{a})^2}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))ds}\right)^2 - \frac{i}{z^2} \\
&= \frac{\frac{\partial}{\partial z}\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial z} - \frac{i}{H_s(z)}\right)\frac{\frac{-\pi^2}{a^2}}{(s(z-1)+\frac{i\pi}{a})^2}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))ds} \\
&\quad - \frac{\eta}{\eta-1}\left(\frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial z} - \frac{i}{H_s(z)}\right)\frac{\frac{-\pi^2}{a^2}}{(s(z-1)+\frac{i\pi}{a})^2}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))ds}\right)^2 - \frac{i}{z^2} \\
&= \frac{\eta}{\eta-1}\frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial z} - \frac{i}{H_s(z)}\right)^2\frac{\frac{\pi^4}{a^4}}{(s(z-1)+\frac{i\pi}{a})^4}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))ds} \\
&\quad + \frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))\left(\frac{\partial^2\varphi_\eta(H_s(z))}{\partial z^2} + \frac{i}{H_s(z)^2}\right)\frac{\frac{\pi^4}{a^4}}{(s(z-1)+\frac{i\pi}{a})^4}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))ds} \\
&\quad + \frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial z} - \frac{i}{H_s(z)}\right)\frac{\frac{2\pi^2 s}{a^2}}{(s(z-1)+\frac{i\pi}{a})^3}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))ds} \\
&\quad - \frac{\eta}{\eta-1}\left(\frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial z} - \frac{i}{H_s(z)}\right)\frac{\frac{-\pi^2}{a^2}}{(s(z-1)+\frac{i\pi}{a})^2}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))ds}\right)^2 - \frac{i}{z^2}.
\end{aligned}
$$

Evaluating at $z = 1$,

$$\frac{\partial^2\tilde{\varphi}_{\eta,\varepsilon}(1)}{\partial z^2} = \frac{\eta}{\eta-1}\frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(1)\left(\frac{\partial\varphi_\eta(1)}{\partial z}-i\right)^2\frac{\frac{\pi^4}{a^4}}{(\frac{i\pi}{a})^4}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(1)ds} + \frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(1)\left(\frac{\partial^2\varphi_\eta(1)}{\partial z^2}+i\right)\frac{\frac{\pi^4}{a^4}}{(\frac{i\pi}{a})^4}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(1)ds}$$
$$+\frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(1)\left(\frac{\partial\varphi_\eta(1))}{\partial z}-i\right)\frac{\frac{2\pi^2 s}{a^2}}{(\frac{i\pi}{a})^3}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(1)ds} - \frac{\eta}{\eta-1}\left(\frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(1)\left(\frac{\partial\varphi_\eta(1)}{\partial z}-i\right)\frac{\frac{-\pi^2}{a^2}}{(\frac{i\pi}{a})^2}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(1)ds}\right)^2 - i$$
$$= \frac{\eta}{\eta-1}\left(\frac{\partial\varphi_\eta(1)}{\partial z}-i\right)^2 + \left(\frac{\partial^2\varphi_\eta(1)}{\partial z^2}+i\right) + \left(\frac{\partial\varphi_\eta(1))}{\partial z}-i\right)\frac{i2as}{\pi}$$
$$-\frac{\eta}{\eta-1}\left(\frac{\partial\varphi_\eta(1)}{\partial z}-i\right)^2 - i$$
$$= \frac{\partial^2\varphi_\eta(1)}{\partial z^2} + \left(\frac{\partial\varphi_\eta(1))}{\partial z}-i\right)\frac{i2as}{\pi}.$$

Using the same argument of continuity on the interior of K_a as before, we can see that we have continuity of $\frac{\partial^2\tilde{\varphi}_{\eta,\varepsilon}}{\partial z^2}$at $z = 1$. Following the same procedure, one can also check that $\frac{\partial^2\tilde{\varphi}_{\eta,\varepsilon}}{\partial\bar{z}^2}$ is also continuous at $z = 1$.

For the mixed second derivative,

$$\frac{\partial^2\tilde{\varphi}_{\eta,\varepsilon}(z)}{\partial z\partial\bar{z}} = \frac{\eta-1}{\eta}\frac{\frac{\partial^2}{\partial z\partial\bar{z}}\tilde{\Phi}_{\eta,M_\varepsilon}(z)\tilde{\Phi}_{\eta,M_\varepsilon}(z) - \left|\frac{\partial}{\partial z}\tilde{\Phi}_{\eta,M_\varepsilon}(z)\right|^2}{\tilde{\Phi}_{\eta,M_\varepsilon}(z)^2}$$
$$= \frac{\fint_{-M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))\left(\frac{\partial^2\varphi_\eta(H_s(z))}{\partial z\partial\bar{z}} - \frac{\eta}{1-\eta}\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial z}-\frac{i}{H_s(z)}\right)\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial\bar{z}}+\frac{i}{\overline{H_s(z)}}\right)\right)\left|\frac{\partial H_s(z)}{\partial z}\right|^2 ds}{\fint_{-M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))ds}$$
$$+\frac{\eta}{1-\eta}\left|\frac{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))\left(\frac{\partial\varphi_\eta(H_s(z))}{\partial z}-\frac{i}{H_s(z)}\right)\frac{\frac{-\pi^2}{a^2}}{(s(z-1)+\frac{i\pi}{a})^2}ds}{\fint_{M_\varepsilon}^{M_\varepsilon}\Phi_\eta(H_s(z))ds}\right|^2.$$

Evaluating at $z = 1$:

$$\begin{aligned}
\frac{\partial^2 \tilde{\varphi}_{\eta,\varepsilon}(1)}{\partial z \partial \bar{z}} &= \frac{\fint_{-M_\varepsilon}^{M_\varepsilon} \Phi_\eta(1) \left(\frac{\partial^2 \varphi_\eta(1)}{\partial z \partial \bar{z}} - \frac{\eta}{1-\eta} \left(\frac{\partial \varphi_\eta(1)}{\partial z} - i \right) \left(\frac{\partial \varphi_\eta(1)}{\partial \bar{z}} + i \right) \right) ds}{\fint_{-M_\varepsilon}^{M_\varepsilon} \Phi_\eta(1) ds} \\
&\quad + \frac{\eta}{1-\eta} \left| \frac{\fint_{M_\varepsilon}^{M_\varepsilon} \Phi_\eta(1) \left(\frac{\partial \varphi_\eta(1)}{\partial z} - i \right) ds}{\fint_{M_\varepsilon}^{M_\varepsilon} \Phi_\eta(1) ds} \right|^2 \\
&= \frac{\partial^2 \varphi_\eta(1)}{\partial z \partial \bar{z}} - \frac{\eta}{1-\eta} \left| \frac{\partial \varphi_\eta(1)}{\partial z} - i \right|^2 + \frac{\eta}{1-\eta} \left| \frac{\partial \varphi_\eta(1)}{\partial z} - i \right|^2 \\
&= \frac{\partial^2 \varphi_\eta(1)}{\partial z \partial \bar{z}}.
\end{aligned}$$

And we can once again take the limits as we approach $z = 1$ to see that $\frac{\partial^2 \tilde{\varphi}_{\eta,\varepsilon}(z)}{\partial z \partial \bar{z}}$ is continuous as $z = 1$, which shows that $\tilde{\varphi}_{\eta,\varepsilon} \in C^2(K_a)$.

Now, we show that $\tilde{\varphi}_{\eta,\varepsilon}(z)$ satisfies property 1 of the hypothesis. Applying the operator P_η to $\tilde{\varphi}_{\eta,\varepsilon}(z)$, we see;

$$\begin{aligned}
P_\eta \tilde{\varphi}_{\eta,\varepsilon}(z) &= \frac{\eta - 1}{\eta} \frac{\frac{\partial^2}{\partial z \partial \bar{z}} \tilde{\Phi}_{\eta,\varepsilon}(z) \tilde{\Phi}_{\eta,\varepsilon}(z) - \left| \frac{\partial}{\partial z} \tilde{\Phi}_{\eta,\varepsilon}(z) \right|^2}{\tilde{\Phi}_{\eta,\varepsilon}(z)^2} \\
&\quad - \frac{\eta}{1-\eta} \left(\frac{\eta - 1}{\eta} \frac{\frac{\partial}{\partial z} \tilde{\Phi}_{\eta,\varepsilon}(z)}{\tilde{\Phi}_{\eta,\varepsilon}(z)} \right) \left(\frac{\eta - 1}{\eta} \frac{\frac{\partial}{\partial \bar{z}} \tilde{\Phi}_{\eta,\varepsilon}(z)}{\tilde{\Phi}_{\eta,\varepsilon}(z)} \right) \\
&= \frac{\eta - 1}{\eta} \frac{\frac{\partial^2}{\partial z \partial \bar{z}} \tilde{\Phi}_{\eta,\varepsilon}(z) \tilde{\Phi}_{\eta,\varepsilon}(z) - \left| \frac{\partial}{\partial z} \tilde{\Phi}_{\eta,\varepsilon}(z) \right|^2}{\tilde{\Phi}_{\eta,\varepsilon}(z)^2} + \frac{\eta - 1}{\eta} \frac{\left| \frac{\partial}{\partial z} \tilde{\Phi}_{\eta,\varepsilon}(z) \right|^2}{\tilde{\Phi}_{\eta,\varepsilon}(z)^2} \\
&= \frac{\eta - 1}{\eta} \frac{\frac{\partial^2}{\partial z \partial \bar{z}} \tilde{\Phi}_{\eta,\varepsilon}(z)}{\tilde{\Phi}_{\eta,\varepsilon}(z)}.
\end{aligned}$$

Since $\tilde{\Phi}_{\eta,\varepsilon}(z) < 0$ and is strictly subharmonic, the second term is negative, but so is $\frac{\eta-1}{\eta}$, and so the entire term is strictly positive. Because $P_\eta \tilde{\varphi}_{\eta,\varepsilon}$ is composed of functions shown to be continuous on K_a, we also know that $P_\eta \tilde{\varphi}_{\eta,\varepsilon}$ itself is continuous on K_a. □

We will now finish with the proof of Theorem 1.3.

Proof of Theorem 1.3. Following a similar proof as that of Theorem 1.1, given $\tilde{\varphi}_{\eta,\varepsilon}$ as defined in Lemma 6.1, there exists an extension of $\rho_\varepsilon = \delta e^{-\tilde{\varphi}_{\eta,\varepsilon}}$ to a neighborhood of Ω following the argument of the proof of Theorem 2.11 (1) in section 5 of [12] where $-(-\rho_\varepsilon)^\eta$ is plurisubharmonic on Ω. The rest of the properties follow directly from Lemma 6.1. $\square$

7 Closing Statements and Future Work

Given the number of parameters involved in defining φ_η, the methods used in this thesis are insufficient in finding a sharp upper bound on its gradient. A future course of action would be trying different approaches to find sharper estimates that don't have magnitudes like that of $E_{\eta,2}$, which may reveal that we do achieve the Regularized Diederich-Fornæss index of 1 on our domains containing crescents in the boundary. Numerical methods may be explored to account for the many possible parameters over our crescent regions. We may also take the method of averaging functions over a family of automorphisms to examine other regions with interesting geometry. As this method has been useful in the study of worm domains as well as our domains with crescents in the boundary, it could be useful to examine the method over different regions with their own families of automorphisms as well. As the larger objective of this thesis was to find a deeper understanding of the relationship between good vector fields and the Regularized Diederich-Fornæss index, future work could be done to try and relate the Regularized index to other sufficient conditions for global regularity of the Bergman Projection. Some of these conditions include the Steinness Index developed in [16], or the Levi Core which has been compared with the Regularized index on certain domains in [7].

8 References

[1] D. E. Barrett, *Behavior of the Bergman projection on the Diederich-Fornæss worm*, Acta Math. **168** (1992), 1–10.

[2] S. Bell and E. Ligocka, *A simplification and extension of Fefferman's theorem on biholomorphic mappings*, Invent. Math. **57** (1980), 283–289.

[3] H. P. Boas and E. J. Straube, *de Rham cohomology of manifolds containing the points of infinite type, and Sobolev estimates for the $\overline{\partial}$-Neumann problem*, J. Geom. Anal. **3** (1993), 225–235.

[4] ———, *Global regularity of the $\overline{\partial}$-Neumann problem: a survey of the L^2-Sobolev theory*, Several complex variables (Berkeley, CA, 1995–1996), Math. Sci. Res. Inst. Publ., vol. 37, Cambridge Univ. Press, Cambridge, 1999, pp. 79–111.

[5] S.-C. Chen and M.-C. Shaw, *Partial differential equations in several complex variables*, AMS/IP Studies in Advanced Mathematics, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001.

[6] M. Christ, *Global C^∞ irregularity of the $\overline{\partial}$-Neumann problem for worm domains*, J. Amer. Math. Soc. **9** (1996), 1171–1185.

[7] G. M. Dall'Ara and S. Mongodi, *The core of the Levi distribution*, arXiv:2109.04763, 2021.

[8] J. P. D'Angelo, *Several complex variables and the geometry of real hypersurfaces*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1993.

[9] K. Diederich and J. E. Fornaess, *Pseudoconvex domains: an example with nontrivial Nebenhülle*, Math. Ann. **225** (1977), 275–292.

[10] ———, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), 129–141.

[11] P. S. Harrington, *Global regularity for the $\overline{\partial}$-Neumann operator and bounded plurisubharmonic exhaustion functions*, Adv. Math. **228** (2011), 2522–2551.

[12] ———, *The Diederich-Fornæss index and good vector fields*, Indiana Univ. Math. J. **68** (2019), 721–760.

[13] J. J. Kohn, *Quantitative estimates for global regularity*, Analysis and geometry in several complex variables (Katata, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 1999, pp. 97–128.

[14] B. Liu, *The Diederich-Fornæss index I: For domains of non-trivial index*, Adv. Math. **353** (2019), 776–801.

[15] E. J. Straube, *Lectures on the L^2-Sobolev theory of the $\overline{\partial}$-Neumann problem*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2010.

[16] J. Yum, *On the Steinness index*, J. Geom. Anal. **29** (2019), 1583–1607.