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Cohomology of the Symmetric Group with Twisted Coefficients and Quotients of the Braid Group

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Cohomology of the Symmetric Group with Twisted Coefficients and Quotients of the Braid Group

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Abstract

In 2014 Brendle and Margalit proved the level 4 congruence subgroup of the braid group, $B_n[4]$, is the subgroup of the pure braid group generated by squares of all elements, PB_n^2. We define the mod 4 braid group, $\mathcal{Z}_n$, to be the quotient of the braid group by the level 4 congruence subgroup, $B_n/B_n[4]$. In this dissertation we construct a group presentation for $\mathcal{Z}_n$ and determine a normal generating set for $B_n[4]$ as a subgroup of the braid group.

Further work by Kordek and Margalit in 2019 proved $\mathcal{Z}_n$ is an extension of the symmetric group, S_n, by $\mathbb{Z}_2^{\binom{n}{2}}$. A classical result of Eilenberg and MacLane classifies group extensions by classes in the second group cohomology with twisted coefficients. We first construct a representative for the cohomology class, $[\phi]$, of $H^2(S_n; \mathbb{Z}^{\binom{n}{2}})$ classifying the extension, G_n, of the symmetric group by the abelianization of the pure braid group. We then show a representative for the cohomology class in $H^2(S_n; \mathbb{Z}_2^{\binom{n}{2}})$ classifying $\mathcal{Z}_n$ is the composition of $[\phi]$ with the mod 2 reduction of integers.

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1 Introduction

Brendle and Margalit proved the level 4 congruence subgroup, $B_n[4]$, of the braid group, B_n, can be described as the subgroup of the pure braid group generated squares of all elements, PB_n^2 [6]. We define the mod 4 braid group, denoted $\mathcal{Z}_n$, to be the quotient of the braid group by $B_n[4] = PB_n^2$. Kordek and Margalit proved $\mathcal{Z}_n$ is an extension of the symmetric group, S_n, by $\mathbb{Z}_2^{\binom{n}{2}}$ in 2018 [12]. Furthermore, a classical result of Eilenberg and MacLane proves a classification of group extensions by low dimensional group cohomology with twisted coefficients (page 337 of [10]). In particular, there exists an element of the second cohomology group of S_n with coefficients in $\mathbb{Z}_2^{\binom{n}{2}}$ twisted by the action of the symmetric group permuting unordered pairs of integers which corresponds to $\mathcal{Z}_n$.

   We represent the action of a group, Q, on an abelian group, K, by a map $\theta : Q \to \text{Aut(K)}$. Let E be an extension of Q by K such that $\iota : K \to E$ and $\pi : E \to Q$ are the inclusion and projection maps respectively, then E gives rise to θ if conjugation of elements in $\iota(K)$ by any $e \in E$ is determined by $\theta(\pi(e))$. We will only consider the standard action of the symmetric group on pairs of $\{1, 2, \ldots, n\}$, therefore we suppress θ in our notation and use $H^2(Q; K)$ to denote the second cohomolgy of Q with coefficients in K twisted by θ. By fixing θ, Eilenberg and MacLane's classification provides a bijection between $H^2(Q; K)$ and the set of equivalence classes of group extensions of Q by K which give rise to θ [10]. In Section 3, we provide the background of group cohomology and an explanation for the classification of extensions.

   In Section 4 we give a more thorough exposition of the braid group. Extensions of the symmetric group naturally arise while studying quotients of the braid group. The mod 4 braid group, $\mathcal{Z}_n = B_n/B_n[4]$, arises as an extension of S_n by $\mathbb{Z}_2^{\binom{n}{2}}$ twisted by the standard action of the symmetric group on unordered pairs on integers between 1 and n. In Section 4.3 we describe the generalization of the level m braid group, $B_n[m]$, as the kernel of a representation $B_n \to GL_n(\mathbb{Z}_m)$. Recent work by Appel, Bloomquist, Gravel, and Holden

prove $\mathcal{Z}_n \approx B_n[m]/B_n[4m]$ for any odd m [2]. We will construct both a group presentation and a representative for the corresponding cohomology class of $\mathcal{Z}_n$. Our first result states a representative of the cohomology class corresponding to $\mathcal{Z}_n$ can be determined by a representative for an extension of S_n by the abelianization of the pure braid group.

Theorem 1.1. *Suppose $n \geq 1$ and let $[\kappa] \in H^2(S_n; \mathbb{Z}_2^{\binom{n}{2}})$ correspond to the mod 4 braid group. If $[\phi] \in H^2(S_n; \mathbb{Z}^{\binom{n}{2}})$ corresponds to the extension of S_n by the abelianization of the pure braid group, then a representative of $[\kappa]$ is the mod 2 reduction of a representative for $[\phi]$. Furthermore, $[\phi]$ is order 2 in $H^2(S_n; \mathbb{Z}^{\binom{n}{2}})$.*

Since group cohomology with twisted coefficients is equivalent to the cohomology of an Eilenberg-MacLane complex, we construct a representative for the cohomology classes in Theorem 1.1 using the resolution given by the universal cover of an Eilenberg-MacLane complex for the symmetric group, $K(S_n, 1)$ complex. In particular we construct a low dimensional approximation for the cellular chain complex determined by the universal cover of the presentation complex for S_n. The 2-chains in this resolution are generated by the S_n orbits of the following three classes of elements:

- $\tilde{c}_{i,j}$ for all $1 \leq i < j \leq n$.

- $\tilde{d}_{i,j,k,\ell}$ for all $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$ such that $\{i, j\} \cap \{k, \ell\} = \emptyset$.

- $\tilde{e}_{i,j,k}$ for all distinct triples i, j, k in $\{1, 2, \ldots, n\}$.

Each class above corresponds to a relation in the symmetric group: the squaring, commuting, and braid relations respectively. Notice $\tilde{e}_{i,j,k}$ and $\tilde{d}_{i,j,k,\ell}$ are generators when $n \geq 3$ or $n \geq 4$ respectively. Therefore, to define a representative for $[\phi]$ or $[\kappa]$, it suffices to define the image on these three classes. We use the following presentation to define the image of our 2-cocycle corresponding to $\mathcal{Z}_n$:

$$\mathbb{Z}_2^{\binom{n}{2}} = \langle \{\bar{g}_{i,j}\}_{1 \leq i < j \leq n} \mid \bar{g}_{i,j}^2 = 1, \ [\bar{g}_{i,j}, \bar{g}_{k,\ell}] = 1 \rangle. \tag{1}$$

Theorem 1.2. *Suppose $n \geq 1$ and $\{\tilde{c}_{i,j}\}$, $\{\tilde{d}_{i,j,k,\ell}\}$, and $\{\tilde{e}_{i,j,k}\}$ have the conditions on indices above. Then a representative for the cocycle classifying $\mathcal{Z}_n$ as an extension of S_n by $\mathbb{Z}_2^{\binom{n}{2}}$ is given by:*

$$\kappa(\tilde{c}_{i,j}) = \bar{g}_{i,j}$$

$$\kappa(\tilde{d}_{i,j,k,\ell}) = \begin{cases} \bar{g}_{i,k} + \bar{g}_{i,\ell} + \bar{g}_{j,k} + \bar{g}_{j,\ell} & \text{if } i < k < j < \ell \text{ or } k < i < \ell < j \\ 0 & \text{otherwise} \end{cases}$$

$$\kappa(\tilde{e}_{i,j,k}) = \begin{cases} \bar{g}_{i,j} + \bar{g}_{j,k} & \text{if } i < k < j, \ k < j < i, \ \text{or } j < i < k \\ 0 & \text{otherwise} \end{cases}$$

In the construction of the cocycle described by Theorem 1.2 we build a group presentation for $\mathcal{Z}_n$ with relations given in Table 2. Considering Artin's original presentation for the braid group given by:

$$B_n = \left\langle b_1, \ldots, b_n \;\middle|\; \begin{array}{l} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \text{ for all } i \\ b_i b_j = b_j b_i \text{ if } |i - j| > 1 \end{array} \right\rangle \tag{2}$$

we use Tietze transformations to determine relations of $\mathcal{Z}_n$ which are not true in B_n. Since $\mathcal{Z}_n = B_n / B_n[4]$, these extra relations determine a normal generating set for $B_n[4]$.

Theorem 1.3. *Suppose $n \geq 1$. As a subgroup of B_n, the level 4 braid group is normally generated by elements of the form:*

1. *$[b_i^2, b_{i+1}^2]$ for all $1 \leq i \leq n - 1$.*

2. *$[b_i b_{i+1}^2 b_i^{-1}, b_{i+1} b_{i+2}^2 b_{i+1}^{-1}]$ for all $1 \leq i \leq n - 3$.*

3. *b_i^4 for all $1 \leq i \leq n - 1$.*

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Remark The results in this dissertation appear in a paper submitted for publication. The version submitted to a journal uses alternative methods to prove the presentation for G_n and order of the corresponding cohomology class.

2   Constructing Presentations via Extensions

Let Q and K be groups, then E *is an extension of Q by K* if the following sequence is exact:

$$1 \longrightarrow K \longrightarrow E \longrightarrow Q \longrightarrow 1.$$

To construct a representative for the cohomology class corresponding to an extension, E, of Q by K we need to lift elements of Q to E and compute products of those lifts in E. However, this requires the group structure of E. In this section, we describe the process to construct a presentation for E given presentations for Q and K.

Remarks   Although the extensions related to the second cohomology group require K to be abelian, the results in this section are proved for the non-abelian case. Furthermore, the results of this section are already known and included for completeness.

Notation   Throughout this section, we will refer to the same presentations for Q and K. Let $\langle \bar{S}_Q | \bar{R}_Q \rangle$ and $\langle S_K | R_K \rangle$ be group presentations for Q and K respectively. Without loss of generality, we may assume the trivial elements of Q and K are not in the respective generating sets. Furthermore, let $\iota : K \to E$ and $\pi : E \to Q$ be the inclusion and projection maps in the diagram:

$$1 \longrightarrow K \xrightarrow{\;\iota\;} E \xrightarrow{\;\pi\;} Q \longrightarrow 1.$$

We will not distinguish between K as a group and $\iota(K)$ as a subgroup of E since ι is assumed to be an embedding.

Generating Set   Since E is an extension of Q by K, $Q = E/K$. A *lift* of an element $q \in Q$ is an element $e_q \in E$ such that $\pi(e_q) = q$. Fix $S_Q \subset E$ to be a choice of lifts from elements in $\bar{S}_Q$ such that π restricted to S_Q is a bijection. We will show $\mathcal{S} = S_K \cup S_Q$ is a

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generating set for E.

Relations Since ι embeds K into E, the relations of K must be included in the presentation for E. Furthermore, K is a normal subgroup of E. Therefore, for each $k \in K$ and $e \in E$ there exists $k' \in K$ such that $eke^{-1} = k'$. In particular, if $k \in S_K$ and $s \in S_Q$, we have $sks^{-1} = k'$ for some $k' \in K$. Notice, the choice of s as a lift of an element in $\bar{S}_Q$ determines k'. Furthermore, k' can be written as a word, w_k, on the generators of K. So, for each $k \in S_K$ and $s \in S_Q$, $sks^{-1} = w_k$ is a relation of E. By a symmetric argument, for each $k \in S_K$ and $s \in S_Q$, there exists a word, u_k on the generators of K such that $s^{-1}ks = u_k$ is a relation of E. For all $k \in S_K$ and $s \in S_Q$, set R' to be the set of relations of the form $sks^{-1} = w_k$ and $s^{-1}ks = u_k$.

Now, any relation in $\bar{R}_Q$ is given by $\bar{s}_1 \cdots \bar{s}_q = 1_Q$ where $q \geq 1$ and $\bar{s}_i \in \bar{S}_Q$ for all $1 \leq i \leq q$. Since, for each i, $s_i \in S_Q$ is defined to be a lift of $\bar{s}_i$ and E is an extension of Q by K, there exists some $k \in K$ such that $s_1 \cdots s_q = k$. Furthermore, the choices of s_i as a lift of $\bar{s}_i$ uniquely determines $k \in K$. Therefore, k is expressed as a word $k_1 \cdots k_t$ on the generators of S_K, for some $t \geq 1$. Hence $s_1 \cdots s_q = k_1 \cdots k_t$ determines the relation $s_1 \cdots s_q k_t^{-1} \cdots k_1^{-1} = 1_E$ in E. Let R_Q be the set of all relations in E determined by lifting relations of $\bar{R}_Q$ using this method. Define $R = R_K \cup R_Q \cup R'$ and $E' = \langle \mathcal{S} \mid R \rangle$.

Goal To show the group E' defined above is a group presentation for E, we must show there exists an isomorphism between E' and E. Define $f : E' \to E$ by $f(s) = s$ for all $s \in \mathcal{S}$. Since each element of $\mathcal{S}$ is chosen from the elements of E and by the construction of R in the preceding paragraph, f is a well defined homomorphism. To prove f is an isomorphism we use the 5-Lemma after constructing maps ι' and π' which make the

following diagram commute:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & Q & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}_K} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_Q} & & \\
1 & \longrightarrow & K & \xrightarrow{\iota} & E & \xrightarrow{\pi} & Q & \longrightarrow & 1
\end{array}
\tag{3}
$$

Inclusion and Projection Maps We begin by defining the inclusion map $\iota' : K \to E'$.

For every $k \in K$, then there exists a word, $k_1 k_2 \cdots k_t$ on the generators of S_K representing

k. Since $S_K \subset \mathcal{S}$ and $R_K \subseteq R$, the map $\iota' : K \to E$ defined by $\iota'(k) = k_1 k_2 \cdots k_t$ is a well

defined homomorphism. Furthermore, notice that $f \circ \iota'(k) = \iota(k)$ for all $k \in K$. Therefore

ι' is injective since ι is injective.

Recall the generating set for E' is $\mathcal{S} = S_K \cup S_Q$ where S_Q is a lift of the generating set

for Q. In particular, for each $s \in S_Q$, there exists an $\bar{s} \in \bar{S}_Q$ such that $\pi(s) = \bar{s}$. Define

$\pi' : E' \to Q$ on the generators of E' by the following:

$$
\pi'(s) = \begin{cases} \bar{s} & \text{if } s \in S_Q \\ 1_Q & \text{if } s \in S_K \end{cases}
$$

Notice that if $r = 1$ is a relation of R_K, then r can be written as a word on the generators

of K. By definition, $\pi'(r) = 1_Q$ and π' preserves the relations in R_K. If $r = 1$ is a relation

of R_Q, then by the construction of R_Q as a lift of $\bar{R}_Q$, r can be expressed as a word:

$$
s_1 s_2 \cdots s_p k_t^{-1} k_{t-1}^{-1} \cdots k_1 = 1
$$

on the generators of E' where $\bar{s}_1 \bar{s}_2 \cdots \bar{s}_p = 1_Q$. By definition, $\pi'(r) = \bar{s}_1 \bar{s}_2 \cdots \bar{s}_p = 1_Q$.

Furthermore, if $r = 1$ is a relation in R', then r can be expressed as $sks^{-1}w_k^{-1}$ or $s^{-1}ksu_k^{-1}$

where $s \in S_Q$, $k \in S_K$, and w_k^{-1} and u_k^{-1} are in K. So w_k^{-1} and u_k^{-1} can be expressed as a

word on the generators of S_K and $\pi'(r) = \bar{s}\bar{s}^{-1} = 1_Q$ or $\pi'(r) = s^{-1}s = 1_Q$. Therefore π'

preserves the relations of E' and is a well defined homomorphism. Furthermore, for any

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$q \in Q$, $q = \bar{s}_1 \bar{s}_2 \cdots \bar{s}_t$ where $\bar{s}_i \in \bar{S}_Q$ for all i. By the choice of S_Q as a lift of $\bar{S}_Q$, $\pi'(s_1 s_2 \cdots s_t) = q$. Therefore π' is surjective.

Lemma 2.1. *Let K and Q be groups with presentations defined above. Let E' be the group with generators and relations described in the preceding paragraphs. Then any element $e \in E$ can be described by $w_1 w_2$ where w_1 and w_2 are words in the generators of Q and K respectively.*

Proof. Since E' is generated by $\mathcal{S}$, any element of E' can be described by a word $s_1 s_2 \cdots s_t$ where $s_i \in \mathcal{S}$ and $t \geq 1$. We induct on t, beginning with $t = 2$. Suppose $e = s_1 s_2$ where $s_i \in \mathcal{S}$ for $i = 1, 2$. If $s_1 \in S_Q$ or both $s_1, s_2 \in S_K$ then the lemma is trivial. Therefore, suppose $s_1 \in S_K$ and $s_2 \in S_Q$. By the relations of R', there exists a word, u_{s_2}, on the generators of K such that $s_2^{-1} s_1 s_2 = u_{s_2}$. Therefore:

$$s_1 s_2 = s_2 s_2^{-1} s_1 s_2$$

$$= s_2 u_{s_2}.$$

Since u_{s_2} and s_2 are both in S_K, the lemma holds for $t = 2$.

Now, suppose the lemma is true for any element of E' that can be represented by a reduced word of length $t - 1$ in the generators of $\mathcal{S}$. Suppose $e = s_1 \cdots s_t$, then by induction $s_1 \cdots s_{t-1} = w_1 w_2$ where w_1 and w_2 are words in the generators of S_Q and S_K respectively. If $s_t \in S_K$, then we are done. Therefore assume $s_t \in S_Q$, then we have $e = w_1 w_2 s_t$. In particular, $w_2 = k_1 \cdots k_\ell$ where $k_1, \ldots, k_\ell \in S_K$. By the relations of R' there exists a word u_i on the generators K such that $s_t^{-1} k_i s_t = u_i$ for every i such that $1 \leq i \leq \ell$.

Thus we have:

$$s_1 \cdots s_t = w_1 w_2 s_t$$
$$= w_1 k_1 \cdots k_\ell s_t$$
$$= w_1 k_1 \cdots k_{\ell-1} s_t s_t^{-1} k_\ell s_t$$
$$= w_1 k_1 \cdots k_{\ell-2} s_t s_t^{-1} k_{\ell-1} s_t u_\ell$$
$$\vdots$$
$$= w_1 s_t u_1 \cdots u_\ell.$$

Since each u_i is a word on the generators of K for all i, the product of the u_i's is a word on the generators of K and the lemma is proven. □

Remark In the paragraphs constructing ι' and π' we showed ι' is injective and π' is surjective. To apply the 5-Lemma to (3), it remains to show $\operatorname{im} \iota' = \ker \pi'$.

Theorem 2.2. *Let K, Q, and E' be groups with the presentations described above. Furthermore, let $\iota' : K \to E'$ and $\pi' : E' \to Q$ be the homomorphisms defined previously, then E' is an extension of Q by K.*

Proof. Suppose ι' and π' are the inclusion and projection maps defined above. For any $k \in K$ there exists a word $k_1 k_2 \cdots k_t$ representing k such that $k_i \in S_K$ for all i. By the definitions of ι' and π':

$$\pi' \circ \iota'(k) = \pi'(\iota'(k))$$
$$= \pi'(k_1 k_2 \cdots k_t)$$
$$= \pi'(k_1) \pi'(k_2) \cdots \pi'(k_t)$$
$$= 1_Q.$$

Therefore im $\iota' \subseteq \ker \pi'$. It remains to show $\ker \pi' \subseteq \text{im } \iota'$. Suppose $e \in \ker \pi'$, by Lemma 2.1 there exists words w_1 on the generators of S_Q and w_2 on the generators of S_K such that $e = w_1 w_2$. Let $w_1 = s_1 \cdots s_q$ where $s_i \in S_Q$ and for all i. By the definition of π':

$$\pi'(e) = \pi'(s_1 \cdots s_q k_1 \cdots k_t)$$
$$\pi'(e) = \bar{s}_1 \cdots \bar{s}_q$$
$$= 1_Q.$$

Therefore $\bar{s}_1 \cdots \bar{s}_q = 1_Q$ is a relation of Q. Since $\bar{R}_Q$ normally generates all relations of Q, there exists words $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_q$ on the generators of $\bar{S}_Q$ such that:

$$\bar{s}_1 \cdots \bar{s}_q = \bar{x}_1 \bar{r}_1 \bar{x}_1^{-1} \bar{x}_2 \bar{s}_2 \bar{x}_2^{-1} \cdots \bar{x}_q \bar{s}_q \bar{x}_q^{-1}$$

where $\bar{r}_i \in \bar{R}_Q$ for all i. For each i, $\bar{x}_i$ lifts to a word x_i on the generators of S_Q and $\bar{r}_i$ lifts to the relation $r_i = k_i$ for some $k_i \in K$. Therefore:

$$w_1 = x_1 k_1 x_1^{-1} x_2 k_2 x_2^{-1} \cdots x_t k_t x_t^{-1}.$$

Furthermore, for each i, k_i can be represented by $k_{i,1} k_{i,2} \cdots k_{i,\ell_i}$ where each $k_{i,j} \in S_K$. Therefore, for each i, $x_i k_i x_i^{-1}$ can be represented by:

$$x_i k_{i,1} x_i^{-1} x_i k_{i,2} x_i^{-1} \cdots x_i k_{i,\ell_i} x_i^{-1}.$$

Since each x_i is a word on the generators of S_Q, applying relations of R', there exists $k'_{i,j} \in K$ such that for all i, j:

$$x_i k_{i,j} x_i = k'_{i,j}.$$

For each i, $k_i = k_{i,1} k_{i,2} \cdots k_{i,\ell_i} = k'_i$ where $k'_i \in K$. Therefore $e = k'_1 k'_2 \cdots k'_q w_2$ where k_i and

w_2 are words on the generators of S_K. So e can be expressed as a word on the generators of S_K and $e \in K$. Thus $\ker \pi' \subseteq \operatorname{im} \iota'$ and E' is an extension of Q by K. $\qquad\square$

Remark Recall the diagram from (3):

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \xrightarrow{\;\iota'\;} & E' & \xrightarrow{\;\pi'\;} & Q & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \operatorname{id}_K} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \operatorname{id}_Q} & & \\
1 & \longrightarrow & K & \xrightarrow{\;\iota\;} & E & \xrightarrow{\;\pi\;} & Q & \longrightarrow & 1
\end{array}
$$

Theorem 2.2 proves the top row of (3) is exact. Furthermore, $f : E' \to E$ is defined by $f(s) = s$ for all $s \in \mathcal{S}$. Notice that ι' and π' are constructed such that $f \circ \iota'(s) = \iota(s)$ and $\pi' \circ f(s) = \pi(s)$ for all $s \in \mathcal{S}$. Therefore (3) commutes and f is an isomorphism by the 5-Lemma. This completes the proof of the following theorem:

Theorem 2.3. *Let E be an extension of Q by K with presentations as above. If $\mathcal{S}$ and R are as defined in the paragraphs on generators and relations, then $E \approx \langle \mathcal{S} \mid R \rangle$.*

3   Group Cohomology

Group cohomology provides information about a group by studying the actions of the group on modules. In particular, Eilenberg and MacLane proved group extensions of G by an abelian group A can be classified by elements of the second cohomology group of G with coefficients in A [10]. However, the structure of cohomology groups is highly dependent on the action of G on A. In this section we will provide the background on group cohomology required to construct a representative for the cohomology class corresponding to a group extension, most of which can be found in chapters I, III, and IV of Brown's text [7].

Definition   Let G be a group, then a topological space, X, is an *Eilenberg-MacLane space* of type $(G, 1)$, denoted $K(G, 1)$-*space*, if it satisfies the following three properties:

- X is connected.

- The fundamental group of X, $\pi_1(X)$, is isomorphic to G.

- The universal cover of X is contractible.

If X is a *CW*-complex, then X is often referred to as a $K(G, 1)$-*complex*.

Topological Interpretation   Let G be a group and A, an abelian group on which G acts. Suppose X is a $K(G, 1)$-space with universal cover $\tilde{X}$ and let $\alpha_{x,y}$ be a path in X which begins at x and ends at y. Let $\{A_x\}_{x \in X}$ be a set of groups which satisfies the following three conditions:

- As a group, A_x is isomorphic to A for all $x \in X$.

- For each class of homotopy equivalent paths relative to endpoint, $\alpha_{x,y}$, there is an isomorphism $A_x \to A_y$ given by $a \cdot \alpha_{x,y}$ for each $a \in A_x$.

- For any two paths $\alpha_{x,y}$ and $\beta_{y,z}$ in X with $a \in A_x$, $(a \cdot \alpha_{x,y})\beta_{y,z} = a \cdot (\alpha_{x,y}\beta_{y,z})$.

If we take x_0 to be the basepoint of x, then the fundamental group of X acts as a right action on A_{x_0} by $a \mapsto a \cdot \alpha$ for any $\alpha \in \pi_1(X)$. Since X is a $K(G, 1)$-space, $\pi_1(X) = G$ and we get an action of G on A. We call $\{A_x\}_{x \in X}$ the *local coefficients system determined by A*, denoted $\mathcal{M}_A$, if the right action of $\pi_1(X)$ on A_{x_0} induces the given action of G on A. Eilenberg proved the following theorem which gives the topological interpretation of group cohomology [9]:

Theorem 3.1. *Let G be a group and suppose A is an abelian group such that G acts on A. If $\mathcal{M}_A$ is the local coefficient system determined by A for a $K(G, 1)$-space, then:*

$$H^i(G; A) = H^i(K(G, 1); \mathcal{M}_A).$$

3.1 Normalized Bar Resolution

Let G be a group. The *integral group ring*, denoted $\mathbb{Z}G$, is the set of all finite linear combinations of elements of G. Under addition, consider $\mathbb{Z}G$ as the free $\mathbb{Z}$-module with the elements of G as basis. We extend multiplication in G to multiplication in $\mathbb{Z}G$ by satisfying distribution laws. A resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ is an exact sequence of $\mathbb{Z}G$-modules:

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

A resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ is called *projective*, or *free*, if P_i is a projective, or free, $\mathbb{Z}G$-module, respectively, for all i.

Normalized Bar Resolution   Let G be a group and let P_t be the free $\mathbb{Z}$ module generated by $t + 1$ tuples $(g_0, g_1, \ldots, g_t)$ where $g_i \in G$ for all i. To describe P_t as a $\mathbb{Z}G$ module, consider the action of G on P_t given by $g \cdot (g_0, g_1, \ldots, g_t) = (g \cdot g_0, g \cdot g_1, \ldots, g \cdot g_t)$.

Then, as a $\mathbb{Z}G$ module, P_t is freely generated by the G orbits of $(1, g_1, \ldots, g_t)$. Set:

$$[g_1 \mid g_2 \mid \cdots \mid g_t] = (1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_t).$$

Then the set of all $[g_1 \mid g_2 \mid \cdots \mid g_t]$ generate P_t as a free $\mathbb{Z}G$ module. Let $[\ ]$ denote the generator of P_0, define $\epsilon_P : P_0 \to \mathbb{Z}$ by $\epsilon_P([\ ]) = 1$. For $t \geq 1$, define $\partial_t^P : P_t \to P_{t-1}$ by $\partial_t^P([g_1 \mid g_2 \mid \cdots \mid g_t]) = \sum_{i=0}^{t} (-1)^i d_i$ where:

$$d_i[g_1 \mid \cdots \mid g_t] = \begin{cases} g_1[g_2 \mid \cdots \mid g_t] & i = 0 \\ [g_1 \mid \cdots \mid g_{i-1} \mid g_i g_{i+1} \mid g_{i+2} \mid \cdots \mid g_t] & 0 < i < t \\ [g_1 \mid \cdots \mid g_{t-1}] & i = t \end{cases}$$

Then the *bar resolution of G* is the following free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$\mathcal{P} : \qquad \cdots \longrightarrow P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\varepsilon_P} \mathbb{Z} \longrightarrow 0.$$

Let D_t be the subcomplex of P_t generated by elements of the form $[g_1 \mid g_2 \mid \cdots \mid g_t]$ where $g_i = g_{i+1}$ for some i between 1 and n. The resolution $\bar{\mathcal{P}}$ determined by $\bar{P}_t = P_t/D_t$ is the *normalized bar resolution*.

Remarks For a topological interpretation of the bar resolution, consider the contractible simplicial complex, X, in which the vertices are elements of G and every finite subset of G is a simplex of X. Note that if G is finite, X is a simplex; otherwise X is infinite dimensional. We get a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ by taking the ordered chain complex with the basis of $(t+1)$-tuples of vertices $(g_0, g_1, \ldots, g_t)$ such that $g_i \in G$ for all i. The boundary operator on this ordered chain complex is the alternating sum from $i = 0$ to t of the map which forgets g_i.

The generating set of P_t is of size $|G|^t$. To define a homomorphism from P_t to another

module, we would need to define the image on each generated. We will use the fact that any two projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ are chain homotopy equivalent (Section I.7 of [7]).

3.2 CLASSIFICATION OF GROUP EXTENSIONS

Equivalent extensions   Let K be an abelian group and let Q be a group which acts on K by the map $\theta : Q \to Aut(K)$. An extension, E, of Q by K, with inclusion and projection maps ι and π respectively, is said to *give rise to* θ if it satisfies the following property: for any $k \in K$ and $\tilde{g} \in E$ such that $\pi(\tilde{g}) = g \in G$, then $\tilde{g}\iota(k)\tilde{g}^{-1} = \iota(\theta(g)(k))$. Two group extensions, E_1 and E_2, are equivalent if there exists an isomorphism $\varphi : E_1 \to E_2$ such that the diagram:

$$
\begin{array}{ccc}
 & & E_1 \\
 & \nearrow & \downarrow \varphi \searrow \\
1 \longrightarrow K & & Q \longrightarrow 1 \\
 & \searrow & \nearrow \\
 & & E_2
\end{array}
$$

commutes. We denote the set of equivalence classes of group extensions giving rise to θ by $\mathcal{E}(Q; K)$.

Constructing 2-cocycles   Let K be an abelian group. Suppose E is an extension of Q by K giving rise to θ. Let $\iota : K \to E$ and $\pi : E \to Q$ be the inclusion and projection maps. A *section*, s, is a function $s : Q \to E$ such that $\pi \circ s = \mathrm{id}_Q$; furthermore we say s is *normalized* if $s(1_Q) = 1_E$. Since s may not be a homomorphism, $s(p_1)s(p_2)$ is may not equal $s(p_1 p_2)$ in E. Therefore we can measure the failure of s to be a homomorphism by some function $\kappa \in Hom_Q(P_2, K)$ such that $s(p_1)s(p_2) = \iota(\kappa([p_1 \mid p_2]))s(p_1 p_2)$. If we know the structure of E, we have a formula:

$$
\kappa([p_1 \mid p_2]) = s(p_1)s(p_2)s(p_1 p_2)^{-1}
$$

which determines a cocycle corresponding to E as an extension of Q by K. A thorough explanation that κ satisfies the cocycle condition can be found in chapter IV, Section 3 of Brown [7].

Corresponding Extensions   Let κ be a representative for a cohomology class in $H^2(Q; K)$ determined by the normalized standard resolution. Suppose Q acts on K by the action θ, define E_κ to be the twisted semi-direct product $K \ltimes_\kappa Q$ with multiplication defined by:

$$(a, g) \cdot (b, h) = (a + g \cdot b + \kappa(g, h), gh).$$

Note that multiplication in E_κ satisfies associativity since κ is a cocycle (page 92 of [7]). Thus E_κ is a representative for the equivalence class of group extensions of Q by K corresponding to $[\kappa]$.

The following theorem by Eilenberg and MacLane provides the classification of group extensions by 2-cocycles constructed above [10].

Theorem 3.2 (Eilenberg MacLane 1947). *Suppose Q and K are groups with K abelian such that Q acts on K by θ. There exists a natural bijection between $\mathcal{E}(Q, K)$ and $H^2(Q; K)$.*

Remarks   The choice of section determines the representative of the cohomology class in $H^2(Q; K)$. By definition, E is a split extension of Q by K if there exists a section which is a homomorphism. Since a representative of the cohomology class measures the failure of the section to split, the semi-direct product $K \rtimes_\theta Q$ corresponds to the trivial cohomology class.

Furthermore, changing the choice of projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ yields a corresponding representative in an isomorphic cohomology group. In Section 5 we build the resolution used to define our cocycle in Theorem 1.2.

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4   Braid group

4.1   DEFINITIONS

Strand Diagrams   Let $\{x_1, x_2, \ldots, x_n\}$ be a collection of distinct, marked points in $\mathbb{C}$.

We consider an element of the braid group, called a *braid*, to be a collection of n

non-colliding paths $f_i : [0, 1] \to \mathbb{C} \times [0, 1]$, called *strands*, such that $f_i(0) = x_i$, $f_i(1) = x_j$ for

some j, and $f_i(t) \in \mathbb{C} \times \{t\}$. Note that since the paths f_i and f_j are non-colliding,

$f_i(1) = f_j(1)$ if and only if $i = j$. An *isotopy* of any surface, S, is a homotopy

$H : S \times [0, 1] \to S$ such that H restricted to any $t \in [0, 1]$ is a homeomorphism. Two braids

$f = (f_1, f_2, \cdots, f_n)$ and $g = (g_1, g_2, \ldots, g_n)$, are equivalent if there exists an isotopy

$H : (\mathbb{C} \times [0, 1]) \times [0, 1] \to \mathbb{C} \times [0, 1]$ such that $g_i = H \circ f_i$ for all i. Notice that strands

cannot cross each other throughout the isotopy.

To obtain the group structure of the braid group under this description it remains to

define the composition of braids, the identity element, and inverses. We compose two

braids, f and g, by gluing $f_i(1)$ to $g_i(0)$ and scaling by half. The identity element in the

braid group is the braid which is constant on the marked points and the inverse of f is the

reflection through the plane $\mathbb{C} \times \{1\}$.

A *strand diagram* for a braid is a projection of the images of a braid to $\mathbb{R} \times [0, 1]$.

Under a projection to $\mathbb{R} \times [0, 1]$, a rotation between two strands will intersect. In

particular, a clockwise rotation of two adjacent strands should have the lower indexed

strand laying under the higher indexed strand. Therefore we use a gap in the strand which

is laying underneath the other to describe crossings in the strand diagrams. To describe

the multiplication of two braids, f and g, by a strand diagram we do the following: first we

scale both f and g by $1/2$, then we stack the diagrams so that g begins where f ends. We

will omit the re-scaling of the diagrams to improve readability.

For any braid, there exists a homotopy such that for any $t \in [0, 1]$, there is at most one

crossing in the strand diagram. Therefore the braid group is generated by adjacent
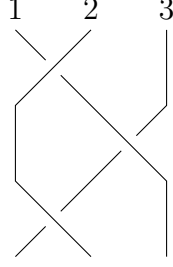
Figure 1: Strand Diagram of $b_{1,3}$.

crossings in the strand diagram. Define a positively oriented twist to be the clockwise rotation between two strands. In Artin's presentation given by (2), b_i represents the positively oriented twist between the ith and $(i+1)^{\text{st}}$ strands. For any $1 \leq i < j \leq n$, define the the half twist between the ith and j^{th} strands, denoted $b_{i,j}$, by the following strand diagram:

- The ith strand passes under all strands between i and j.

- The ith strand passes under the jth strand.

- The jth passes under all strands between i and j.

By the generators given in Artin's presentation:

$$b_{i,j} = b_i b_{i+1} \cdots b_{j-2} b_{j-1} b_{j-2}^{-1} \cdots b_{i+1}^{-1} b_i^{-1}.$$

Note that $b_{i,j}^{-1}$ only changes the description of $b_{i,j}$ by passing the jth strand under the ith strand. Figure 1 gives the strand diagram for the positively oriented half twist between the first and third strands. The choice of $b_{i,j}$ gives us the following presentation for the braid group which is equivalent to the Birman-Ko-Lee presentation [5]:

$$B_n \approx \left\langle \{b_{i,j}\}_{1 \leq i < j \leq n} \ \middle| \ \begin{array}{l} [b_{i,j}, b_{k,\ell}] = 1 \text{ if } (j-k)(j-\ell)(i-k)(i-\ell) > 0 \\ b_{i,j} b_{j,k} b_{i,j}^{-1} = b_{i,k} \text{ if } i < j < k, \ k < i < j, \ j < k < i \end{array} \right\rangle \quad (4)$$

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Remark Artin's presentation describes all braids by a composition of clockwise rotations between adjacent strands. Our presentation in (4) describes all braids by a composition of clockwise rotations between any two strands.

Mapping Class Group Suppose S is an orientable surface. The *mapping class group*, $\mathrm{Mod}(S)$, is the group of isotopy classses of orientation preserving homeomorphisms which restrict to the identity on the boundary of S. Let D_n be the closed disk with n marked points. Let $h : D_n \to D_n$ be a homeomorphism which is invariant on the collection of marked points; note that h may permute the marked points. Forgetting the points are marked, h is isotopic to the identity, therefore there exists a homotopy $H : D^2 \times [0,1] \to D^2$ such that for every $t \in [0,1]$, H restricts to a homeomorphism which is the identity on the boundary. The marked points will move around the interior of the disk and return to their starting position throughout H. Therefore we get a collection of non-colliding paths in $D^2 \times [0,1]$ by following the path of each marked point throughout H. This collection of non-colliding paths describes a braid in the strand diagram definition. Therefore, we can consider the braid group as the mapping class group of D_n:

$$ B_n \approx \mathrm{Mod}(D_n). $$

4.2 Extensions of the Symmetric Group

The *symmetric group of order n*, S_n, is the group of all permutations on the set $\{1, 2, \ldots, n\}$ under composition. Let σ_i be the transposition which swaps i with $i+1$. The standard presentation of the symmetric group generated by adjacent transpositions is given by:

$$ S_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i^2 = 1, \quad [\sigma_i, \sigma_j] = 1 \text{ if } |i - j| > 1 \\[2mm] \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\rangle. \tag{5} $$

Remark Since multiplication in S_n is function composition, words on the generators of S_n are usually read right to left. For consistency with computations outside the symmetric group, we will read words on the generators of S_n from left to right.

Under Artin's presentation (2), the homomorphism which maps b_i to σ_i is surjective. The kernel of this surjection is the pure braid group, PB_n. Since we use the Birman-Ko-Lee presentation (4) for computations, we use a presentation for S_n generated by all transpositions. We denote the transposition which permutes the integers i and j by $\sigma_{i,j}$ and choose as a word on the generators of (5):

$$\sigma_{i,j} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-2} \sigma_{j-1} \sigma_{j-2}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^{-1}.$$

A presentation for the symmetric group of order n generated by all possible transpositions is given by:

$$S_n = \left\langle \{\sigma_{i,j}\}_{1 \leq i < j \leq n} \left| \begin{array}{l} \sigma_{i,j}^2 = 1, \ \ [\sigma_{i,j}, \sigma_{k,\ell}] = 1 \text{ if } \{i,j\} \cap \{k,\ell\} = \emptyset \\[2mm] \sigma_{i,j}\sigma_{j,k}\sigma_{i,j}^{-1} = \sigma_{i,k} \text{ for all } i,j,k \end{array} \right. \right\rangle. \qquad (6)$$

Since the pure braid group is the kernel of the surjection from $B_n \to S_n$, the following sequence is exact and B_n is an extension of S_n by PB_n:

$$1 \longrightarrow PB_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 1$$

Consider the commutator subgroup of the pure braid group, $[PB_n, PB_n]$. A well known fact of group homology is: for any group G, $H_1(G; \mathbb{Z})$ with untwisted coefficients is isomorphic to the abelianization of G (page 36 of [7]). Furthermore, $H_1(PB_n; \mathbb{Z}) \approx \mathbb{Z}^{\binom{n}{2}}$ (page 252 of [11]). Since $[PB_n, PB_n]$ is a characteristic subgroup of PB_n and PB_n is normal in B_n,

$[PB_n, PB_n]$ is a normal subgroup of B_n. Therefore, by the third isomorphism theorem:

$$(B_n/[PB_n, PB_n])/(PB_n/[PB_n, PB_n]) \approx B_n/PB_n \approx S_n.$$

Set $G_n = B_n/[PB_n, PB_n]$, then G_n is an extension of the symmetric group by $H_1(PB_n; \mathbb{Z}) = PB_n/[PB_n, PB_n]$. Recall, in [6], Brendle and Margalit proved $B_n[4]$ is the subgroup of PB_n generated by squares of all elements, PB_n^2. Let $P\mathcal{Z}_n$ be the image of $B_n[4]$ in $H_1(PB_n; \mathbb{Z})$. By the work of Kordek and Margalit [12], $\mathcal{Z}_n$ is also an extension of the S_n by $P\mathcal{Z}_n$. Therefore we get the following diagram where each row is exact:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & PB_n & \longrightarrow & B_n & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(PB_n; \mathbb{Z}) & \longrightarrow & G_n & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P\mathcal{Z}_n & \longrightarrow & \mathcal{Z}_n & \longrightarrow & S_n & \longrightarrow & 1
\end{array}
$$

4.3 LEVEL m BRAID GROUP

Integral Burau Representation The unreduced Burau representation is the representation $\rho : B_n \to GL_n(\mathbb{Z}[t, t^{-1}])$ defined on the presentation given in (2) by:

$$b_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$$

where I_j represents the $j \times j$ identity matrix. The Burau representation maps b_i to the square matrix which is the identity except for the square at the i, $i+1$ row and columns. Evaluating the unreduced Burau representation at $t = -1$ yields the *integral Burau representation*, $B_n \to GL_n(\mathbb{Z})$.

Topologically we can describe the integral Burau representation by considering the two-fold branched cover, X_n of the disk with n marked points, D_n, where the branch locus
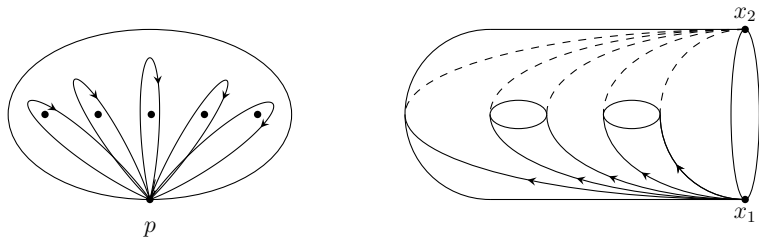
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Figure 2: Double cover of D_5°

are the marked points. Since the fundamental group of the n punctured disk is the free group of rank n, X_n is an orientable surface with genus $2n - 2$. Furthermore, choose a basepoint in the boundary of D_n. Since we are taking the two-fold branch cover, the lift of a loop which encompasses all the marked points in D_n to X_n is a loop if n is even and a path with different endpoints if n is odd. Therefore X_n is a compact, orientable surface of genus $(n-1)/2$ with one boundary component if n is odd and a compact, orientable surface of genus $(n-2)/2$ with two boundary components if n is even. Recall the mapping class group of a space is the set of orientation-preserving homeomorphisms which restrict to a homeomorphism on the boundary equivalent up to homotopy. So each element of the mapping class group of D_n lifts to a unique element of X_n.

Let D_n° be D_n with the marked points removed. Pick a basepoint of $p \in \partial D_n^\circ$ and let $\{x_1, x_2\}$ be the lifts of p in X_n. Any path in the fundamental group of D_n° is a path in D_n, the lifts of which generate $H_1(X_n, \{x_1, x_2\}; \mathbb{Z})$. Figure 2 depicts the lifts of paths in D_5 to paths in X_5. Since B_n is the mapping class group of D_n, the integral Burau representation can be described by the following composition:

$$B_n \longrightarrow \mathrm{Mod}(X_n) \longrightarrow \mathrm{Aut}(H_1(X_n, \{x_1, x_2\}; \mathbb{Z})) \subseteq GL_n(\mathbb{Z}).$$

Furthermore, composing the integral Burau representation with the mod m reduction of integers yields a representation $B_n \to GL_n(\mathbb{Z}_m)$.

Definitions

- The kernel of the representation of the braid group given by the composition of the integral Burau representation with the mod m reduction of integers is the level m braid group, $B_n[m]$:

$$B_n[m] = \ker\{\ B_n \xrightarrow{\ \rho\ } GL_n([\mathbb{Z}[t, t^{-1}]) \xrightarrow{\ t=-1\ } GL_n(\mathbb{Z}) \longrightarrow GL_n(\mathbb{Z}_m)\ \}.$$

- The *mod 4 braid group*, denoted $\mathscr{Z}_n$, is the quotient of the braid group by the level 4 braid group, $B_n/B_n[4]$.

Structure of $B_n[m]$ In general, the algebraic structure of $B_n[m]$ is unknown. Arnol'd first proved $B_n[2] \approx PB_n$ in 1968 [3] while Brendle and Margalit proved $B_n[4] \approx PB_n^2$ in 2014 [6]. The algebraic structure of quotients of level m braid groups are better understood. Stylianakis proved that for each odd prime p, $B_n[p]/B_n[2p] \approx S_n$ in 2018 [15]. In 2020 Appel, Bloomquist, Gravel, and Holden generalized Stylianakis' result to $B_n[\ell]/B_n[2\ell] \approx S_n$ for every positive, odd integer ℓ [2]. Furthermore, they give the following theorem in the same work [2]:

Theorem 4.1 (Appel, Bloomquist, Gravel, Holden). *For each n and each ℓ odd:*

$$B_n[\ell]/B_n[4\ell] \approx \mathscr{Z}_n.$$

5 Universal Cover of an Eilenberg-MacLane space

We begin this section by constructing a truncated resolution of $\mathbb{Z}$ over $\mathbb{Z}S_n$ determined by the universal cover of a $K(S_n, 1)$-space. We first construct the 2-skeleton of the Cayley complex, $\tilde{X}$ for the symmetric group and describe the corresponding cellular chain complex, denoted $\mathcal{C}(\tilde{X})$, in dimensions 0, 1, and 2. Then, for $0 \leq i \leq 2$ we construct homomorphisms $\gamma_i : C_i(\tilde{X}) \to \bar{P}_i$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
C_2(\tilde{X}) & \xrightarrow{\partial_2^C} & C_1(\tilde{X}) & \xrightarrow{\partial_1^C} & C_0 & \xrightarrow{\epsilon_C} & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow{\gamma_2} & & \downarrow{\gamma_1} & & \downarrow{\gamma_0} & & \downarrow{\mathrm{id}} & & \\
\bar{P}_2 & \xrightarrow{\partial_2^P} & \bar{P}_1 & \xrightarrow{\partial_1^P} & \bar{P} & \xrightarrow{\epsilon_P} & \mathbb{Z} & \longrightarrow & 0
\end{array}
$$

The existence of this commuting diagram yields the following theorem:

Theorem 5.1. *Let $0 \to K \to E \to S_n \to 1$ be a group extension which corresponds to $[\kappa] \in H^2(S_n; K)$ under the normalized standard resolution. There exists $\kappa' \in Hom(C_2(\tilde{X}), K)$ such that $[\kappa'] = [\kappa]$ in $H^2(S_n; K)$ defined by $\kappa' = \kappa \circ \gamma_2$.*

Cayley Complex Consider the symmetric group with presentation given by (6). The *presentation complex* of the symmetric group, X, is the CW-complex with one 0-cell, a 1-cell for each generator of S_n, and a 2-cell for each relation in S_n. Let x_0 denote the 0-cell of X and $x_{i,j}$ denote the 1-cell of X corresponding to the generator $\sigma_{i,j}$ of S_n. Note that we only use positive generators to label 1-cells. Let $c_{i,j}$ be the two cell glued by the relation $\sigma_{i,j}^2 = 1$ and $d_{i,j,k,\ell}$ be the two cell glued by the relation $[\sigma_{i,j}, \sigma_{k,\ell}] = 1$ if $\{i, j\} \cap \{k, \ell\} = \emptyset$. Consider the relation $\sigma_{i,j}\sigma_{j,k}\sigma_{i,j}^{-1} = \sigma_{i,k}$ from (6) as $\sigma_{i,j}\sigma_{j,k}\sigma_{i,j}^{-1}\sigma_{i,k}^{-1} = 1$; let $e_{i,k,j}$ be the 2-cell that glues along this relation in X. Note we have one 2-cell, $e_{i,j,k}$, for each distinct, ordered triple in $\{1, 2, \ldots, n\}$, while $c_{i,j}$ and $d_{i,j,k,\ell}$ assume $i < j$ and $k < \ell$.

The *Cayley complex* for the symmetric group, denoted $\tilde{X}$ is the universal cover of X. For each $p \in S_n$, there exists a 0-cell of $\tilde{X}$ denoted $p \cdot \tilde{x}_0$. Choose the basepoint of $\tilde{X}$, $\tilde{x}_0$, to
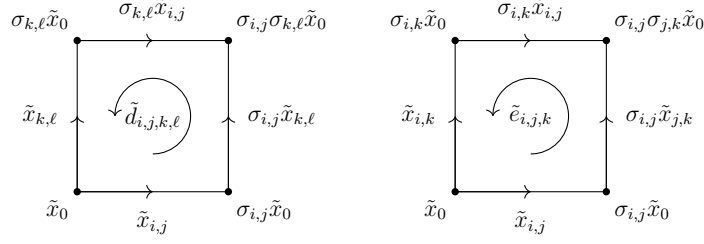
Figure 3: Gluing 2-cells of the Cayley Complex.

be the 0-cell corresponding to $p = 1_{S_n}$. Let $C_0(\tilde{X})$ be the set of all 0-cells of $\tilde{X}$, then S_n acts on $C_0(\tilde{X})$ by the following: for any $p_1, p_2 \in S_n$, p_1 acts on $p_2\tilde{x}$ by $p_1 \curvearrowright p_2\tilde{x}_x = p_2 p_1 \tilde{x}_0$. Therefore $C_0(\tilde{X})$ is generated by $\tilde{x}_0$ as a $\mathbb{Z}S_n$ module.

For every $p \in S_n$ and $1 \leq i < j \leq n$, there exists a 1-cell of $\tilde{X}$, denoted $p \cdot \tilde{x}_{i,j}$, which begins at $p \cdot \tilde{x}$ and ends at $\sigma_{i,j}p \cdot \tilde{x}_0$. Let $C_1(\tilde{X})$ be the set of all 1-cells of $\tilde{X}$, then S_n acts on $C_1(\tilde{X})$ by $p_1 \curvearrowright p_2\tilde{x}_{i,j} = p_2 p_1 \tilde{x}_{i,j}$ for all $p_1, p_2 \in S_n$ and $p_1\tilde{x}_{i,j} \in R_0$. Take $\tilde{x}_{i,j} = 1_{S_n} \cdot \tilde{x}_{i,j}$ for all $1 \leq i < j \leq n$, then the set of all $\tilde{x}_{i,j}$ generate $C_1(\tilde{X})$ as a $\mathbb{Z}S_n$ module.

There are three types of 2-cells glued into $\tilde{X}$. We denote the first type of 2-cell in $\tilde{X}$ by $p \cdot \tilde{c}_{i,j}$ where $1 \leq i < j \leq n$. Then each $p \cdot \tilde{c}_{i,j}$ is glued in by the following loop:

- Begin at $p \cdot \tilde{x}_0$ and follow $p \cdot \tilde{x}_{i,j}$ to $\sigma_{i,j}p \cdot \tilde{x}_0$.

- Follow $p\sigma_{i,j} \cdot \tilde{x}_{i,j}$ back to $p \cdot \tilde{x}_0$.

If $\{i, j\} \cap \{k, \ell\} = \emptyset$ with $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$, there is a second type of 2-cell denoted $p \cdot \tilde{d}_{i,j,k,\ell}$ glued into $\tilde{X}$ by the following loop:

- Begin at $p \cdot \tilde{x}_0$ and follow $\tilde{x}_{i,j}$ to $p\sigma_{i,j} \cdot \tilde{x}_0$.

- Follow $p\sigma_{i,j} \cdot \tilde{x}_{k,\ell}$ to $p\sigma_{k,\ell}\sigma_{i,j} \cdot \tilde{x}_0$.

- Follow $p\sigma_{k,\ell} \cdot \tilde{x}_{i,j}$ in reverse to $p\sigma_{k,\ell} \cdot \tilde{x}_0$.

- Follow $p \cdot \tilde{x}_{k\ell}$ in reverse back to $p \cdot \tilde{x}_0$.

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The third type of 2-cell occurs for all ordered, distinct triples of $\{1, 2, \ldots, n\}$. For each $p \in S_n$, let $p \cdot \tilde{e}_{i,j,k}$ denote the 2-cell glued in by the following:

- Begin at $p \cdot \tilde{x}_0$ and follow $\tilde{x}_{i,j}$ to $p\sigma_{i,j} \cdot \tilde{x}_0$.

- Follow $p\sigma_{i,j} \cdot \tilde{x}_{j,k}$ to $p\sigma_{i,j}\sigma_{j,k} \cdot \tilde{x}_0$.

- Follow $p\sigma_{i,k} \cdot \tilde{x}_{i,j}$ in reverse to $p\sigma_{i,k} \cdot \tilde{x}_0$.

- Follow $p \cdot \tilde{x}_{i,k}$ in reverse back to $p \cdot \tilde{x}_0$.

Notice S_n acts on the 2-cells of $\tilde{X}$ by changing the 0-cell at which each loop begins and ends. Therefore the 2-cells $\tilde{c}_{i,j}$, $\tilde{d}_{i,j,k,\ell}$, and $\tilde{e}_{i,j,k}$ representing the 2-cell with $p = 1$ in their respective classes generate the 2-cells of $\tilde{X}$, $C_2(\tilde{X})$, as a $\mathbb{Z}S_n$ module. The gluing maps of $\tilde{d}_{i,j,k,\ell}$ and $\tilde{e}_{i,j,k}$ are given in Figure 3.

To construct a resolution of $\mathbb{Z}$ over $\mathbb{Z}S_n$ it remains to define maps on the generators of $C_i(\tilde{X})$ for $0 \le i \le 2$. Define the augmentation map $\epsilon_C : C_0(\tilde{X}) \to \mathbb{Z}$ by $\epsilon_C(\tilde{x}_0) = 1$. Since $\tilde{X}$ is a CW-complex, the differential maps are determined by the gluing maps. So $\partial_1^C : C_1(\tilde{X}) \to C_0(\tilde{X})$ is defined by $\partial_1^C(\tilde{x}_{i,j}) = (\sigma_{i,j} - 1)\tilde{x}_0$. For $\partial_2^C : C_2(\tilde{X}) \to C_1(\tilde{X})$ we get:

$$\partial_2^C(\tilde{c}_{i,j}) = (\sigma_{i,j} + 1)\tilde{x}_{i,j}$$

$$\partial_2^C(\tilde{d}_{i,j,k,\ell}) = \tilde{x}_{i,j} + \sigma_{i,j} \cdot \tilde{x}_{k,\ell} - \sigma_{k,\ell} \cdot \tilde{x}_{i,j} - \tilde{x}_{k,\ell}$$

$$\partial_2^C(\tilde{e}_{i,k,j}) = \tilde{x}_{i,j} + \sigma_{i,j} \cdot \tilde{x}_{j,k} - \sigma_{i,k} \cdot \tilde{x}_{i,j} - \tilde{x}_{i,k}$$

For $0 \le i \le 2$, $C_i(\tilde{X})$ is the truncated resolution of $\mathbb{Z}$ over $\mathbb{Z}S_n$ corresponding to the universal cover of a $K(S_n, 1)$-complex. To describe our cocycle by this truncated resolution, $\mathcal{C}(\tilde{X})$, it remains to construct a map from the truncated resolution $\mathcal{C}(\tilde{X})$ determined by the Cayley complex to the normalized bar resolution, $\bar{\mathcal{P}}$. Since $C_i(\tilde{X})$ is a free $\mathbb{Z}S_n$ for $0 \le i \le 2$, it suffices to define $\gamma_i : C_i(\tilde{X}) \to \bar{P}_i$ on the generators of $C_i(\tilde{X})$. Define

$\gamma_0(\tilde{x}_0) = [\;]$ and $\gamma_1(\tilde{x}_{i,j}) = [\sigma_{i,j}]$. By the construction of the bar resolution, γ_0 and γ_1 commute with the differential maps.

Theorem 5.2. *Define $\gamma_2 : C_2(\tilde{X}) \to \bar{P}_2$ by:*

$$\gamma_2(\tilde{c}_{i,j}) = [\sigma_{i,j} \mid \sigma_{i,j}]$$

$$\gamma_2(\tilde{d}_{i,j,k,\ell}) = [\sigma_{i,j} \mid \sigma_{k,\ell}] - [\sigma_{k,\ell} \mid \sigma_{i,j}]$$

$$\gamma_2(\tilde{e}_{i,k,j}) = [\sigma_{i,j} \mid \sigma_{j,k}] - [\sigma_{i,k} \mid \sigma_{i,j}]$$

Then γ_2 commutes with the differentials:

$$
\begin{array}{ccccccccc}
C_2(\tilde{X}) & \xrightarrow{\partial_2^C} & C_1(\tilde{X}) & \xrightarrow{\partial_1^C} & C_0(\tilde{X}) & \xrightarrow{\epsilon_C} & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_0} & & \downarrow{\scriptstyle id} & & \\
\bar{P}_2 & \xrightarrow{\partial_2^P} & \bar{P}_1 & \xrightarrow{\partial_1^P} & \bar{P}_0 & \xrightarrow{\epsilon_P} & \mathbb{Z} & \longrightarrow & 0
\end{array}
$$

Proof. Since γ_2 is defined on the generators of a free $\mathbb{Z}S_n$ module, γ_2 is a well defined homomorphism. Therefore it suffices to show $\gamma_1 \circ \partial_2^C = \partial_2^P \circ \gamma_2$. Recall the generators of $C(\tilde{X})_1$ are $\tilde{x}_{i,j}$ and $\gamma_1(\tilde{x}_{i,j}) = [\sigma_{i,j}]$. Therefore:

$$\gamma_1 \circ \partial_2^C(\tilde{c}_{i,j}) = [\sigma_{i,j}] + \sigma_{i,j}[\sigma_{i,j}]$$

$$\gamma_1 \circ \partial_2^C(\tilde{d}_{i,j,k,\ell}) = [\sigma_{i,j}] + \sigma_{i,j}[\sigma_{k,\ell}] - \sigma_{k,\ell}[\sigma_{i,j}] - [\sigma_{k,\ell}]$$

$$\gamma_2 \circ \partial_2^C(\tilde{e}_{i,k,j}) = [\sigma_{i,j}] + \sigma_{i,j}[\sigma_{j,k}] - \sigma_{i,k}[\sigma_{i,j}] - [\sigma_{i,k}]$$

Since we are using the normalized bar resolution, $[p_1 \mid \cdots \mid p_t] = 0$ if p_i is trivial for any i. Now, since $\partial_2^R([p_1 \mid p_2]) = p_1[p_2] - [p_1 \cdot p_2] + [p_1]$ for any $p_1, p_2 \in S_n$ and ∂_2^P is a $\mathbb{Z}S_n$ module

homomorphism, we have:

$$\partial_2^P([\sigma_{i,j} \mid \sigma_{i,j}]) = \sigma_{i,j}[\sigma_{i,j}] - [\sigma_{i,j}^2] + [\sigma_{i,j}]$$

$$= \sigma_{i,j}[\sigma_{i,j}] - [1] + [\sigma_{i,j}]$$

$$= [\sigma_{i,j}] + \sigma_{i,j}[\sigma_{i,j}].$$

Thus $(\partial_2^P \circ \gamma_2)(\tilde{c}_{i,j}) = (\gamma_1 \circ \partial_2^C)(\tilde{c}_{i,j})$. To compute $\partial_2^P(\gamma_2(\tilde{d}_{i,j,k,\ell}))$, recall $\sigma_{i,j}\sigma_{k,\ell} = \sigma_{k,\ell}\sigma_{i,j}$ since $\{i,j\} \cap \{k, \ell\} = \emptyset$, therefore:

$$\partial_2^P([\sigma_{i,j} \mid \sigma_{k,\ell}] - [\sigma_{k,\ell} \mid \sigma_{i,j}]) = \sigma_{i,j}[\sigma_{k,\ell}] - [\sigma_{i,j}\sigma_{k,\ell}] + [\sigma_{i,j}] - \sigma_{k,\ell}[\sigma_{i,j}] + [\sigma_{k,\ell}\sigma_{i,j}] - [\sigma_{k,\ell}]$$

$$= \sigma_{i,j}[\sigma_{k,\ell}] - [\sigma_{i,j}\sigma_{k,\ell}] + [\sigma_{i,j}] - \sigma_{k,\ell}[\sigma_{i,j}] + [\sigma_{i,j}\sigma_{k,\ell}] - [\sigma_{k,\ell}]$$

$$= \sigma_{i,j}[\sigma_{k,\ell}] + [\sigma_{i,j}] - \sigma_{k,\ell}[\sigma_{i,j}] - [\sigma_{k,\ell}].$$

Finally, to compute $\partial_2^P(\tilde{e}_{i,k,j})$ recall that in S_n, $\sigma_{i,k} = \sigma_{i,j}\sigma_{j,k}\sigma_{i,j}^{-1}$ and therefore $\sigma_{i,k}\sigma_{i,j} = \sigma_{i,j}\sigma_{j,k}$. Hence we have:

$$\partial_2^P([\sigma_{i,j} \mid \sigma_{j,k}] - [\sigma_{i,k} \mid \sigma_{i,j}]) = \sigma_{i,j}[\sigma_{j,k}] - [\sigma_{i,j}\sigma_{j,k}] + [\sigma_{i,j}] - \sigma_{i,k}[\sigma_{i,j}] + [\sigma_{i,k}\sigma_{i,j}] - [\sigma_{i,k}]$$

$$= \sigma_{i,j}[\sigma_{j,k}] - [\sigma_{i,k}\sigma_{j,k}] + [\sigma_{i,j}] - \sigma_{i,k}[\sigma_{i,j}] + [\sigma_{i,j}\sigma_{j,k}] - [\sigma_{i,k}]$$

$$= \sigma_{i,j}[\sigma_{j,k}] + [\sigma_{i,j}] - \sigma_{i,k}[\sigma_{i,j}] - [\sigma_{i,k}].$$

$$\square$$

Recall in Section 3.2, the cocycle constructed from a group extension is given by:

$$\kappa([p_1 \mid p_2]) = s(p_1)s(p_2)s(p_1p_2)^{-1}.$$

Therefore, given $\gamma_2 : C_2(\tilde{X}) \to \bar{P}_2$ we can describe a cocycle on the generators of $C_2(\tilde{X})$ by the image of $C_2(\tilde{X})$ in $\bar{P}_2$. In particular, for the symmetric group we can measure the

failure of an extension to split by determining the failure of the relations to lift. By taking the images of $\tilde{c}_{i,j}$, $\tilde{d}_{i,j,k,\ell}$, and $\tilde{e}_{i,j,k}$ given in Theorem 5.2 we get the following formula for a corresponding cocycle:

Theorem 5.3. *Let K be any S_n module and let E be an extension of S_n by K. Suppose $\kappa \in Hom(\bar{P}_2, K)$ is a representative for the cohomology class in $H^2(S_n; K)$ corresponding to E determined by the normalized bar resolution. Define $\kappa' \in Hom(R_2, K)$ by:*

$$\kappa'(\tilde{c}_{i,j}) = s(\sigma_{i,j})s(\sigma_{i,j})$$

$$\kappa'(\tilde{d}_{i,j,k,\ell}) = s(\sigma_{i,j})s(\sigma_{k,\ell})s(\sigma_{i,j}\sigma_{k,\ell})^{-1} - s(\sigma_{k,\ell})s(\sigma_{i,j})s(\sigma_{k,\ell}\sigma_{i,j})^{-1}$$

$$\kappa'(\tilde{e}_{i,k,j}) = s(\sigma_{i,j})s(\sigma_{j,k})s(\sigma_{i,j}\sigma_{j,k})^{-1} - s(\sigma_{i,k})s(\sigma_{i,j})s(\sigma_{i,k}\sigma_{i,j})^{-1}$$

Then κ' is the 2-cocycle determined by the resolution corresponding to $\tilde{X}$ such that $[\kappa']$ and $[\kappa]$ represent the same group extension.

6 Extension by the Abelianization of the Pure Braid Group

In Section 4.2 we defined G_n to be an extension of the symmetric group by the abelianization of the pure braid group. Furthermore, the abelianization of the pure braid group is isomorphic to $\mathbb{Z}^{\binom{n}{2}}$ (page 252 of [11]), therefore:

$$0 \longrightarrow \mathbb{Z}^{\binom{n}{2}} \longrightarrow G_n \longrightarrow S_n \longrightarrow 1 \ .$$

Notice the action of S_n on $\mathbb{Z}^{\binom{n}{2}}$ is induced by the conjugating pure braids in B_n by half twists. Therefore the action of S_n on the abelianization of the pure braid group permutes the strands of pure braids. So S_n permutes the generators of $\mathbb{Z}^{\binom{n}{2}}$ by acting on the indices of the generators with the standard action of S_n on unordered pairs of integers.

6.1 GROUP PRESENTATION

Generators Under the presentation of B_n given in (4), $b_{i,j}$ represents the positive half twist between the ith and j^{th} strands. Let $\tilde{\sigma}_{i,j}$ be the projection of $b_{i,j}$ in G_n. For $1 \leq i < j \leq n$, let $g_{i,j}$ be the commuting generators of $\mathbb{Z}^{\binom{n}{2}}$. Since $\mathbb{Z}^{\binom{n}{2}} \approx H_1(PB_n; \mathbb{Z})$, each $g_{i,j}$ represents the positively oriented full twist between the ith and j^{th} strands. From Section 2 we need to choose a lift of generators for S_n. For each $\sigma_{i,j}$ a generator in (6), choose $\tilde{\sigma}_{i,j}$ to be the lift in the generating set of G_n. Therefore, for $1 \leq i < j \leq n$, we get the following generating set for G_n:

$$\{g_{i,j}\} \cup \{\tilde{\sigma}_{i,j}\}.$$

Relations By Section 2, to construct a presentation for G_n, we need to include relations of $\mathbb{Z}^{\binom{n}{2}}$, lift relations of S_n, and conjugate generators of $\mathbb{Z}^{\binom{n}{2}}$ by generators of S_n. A full list of all the relations in G_n are given in Table 1. Notice R1 is the included relation of $\mathbb{Z}^{\binom{n}{2}}$, R2-R5 are the lifted relations of S_n, and R6 is the relation determined by conjugating

R1: $\qquad [g_{i,j}, g_{k,\ell}] = 1 \qquad\qquad$ for all i, j, k, ℓ

R2: $\qquad \tilde{\sigma}_{i,j}^2 = g_{i,j} \qquad\qquad$ for all i

R3: $\qquad \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,j}\tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,k} \qquad$ if $\quad k < i < j; \ \ i < j < k; \ \text{ or } \ j < k < i$

R4: $\qquad \tilde{\sigma}_{i,j}^{-1}\tilde{\sigma}_{j,k}\tilde{\sigma}_{i,j} = \tilde{\sigma}_{i,k} \qquad$ if $\quad i < k < j; \ \ j < i < k; \ \text{ or } \ k < j < i$

R5: $\quad [\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,\ell}] = \begin{cases} g_{i,k}g_{i,\ell}^{-1}g_{j,k}^{-1}g_{j,\ell} & i < k < j < \ell \\ g_{k,i}^{-1}g_{k,j}g_{i,\ell}g_{\ell,j}^{-1} & k < i < \ell < j \\ 1 & \text{otherwise} \end{cases}$

R6: $\qquad \tilde{\sigma}_{i,j}g_{k,\ell}\tilde{\sigma}_{i,j}^{-1} = g_{\sigma_{i,j}(k),\sigma_{i,j}(\ell)} \qquad$ for all $i, j, k, \ell \in \{1, \ldots, n\}$

generators of $\mathbb{Z}^{\binom{n}{2}}$ by lifts of generators of S_n. The rest of this section proves the relations in Table 1. Since $\mathbb{Z}^{\binom{n}{2}}$ embeds into G_n, the relation $[g_{i,j}, g_{k,\ell}]$ is preserved in G_n. Therefore R1 holds in G_n as the inclusion of the relation in $\mathbb{Z}^{\binom{n}{2}}$. Notice that since $\tilde{\sigma}_{i,j}$ is represents the half twist between the ith and j^{th} strands while $g_{i,j}$ is the full twist between the ith and j^{th} strands. R2 in Table 1 states applying a half twist twice yields a full twist. Therefore relation R2 holds in G_n and it remains to prove relations R3-R6.

Remark Throughout this section we will remove the assumption $i < j$ to simplify the statements and proofs. Furthermore, in the proof of relation R5 we will use strand diagrams to determine an equivalent braid in B_n, then project to G_n and use R1 to determine the relation. Also, relation R4 is derived from relation R3 by conjugating and relabelling indices but included in the presentation to clarify future computations.

Theorem 6.1. *Suppose $k < i < j$, $i < j < k$, or $j < k < i$. In G_n:*

$$\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,j}\tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,k}.$$

Proof. In (4), $b_{i,j}b_{j,k}b_{i,j}^{-1} = b_{i,k}$ is a relation if $i < j < k$, $k < i < j$, or $j < k < i$. Since G_n is

a quotient of B_n with $\tilde{\sigma}_{i,j}$ representing the projection of $b_{i,j}$ in G_n, $\tilde{\sigma}_{i,j}\tilde{\sigma}_{j,k}\tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,k}$ is a relation of G_n. $\qquad\square$

Corollary 6.1.1. Suppose $i < k < j$, $j < i < k$, or $k < j < i$. In G_n:

$$\tilde{\sigma}_{i,j}^{-1}\tilde{\sigma}_{j,k}\tilde{\sigma}_{i,j} = \tilde{\sigma}_{i,k}.$$

Proof. Suppose $\alpha < \beta < \delta$, $\beta < \delta < \alpha$, or $\delta < \alpha < \beta$, then by Theorem 6.1:

$$\tilde{\sigma}_{\alpha,\delta}\tilde{\sigma}_{\delta,\beta}\tilde{\sigma}_{\alpha,\delta}^{-1} = \tilde{\sigma}_{\alpha,\beta}.$$

Conjugating both sides of the equality by $\tilde{\sigma}_{\alpha,\delta}^{-1}$ yields:

$$\tilde{\sigma}_{\delta,\beta} = \tilde{\sigma}_{\alpha,\delta}^{-1}\tilde{\sigma}_{\alpha,\beta}\tilde{\sigma}_{\alpha,\delta}.$$

By setting $i = \beta$, $j = \alpha$, and $k = \delta$ the proof is complete. $\qquad\square$

Theorem 6.1 and Corollary 6.1.1 prove relations R3 and R4 hold in G_n. Next we prove relations R5.

Theorem 6.2. *Suppose $|\{i,j\} \cap \{k,\ell\}| \neq 1$. Then*

$$[\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,\ell}] = \begin{cases} g_{i,k}g_{i,\ell}^{-1}g_{j,k}^{-1}g_{j,\ell} & i < k < j < \ell \\ g_{k,i}^{-1}g_{k,j}g_{i,\ell}g_{\ell,j}^{-1} & k < i < \ell < j \\ 1 & otherwise \end{cases}$$

Proof. Suppose $\{i,j\} \cap \{k,\ell\} = \emptyset$. If $(j-k)(j-\ell)(i-k)(i-\ell) > 0$, then $[b_{i,j}, b_{k,\ell}] = 1$ is a relation of (4). Since $\tilde{\sigma}_{i,j}$ is the projection of $b_{i,j}$ in a quotient of B_n, it follows $[\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,\ell}] = 1$ if $(j-k)(j-\ell)(i-k)(i-\ell) > 0$.

Suppose $(j-k)(j-\ell)(i-k)(i-\ell) < 0$, since $\{i,j\} \cap \{k,\ell\} = \emptyset$ either $i < k < j < \ell$ or

$k < i < \ell < j$. Begin with the strand diagram for $[b_{i,j}, b_{k,\ell}]$ where $i < k < j < \ell$. We find $[b_{i,j}, b_{k,\ell}]$ as a pure braid by the following sequence of homotopy equivalent rotations:

1. After the first crossing of i and k, introduce a clockwise and counterclockwise half twist of between the ith and j^{th} strands. As a word in the presentation for the braid group, we get:

$$b_{i,k}^2 b_{i,k}^{-1} b_{k,j} b_{i,k}^{-1} b_{k,\ell} b_{i,j}^{-1} b_{k,\ell}^{-1}.$$

2. After the first positive half twist between the ith and j^{th} strand, rotate the ith strand clockwise until it lays to the left of the kth strand in the projection to $\mathbb{R} \times [0,1]$. Then rotate the ith strand counterclockwise around to its starting position after the positive half twist between the jth and ℓ^{th} strands. As a word in B_n we have:

$$b_{i,k}^2 b_{i,k}^{-1} b_{k,j}^2 b_{i,k} b_{k,j}^{-1} b_{j,\ell} b_{k,j} b_{i,k}^{-1} b_{k,j}^{-2} b_{i,k}^{-1} b_{k,\ell}^{-1}.$$

3. After the full twist between strands i and k, rotate the kth strand clockwise around the jth strand until it returns to the starting position of the ith strand. To preserve the isotopy class, after the positive full twist between the ith and j^{th} strands, rotate the kth strand counterclockwise around the jth strand until it returns to its starting position. Then we have:

$$b_{i,k}^2 b_{k,j}^2 b_{i,j}^2 b_{k,j}^{-2} b_{k,j}^{-1} b_{j,\ell} b_{k,j} b_{i,k}^{-1} b_{k,j}^{-2} b_{i,k}^{-1} b_{k,\ell}^{-1}.$$

4. After the positive half twist between the kth and ℓ^{th} strands, rotate the kth strand clockwise until it reaches the starting position of the ith strand. To maintain the same isotopy class, rotate the kth strand counterclockwise back to the starting position of the ℓth strand after the final negative half twist between the jth and ℓ^{th}

strands. On the generators of B_n, we get:

$$b_{i,k}^2 b_{k,j}^2 b_{i,j}^2 b_{k,j}^{-2} b_{k,j}^{-1} b_{j,\ell}^2 b_{k,j} b_{i,k} b_{j,\ell} b_{k,j}^{-1} b_{j,\ell}^{-2} b_{k,j}^{-1} b_{j,\ell} b_{i,k}^{-1} b_{k,j}^{-1}, b_{j,\ell}^{-2} b_{k,j}^{-1}.$$

5. After the negative full twist between the kth and j^{th} strands, rotate the jth strand clockwise around the ℓth strand. Continue to rotate the jth strand clockwise around the ℓth and k^{th} strands. To maintain the equivalence class, rotate the jth strand counterclockwise around the kth and ℓ^{th} strands after the full twist between the kth and ℓ^{th} strands. Continue the counterclockwise rotation of the jth strand until it returns to the starting position. As a reduced word in B_n, we get:

$$b_{i,k}^2 b_{k,j}^2 b_{i,j}^2 b_{k,j}^{-2} b_{j,\ell}^2 b_{k,\ell}^2 b_{j,\ell}^{-2} b_{i,k} b_{j,\ell} b_{k,j}^{-1} b_{j,\ell}^{-2} b_{k,j}^{-1} b_{j,\ell} b_{i,k}^{-1} b_{k,j}^{-1} b_{j,\ell}^{-2} b_{k,j}^{-1}.$$

6. After the first positive half twist between the ith and ℓ^{th} strands, rotate the ℓth strand clockwise around both the ith and j^{th} strands. After the following full negative twist between the ith and j^{th} strands, rotate the ℓth strand counterclockwise back to the starting position. As a word in B_n we get:

$$b_{i,k}^2 b_{k,j}^2 b_{i,j}^2 b_{k,j}^{-2} b_{j,\ell}^2 b_{k,\ell}^2 b_{j,\ell}^{-2} b_{i,k} b_{j,\ell}^2 b_{k,j}^{-2} b_{j,\ell} b_{i,k}^{-1} b_{k,j}^{-1} b_{j,\ell}^{-2} b_{k,j}^{-1}.$$

7. After the first negative full twist between the jth and ℓ^{th} strands, swap the order of the half twist between the ith and k^{th} strands with the following positive full twist between the jth and ℓ^{th} strand. We get a negative full twist between the jth and ℓ^{th} strands followed by a positive full twist between the jth and ℓ^{th} strands, so we omit them in the diagram.

Next, before the negative full twist between the ith and j^{th} strands, insert a negative and positive half twist between the ith and k^{th} strands. Then, immediately before the

negative full twist between the ith and ℓ^{th} strands, rotate the jth strand clockwise around the ith and ℓ^{th} strands. To maintain the same isotopy class, we rotate the jth strand counterclockwise around the ith and ℓ^{th} strands after the negative full twist between the ith and ℓ^{th} strands. As a word in B_n, we get:

$$b_{i,k}^2 b_{k,j}^2 b_{i,j}^2 b_{k,j}^{-2} b_{j,\ell}^2 b_{k,\ell}^2 b_{i,j}^{-2} b_{i,\ell}^{-2} b_{k,j}^{-1} b_{j,\ell}^{-2} b_{k,j}^{-1}.$$

8. Before full negative twist between the kth and ℓ^{th} strands we rotate the jth strand clockwise around the ℓth strand. Continue rotating the kth strand clockwise around both the kth and ℓ^{th} strands. To maintain an equivalent isotopy class, after the negative full twist between the kth and ℓ^{th} strands, rotate the Kth strand counterclockwise around both the kth and ℓ^{th} strands. Continue rotating the kth strand counterclockwise until it returns to its starting position. As a word in B_n, we get:

$$b_{i,k}^2 b_{k,j}^2 b_{i,j}^2 b_{k,j}^{-2} b_{j,\ell}^2 b_{k,\ell}^2 b_{i,j}^{-2} b_{i,\ell}^{-2} b_{j,\ell}^2 b_{k,\ell}^{-2} b_{j,\ell}^{-2} b_{k,j}^{-2}.$$

Under the projection $B_n \to G_n$, $g_{i,j}$ is the image of a pure braid. By the choice of $\tilde{\sigma}_{i,j}$ and since the $g_{i,j}$'s commute in G_n:

$$\begin{aligned}
[\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,\ell}] &= g_{i,k} g_{k,j} g_{i,j} g_{k,j}^{-1} g_{j,\ell} g_{k,\ell} g_{i,j}^{-1} g_{i,\ell}^{-1} g_{j,\ell} g_{k,\ell}^{-1} g_{j,\ell}^{-1} g_{k,j}^{-1} \\
&= g_{i,k} g_{i,\ell}^{-1} g_{k,j}^{-1} g_{j,\ell}.
\end{aligned}$$

Now, suppose $k < i < j < \ell$, then $[\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,\ell}] = [\tilde{\sigma}_{k,\ell}, \tilde{\sigma}_{i,j}]^{-1}$. By the previous case relation R4 in Table 1 is proven. □

Remark   Since the proof of Theorem 6.2 requires eight steps, the strand diagrams are found in Appendix A.

Theorem 6.3. *Suppose $i, j, k, \ell \in \{1, 2, \ldots, n\}$. In G_n:*

$$\tilde{\sigma}_{i,j} g_{k,\ell} \tilde{\sigma}_{i,j}^{-1} = g_{\sigma_{i,j}(k), \sigma_{i,j}(\ell)}.$$

Proof. By relation R2 we have:

$$\tilde{\sigma}_{i,j} g_{k,\ell} \tilde{\sigma}_{i,j}^{-1} = \tilde{\sigma}_{i,j} \tilde{\sigma}_{k,\ell}^2 \tilde{\sigma}_{i,j}^{-1}.$$

If $\{i, j\} = \{k, \ell\}$, then the statement holds trivially. Suppose $|\{i, j\} \cap \{k, \ell\}| = 1$. Without loss of generality, assume $j = \ell$. If $k < i < j$, $i < j < k$, or $j < k < i$ apply relation R3 to obtain the result. Otherwise, use relation R4 to substitute $\tilde{\sigma}_{i,j}^2$ with $\tilde{\sigma}_{i,j}^{-1} \tilde{\sigma}_{i,k}^2 \tilde{\sigma}_{i,j}^{-1}$ to obtain the result.

Suppose $\{i, j\} \cap \{k, \ell\} = \emptyset$. If $(i - k)(i - \ell)(j - k)(j - \ell) > 0$, then R5 implies $\tilde{\sigma}_{i,j}$ and $\tilde{\sigma}_{k,\ell}$ commute. Therefore the theorem holds. Now, suppose $i < k < j < \ell$, then by relation R3 and the previous case:

$$
\begin{aligned}
\tilde{\sigma}_{i,j} \tilde{\sigma}_{k,\ell}^2 \tilde{\sigma}_{i,j}^{-1} &= \tilde{\sigma}_{i,k} \tilde{\sigma}_{k,j} \tilde{\sigma}_{i,k}^{-1} g_{k,\ell} \tilde{\sigma}_{i,k} \tilde{\sigma}_{k,j}^{-1} \tilde{\sigma}_{i,k}^{-1} \\
&= \tilde{\sigma}_{i,k} \tilde{\sigma}_{k,j} g_{i,\ell} \tilde{\sigma}_{k,j}^{-1} \tilde{\sigma}_{i,k}^{-1} \\
&= \tilde{\sigma}_{i,k} g_{i,\ell} \tilde{\sigma}_{i,k}^{-1} \\
&= g_{k,\ell}.
\end{aligned}
$$

By a symmetric argument, the theorem holds for the case $k < i < \ell < j$ as well. $\square$

Remark Now that we have a presentation for G_n, we can define the inclusion and projection maps. By the constructions given in Section 2, the inclusion map $\iota_1 : \mathbb{Z}^{\binom{n}{2}} \to G_n$

by $\iota_1(g_{i,j}) = g_{i,j}$ for all $1 \leq i < j \leq n$. The projection map $\pi_1 : G_n \to S_n$ is given by:

$$\pi_1(s) = \begin{cases} \sigma_{i,j} & \text{if } s \in \{\tilde{\sigma}_{i,j}\}_{1 \leq i < j \leq n} \\ 1_{S_n} & \text{if } s \in \{g_{i,j}\}_{1 \leq i < j \leq n} \end{cases}$$

6.2   Corresponding Cohomology Class

Now that we have constructed a presentation for G_n, it remains to choose a normalized

section, s, to measure the failure of G_n to split. The choice of s will determine the image of

the cocycle constructed in Theorem 5.3. We define our section $s : S_n \to G_n$ by $s(\sigma_{i,j}) = \tilde{\sigma}_{i,j}$.

Normal Forms of Permutations   Since our normalized section is only defined on the

generators of S_n given in (6), we must choose a normal forms to represent each

permutation. To be consistent with multiplication in G_n, consider multiplication in S_n

from left to right. Let $p \in S_n$ such that $p(n) = k_n$, then $p \cdot \sigma_{k_n,n} \in S_{n-1}$. Suppose

$p \cdot \sigma_{k_n,n}(n-1) = k_{n-1}$, then $p \cdot \sigma_{k_n,n} \cdot \sigma_{k_{n-1},n-1} \in S_{n-2}$. Inductively:

$$p \cdot \sigma_{k_n,n} \cdot \sigma_{k_{n-1},n-1} \cdots \sigma_{1,k_1} = 1.$$

Therefore $p = \sigma_{1,k_1} \cdot \sigma_{2,k_2} \cdots \sigma_{k_n,n}$. Define the normal section $s : S_n \to G_n$ by:

$$s(p) = \tilde{\sigma}_{1,k_1} \tilde{\sigma}_{2,k_2} \cdots \tilde{\sigma}_{k_n,n}.$$

This choice of normal forms give the following algorithm to describe a permutation in S_n:

- Rewrites p as a word in the generators of S_n given in (6).

- For all $1 \leq i < j \leq n$, replaces $\sigma_{i,j}^2$ with 1_{S_n}.

- If $\{i,j\} \cap \{k,\ell\} = \emptyset$ and $\max\{k,\ell\} < \max\{i,j\}$, replaces $\sigma_{i,j}\sigma_{k,\ell}$ with $\sigma_{k,\ell}\sigma_{i,j}$.

- If $j = \max\{i,j,k\}$, replaces $\sigma_{i,j}\sigma_{j,k}$ with $\sigma_{i,k}\sigma_{i,j}$.

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- If $i = \max\{i, j, k\}$, replaces $\sigma_{i,j}\sigma_{j,k}$ with $\sigma_{j,k}\sigma_{i,k}$.

First we determine the normal forms for products of any two transpositions.

Lemma 6.4. *Suppose $\{i, j\} \cap \{k, \ell\} = \emptyset$, then*

$$s(\sigma_{k,\ell}\sigma_{i,j}) = s(\sigma_{i,j}\sigma_{k,\ell}) = \begin{cases} \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell} & j < \ell \\[2mm] \tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j} & \ell < j \end{cases}$$

Proof. Notice $\sigma_{k,\ell}\sigma_{i,j}$ and $\sigma_{i,j}\sigma_{k,\ell}$ describe the same permutation, therefore they have the same normal form. If $\ell < j$ then we take $\sigma_{i,j}\sigma_{k,\ell}\sigma_{i,j}\sigma_{k,\ell} = 1$ and $\sigma_{i,j}\sigma_{k,\ell}$ becomes $\sigma_{k,\ell}\sigma_{i,j}$. Therefore the normal form of both $\sigma_{i,j}\sigma_{k,\ell}$ and $\sigma_{k,\ell}\sigma_{i,j}$ is $\sigma_{k,\ell}\sigma_{i,j}$. Therefore $s(\sigma_{i,j}\sigma_{k,\ell}) = \tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j}$. Now, suppose $j < \ell$. Then we take the transposition fixing k first, yielding $\sigma_{i,j}\sigma_{k,\ell}\sigma_{k,\ell}\sigma_{i,j}$ and thus $s(\sigma_{i,j}\sigma_{k,\ell}) = \tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j}$. $\qquad\square$

Lemma 6.5. *The normal form for products of transpositions with intersection are:*

$$s(\sigma_{i,k}\sigma_{i,j}) = s(\sigma_{i,j}\sigma_{j,k}) = \begin{cases} \tilde{\sigma}_{i,k}\tilde{\sigma}_{i,j} & \max\{i, k, j\} = j \\[2mm] \tilde{\sigma}_{i,j}\tilde{\sigma}_{j,k} & \max\{i, k, j\} = k \\[2mm] \tilde{\sigma}_{k,j}\tilde{\sigma}_{k,i} & \max\{i, k, j\} = i \end{cases}$$

The proof of Lemma 6.5 is a computation similar to the proof of Lemma 6.4 and omitted. Now that we have the normal forms for products of generators, it is possible to compute a representative for the cohomology class describing G_n as an extension of S_n by $\mathbb{Z}^{\binom{n}{2}}$.

Theorem 6.6. *Let:*

$$0 \longrightarrow \mathbb{Z}^{\binom{n}{2}} \xrightarrow{\iota_1} G_n \xrightarrow{\pi_1} S_n \longrightarrow 1$$

be the group extension where the action of S_n on $\mathbb{Z}^{\binom{n}{2}}$ is determined by the conjugation of

pure braids by half twists in B_n. The cocycle, ϕ, defined by:

$$\phi(\tilde{c}_{i,j}) = g_{i,j}$$

$$\phi(\tilde{d}_{i,j,k,\ell}) = \begin{cases} g_{i,k} - g_{i,\ell} - g_{k,j} + g_{j,\ell} & i < k < j < \ell \\ -g_{i,k} + g_{k,j} + g_{i,\ell} - g_{\ell,j} & k < i < \ell < j \\ 0 & otherwise \end{cases}$$

$$\phi(\tilde{e}_{i,k,j}) = \begin{cases} g_{i,j} - g_{k,j} & i < k < j, \ j < i < k, \ k < j < i \\ 0 & otherwise \end{cases}$$

is a representative for the cohomology class of $H^2(S_n; \mathbb{Z}^{\binom{n}{2}})$ corresponding to this extension.

Notation The image of ϕ is contained in the abelian group $\mathbb{Z}^{\binom{n}{2}}$ with additive notation. However the extension G_n non-abelian. The computations in the following proof use additive and multiplicative notation to distinguish between the abelian and non-abelian settings.

Proof. By Theorem 5.3 we have

$$\phi(\tilde{c}_{i,j}) = s(\sigma_{i,j})s(\sigma_{i,j})$$

$$= \tilde{\sigma}_{i,j}\tilde{\sigma}_{i,j}$$

$$= g_{i,j}.$$

Without loss of generality assume $i < j$ and $k < \ell$, then consider $\tilde{d}_{i,j,k,\ell}$. By Lemma 4.8 $s(\sigma_{k,\ell}\sigma_{i,j}) = s(\sigma_{i,j}\sigma_{k,\ell})$ depends on $\max\{j, \ell\}$. If $j < \ell$, then by Lemma 6.4,

$s(\sigma_{k,\ell}\sigma_{i,j}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell}$. Therefore by Theorem 5.3 and relation R5

$$\phi(\tilde{d}_{i,j,k,\ell}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell})^{-1} - \tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell})^{-1}$$

$$= -\tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell}^{-1}\tilde{\sigma}_{i,j}^{-1}$$

$$= -\begin{cases} g_{i,k}^{-1}g_{i,\ell}g_{k,j}g_{j,\ell}^{-1} & i < k < j < \ell \\ 0 & \text{otherwise} \end{cases}$$

$$= -\begin{cases} -g_{i,k} + g_{i,\ell} + g_{k,j} - g_{j,\ell} & i < k < j < \ell \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} g_{i,k} - g_{i,\ell} - g_{k,j} + g_{j,\ell} & i < k < j < \ell \\ 0 & \text{otherwise} \end{cases}$$

For $\phi(\tilde{d}_{i,j,k,\ell})$, it remains to show the case $\ell < j$. By Lemma 6.4, Theorem 5.3, and R5

$$\phi(\tilde{d}_{i,j,k,\ell}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell}(\tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j})^{-1} - \tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j}(\tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j})^{-1}$$

$$= \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,\ell}\tilde{\sigma}_{i,j}^{-1}\tilde{\sigma}_{k,\ell}^{-1}$$

$$= \begin{cases} g_{k,i}^{-1}g_{k,j}g_{i,\ell}g_{\ell,j}^{-1} & k < i < \ell < j \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} -g_{k,i} + g_{k,j} + g_{i,\ell} - g_{\ell,j} & k < i < \ell < j \\ 0 & \text{otherwise} \end{cases}$$

By Lemma 6.5, to determine $\phi(\tilde{e}_{i,j,k})$, there are three cases for the normal form of the permutation $\sigma_{i,k}\sigma_{i,j} = \sigma_{i,j}\sigma_{j,k}$ which depend on $\max\{i, k, j\}$. Suppose $\max\{i, k, j\} = j$,

then by Theorem 5.3 and Lemma 6.5 we have:

$$\phi(\tilde{e}_{i,k,j}) = s(\sigma_{i,j})s(\sigma_{j,k})s(\sigma_{i,j}\sigma_{k,j})^{-1} - s(\sigma_{i,k})s(\sigma_{i,j})s(\sigma_{i,k}\sigma_{i,j})^{-1}$$

$$= \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,j}(\tilde{\sigma}_{i,k}\tilde{\sigma}_{i,j})^{-1} - \tilde{\sigma}_{i,k}\tilde{\sigma}_{i,j}(\tilde{\sigma}_{i,k}\tilde{\sigma}_{i,j})^{-1}$$

$$= \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,j}\tilde{\sigma}_{i,j}^{-1}\tilde{\sigma}_{i,k}^{-1}.$$

By relations R3 and R4 together with Theorem 6.3 we get:

$$\phi(\tilde{e}_{i,k,j}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{k,j}\tilde{\sigma}_{i,j}^{-1}\tilde{\sigma}_{i,k}^{-1}$$

$$= \begin{cases} \tilde{\sigma}_{i,k}\tilde{\sigma}_{i,k}^{-1} & k < i < j \\[2mm] g_{i,j}\tilde{\sigma}_{i,j}^{-1}\tilde{\sigma}_{k,j}\tilde{\sigma}_{i,j}g_{i,j}^{-1}\tilde{\sigma}_{i,k}^{-1} & i < k < j \end{cases}$$

$$= \begin{cases} 0 & k < i < j \\[2mm] g_{i,j}\tilde{\sigma}_{i,k}g_{i,j}^{-1}\tilde{\sigma}_{i,k}^{-1} & i < k < j \end{cases}$$

$$= \begin{cases} 0 & k < i < j \\[2mm] g_{i,j}g_{k,j}^{-1} & i < k < j \end{cases}$$

$$= \begin{cases} 0 & k < i < j \\[2mm] g_{i,j} - g_{k,j} & i < k < j \end{cases}$$

Hence we are done if $\max\{i, k, j\} = j$.

Now, let $\max\{i, k, j\} = k$, then by Lemma 6.5 $s(\sigma_{i,k}\sigma_{i,j}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{j,k}$. Therefore, as above we have

$$\phi(\tilde{e}_{i,k,j}) = \tilde{\sigma}_{i,j}\tilde{\sigma}_{j,k}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{j,k})^{-1} - \tilde{\sigma}_{i,k}\tilde{\sigma}_{i,j}(\tilde{\sigma}_{i,j}\tilde{\sigma}_{j,k})^{-1}$$

$$= -\tilde{\sigma}_{i,k}\tilde{\sigma}_{i,j}\tilde{\sigma}_{j,k}^{-1}\tilde{\sigma}_{i,j}^{-1}.$$

Now, if $i < j < k$ then R3 applies to $\tilde{\sigma}_{i,j}\tilde{\sigma}_{j,k}^{-1}\tilde{\sigma}_{i,j}^{-1}$ and if $j < i < k$ then R4 applies:

$$\phi(\tilde{e}_{i,k,j}) = - \begin{cases} \tilde{\sigma}_{i,k}\tilde{\sigma}_{i,k}^{-1} & i < j < k \\[2mm] \tilde{\sigma}_{i,k}g_{i,j}\tilde{\sigma}_{i,j}^{-1}\tilde{\sigma}_{j,k}^{-1}\tilde{\sigma}_{i,j}g_{i,j}^{-1} & j < i < k \end{cases}$$

$$= - \begin{cases} 0 & i < j < k \\[2mm] \tilde{\sigma}_{i,k}g_{i,j}\tilde{\sigma}_{i,k}^{-1}g_{i,j}^{-1} & j < i < k \end{cases}$$

$$= - \begin{cases} 0 & i < j < k \\[2mm] g_{i,j}^{-1}g_{k,j} & j < i < k \end{cases}$$

$$= \begin{cases} 0 & i < j < k \\[2mm] g_{i,j} - g_{k,j} & j < i < k \end{cases}$$

It remains to prove the result if $\max\{i,k,j\} = i$. By Lemma 6.5, $\max\{i,k,j\} = i$ implies $s(\sigma_{i,j}\sigma_{i,j}) = \tilde{\sigma}_{k,j}\tilde{\sigma}_{k,i}$. Thus

$$\phi(\tilde{e}_{i,k,j}) = \tilde{\sigma}_{j,i}\tilde{\sigma}_{j,k}(\tilde{\sigma}_{j,k}\tilde{\sigma}_{k,i})^{-1} - \tilde{\sigma}_{k,i}\tilde{\sigma}_{j,i}(\tilde{\sigma}_{j,k}\tilde{\sigma}_{k,i})^{-1}$$

$$= \tilde{\sigma}_{j,i}\tilde{\sigma}_{j,k}\tilde{\sigma}_{k,i}^{-1}\tilde{\sigma}_{j,k}^{-1} - \tilde{\sigma}_{k,i}\tilde{\sigma}_{i,j}\tilde{\sigma}_{k,i}^{-1}\tilde{\sigma}_{j,k}^{-1}.$$

If $j < k < i$, then relation R3 applies to $\tilde{\sigma}_{j,k}\tilde{\sigma}_{k,i}^{-1}\tilde{\sigma}_{j,k}^{-1}$ while R4 applies if $k < j < i$. Therefore,

$$\tilde{\sigma}_{j,i}\tilde{\sigma}_{j,k}\tilde{\sigma}_{k,i}^{-1}\tilde{\sigma}_{j,k}^{-1} = \begin{cases} \tilde{\sigma}_{j,i}\tilde{\sigma}_{j,i}^{-1} & j < k < i \\[2mm] \tilde{\sigma}_{j,i}g_{k,j}\tilde{\sigma}_{k,j}^{-1}\tilde{\sigma}_{k,i}^{-1}\tilde{\sigma}_{k,j}g_{k,j}^{-1} & k < j < i \end{cases}$$

$$= \begin{cases} 0 & j < k < i \\[2mm] \tilde{\sigma}_{j,i}g_{k,j}\tilde{\sigma}_{j,i}^{-1}g_{k,j}^{-1} & k < j < i \end{cases}$$

$$= \begin{cases} 0 & j < k < i \\[2mm] g_{k,i}g_{k,j}^{-1} & k < j < i \end{cases}$$

Furthermore, if $j < k < i$ then R3 applies to $\tilde{\sigma}_{k,i}\tilde{\sigma}_{j,i}\tilde{\sigma}_{k,i}^{-1}$ while R4 applies if $k < j < i$. Hence

$$\tilde{\sigma}_{k,i}\tilde{\sigma}_{j,i}\tilde{\sigma}_{k,i}^{-1}\tilde{\sigma}_{j,k}^{-1} = \begin{cases} \tilde{\sigma}_{j,k}\tilde{\sigma}_{j,k}^{-1} & j < k < i \\[2mm] g_{k,i}\tilde{\sigma}_{k,i}^{-1}\tilde{\sigma}_{j,i}\tilde{\sigma}_{k,i}g_{k,i}^{-1}\tilde{\sigma}_{k,j}^{-1} & k < j < i \end{cases}$$

$$= \begin{cases} 0 & j < k < i \\[2mm] g_{k,i}\tilde{\sigma}_{k,j}g_{k,i}^{-1}\tilde{\sigma}_{k,j}^{-1} & k < j < i \end{cases}$$

$$= \begin{cases} 0 & j < k < i \\[2mm] g_{k,i}g_{j,i}^{-1} & k < j < i \end{cases}$$

Therefore:

$$\phi(\tilde{e}_{i,k,j}) = \begin{cases} 0 - 0 & j < k < i \\ g_{k,i}g_{k,j}^{-1} - g_{k,i}g_{j,i}^{-1} & k < j < i \end{cases}$$

$$= \begin{cases} 0 & j < k < i \\ g_{k,i} - g_{k,j} - (g_{k,i} - g_{j,i}) & k < j < i \end{cases}$$

$$= \begin{cases} 0 & j < k < i \\ g_{j,i} - g_{k,j} & k < j < i \end{cases}$$

$\square$

6.3 ORDER OF COHOMOLOGY CLASS

To determine the order of $[\phi]$ we build the extensions G_n^2 of S_n by $\mathbb{Z}^{\binom{n}{2}}$ which correspond to the class of $2 \cdot [\phi]$ and show that G_n^2 also corresponds to the trivial cohomology class. To construct G_n^2 we use the pull-back construction described in Adem and Milgram's text [1].

Definition Let E_1 and E_2 be extensions of Q by K with projections maps $\pi_1 : E_1 \to Q$ and $\pi_2 : E_2 \to Q$. Define the *pull-back* of E_1 and E_2 to be the set $E_1 \times_Q E_2 = \{(e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2)\}$. A proof the pull-back construction is a group can be found on page 20 of [1]. If $\iota_1 : K \to E_1$ and $\iota_2 : K \to E_2$ are the inclusion maps, define $\Delta_K = \{(\iota_1(k), \iota_2(k)^{-1}) \in E_1 \times E_2 \mid k \in K\}$ to be the *skew diagonal* of the pull-back.

Remark Notice that Δ_K is a subgroup of $E_1 \times_Q E_2$, however Δ_K may not be normal. If both the kernel and cokernel of the extension are abelian, then the quotient of the pull-back by the skew diagonal of the kernel is a group extension and referred to as the *Baer sum* of two extensions. In the abelian setting, the set of equivalence classes of group

extensions form a commutative group with Baer sums as the binary operator. In this section, we will describe the pull-back of G_n with itself and prove the quotient by the skew diagonal is an extension of the symmetric group with a corresponding cohomology class represented by a multiple of ϕ.

Lemma 6.7. *Suppose E_1 and E_2 are extensions of Q by an abelian group K. If E_1 and E_2 give rise to the same action of Q on K, then the skew diagonal of the pull-back is a normal subgroup.*

Proof. Suppose E_1 and E_2 are extensions of Q by K which give rise to the action $\theta : Q \to \text{Aut}(K)$. Let $(e_1, e_2) \in E_1 \times_Q E_2$ and $(\iota_1(k), \iota_2(k)^{-1}) \in \Delta_K$. Since multiplication in $E_1 \times_Q E_2$ is defined coordinate-wise:

$$(e_1, e_2)(\iota_1(k), \iota_2(k)^{-1})(e_1, e_2)^{-1} = (e_1 \iota_1(k) e_1^{-1}, e_2 \iota_2(k)^{-1} e_2^{-1}).$$

Since $(e_1, e_2) \in E_1 \times_Q E_2$, $\pi_1(e_1) = \pi_2(e_2)$. Furthermore, since E_1 and E_2 give rise to θ, $\theta(\pi_1(e_1)) = \theta(\pi_2(e_2))$. Let $k' = \theta_{\pi_1(e_1)}$, then:

$$
\begin{aligned}
(e_1 \iota_1(k) e_1^{-1}, e_2 \iota_2(k)^{-1} e_2^{-1}) &= (e_1 \iota_1(k) e_1^{-1}, (e_2 \iota_2(k) e_2^{-1})^{-1}) \\
&= \left(\iota_1(\theta_{\pi_1(e_1)}(k)), \iota_2(\theta_{\pi_2(e_2)}(k))^{-1} \right) \\
&= \left(\iota_1(\theta_{\pi_1(e_1)}(k)), \iota_2(\theta_{\pi_1(e_1)}(k))^{-1} \right) \\
&= (\iota_1(k'), \iota_2(k')^{-1}).
\end{aligned}
$$

Since $k' \in K$, $(\iota_1(k'), \iota_2(k')^{-1}) \in \Delta_K$ and by definition Δ_K is a normal subgroup of $E_1 \times_Q \times E_2$. $\square$

The previous lemma gives a necessary condition for the skew diagonal to be a normal subgroup of the pull-back for non-abelian kernels. The following Lemma shows the quotient of the pull-back by the skew diagonal is an extension if the skew diagonal is normal.

Lemma 6.8. *Let E_1 and E_2 be extensions of Q by K. If Δ_K is a normal subgroup of $E_1 \times_Q E_2$, then $(E_1 \times_Q E_2)/\Delta_K$ is an extension of Q by K.*

Proof. Suppose Δ_K is a normal subgroup of $E_1 \times_Q E_2$. Define $\iota : K \to E_1 \times_Q E_2$ by $\iota(k) = (\iota_1(k), 1_{E_2})$ and $\pi : E_1 \times_Q E_2 \to Q$ by $\pi((e_1, e_2)) = \pi_1(e_1) = \pi_2(e_2)$. By construction $\pi \circ \iota(k) = \pi_1(\iota_1(k))$. Since E_1 is an extension of Q by K, $\pi_1(\iota_1(k)) = 1_Q$ for all $k \in K$. Therefore $\pi \circ \iota(k) = 1_Q$ for all $k \in K$ and $\operatorname{im} \iota \subseteq \ker \pi$. It remains to show $\ker \pi \subseteq \operatorname{im} \iota$.

Suppose $(e_1, e_2) \in \ker \pi$. Then $e_1 \in \ker \pi_1$ and, since $\pi_1(e_1)\pi_2(e_2)$, $e_2 \in \ker \pi_2$. Since E_1 and E_2 are extensions of Q by K, there exists $k_1, k_2 \in K$ such that $\iota_1(k_1) = e_1$ and $\iota_2(k_2) = e_2$. Since K embeds into both E_1 and E_2:

$$
\begin{aligned}
(\iota_1(k_1), \iota_2(k_2)) &= (\iota_1(k_1) \cdot \iota_1(k_2) \cdot \iota_1(k_2)^{-1}, 1_{E_2} \cdot \iota_2(k_2)) \\
&= (\iota_1(k_1) \cdot \iota_1(k_2), 1_{E_2}) \cdot (\iota_1(k_2)^{-1}, \iota_2(k_2)) \\
&= (\iota_1(k_1 k_2), 1_{E_2}) \cdot (\iota_1(k_2)^{-1}, \iota_2(k_2)) \\
&= \iota(k_1 k_2).
\end{aligned}
$$

Therefore $(e_1, e_2) = \iota(k_1 k_2)$ in $(E_1 \times_Q E_2)/\Delta_K$. So $\ker \pi \subseteq \operatorname{im} \iota$ and $(E_1 \times_Q E_2)/\Delta_K$ is an extension of Q by K. $\qquad \square$

Remark Let G_n^2 to be the quotient of the pull-back $G_n \times_{S_n} G_n$ by the skew diagonal, Δ, of $\mathbb{Z}^{\binom{n}{2}}$. Lemmas 6.7 and 6.8 imply G_n^2 is an extension of S_n by $\mathbb{Z}^{\binom{n}{2}}$. Next we will show G_n^2 corresponds to the cohomology class $2[\phi]$ in $H^2(\mathbb{Z}^{\binom{n}{2}})$.

Recall $\iota_1 : \mathbb{Z}^{\binom{n}{2}} \to G_n$ is defined by $\iota_1(g_{i,j}) = g_{i,j}$ while $\pi_1 : G_n \to S_n$ is defined by $\pi_1(\tilde{\sigma}_{i,j}) = \sigma_{i,j}$ and $\pi_1(g_{i,j}) = 1_{S_n}$ for all $1 \le i < j \le n$. Furthermore, in Section 6.2 we chose the section $s : S_n \to G_n$ which lifted permutations by rewriting them as a product of 2-cycles, then lifting each 2-cycle to the projection of the corresponding half twist in G_n.

Theorem 6.9. *Let ϕ be the representative of the cohomology class corresponding to G_n constructed in Theorem 6.9. G_n^2 corresponds to the cohomology class $2[\phi]$ in $H^2(S_n; \mathbb{Z}^{\binom{n}{2}})$.*

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Proof. Consider the pull-back $G_n \times_{S_n} G_n$ with the skew diagonal

$\Delta = \{(\iota(k), \iota(k)^{-1}) \in G_n \times G_n \mid k \in \mathbb{Z}^{\binom{n}{2}}\}$. By Lemma 6.7, Δ is a normal subgroup of

$G_n \times_{S_n} G_n$. Furthermore, $G_n^2 = (G_n \times_{S_n} G_n)/\Delta$ is an extension of S_n by $\mathbb{Z}^{\binom{n}{2}}$ by Lemma

6.7. Recall from Section 3.2, the representative for the corresponding cocycle of an

extension depends on the chosen section from the quotient group to the extension. For any

$p \in S_n$, define $s_2 : S_n \to G_n^2$ by:

$$s_2(p) = (s(p), s(p))$$

where $s : S_n \to G_n$ is from Section 6.2. Applying Theorem 5.3, the corresponding

cohomology class for G_n^2, denoted ϕ_2, is determined by the image of $\tilde{c}_{i,j}$, $\tilde{d}_{i,j,k,\ell}$, and $\tilde{e}_{i,j,k}$.

Consider elements of $\mathbb{Z}^{\binom{n}{2}}$ under the inclusion to G_n and computing the formula from

Theorem 5.3, we get:

$$\phi_2(\tilde{c}_{i,j}) = (\phi(\tilde{c}_{i,j}), \phi(\tilde{c}_{i,j}))$$

$$\phi_2(\tilde{d}_{i,j,k,\ell}) = (\phi(\tilde{d}_{i,j,k,\ell}), \phi(\tilde{d}_{i,j,k,\ell}))$$

$$\phi_2(\tilde{e}_{i,j,k}) = (\phi(\tilde{e}_{i,j,k}), \phi(\tilde{e}_{i,j,k}))$$

For any $(g, h) \in G_n^2$, notice:

$$(g, h) = (gh, 1_{G_n}) \cdot (h^{-1}, h).$$

If $h \in \iota(\mathbb{Z}^{\binom{n}{2}})$, then (h^{-1}, h) is equivalent to $(1_{G_n}, 1_{G_n})$ in G_n^2 and $(g, h) = (gh, 1_{G_n})$. In

particular, $\phi(x) \in \mathbb{Z}^{\binom{n}{2}}$ for all $x \in \{\tilde{c}_{i,j},\ \tilde{d}_{i,j,k,\ell},\ \tilde{e}_{i,j,k}\}$. From the proof of Lemma 6.8,

$\iota_2 : \mathbb{Z}^{\binom{n}{2}}$ is defined by $\iota_2(k) = (k, 1_{G_n})$. Furthermore:

$$\phi_2(x) = (\phi(x), \phi(x))$$
$$= (\phi(x)^2, 1_{G_n})$$
$$= \iota_2(\phi(x))^2$$
$$= \iota_2(\phi(x) + \phi(x)).$$

In $H^2(S_n; \mathbb{Z}^{\binom{n}{2}})$, $[\phi_2] = [\phi + \phi] = 2[\phi]$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark It remains to show $[\phi_2]$ is equivalent to the trivial cohomology class. In Section 6.1 we chose the lift of $\sigma_{i,j}$ to be $\tilde{\sigma}_{i,j}$, the image of $b_{i,j}$ in $B_n/[PB_n, PB_n]$, where $b_{i,j} = b_i \cdots b_{j-2} b_{j-1} b_{j-2}^{-1} \cdots b_i^{-1}$. As an element of the braid group, $b_{i,j}$ pulls the ith strand under all strands between i and j, rotates the ith and j^{th} strands clockwise, then pulls the jth strand under the other strands until it reaches the ith strand's starting position.

Alternatively, consider the choice $\mathfrak{b}_{i,j} = b_i^{-1} \cdots b_{j-2}^{-1} b_{j-1} b_{j-2} \cdots b_i^{-1}$. Notice that $\mathfrak{b}_{i,j}$ is the half twist between the ith and j^{th} strands which pulls the ith strand over all strands between i and j, rotates the ith and j^{th} strands clockwise, then pulls the jth strand over the other strands until it reaches the ith strand's starting position. In particular, $\mathfrak{b}_{i,j}^{-1}$ is the braid in which the strands cross in the same order as $b_{i,j}$ with reverse orientation. Let $\tilde{\lambda}_{i,j}$ be the projection of $\mathfrak{b}_{i,j}$ in G_n. Consider the section $\mathfrak{s} : S_n \to G_n$ defined by $\mathfrak{s}(\sigma_{i,j}) = \tilde{\lambda}_{i,j}^{-1}$ with the same choice of normal forms as in Section 6.2.

Theorem 6.10. *G_n^2 corresponds to the trivial cohomology class in $H^2(S_n; \mathbb{Z}^{\binom{n}{2}})$.*

Proof. Since the representative for the corresponding cohomology class is determined by the choice of section from S_n to G_n^2, we define a section $\tilde{s} : S_n \to G_n^2$ which determines a cocycle, τ, which is trivial on the generators $\tilde{c}_{i,j}$, $\tilde{d}_{i,j,k,\ell}$, and $\tilde{e}_{i,j,k}$. Define $\tilde{s}$ by:

$$\tilde{s}(\sigma_{i,j}) = (\tilde{\sigma}_{i,j}, \tilde{\lambda}_{i,j}^{-1})$$

Applying Theorem 5.3 with Lemmas 6.4 and 6.5 we get:

$$\tau(\tilde{c}_{i,j}) = (\tilde{\sigma}_{i,j}^2, \ \tilde{\lambda}_{i,j}^{-2})$$

$$\tau(\tilde{d}_{i,j,k,\ell}) = ([\tilde{\sigma}_{i,j}, \tilde{\sigma}_{k,\ell}], \ [\tilde{\lambda}_{i,j}^{-1}, \tilde{\lambda}_{k,\ell}^{-1}])$$

$$\tau(\tilde{e}_{i,j,k}) = (\tilde{\sigma}_{i,j}\tilde{\sigma}_{j,k}\tilde{\sigma}_{i,j}^{-1}\tilde{\sigma}_{i,k}^{-1}, \ \tilde{\lambda}_{i,j}^{-1}\tilde{\lambda}_{j,k}^{-1}\tilde{\lambda}_{i,j}\tilde{\lambda}_{i,k})$$

Since $\tilde{\lambda}_{i,j}$ reverses the orientation of every crossing for $\tilde{\sigma}_{i,j}$, the lift or relations in S_n under $\mathfrak{s}$ yields the same product of pure braids as s with reverse orientation at the crossing of every strand. Since the pure braids commute in G_n, reversing the orientation of every crossing of a pure braid yields the inverse element. Therefore we have the following:

$$\tau(x) = (\phi(x), -\phi(x))$$

for all $x \in \{\tilde{c}_{i,j}, \tilde{d}_{i,j,k,\ell}, \tilde{e}_{i,j,k}\}$. In particular, $\tau(x) = 0$ for all $x \in \{\tilde{c}_{i,j}, \tilde{d}_{i,j,k,\ell}, \tilde{e}_{i,j,k}\}$ since $(\phi(x), -\phi(x)) \in \Delta$. Therefore $[\tau]$ is the trivial class in $H^2(S_n; \mathbb{Z}^{\binom{n}{2}})$. Thus G_n^2 corresponds to the trivial cohomology class. $\square$

Remark Since τ and ϕ_2 correspond to G_n but are constructed from different choices of sections, they represent the same cohomology class in $H^2(S_n; \mathbb{Z}^{\binom{n}{2}})$. From Theorem 6.9 we get $[\phi_2] = 2[\phi]$ where $[\phi]$ corresponds to G_n. Since τ represents the trivial cohomology class and $[\tau] = [\phi_2]$, $[\phi]$ is an order two element of $H^2(S_n; \mathbb{Z}^{\binom{n}{2}})$.

7   Mod 4 Braid Group

Recall the mod 4 braid group, $\mathcal{Z}_n$, is the quotient of the braid group by the level 4 braid group, $B_n/B_n[4]$. Furthermore, Brendle and Margalit proved the image of $B_n[4]$ in PB_n, denoted $\mathcal{PZ}_n$, is isomorphic to $\mathbb{Z}^{\binom{n}{2}}$ (recall Section 4.2). Therefore we have the following short exact sequence:

$$0 \longrightarrow \mathbb{Z}_2^{\binom{n}{2}} \xrightarrow{\;\iota\;} \mathcal{Z}_n \xrightarrow{\;\pi\;} S_n \longrightarrow 1.$$

Note the action of S_n on $\mathbb{Z}_2^{\binom{n}{2}}$ is induced by the conjugation action of B_n on PB_n. Since, $\mathbb{Z}_2^{\binom{n}{2}} \approx \mathcal{PZ}_n$, each $\bar{g}_{ij}$ in (1) corresponds to the pure braid between strands i and j in $PB_n/B_n[4]$.

7.1   Classification by Cohomology Class

Theorem 7.1. *Let H_n be the subgroup of G_n normally generated by $\{g_{i,j}^2\}_{1 \leq i < j \leq n}$. Then $G_n/H_n \approx \mathcal{Z}_n$.*

Proof. Since H_n is generated by $g_{i,j}^2$ for $1 \leq i < j \leq n$ and $g_{i,j}$ corresponds to the pure braid between the ith and j^{th} strands, H_n is the subgroup of G_n generated by squares of standard generators of pure braids. Since the $g_{i,j}$'s commute in G_n, the image of PB_n is an abelian subgroup of G_n. Therefore the set of all squares of pure braids in G_n is generated by squares of generators of the pure braid group. Since the image of $B_n[4]$ in PB_n is the subgroup generated by squares of all elements, H_n is the image of $B_n[4]$ in G_n. Thus $G_n/H_n \approx \mathcal{Z}_n$. $\qquad\qquad\square$

Presentation of $\mathcal{Z}_n$ Let $\bar{\sigma}_{i,j}$ be the image of $b_{i,j}$ in $\mathcal{Z}_n$. Since A_n is the image of $B_n[4]$ in G_n, we get a presentation for $\mathcal{Z}_n$ by adding the relation $g_{i,j}^2 = 1$ to the presentation of G_n. Furthermore, the projection of $g_{i,j}$ to $\mathcal{Z}_n$ induces the mod 2 reduction of integers. Therefore $\bar{g}_{i,j}$ represents the image of a pure braid between the ith and j^{th} strands in $\mathcal{Z}_n$.

$\mathfrak{R}0$: $\bar{g}_{i,j}^2 = 1$ for all i, j

$\mathfrak{R}1$: $[\bar{g}_{i,j}, \bar{g}_{k,\ell}] = 1$ for all i, j, k, ℓ

$\mathfrak{R}2$: $\bar{\sigma}_{i,j}^2 = \bar{g}_{i,j}$ for all i, j

$\mathfrak{R}3$: $\bar{\sigma}_{i,j}\bar{\sigma}_{k,j}\bar{\sigma}_{i,j}^{-1} = \bar{\sigma}_{i,k}$ if $k < i < j;$ $i < j < k;$ or $j < k < i$

$\mathfrak{R}4$ $\bar{\sigma}_{i,j}^{-1}\bar{\sigma}_{j,k}\bar{\sigma}_{i,j} = \bar{\sigma}_{i,k}$ if $i < k < j;$ $j < i < k;$ or $k < j < i$

$\mathfrak{R}5$ $[\bar{\sigma}_{i,j}, \bar{\sigma}_{k,\ell}] = \begin{cases} \bar{g}_{i,k}\bar{g}_{i,\ell}\bar{g}_{j,k}\bar{g}_{j,\ell} & i < k < j < \ell \text{ or } k < i < \ell < j \\ 1 & \text{otherwise} \end{cases}$

$\mathfrak{R}6$: $\bar{\sigma}_{i,j}\bar{g}_{k,\ell}\bar{\sigma}_{i,j}^{-1} = \bar{g}_{\sigma_{i,j}(k), \sigma_{i,j}(\ell)}$ for all $i, j, k, \ell \in \{1, \ldots, n\}$

Therefore we get the following generating set for $\mathcal{Z}_n$:

$$\{\bar{g}_{i,j}\} \cup \{\bar{\sigma}_{i,j}\}.$$

where $1 \leq i < j \leq n$. A full list of relations is given in Table 2. Notice that the relation $\mathfrak{R}4$ can be determined by relation $\mathfrak{R}3$. The following theorem completes the proofs of both Theorem 1.1 and Theorem 1.2.

Theorem 7.2. *Let $\kappa \in Hom(R_2, \mathbb{Z}_2^{\binom{n}{2}})$ be the represenative for the cohomology class in $H^2(S_n; \mathbb{Z}_2^{\binom{n}{2}})$ corresponding to $\mathcal{Z}_n$ as an extension of S_n by $\mathbb{Z}_2^{\binom{n}{2}}$ given by the usual construction. Let $\eta : \mathbb{Z}^{\binom{n}{2}} \to \mathbb{Z}_2^{\binom{n}{2}}$ be the mod 2 reduction of integers. Then $\kappa = \eta \circ \phi$ where ϕ is the representative for G_n defined in Theorem 6.6.*

Proof. Define $f : G_n \to \mathcal{Z}_n$ by $f(\tilde{\sigma}_{i,j}) = \bar{\sigma}_{i,j}$ and $f(g_{i,j}) = \bar{g}_{i,j}$ for all i, j such that $1 \leq i < j \leq n$. Then f is a well defined group homomorphism which commutes with the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{\binom{n}{2}} & \xrightarrow{\iota_1} & G_n & \xrightarrow{\pi_1} & S_n & \longrightarrow & 1 \\
& & \Big\downarrow{\eta} & & \Big\downarrow{f} & & \Big\downarrow{\text{id}} & & \\
0 & \longrightarrow & \mathbb{Z}_2^{\binom{n}{2}} & \xrightarrow{\iota} & \mathcal{Z}_n & \xrightarrow{\pi} & S_n & \longrightarrow & 1
\end{array}
$$

Choose a normalized section, $s' : S_n \to \mathcal{Z}_n$ by $s'(\sigma_{i,j}) = \bar{\sigma}_{i,j}$. Notice $s'(\sigma_{i,j}) = f \circ s(\sigma_{i,j})$. By Theorem 5.3

$$\kappa(\tilde{c}_{i,j}) = s'(\sigma_{i,j})s'(\sigma_{i,j})$$

$$\kappa(\tilde{d}_{i,j,k,\ell}) = = s'(\sigma_{i,j})s'(\sigma_{k,\ell})s'(\sigma_{i,j}\sigma_{k,\ell})^{-1} - s'(\sigma_{k,\ell})s'(\sigma_{i,j})s'(\sigma_{k,\ell}\sigma_{i,j})^{-1}$$

$$\kappa'(\tilde{e}_{i,k,j}) = s'(\sigma_{i,j})s'(\sigma_{j,k})s'(\sigma_{i,j}\sigma_{j,k})^{-1} - s'(\sigma_{i,k})s'(\sigma_{i,j})s'(\sigma_{i,k}\sigma_{i,j})^{-1}$$

So $\kappa = \phi \circ f$. Furthermore, f induces the mod 2 reduction of integers on the cohomology groups. $\qquad\square$

7.2   Normal generating set for the level 4 braid group

We begin with a well known consequence of Schreier's formula which guarantees the existence of a finite generating set for $B_n[4]$. The proof of the following proposition can be found in Lyndon and Schupp's text (page 164 of [13]).

Proposition 7.3. If G is a group with a generating set of size j and H is a subgroup of finite index k, then there exists a finite generating set for H of size at most $k(j-1)+1$.

Note that $\mathcal{Z}_n$ is finite of order $n! \cdot 2^{\binom{n}{2}}$ since $\mathcal{Z}_n$ is a group extension of finite groups. Furthermore, B_n is finitely generated with a generating set of size $n-1$, therefore there exists a generating set for $B_n[4]$ of size at most:

$$n! \cdot 2^{\binom{n}{2}} \cdot (n-2) + 1.$$

Theorem 1.3. *Suppose $n \geq 1$. As a subgroup of B_n, the level 4 braid group is normally generated by elements of the form:*

1. *$[b_i^2, b_{i+1}^2]$ for all $1 \leq i \leq n-1$.*

2. *$[b_i b_{i+1}^2 b_i^{-1}, b_{i+1} b_{i+2}^2 b_{i+1}^{-1}]$ for all $1 \leq i \leq n-3$.*

3. b_i^4 for all $1 \leq i \leq n - 1$.

Proof of Theorem 1.3. Consider the quotient of the Artin presentation of B_n given in (2) by the normal subgroup generated by b_i^4, $[b_i^2, b_{i+1}^2]$, and $[b_{i,i+2}^2, b_{i+1,i+3}^2]$. Then this quotient has a group presentation:

$$\left\langle b_1, \ldots, b_{n-1} \, \middle| \, \begin{array}{l} b_i^4 = 1, \ \ [b_i^2, b_{i+1}^2] = 1, \ \ [b_{i,i+2}^2, b_{i+1,i+3}^2] = 1 \\[2mm] b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \ \ b_i b_j = b_j b_i \text{ if } |i - j| > 1 \end{array} \right\rangle. \tag{7}$$

Now, consider the presentation for $\mathcal{Z}_n$ defined above. Replace $\bar{g}_{i,j}$ with $\bar{\sigma}_{i,j}^2$ for all $1 \leq i < j \leq n$ using relation $\mathfrak{R}2$. Therefore we can reduce the generating set of $\mathcal{Z}_n$ to $\{\bar{\sigma}_{i,j}\}$ where $1 \leq i < j \leq n$. Considering the case of $\mathfrak{R}3$ where $i < j < k$, the set $\{\bar{\sigma}_{i,i+1}\}$ generates the set of all $\bar{\sigma}_{i,j}$'s. Thus we can take a generating set of $\{\bar{\sigma}_{i,i+1}\}_{1 \leq i \leq n-1}$ for $\mathcal{Z}_n$.

Furthermore, replace $\mathfrak{R}0$ and $\mathfrak{R}2$ with $\bar{\sigma}_{i,i+1}^4 = 1$ in $\mathcal{Z}_n$ since since $\bar{\sigma}_{i,j}^2 = \bar{g}_{i,j}$ and $\bar{g}_{i,j}^2 = 1$. By relation $\mathfrak{R}3$ and $\mathfrak{R}4$, we have $\bar{\sigma}_{i,i+1} \bar{\sigma}_{i+1,i+2} \bar{\sigma}_{i,i+1}^{-1} = \bar{\sigma}_{i,i+2}$ and $\bar{\sigma}_{i+1,i+2}^{-1} \bar{\sigma}_{i,i+1} \bar{\sigma}_{i+1,i+2} = \bar{\sigma}_{i,i+2}$ respectively. Therefore:

$$\bar{\sigma}_{i,i+1} \bar{\sigma}_{i+1,i+2} \bar{\sigma}_{i,i+1}^{-1} = \bar{\sigma}_{i+1,i+2}^{-1} \bar{\sigma}_{i,i+1} \bar{\sigma}_{i+1,i+2}$$

and we can replace relations $\mathfrak{R}3$ and $\mathfrak{R}4$ with the relation braid relation. Furthermore, $\mathfrak{R}5$ implies $[\bar{\sigma}_{i,i+1}, \bar{\sigma}_{j,j+1}] = 1$ for $|i - j| > 1$. Replacing $\bar{g}_{i,j}$ with $\bar{\sigma}_{i,j}^2$ in relation $\mathfrak{R}1$ and restricting to $\{\bar{\sigma}_{i,i+1}\}_{1 \leq i \leq n-1}$ we get $[\bar{\sigma}_{i,i+1}^2, \bar{\sigma}_{i+1,i+2}^2] = 1$ and $[\bar{\sigma}_{i,i+2}^2, \bar{\sigma}_{i+1,i+3}^2] = 1$. Thus we get the same presentation as (7) with $\bar{\sigma}_{i,i+1}$ replacing b_i. Since $\mathcal{Z}_n = B_n / B_n[4]$, the relations of (7) which are not relations of (2) normally generate $B_n[4]$ as a subgroup of B_n. □

Remark By Proposition 7.3 we know $B_n[4]$ has a generating set which grows faster than exponentially in n. Brendle and Margalit pose the question [6]: does there exist a generating set of $B_n[4]$ with polynomial growth in n? In future research we hope to build a generating set by taking the normal generating set constructed in Theorem 1.3 and

determine which generators are missing. Cohen, Falk, and Randell proved there exists a surjection from PB_n to the free group of rank 2 [8], therefore the commutator subgroup of PB_n is not finitely generated. In particular, we first must determine the intersection of the subgroup of B_n normally generated by squares of pure braids and the subgroup of B_n normally generated by commutators of pure braids.
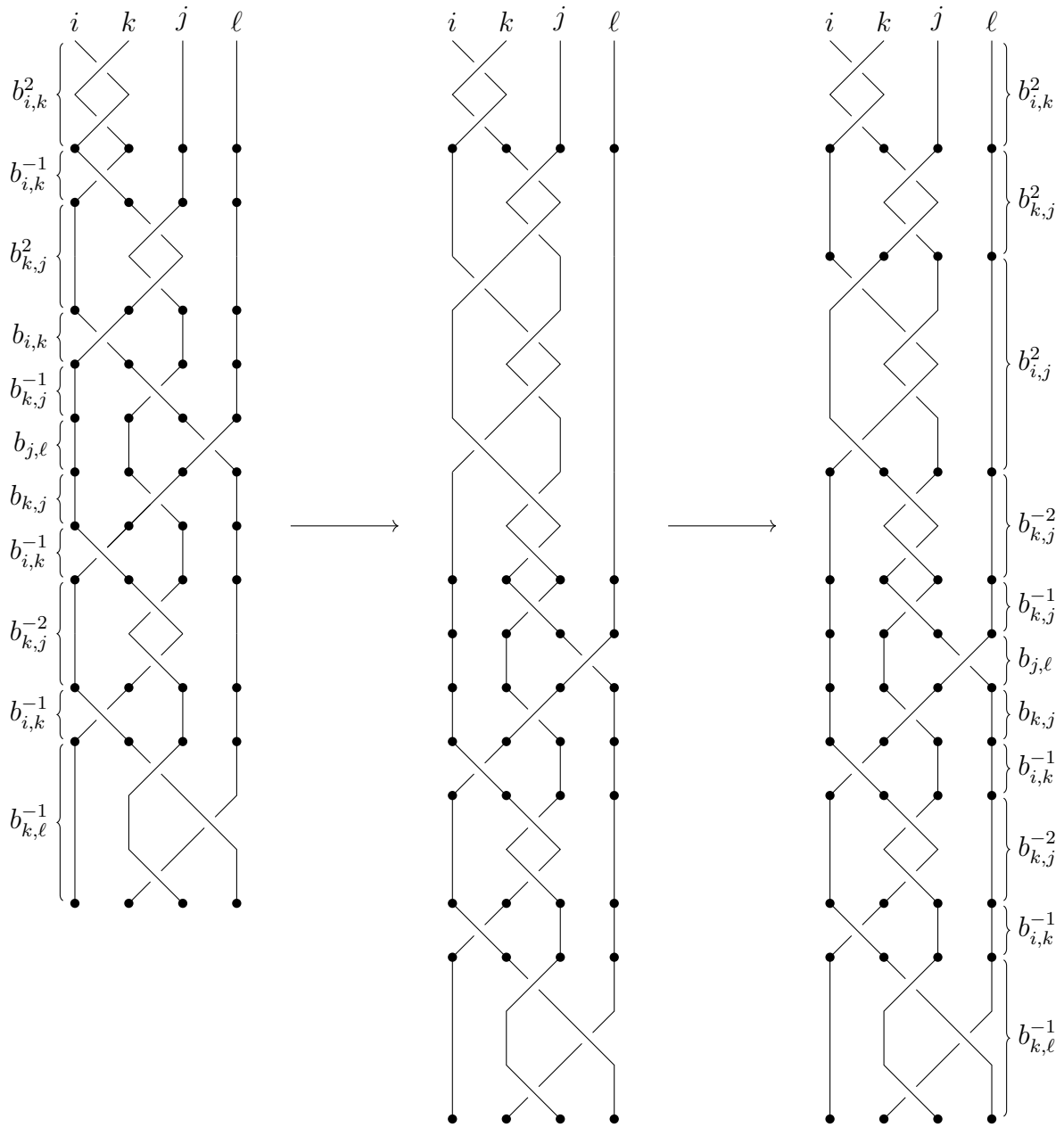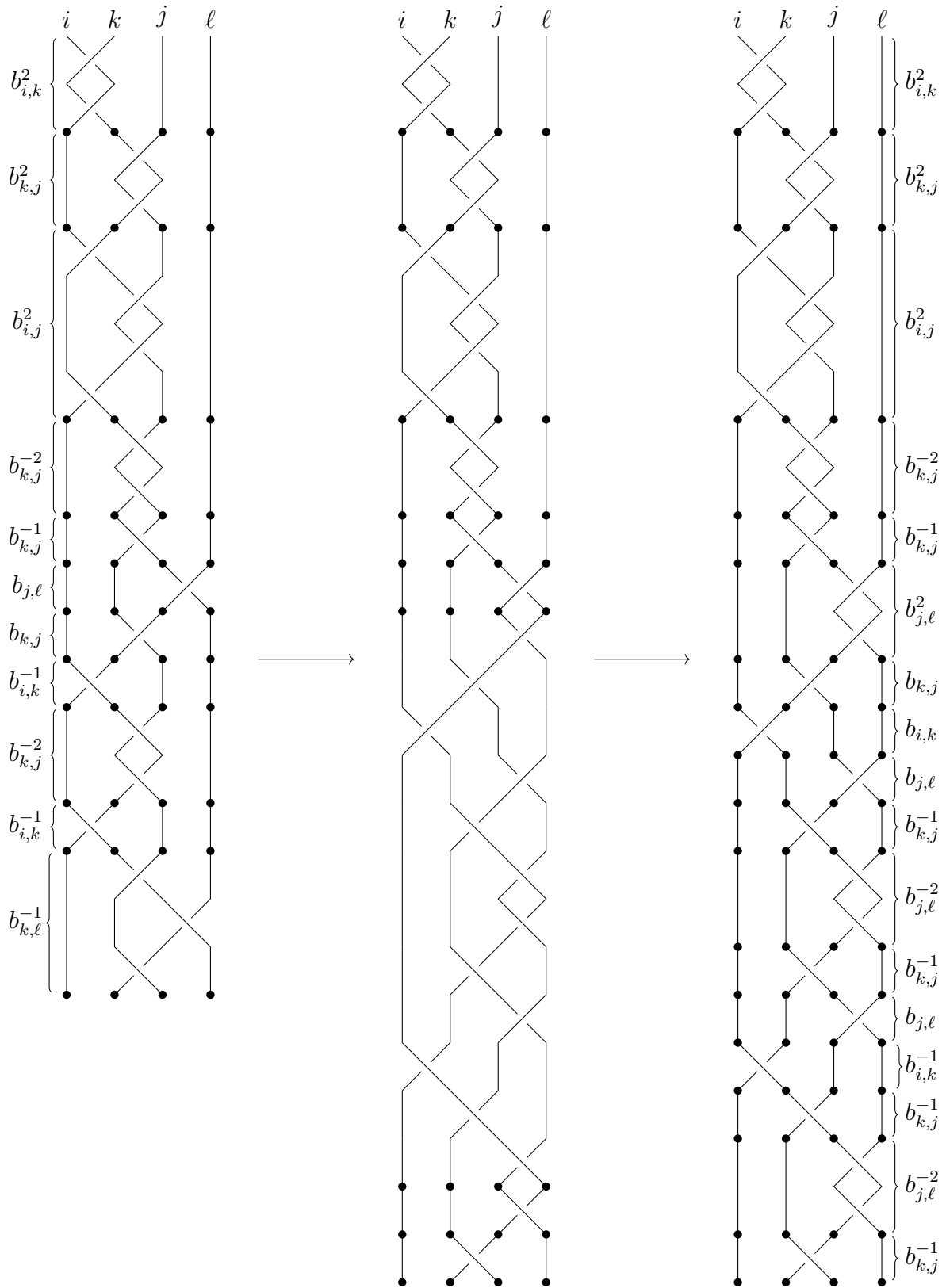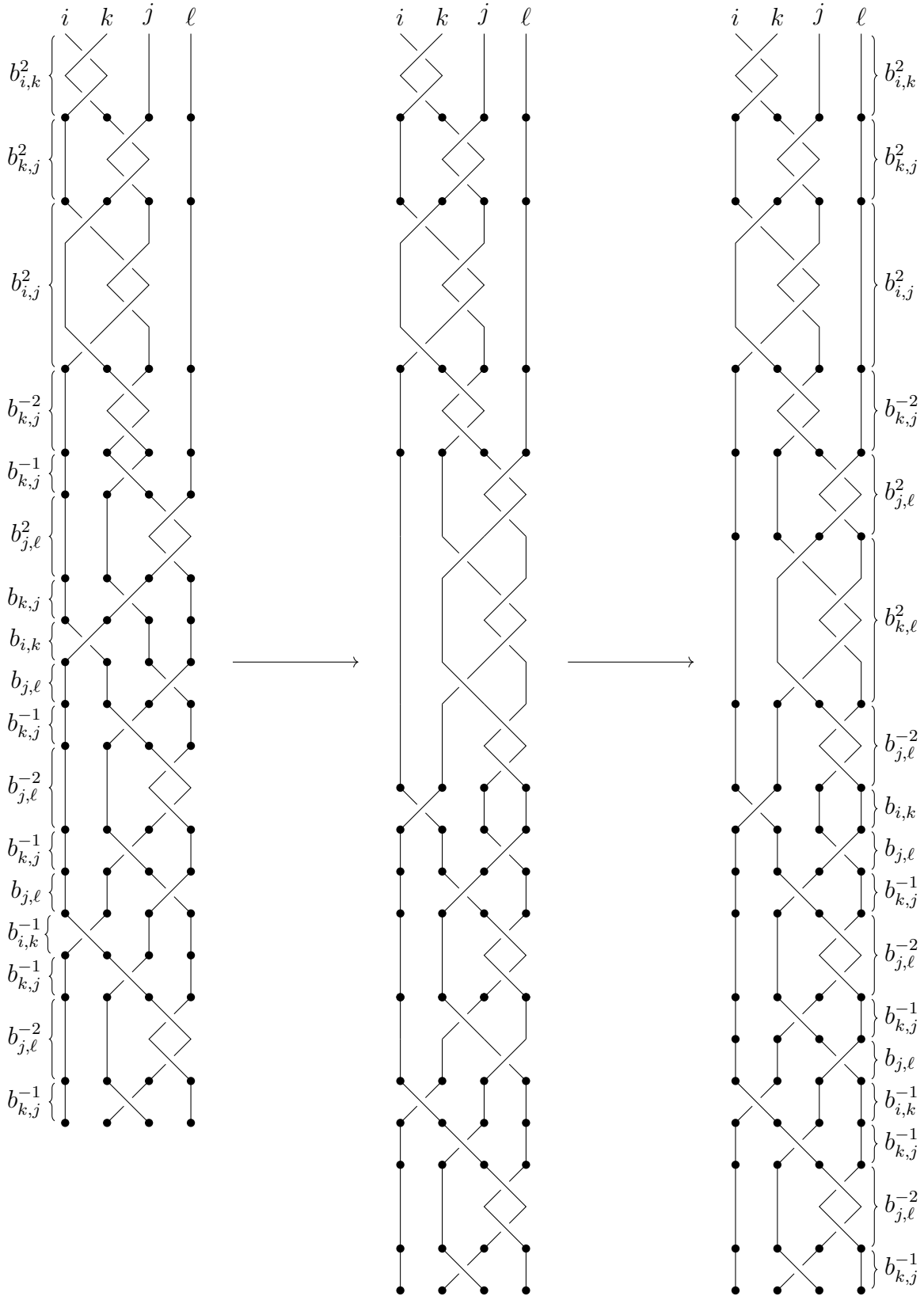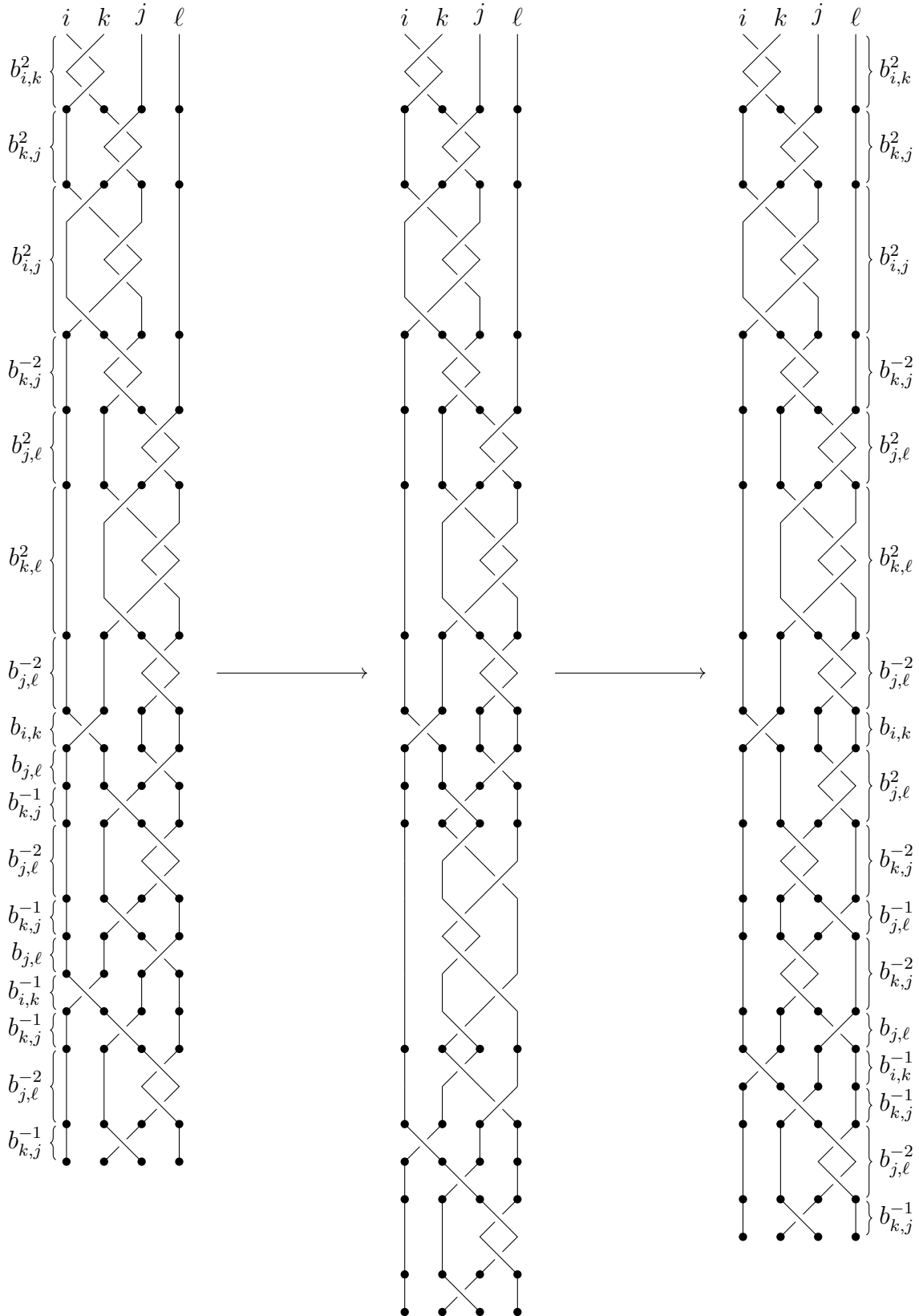
A   Diagrams for Theorem 6.2

STEP 1

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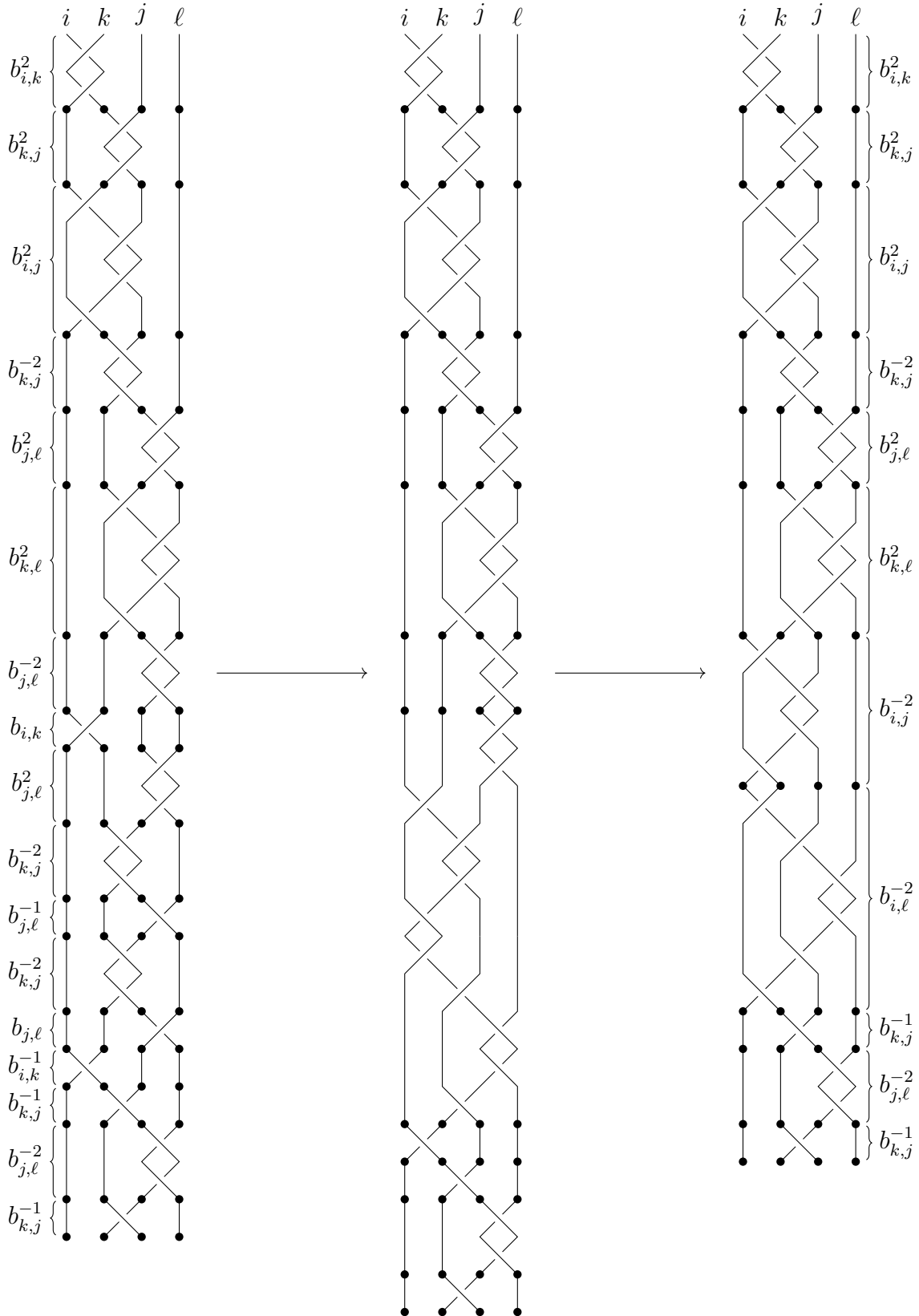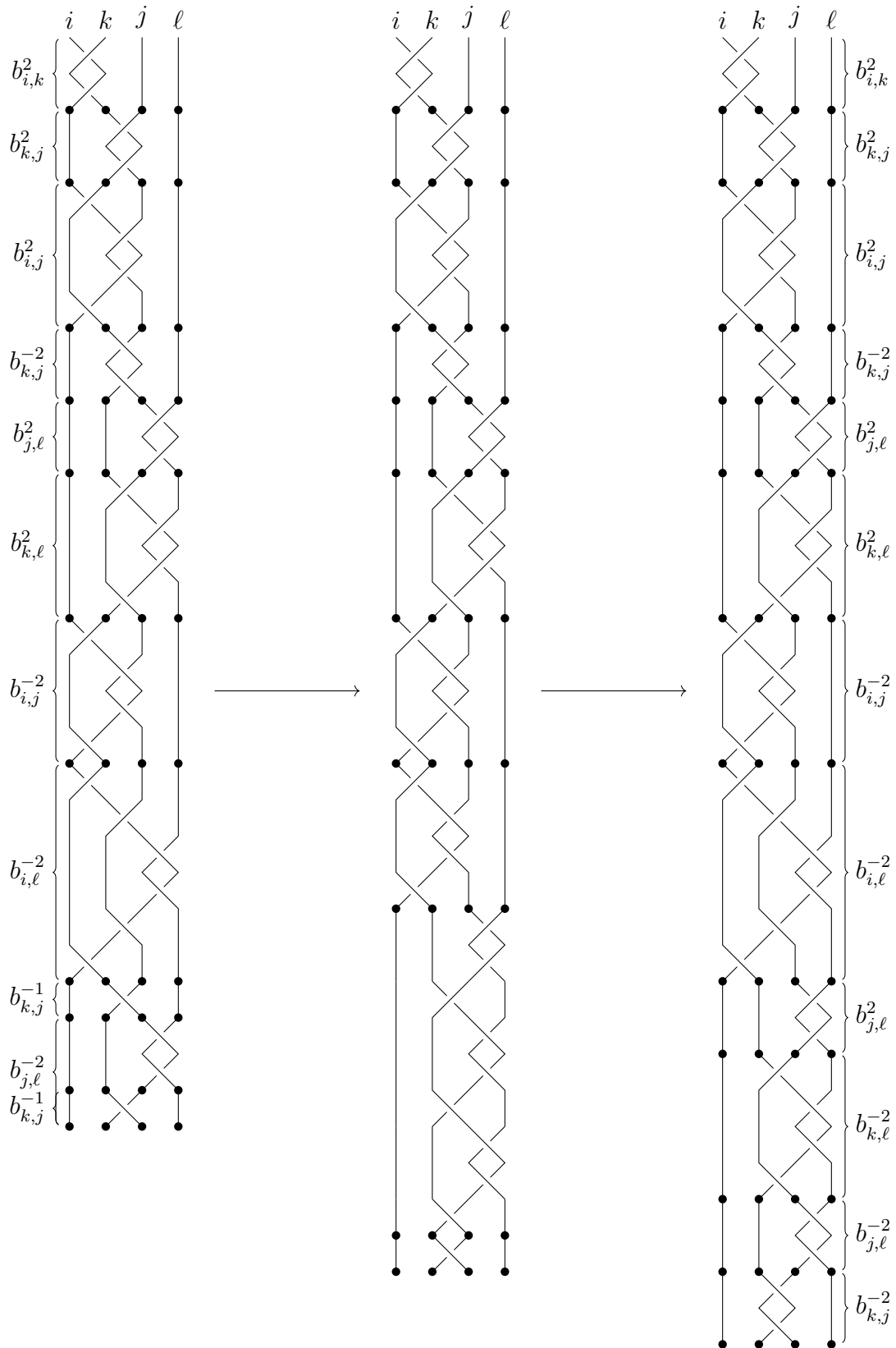
STEP 4

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STEP 6

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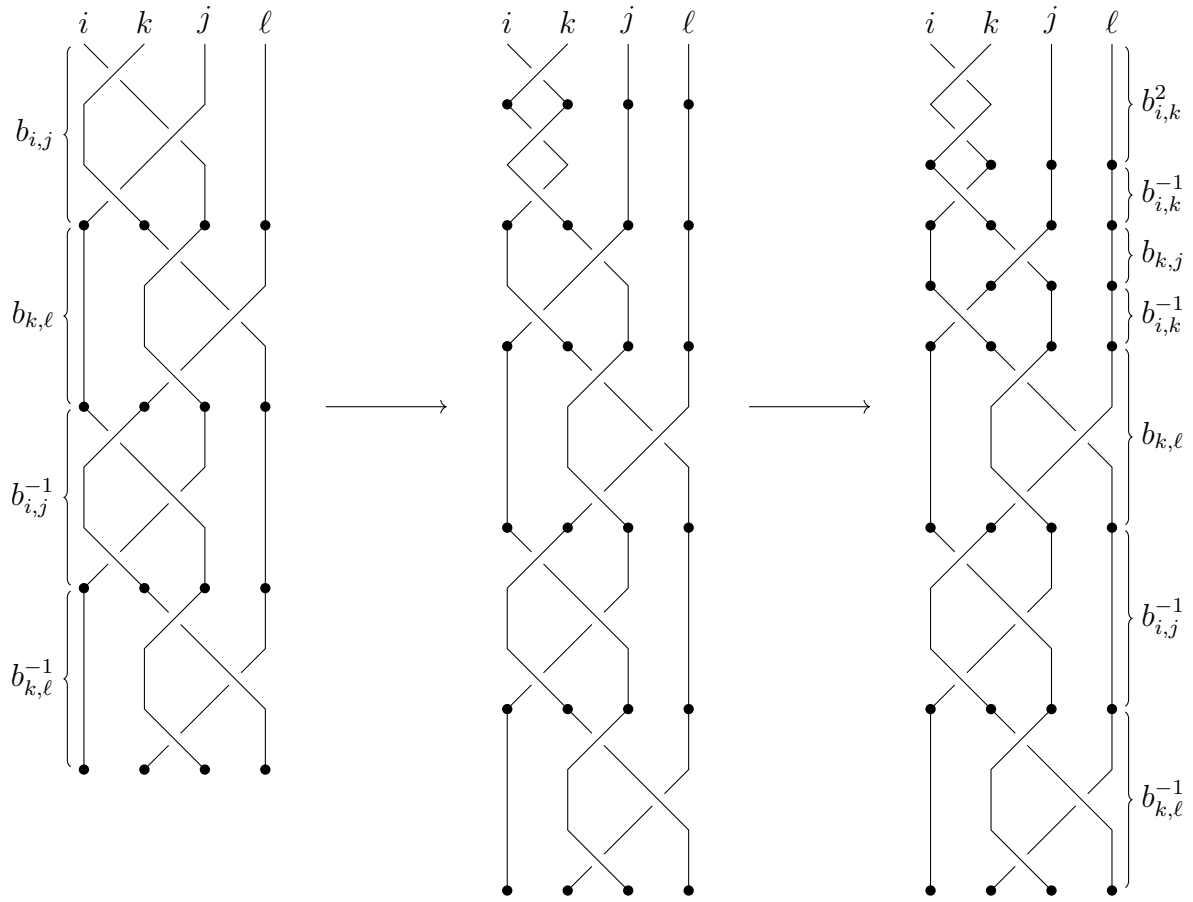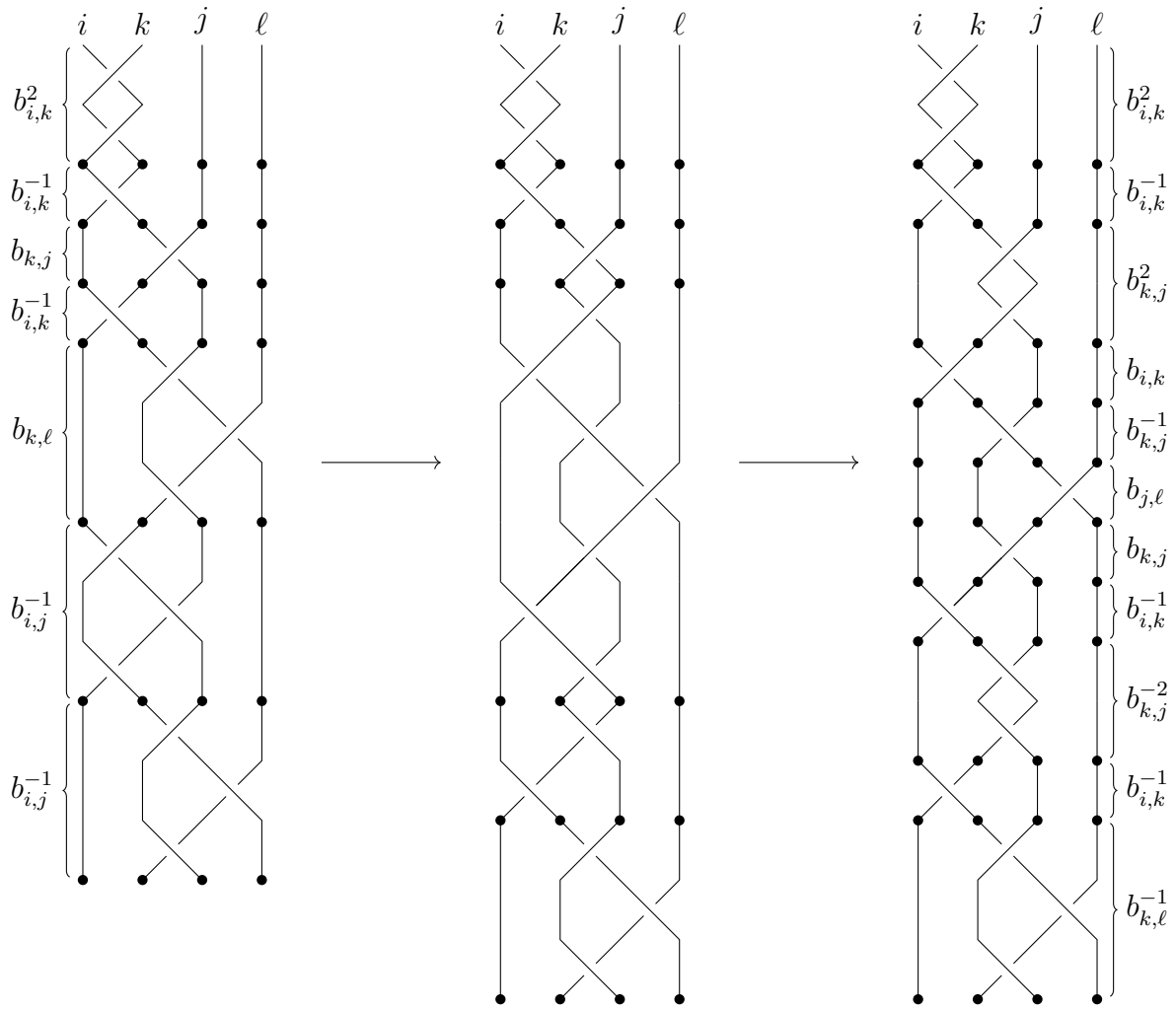
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