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Weakly q-Convex Domains and Bounded q-Subharmonic Exhaustion Functions

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Abstract

We generalize the Diederich-Fornaess index to bounded weakly q-convex domains with bounded q-subharmonic exhaustion functions. Sufficient conditions for this generalized Diederich-Fornæss index to have a given lower bound are proved. We show this generalized index is positive on bounded weakly q-convex domains with C^3 boundaries. Additionally, we prove sufficient conditions for this generalized index to equal one. For example, we show that if the domain has Property $\left(\widetilde{P_q}\right)$ then the domain has high hyperconvexity.

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1 Introduction

Hartogs opened the door to studying pseudoconvex domains with his discovery of Hartogs' Extension Phenomenon, see [28]. This phenomenon shows that for a bounded domain in $\mathbb{C}^n$ with $n \geq 2$, every holomorphic function defined in a neighborhood of the boundary can be extended to the entire domain. For more background on Hartogs' Extension Phenomenon and its relation with pseudoconvex domains, see Section 3.3 of [6] or [28]. When the Hartogs' Extension Phenomenon does not hold the domain is called a domain of holomorphy.

More precisely, a domain of holomorphy is a domain Ω for which we cannot find two nonempty sets, A_1 and A_2, where A_1 is not contained in Ω and A_2 is contained in $\Omega \cup A_1$ such that every holomorphic function on Ω extends to a holomorphic function on $\Omega \cup A_1$ and the holomorphic functions are equal on A_2. Every domain of holomorphy satisfies a geometric condition called pseudoconvexity (see, for example, Theorem 3.5.5 in [6]). Showing that a pseudoconvex domain is a domain of holomorphy presents more of a challenge. This is the Levi problem.

To define pseudoconvexity we will need the Levi form, a feature of the domain which is independent of the defining function. A definition of the Levi form is given in Definition 2.2. For now, we note that the Levi form is a Hermitian quadratic form with $(n-1)$ real-valued eigenvalues. A pseudoconvex domain is a domain where the Levi form is positive semi-definite for all points in the boundary of the domain. A definition of pseudoconvexity for domains which are not C^3 can be found in Section 3.4 of [6] for example. We will use a definition that is equivalent for C^3 domains, see Definition 2.3 below.

Pseudoconvex domains have plurisubharmonic exhaustion functions, functions that are subharmonic on any complex line (see Definition 2.5 when $q = 1$ for an equivalent definition for C^2 functions), as shown in Theorem 3.4.4 of [6]. The exhaustion functions are not required to be bounded. However, when a bounded plurisubharmonic exhaustion function exists, the domain is called hyperconvex. The Diederich-Fornæss index was presented by Diederich and Fornæss in [11]. They proved for a bounded domain Ω with a C^2 boundary that there exists $0 < \eta < 1$ and a C^2 defining function ρ on Ω so that $-(-\rho)^\eta$ is strictly plurisubharmonic on Ω. The Diederich-Fornæss index for Ω is the supremum of all η for all defining functions of Ω. An important result they showed was that the index is positive on all pseudoconvex domains with C^2 boundaries, [11]. As we would like to find a similar such value for our situation we will also be using η in a similar fashion. The Diederich-Fornæss index is a way of quantifying the hyperconvexity. A domain with η close to zero is said to have low hyperconvexity. A domain with η close to 1 is said to have high hyperconvexity. Range in [27] has a simpler proof of Diederich and Fornæss's result for C^3 domains. We will be adapting Range's proof for this thesis.

The pseudoconvex domains as defined by the Levi form are thus related to eigenvalues of this form. Less is known about the theory for $(0, q)$-forms when $q > 1$. Early work of Andreotti and Grauert [1] gives foundational results. A domain with a C^2 boundary is said to be weakly q-pseudoconvex if and only if the number of negative eigenvalues of the Levi form is less than or equal to q, as defined in [12], see the references therein. Eastwood and Suria in [12] using that definition of q-pseudoconvexity made several connections between q-pseudoconvexity and the extension of cohomology classes which is analogous to the Hartogs' Extension Phenomenon when $q > 0$.

A domain is said to be weakly q-convex when the sum of the q-smallest eigenvalues of the Levi form is nonnegative as in Definition 2.4 and [26]. Note that weak q-convexity actually implies $(q - 1)$-pseudoconvexity in the sense of Eastwood and Suria. The characterization of weakly q-convex domains we will be using was presented by Ho in [20]. In the same paper, Ho provided a definition for a q-subharmonic function on $U \in \mathbb{C}^n$. Our definition of q-subharmonic, see Definition 2.5 below, is equivalent to Ho's definition when the function is C^2. Pinton and Zampieri expanded on Ho's work as part of their study in [26]. The motivation for our main theorem comes from their discussion of plurisubharmonic defining functions of Section 4. Our later examination of the connection of a weakly q-convex domain, the Diederich-Fornæss index, and q-subharmonic functions was sparked by their statement that such a connection should exist (note that [26] uses q-plurisubharmonic in place of q-subharmonic; we will use Ho's terminology from [20] for the sake of consistency with earlier definitions of q-plurisubharmonicity).

The Bergman Projection is the orthogonal projection from the space of square integrable functions of a domain in $\mathbb{C}^n$ onto the space of holomorphic square integrable functions for that domain. A domain with the Diederich-Fornæss index near one implies regularity for the Bergman Projection in Sobolev spaces (See [22], [3], [26], [15], and [23]). Kohn in [22] showed that the Bergman Projection is regular in the Sobolev space $W^s(\Omega)$ for a range of s depending on the Diederich-Fornæss index and some other hypotheses, generalizing an earlier result of Boas and Straube [4]. Berndtsson and Charpentier in [3] showed that the Bergman Projection is continuous in $W^s(\Omega)$ for all $s \geq 0$ less than one half of the Diederich-Fornæss index. Harrington in [15] proves that a Diederich-Fornæss index of one with some additional hypotheses (referred to as a regularized Diederich-Fornæss index of one by Dall'Ara and Mongodi [7]), implies that the Bergman Projection is continuous in $W^s(\Omega)$ for all $s \geq 0$.

The conditions q-pseudoconvexity and q-plurisubharmonicity are invariant under holomorphic changes of coordinates, but q-convexity and q-subharmonicity depend on the choice of holomorphic coordinates. This is because eigenvalues of congruent matrices, see Definition 4.5.4 in [21], have the same signs but may have different relative sizes. Our proof of the main theorem will only use orthonormal changes of coordinates. Thus our conclusions will show q-subharmonicity.

A domain is said to be hyper q-convex when the sum of the q-smallest eigenvalues of the Levi form is positive as in Definition 2.4 (see, for example, Wu [30]). We note that by this definition every hyper q-convex domain is also a weakly q-convex domain. Thus all of our later discussion for weakly q-convex domains trivially applies to hyper q-convex domains. Hyper q-convexity is related to weak q-convexity in the same way that strict pseudoconvexity is related to pseudoconvexity.

In this thesis, we will generalize the Diederich-Fornæss Index to bounded q-plurisubharmonic exhaustion functions on bounded weakly q-convex domains. Our main theorem is a sufficient condition for this generalized Diederich-Fornæss index to have a given lower bound. In addition to q, this lower bound depends on two quantities $\mathcal{B}$ and $\mathcal{A}$ derived from an arbitrary defining function and a quantity ξ derived from a strictly q-subharmonic weight function. The main theorem will be followed by three corollaries. The first corollary will be generalizing work of Diederich and Fornæss and also Range to bounded weakly q-convex domains. It will show that in the situation of low hyperconvexity the main theorem's conclusions hold.

The second corollary generalizes works of Liu [24], Yum [31], and Harrington [16]. Due to their work it is known that the Diederich-Fornæss index has high hyperconvexity when $\alpha = 0$ and when restricted to weakly pseudoconvex directions. This α, an important real-valued 1-form, was first defined by D'Angelo in [8]. In [9], D'Angelo further developed α and its connection to pseudoconvex domains. For the second corollary, we need a comparable condition to the vanishing of α in weakly pseudoconvex directions that applies to $(0, q)$-forms. The quantity $\mathcal{A}$ defined by (6) represents this. In future research, we hope to relate $\mathcal{A}$ to the vanishing of some q-dimensional cohomology class, as is true in the $q = 1$ case. The second corollary to the main theorem below shows conditions where the main theorem's conclusion holds for a domain with high hyperconvexity.

The third corollary is a generalization of work done by Harrington for Property (P) from Corollary 6.1 in [16] and for Property $\left(\widetilde{P}\right)$ from Corollary 6.2 in [16]. Property (P) is a boundary condition introduced by Catlin in [5]. A domain satisfies Property (P) if for every positive number M there is a uniformly bounded (i.e., with a bound independent of M) plurisubharmonic function on the domain for which there is a lower bound of M for each eigenvalue of the Levi form for all points in the boundary. Catlin also proved that for a smoothly bounded pseudoconvex domain that satisfies Property (P) that a compactness estimate holds for the $\overline{\partial}$-Neumann problem on that domain, Theorem 1 of [5]. Later authors (see, e.g., Chapter 4 in [29] for references) showed that a compactness estimate for $(0, q)$-forms holds whenever Property (P_q) holds. Property (P_q) replaces the eigenvalue bound with a bound on the sum of any q eigenvalues. Property $\left(\widetilde{P_q}\right)$, a possible generalization of Property (P_q), was introduced by McNeal in [25]. He introduces two conditions to build Property $\left(\widetilde{P_q}\right)$, the self-bounded complex gradient and a condition analogous to the eigenvalue condition in Property (P_q). The formal definition of Property $\left(\widetilde{P_q}\right)$ is given in Definition 2.12. The third corollary to the main theorem below shows that if the domain has Property $\left(\widetilde{P_q}\right)$ then the domain has high hyperconvexity.

We conclude with some examples demonstrating that it is possible to be weakly q-convex without having hyper q-convexity. In our examples $\mathcal{B} = \mathcal{A} = 0$, as defined in (5) and (6), but there do not appear to be local obstructions to the vanishing of these. To obtain examples where these do not vanish for any choice of defining function, it seems that it will be necessary to study domains with q-dimensional complex submanifolds in the boundary with non-trivial q-cohomology, and this is likely to be a difficult question. When $q = 1$, the classic example is the worm domain of Diederich and Fornæss, which has an annulus in the boundary and arbitrarily small Diederich-Fornæss Index.

2 Definitions

Let $\Lambda^{(0,q)}$ be the space of $(0,q)$-forms at $z \in \mathbb{C}^n$ with $1 \leq q \leq n$ as in the discussion before Lemma 4.7 in [29]. For $u \in \Lambda^{(0,q)}$, we write $u = \sum_J{}' u_J d\bar{z}^J$ where $\sum'$ indicates the sum over all increasing multi-indices of length q.

Definition 2.1. A domain Ω in $\mathbb{C}^n$, $n \geq 2$, is said to have a C^k boundary, $1 \leq k \leq \infty$, at the point $p \in \partial\Omega$ if there exists a C^k function $r(z)$ defined in an open neighborhood U of p such that $\Omega \cap U = \{z \in U | r(z) < 0\}$, $\partial\Omega \cap U = \{z \in U | r(z) = 0\}$, and $\partial r(z) \neq 0$ on $\partial\Omega \cap U$. If U is an open neighborhood of $\overline{\Omega}$, then $r(z)$ is a defining function for Ω, see Definition 1.1.2 in [6].

Definition 2.2. Let Ω be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$ and let $r(z)$ be a C^2 defining function for Ω. The Levi form of the function r at the point $p \in \partial\Omega$, denoted L_p, is the

Hermitian form

$$L^r(u,v) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) u_j \overline{v_k},$$

defined for all $u, v \in \mathbb{C}^n$ with $\sum_{j=1}^{n} u_j \frac{\partial r}{\partial z_j}(p) = 0$ and $\sum_{j=1}^{n} v_j \frac{\partial r}{\partial z_j}(p) = 0$, see Definition 3.3.1 in [6].

Definition 2.3. Let Ω be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$ and let $r(z)$ be a C^2 defining function for Ω. The domain Ω is called pseudoconvex at $p \in \partial\Omega$ if the Levi form is nonnegative for all $u \in T_p^{1,0}(\partial\Omega)$ as in Definition 3.4.1 of [6]. The domain is called a pseudoconvex domain if it is pseudoconvex for every $p \in \partial\Omega$.

Definition 2.4. A domain Ω with a C^2 boundary is said to be weakly q-convex when the sum of the q smallest eigenvalues of the Levi form is nonnegative. This is equivalent to Ho's definition by Lemma 2.2 in [20]. A domain Ω with a C^2 boundary is said to be hyper q-convex when the sum of the q smallest eigenvalues of the Levi form is positive as in [30].

Definition 2.5. A C^2 function ρ is said to be q-subharmonic on a weakly q-convex domain Ω if the sum of the first q eigenvalues of the complex Hessian, $\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k}$ for all $t \in \mathbb{C}^n$, is nonnegative as in Definition 1.1 and Theorem 1.4 in [20].

Let $\Omega \subset \mathbb{C}^n$ be domain with a C^2 defining function r. Choose $p \in \partial\Omega$ and rotate coordinates so that $\forall 1 \leq j \leq n-1$ we have that $\frac{\partial r}{\partial z_j}(p) = 0$, $\mathrm{Re}\,\frac{\partial r}{\partial z_n}(p) = 0$, and $\mathrm{Im}\,\frac{\partial r}{\partial z_n}(p) > 0$.

Definition 2.6. Let $u \in \Lambda^{(0,q)}$ with $u^\tau = \sum_{J, n \notin J}{}' u_J d\overline{z}^J$ and $u^\nu = \sum_{J, n \in J}{}' u_J d\overline{z}^J$. For $p \in \partial\Omega$ and coordinates as above, define $T_p^{0,q}(\partial\Omega)$ be the complex tangent space of $(0,q)$-forms where $u = \sum_{J, n \notin J}{}' u_J d\overline{z}^J$.

Definition 2.7. For $u, v \in \Lambda^{(0,q)}$, let $L_q^r(u,v) = \sum_H{}' \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) u_{jH} \overline{v_{kH}}$. When we restrict u and v to $T_p^{0,q}(\partial\Omega)$ on a weakly q-convex domain Ω, $L_q^r(u,v)$ is a Hermitian semi-inner product with semi-norm $\sqrt{L_q^r(u,u)}$. Now we define

$$\mathcal{N}_p^q = \left\{ u \in T_p^{0,q}(\partial\Omega) \middle| \sum_H{}' \sum_{j,k=1}^{n-1} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) u_{jH} \overline{u_{kH}} = 0 \right\}.$$

We will show that a Cauchy-Schwarz inequality applies to the space represented by $\mathcal{N}_p^q$ and also that $\mathcal{N}_p^q$ is a vector space in Lemma 4.1 and Lemma 4.2 below.

Definition 2.8. Let Ω be a domain in $\mathbb{C}^n$. The reach of $\partial\Omega$, denoted reach $(\partial\Omega)$, is the supremum of ϵ such that if $z \in \mathbb{C}^n$ and the distance from z to $\partial\Omega$ is smaller than ϵ, then $\partial\Omega$ contains a unique point, p_z, nearest to z as in [13]. If $\partial\Omega$ is C^2 and compact then $\partial\Omega$ has positive reach.

Definition 2.9. A continuously differentiable complex-valued C^1 function $f(z)$ defined on an open subset D of $\mathbb{C}^n$ is called holomorphic, denoted $f \in \mathcal{O}(D)$, if $f(z)$ is holomorphic if $\frac{\partial f}{\partial \bar{z}} = 0$ at every point in D as in [14].

Definition 2.10. A domain D in $\mathbb{C}^n$ is called a domain of holomorphy if we cannot find two nonempty open sets D_1 and D_2 in $\mathbb{C}^n$ with the following properties:

- D_1 is connected, $D_1 \not\subseteq D$ and $D_2 \subset D_1 \cap D$
- For every $f \in \mathcal{O}(D)$ there is a $\tilde{f} \in \mathcal{O}(D_1)$ satisfying $f = \tilde{f}$ on D_2.

Definition 2.11. A domain D in $\mathbb{C}^n$ is a q-domain of holomorphy if and only if, for all $p \in \partial D$, there exists $m \leq q$ and a cohomology class $[\alpha] \in H^m(D, \mathcal{O})$ which does not extend though p as in [12].

In [25], McNeal defines Property $\left(\widetilde{P_q}\right)$.

Definition 2.12. Let Ω be a domain in $\mathbb{C}^n$. Property $\left(\widetilde{P_q}\right)$ is said to be satisfied by Ω if the following holds: there exists a constant C such that for all $M > 0$ there exists a C^2 function f in a neighborhood U_M of $\partial\Omega$ with

$$\sum_{|H|=q-1}{}' \left| \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) u_{jH} \right|^2 \leq C \sum_{|H|=q-1}{}' \sum_{j,k} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(z) u_{jH} \overline{u_{kH}} \tag{1}$$

and

$$\sum_{|H|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(z) u_{jH} \overline{u_{kH}} \geq M|u|^2 \tag{2}$$

for $z \in U_M$ and $u \in \Lambda^{(0,q)}(z)$ as in [29].

When (1) holds, we say that f has a self-bounded complex gradient.

3 Theorem and Corollaries

Let Ω be a weakly q-convex domain with a smooth boundary and r a C^3 defining function. For real values a and η with $a > 0$ and $0 < \eta < 1$, we will show the following conditions are sufficient so that for a C^3 defining function ρ, defined by $\rho = re^{-a\varphi(z)}$ on a neighborhood of $\partial\Omega$ in Ω, we have that $-(-\rho)^\eta$ is q-subharmonic on Ω.

Theorem 3.1. *Let Ω be a bounded weakly q-convex domain in $\mathbb{C}^n$ with a C^3 boundary and a C^3 defining function r. Let $0 < \eta < 1$ and let φ be a $C^3\left(\overline{\Omega}\right)$ real valued function such that for a real nonnegative number ξ the following hold*

$$\sum_{|H|=q-1}{}' \left| \sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j}(z) u_{jH} \right|^2 \leq \xi \sum_{|H|=q-1}{}' \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}}(z) u_{jH} \overline{u_{kH}} \tag{3}$$

and

$$\sum_{|H|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}}(z) u_{jH} \overline{u_{kH}} \geq q|u|^2 \tag{4}$$

for all $(0,q)$ forms u on a neighborhood of $\partial\Omega$ in Ω. Let $L_q^r(u,v)$ and $\mathcal{N}_p^q$ be defined as in Definition 2.7. For $p \in \partial\Omega$ and $\mu > 0$, let

$$U_{p,\mu} = \{z \in \Omega : (-r(z)) < \mu, |z-p| = \operatorname{dist}(z, \partial\Omega)\}.$$

Let

$$\mathcal{B} = \lim_{\mu \to 0} \sup_{p \in \partial\Omega} \sup_{z \in U_{p,\mu}} \sup_{\substack{v \in \mathcal{N}_p^q \\ v \neq 0}} \left(-\left((-r(z))|v|^2\right)^{-1} L_q^r(v,v) \right). \tag{5}$$

When $\mathcal{N}_p^q = \{0\}$ we define $\mathcal{B} = 0$. Let

$$\mathcal{A} = \sup_{p\in\partial\Omega} \sup_{w\in\Lambda^{(0,q-1)}} \sup_{\substack{v\in\mathcal{N}_p^q \\ v\neq 0}} \left(|\partial r(p)|^{-1} \left| L_q^r\left(\frac{v}{|v|}, \frac{\partial r \wedge w}{|\partial r \wedge w|} \right) \right| \right). \tag{6}$$

When the set of $v \in \mathcal{N}_p^q$ such that $v \neq 0$ is empty we define $\mathcal{A} = 0$. If

$$\eta < \frac{q^{\frac{1}{2}}}{2\mathcal{A}\xi^{\frac{1}{2}} + q^{\frac{1}{2}}} \tag{7}$$

and

$$\mathcal{B} < \frac{q(1-\eta)}{4\xi\eta} - \frac{q^{\frac{1}{2}}}{\xi^{\frac{1}{2}}}\mathcal{A} - \mathcal{A}^2 \tag{8}$$

then there exists a nonnegative real valued number a and a C^3 defining function ρ such that $\rho = re^{-a\varphi(z)}$ on a neighborhood of $\partial\Omega$ in Ω and $-(-\rho)^\eta$ is q-subharmonic on Ω.

We have three corollaries to the main theorem. Note that Corollaries 3.2 and 3.3 are similar to Corollary 6.1 in [16] (up to a constant multiple of $\mathcal{A}$) when $q = 1$.

Corollary 3.2. *Let Ω be a bounded weakly q-convex domain in $\mathbb{C}^n$ with a C^3 boundary. If $0 < \eta < 1$ is small enough then there exists a C^3 defining function ρ such that $-(-\rho)^\eta$ is q-subharmonic on Ω.*

On hyper q-convex domains, $\mathcal{A} = 0$ by definition, and we can always choose r so that $\mathcal{B} \leq 0$ (see Lemma 4.4), so the next corollary, Corollary 3.3, always applies to hyper q-convex domains.

Corollary 3.3. *Let Ω be a bounded weakly q-convex domain in $\mathbb{C}^n$ with a C^3 boundary. Suppose there exists a C^3 defining function r for Ω such that $\mathcal{A} = 0$ and $\mathcal{B} \leq 0$ where $\mathcal{A}$ is defined by (6) and $\mathcal{B}$ is defined by (5). Then for all $0 < \eta < 1$ there exists a C^3 defining function ρ such that $-(-\rho)^\eta$ is q-subharmonic on Ω.*

Next we have a corollary when our domain satisfies Property $\left(\widetilde{P_q}\right)$. We will see in the proof that Corollary 3.4 is quantitatively stronger than Corollary 6.2 in [16] when $\mathcal{A}$ is large and $q = 1$ even though the results are qualitatively the same.

Corollary 3.4. *Let Ω be a bounded weakly q-convex domain in $\mathbb{C}^n$ with a C^3 boundary. Suppose Ω has Property $\left(\widetilde{P_q}\right)$ as defined in Definition 2.12. Then for any $0 < \eta < 1$ there exists a C^3 defining function ρ such that $-(-\rho)^\eta$ is q-subharmonic on Ω.*

4 Basic Properties

In this section we will prove some basic properties we have found. Let $L_q^r(u, v)$ and $\mathcal{N}_p^q$ be defined as in Definition 2.7 for all lemmas in this section. First, we claim that a Cauchy-Schwarz inequality applies to the space represented by $\mathcal{N}_p^q$.

Lemma 4.1. *Let $\Omega \in \mathbb{C}^n$ be a weakly q-convex domain with C^2 boundary. Let $u, v \in T_p^{0,q}(\partial\Omega)$. Then $|L_q^r(u, v)| \le \sqrt{L_q^r(u, u)}\sqrt{L_q^r(v, v)}$.*

Proof. Let $L_q^r(u, v)$ be defined as above and let $t \in \mathbb{R}$. Now,

$$\begin{aligned} 0 &\le L_q^r(u + tv, u + tv) \\ &= L_q^r(u, u) + tL_q^r(v, u) + tL_q^r(u, v) + t^2 L_q^r(v, v) \\ &= L_q^r(u, u) + 2t \operatorname{Re} L_q^r(u, v) + t^2 L_q^r(v, v). \end{aligned}$$

We will first prove $|\operatorname{Re} L_q^r(u, v)| \le \sqrt{L_q^r(u, u)}\sqrt{L_q^r(v, v)}$. To see this, we will consider two cases.

Case 1: $L_q^r(v, v) = 0$

Then $0 \le L_q^r(u, u) + 2t \operatorname{Re} L_q^r(u, v)$ for all $t \in \mathbb{R}$. Thus $\operatorname{Re} L_q^r(u, v) = 0$. Hence

$$|\operatorname{Re} L_q^r(u, v)| \le \sqrt{L_q^r(u, u)}\sqrt{L_q^r(v, v)}.$$

Case 2: $L_q^r(v, v) \ne 0$. Since Ω is weakly q-convex $L_q^r(v, v) > 0$ in this case.

Let $t = -\frac{\operatorname{Re} L_q^r(u,v)}{L_q^r(v,v)}$. Then

$$0 \le L_q^r(u,u) - \frac{2\big(\operatorname{Re} L_q^r(u,v)\big)^2}{L_q^r(v,v)} + \frac{\big(\operatorname{Re} L_q^r(u,v)\big)^2}{L_q^r(v,v)}$$
$$= L_q^r(u,u) - \frac{\big(\operatorname{Re} L_q^r(u,v)\big)^2}{L_q^r(v,v)}.$$

Thus $\big(\operatorname{Re} L_q^r(u,v)\big)^2 \le L_q^r(u,u) L_q^r(v,v)$. Since this is true for all $u, v \in T_p^{(0,q)}$, we have that

$$|\operatorname{Re} L_q^r(u,v)| \le \sqrt{L_q^r(u,u)}\sqrt{L_q^r(v,v)}.$$

Now, suppose $L_q^r(u,v) = se^{i\theta}$ for $s \ge 0$. Then we have that $L_q^r\big(e^{-i\theta}u, v\big) = s$. We have shown that

$$|\operatorname{Re} L_q^r\big(e^{-i\theta}u, v\big)| \le \sqrt{L_q^r(e^{-i\theta}u, e^{-i\theta}u)}\sqrt{L_q^r(v,v)}.$$

So

$$s \le \sqrt{L_q^r(e^{-i\theta}u, e^{-i\theta}u)}\sqrt{L_q^r(v,v)}$$
$$= \sqrt{L_q^r(u,u)}\sqrt{L_q^r(v,v)}.$$

Hence,

$$|L_q^r(u,v)| \le \sqrt{L_q^r(u,u)}\sqrt{L_q^r(v,v)}.$$

□

Now, since $\sqrt{L_q^r(u,u)}\sqrt{L_q^r(v,v)} = 0$ in $\mathcal{N}_p^q$, we can show that $\mathcal{N}_p^q$ is a vector space.

Lemma 4.2. *Let Ω be a bounded weakly q-convex domain with a C^2 boundary. Then $\mathcal{N}_p^q$ is a vector space.*

Proof. Let $u, v \in \mathcal{N}_p^q$ and $L_q^r(u, v)$ be defined as above. Let $a, b \in \mathbb{C}$. Now

$$\begin{aligned}|L_q^r(au+bv, au+bv)| &= ||a|^2 L_q^r(u,u) + a\bar{b}L_q^r(u,v) + b\bar{a}L_q^r(v,u) + |b|^2 L_q^r(v,v)| \\ &= |a\bar{b}L_q^r(u,v) + b\bar{a}L_q^r(v,u)|.\end{aligned}$$

By the triangle inequality and Lemma 4.1,

$$\begin{aligned}|L_q^r(au+bv, au+bv)| &\le |a||b||L_q^r(u,v)| + |a||b||L_q^r(v,u)| \\ &\le 2|a||b|\sqrt{L_q^r(u,u)}\sqrt{L_q^r(v,v)} \\ &= 0.\end{aligned}$$

Thus $au + bv \in \mathcal{N}_p^q$. Hence $\mathcal{N}_p^q$ is a vector space.

□

Lemma 4.3. *Let Ω be a bounded weakly q-convex domain with a C^3 boundary and a C^3 defining function r. For $p \in \partial\Omega$ and $\mu > 0$, let*

$$U_{p,\mu} = \{z \in \Omega : (-r(z)) < \mu, |z-p| = \operatorname{dist}(z, \partial\Omega)\}.$$

If $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative, then

$$\mathcal{B} = \sup_{p\in\partial\Omega} \sup_{\substack{u\in\mathcal{N}_p^q \\ u\neq 0}} \frac{\sum_H' \sum_{j,k=1}^n \left(-\frac{\partial^3 r}{\partial\nu\partial\bar{z}_k\partial z_j}(p)\right) u_{jH}\overline{u_{kH}}}{\frac{\partial r}{\partial\nu}(p)|u|^2}. \tag{9}$$

Proof. Let $p \in \partial\Omega$, $z \in \Omega$ such that $|z-p| = \operatorname{dist}(z, \partial\Omega)$, and $z_t = tz + (1-t)p$ for all $0 < t \le 1$. Choose local coordinates in a neighborhood of $p \in \partial\Omega$ so that $\forall 1 \le j \le n-1$ we have that $\frac{\partial r}{\partial z_j}(p) = 0$, $\operatorname{Re}\frac{\partial r}{\partial z_n}(p) = 0$, and $\operatorname{Im}\frac{\partial r}{\partial z_n}(p) > 0$. Also, let $u \in \mathcal{N}_p^q$. Then

$$\lim_{t\to 0} \frac{L_q^r(u,u)}{(-r(z_t))|u|^2} = \lim_{t\to 0} \frac{\sum_H' \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(z_t) u_{jH}\overline{u_{kH}}}{(-r(z_t))|u|^2}$$
$$= \lim_{t\to 0} \frac{\sum_H' \sum_{j,k,i=1}^n \left(\frac{\partial^3 r}{\partial z_j \partial \overline{z_k} \partial z_i}(z_t)(z_i - p_i) u_{jH}\overline{u_{kH}} + \frac{\partial^3 r}{\partial z_j \partial \overline{z_k} \partial \overline{z_i}}(z_t)(\overline{z_i} - \overline{p_i}) u_{jH}\overline{u_{kH}}\right)}{-\left(\frac{\partial r}{\partial z_i}(z_t)(z_i - p_i)|u|^2 + \frac{\partial r}{\partial \overline{z_i}}(z_t)(\overline{z_i} - \overline{p_i})|u|^2\right)}.$$

Since the numerator and denominator are both approaching zero in the first line above, we employ L'Hôpital's rule to get to the second line above. When $z \in U_{p,\mu}$ we have that $|z - p| = \operatorname{dist}(z, \partial\Omega)$. Thus $z_i = p_i$ for all $1 \le i \le n-1$ and $\operatorname{Re} z_n = \operatorname{Re} p_n$. Additionally, we have $\frac{\partial r}{\partial z_i} \to 0$ when $i \neq n$. Hence

$$\lim_{t\to 0} \frac{L_q^r(u,u)}{(-r(z_t))|u|^2}$$
$$= \frac{\sum_H' \sum_{j,k=1}^n \left(\frac{\partial^3 r}{\partial z_j \partial \overline{z_k} \partial z_n}(p)(z_n - p_n) u_{jH}\overline{u_{kH}} + \frac{\partial^3 r}{\partial z_j \partial \overline{z_k} \partial \overline{z_n}}(p)(\overline{z_n} - \overline{p_n}) u_{jH}\overline{u_{kH}}\right)}{-\left(\frac{\partial r}{\partial z_n}(p)(z_n - p_n)|u|^2 + \frac{\partial r}{\partial \overline{z_n}}(p)(\overline{z_n} - \overline{p_n})|u|^2\right)}$$
$$= \frac{\sum_H' \sum_{j,k=1}^n \left(\frac{\partial^3 r}{\partial z_j \partial \overline{z_k} \partial z_n}(p)(i \operatorname{Im} z_n - i \operatorname{Im} p_n) u_{jH}\overline{u_{kH}} + \frac{\partial^3 r}{\partial z_j \partial \overline{z_k} \partial \overline{z_n}}(p)(i \operatorname{Im} \overline{z_n} - i \operatorname{Im} \overline{p_n}) u_{jH}\overline{u_{kH}}\right)}{-\left(\frac{\partial r}{\partial z_n}(p)(i \operatorname{Im} z_n - i \operatorname{Im} p_n)|u|^2 + \frac{\partial r}{\partial \overline{z_n}}(p)(i \operatorname{Im} \overline{z_n} - i \operatorname{Im} \overline{p_n})|u|^2\right)}$$
$$= \frac{\sum_H' \sum_{j,k=1}^n \left(\frac{\partial^3 r}{\partial z_j \partial \overline{z_k} \partial z_n}(p) - \frac{\partial^3 r}{\partial z_j \partial \overline{z_k} \partial \overline{z_n}}(p)\right) u_{jH}\overline{u_{kH}}}{-\left(\frac{\partial r}{\partial z_n}(p) i - \frac{\partial r}{\partial \overline{z_n}}(p) i\right)|u|^2}$$
$$= \frac{\sum_H' \sum_{j,k=1}^n \left(\frac{\partial^3 r}{\partial z_j \partial \overline{z_k} \partial y_n}(p)\right) u_{jH}\overline{u_{kH}}}{-\frac{\partial r}{\partial y_n}(p)|u|^2}.$$

Note that $\frac{\partial}{\partial y_n}$ is the unit outward normal at p. Thus when we take the supremum over all p and u such that $p \in \partial\Omega$, $u \in \mathcal{N}_p^q$ and $u \neq 0$ we obtain

$$\mathcal{B} = \sup_{p \in \partial\Omega} \sup_{\substack{u \in \mathcal{N}_p^q \\ u \neq 0}} \frac{\sum_H' \sum_{j,k=1}^n \left(-\frac{\partial^3 r}{\partial\nu \partial\bar{z}_k \partial z_j}(p)\right) u_{jH} \overline{u_{kH}}}{\frac{\partial r}{\partial\nu}(p)|u|^2}.$$

□

Lemma 4.4. *Let Ω be a weakly q-convex domain with a C^3 boundary. Let r be the signed distance function. Let $\mathcal{B}$ be defined by (5). Then $\mathcal{B} \leq 0$.*

Proof. The signed distance function is also C^3. For $p \in \partial\Omega$ and $\mu > 0$, by Lemma 4.3 we have that

$$\mathcal{B} = \sup_{p \in \partial\Omega} \sup_{\substack{u \in \mathcal{N}_p^q \\ u \neq 0}} \frac{\sum_H' \sum_{j,k=1}^n \left(-\frac{\partial^3 r}{\partial\nu \partial\bar{z}_k \partial z_j}(p)\right) u_{jH} \overline{u_{kH}}}{\frac{\partial r}{\partial\nu}(p)|u|^2}.$$

In [16], (5.3) provides justification that sign of our numerator is negative. Hence $\mathcal{B} \leq 0$. □

5 Proof of Main Theorem

In this section we will be proving the main theorem, Theorem 3.1.

Proof. Choose orthonormal local coordinates in a neighborhood of $p \in \partial\Omega$ so that for $1 \leq j \leq n-1$ we have that $\frac{\partial r}{\partial z_j}(p) = 0$, $\operatorname{Re} \frac{\partial r}{\partial z_n}(p) = 0$, and $\operatorname{Im} \frac{\partial r}{\partial z_n}(p) > 0$. By (8), we can find a real number R such that

$$\mathcal{B} < R < \frac{q(1-\eta)}{4\xi\eta} - \frac{q^{\frac{1}{2}}}{\xi^{\frac{1}{2}}}\mathcal{A} - \mathcal{A}^2.$$

Consider the function

$$f(x) = \frac{qx}{2} - \mathcal{A}^2\left(\frac{1}{2\xi} - \frac{\eta x}{1-\eta}\right)^{-1}\left(\frac{\frac{1}{2\xi} - \eta x}{1-\eta}\right) - R.$$

For

$$a = \frac{1-\eta}{2\xi\eta} - \frac{\mathcal{A}}{(q\xi)^{\frac{1}{2}}}$$

we have that a is nonnegative based on our assumption that $\eta < \frac{q^{\frac{1}{2}}}{2\mathcal{A}\xi^{\frac{1}{2}}+q^{\frac{1}{2}}}$. Then by (8), $f(a) > 0$. Notice that $x = a$ maximizes $f(x)$ over $0 \le x < \frac{1-\eta}{2\xi\eta}$. The inequality $f(a) > 0$ can be rearranged to be

$$-R - \mathcal{A}^2\left(\frac{1}{2\xi} - \frac{\eta a}{1-\eta}\right)^{-1}\left(\frac{\frac{1}{2\xi} - \eta a}{1-\eta}\right) + \frac{qa}{2} > 0. \tag{10}$$

Let u be a $(0,q)$-form. Let u^τ and u^ν be as in Definition 2.6 and $u^\tau = u_1^\tau + u_2^\tau$ where $u_1^\tau \in \mathcal{N}_p^q$ at $p \in \partial\Omega$ and $u_2^\tau \in \left(\mathcal{N}_p^q\right)^\perp$ at $p \in \partial\Omega$. Fix $\varepsilon > 0$, $c > 0$, and $\delta > 0$ such that

$$L_q^r(u^\tau, u^\tau) \ge \varepsilon|u_2^\tau|^2 - R(-r)|u_1^\tau|^2 - c(-r)|u_1^\tau||u_2^\tau| + \delta(-r)|u^\tau|^2 \tag{11}$$

on $U_{p,\mu}$ for μ sufficiently small. Let

$$l = \sup_{p\in\partial\Omega}\ \sup_{w\in\Lambda^{(0,q-1)}}\ \sup_{v\in\left(\mathcal{N}_p^q\right)^\perp}\left(|\partial r(p)|^{-1}\left|L_q^r\left(\frac{v}{|v|}, \frac{\partial r \wedge w}{|\partial r \wedge w|}\right)\right|\right)$$

and suppose $u^\tau \neq 0$. Define a $(0, q-1)$ form ζ by

$$\zeta = |u^\tau|^{-1}\sum_{j=1}^{n}\sum_{H}{}' \frac{\partial\varphi}{\partial z_j} u_{jH}^\tau d\overline{z_H}.$$

Let

$$\begin{aligned}
A &= \left(-R - \mathcal{A}^2\left(\frac{1}{2\xi} - \frac{\eta a}{1-\eta}\right)^{-1}\left(\frac{\frac{1}{2\xi} - \eta a}{1-\eta}\right) + \frac{qa}{2}\right)\\
B &= \left(-c - 2\mathcal{A}l\left(\frac{1}{2\xi} - \frac{\eta a}{1-\eta}\right)^{-1}\left(\frac{\frac{1}{2\xi} - \eta a}{1-\eta}\right)\right)\\
C &= \left(\frac{qa}{2} - l^2\left(\frac{1}{2\xi} - \frac{\eta a}{1-\eta}\right)^{-1}\left(\frac{\frac{1}{2\xi} - \eta a}{1-\eta}\right)\right).
\end{aligned}$$

By (10), $A > 0$. Thus for $z \in U_{p,\mu}$ sufficiently close to the boundary with a distance independent of p we have $4A\varepsilon + (-r)(4AC - B^2) > 0$. Rearranging terms we have

$$4A(-r)(\varepsilon + C(-r)) > B^2(-r)^2.$$

Hence $A(-r)|u_1^\tau|^2 + B(-r)|u_1^\tau||u_2^\tau| + (\varepsilon + C(-r))|u_2^\tau|^2 > 0$ since the discriminant of this quadratic form is negative and so

$$\begin{aligned} &\varepsilon|u_2^\tau|^2 - R(-r)|u_1^\tau|^2 - c(-r)|u_1^\tau||u_2^\tau| \\ &\quad > (-r)\left(\frac{1}{2\xi} - \frac{\eta a}{1-\eta}\right)^{-1}\left(\frac{\frac{1}{2\xi} - \eta a}{1-\eta}\right)(\mathcal{A}|u_1^\tau| + l|u_2^\tau|)^2 - (-r)\frac{qa}{2}\left(|u_1^\tau|^2 + |u_2^\tau|^2\right). \end{aligned} \tag{12}$$

Since u_1^τ and u_2^τ are orthogonal, we have $|u^\tau|^2 = |u_1^\tau|^2 + |u_2^\tau|^2$. So these along with (12) imply the following

$$\begin{aligned} &\varepsilon|u_2^\tau|^2 - R(-r)|u_1^\tau|^2 - c(-r)|u_1^\tau||u_2^\tau| \\ &\quad > (-r)\left(\frac{\frac{1}{2\xi} - \eta a}{1-\eta}\right)\left(\frac{1}{2\xi} - \frac{\eta a}{1-\eta}\right)^{-1}(\mathcal{A}|u_1^\tau| + l|u_2^\tau|)^2 - (-r)\frac{qa}{2}|u^\tau|^2. \end{aligned} \tag{13}$$

Next we recall our bounds for $\mathcal{A}$ and l. Since $|\partial r(p)| = \left|\frac{\partial r}{\partial z_n}(p)\right|$, for $z \in U_{p,\mu}$, we may assume $|r(z)|$ is sufficiently small so that

$$\begin{aligned} &(-r)\left(\frac{\frac{1}{2\xi} - \eta a}{1-\eta}\right)\left(\frac{1}{2\xi} - \frac{\eta a}{1-\eta}\right)^{-1}(\mathcal{A}|u_1^\tau| + l|u_2^\tau|)^2 - (-r)\frac{qa}{2}|u^\tau|^2 \\ &\quad > (-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\left(\left|L_q^r\left(u_1^\tau, \frac{u^\nu}{|u^\nu|}\right)\right| + \left|L_q^r\left(u_2^\tau, \frac{u^\nu}{|u^\nu|}\right)\right|\right)^2\left(\frac{\frac{1}{2\xi} - \eta a}{1-\eta}\right)\left(\frac{a}{2\xi} - \frac{\eta a^2}{1-\eta}\right)^{-1} \\ &\qquad - (-r)\frac{qa}{2}|u^\tau|^2 - \delta(-r)|u^\tau|^2. \end{aligned} \tag{14}$$

Notice that

$$\begin{aligned}\left(\frac{\frac{1}{2\xi}-\eta a}{1-\eta}\right)\left(\frac{1}{2\xi}-\frac{\eta a}{1-\eta}\right)^{-1} &= \left(-\frac{\eta a}{1-\eta}+\frac{1}{2\xi(1-\eta)}\right)\left(\frac{1}{2\xi}-\frac{\eta a}{1-\eta}\right)^{-1}\\ &= \left(\frac{\eta a^2(\eta-1)}{(1-\eta)^2}+\frac{a}{2\xi(1-\eta)}\right)\left(\frac{a}{2\xi}-\frac{\eta a^2}{1-\eta}\right)^{-1}\\ &= \left(\frac{\eta^2 a^2}{(1-\eta)^2}-\frac{\eta a^2}{(1-\eta)^2}+\frac{a}{2\xi(1-\eta)}\right)\left(\frac{a}{2\xi}-\frac{\eta a^2}{1-\eta}\right)^{-1}\\ &= \left(\frac{\eta^2 a^2}{(1-\eta)^2}+\frac{1}{1-\eta}\left(\frac{a}{2\xi}-\frac{\eta a^2}{1-\eta}\right)\right)\left(\frac{a}{2\xi}-\frac{\eta a^2}{1-\eta}\right)^{-1}\\ &= \left(\frac{\eta a}{(1-\eta)}\right)^2\left(\frac{a}{2\xi}-\frac{\eta a^2}{1-\eta}\right)^{-1}+(1-\eta)^{-1}. \quad (15)\end{aligned}$$

Let $\Phi=\left|L_q^r\left(u_1^\tau,\frac{u^\nu}{|u^\nu|}\right)\right|+\left|L_q^r\left(u_2^\tau,\frac{u^\nu}{|u^\nu|}\right)\right|$. Then from the first term on the right hand side of (14) we may write

$$\begin{aligned}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}&\left(\left|L_q^r\left(u_1^\tau,\frac{u^\nu}{|u^\nu|}\right)\right|+\left|L_q^r\left(u_2^\tau,\frac{u^\nu}{|u^\nu|}\right)\right|\right)^2\left(\frac{\frac{1}{2\xi}-\eta a}{1-\eta}\right)\left(\frac{a}{2\xi}-\frac{\eta a^2}{1-\eta}\right)^{-1}\\ &= (-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2\left(\left(\frac{\eta a}{1-\eta}\right)^2\left(\frac{a}{2\xi}-\frac{\eta a^2}{1-\eta}\right)^{-1}+(1-\eta)^{-1}\right).\end{aligned} \quad (16)$$

Expanding the right-hand side of (16) we have

$$\begin{aligned}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}&\Phi^2\left(\left(\frac{\eta a}{1-\eta}\right)^2\left(\frac{a}{2\xi}-\frac{\eta a^2}{1-\eta}\right)^{-1}+(1-\eta)^{-1}\right)\\ &= (-r)\left(\frac{\eta a}{1-\eta}\right)^2\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)^{-1}\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2+(-r)(1-\eta)^{-1}\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2. \quad (17)\end{aligned}$$

Notice that

$$-(-r)\left(\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)^{\frac{1}{2}}|u^\tau||\zeta|-\eta a(1-\eta)^{-1}\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)^{-\frac{1}{2}}\left|\frac{\partial r}{\partial z_n}\right|^{-1}\Phi\right)^2 \leq 0.$$

If we add this to the right-hand side of (17) we obtain the inequality

$$\begin{aligned}(-r)\left(\frac{\eta a}{1-\eta}\right)^2\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)^{-1}\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2+(-r)(1-\eta)^{-1}\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2\\ \geq -(-r)\left(\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)^{\frac{1}{2}}|u^\tau||\zeta|\right.\\ \left.-\eta a(1-\eta)^{-1}\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)^{-\frac{1}{2}}\left|\frac{\partial r}{\partial z_n}\right|^{-1}\Phi\right)^2\\ +(-r)\left(\frac{\eta a}{1-\eta}\right)^2\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)^{-1}\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2\\ +(-r)(1-\eta)^{-1}\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2. \quad (18)\end{aligned}$$

Expanding the square on the right-hand side of (18) we have

$$\begin{aligned}-(-r)\left(\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)^{\frac{1}{2}}|u^\tau||\zeta|-\eta a(1-\eta)^{-1}\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)^{-\frac{1}{2}}\left|\frac{\partial r}{\partial z_n}\right|^{-1}\Phi\right)^2\\ +(-r)\left(\frac{\eta a}{1-\eta}\right)^2\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)^{-1}\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2+(-r)(1-\eta)^{-1}\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2\\ =-\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)|\zeta|^2(-r)|u^\tau|^2+2\eta a(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-1}|\zeta||u^\tau|\Phi\\ +(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2. \quad (19)\end{aligned}$$

Rearranging the $-\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)$ on the first term, we have

$$-\left(\frac{a}{2\xi}-\eta a^2(1-\eta)^{-1}\right)|\zeta|^2(-r)|u^\tau|^2+2\eta a(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-1}|\zeta||u^\tau|\Phi$$
$$+(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2$$
$$=|\zeta|^2(-r)|u^\tau|^2\left(\eta^2a^2(1-\eta)^{-1}-\left(\frac{a}{2\xi}-\eta a^2\right)\right)$$
$$+2\eta a(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-1}|\zeta||u^\tau|\Phi+(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2. \quad (20)$$

Completing the square for Φ, we have

$$|\zeta|^2(-r)|u^\tau|^2\left(\eta^2a^2(1-\eta)^{-1}-\left(\frac{a}{2\xi}-\eta a^2\right)\right)$$
$$+2\eta a(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-1}|\zeta||u^\tau|\Phi+(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\Phi^2$$
$$=(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\left(\eta a|u^\tau||\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\Phi\right)^2-(-r)\left(\frac{a}{2\xi}-\eta a^2\right)|\zeta|^2|u^\tau|^2. \quad (21)$$

Then from (13)-(21) and substituting the expression that Φ represents back in, we can write,

$$\varepsilon|u_2^\tau|^2-R(-r)|u_1^\tau|^2-c(-r)|u_1^\tau||u_2^\tau|+\delta(-r)|u^\tau|^2$$
$$>(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\left(\eta a|u^\tau||\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L_q^r\left(u_1^\tau,\frac{u^\nu}{|u^\nu|}\right)\right|+\left|L_q^r\left(u_2^\tau,\frac{u^\nu}{|u^\nu|}\right)\right|\right)^2$$
$$-(-r)\left(\frac{a}{2\xi}-\eta a^2\right)|\zeta|^2|u^\tau|^2-(-r)\frac{qa}{2}|u^\tau|^2. \quad (22)$$

Note, we have

$$L_q^r\left(u^\tau,\frac{u^\nu}{|u^\nu|}\right)=L_q^r\left(u_1^\tau,\frac{u^\nu}{|u^\nu|}\right)+L_q^r\left(u_2^\tau,\frac{u^\nu}{|u^\nu|}\right).$$

By (11), the inequality (22) implies

$$L_q^r(u^\tau, u^\tau) > (1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\left(\eta a|u^\tau||\zeta|\left|\frac{\partial r}{\partial z_n}\right| + \left|L_q^r\left(u^\tau, \frac{u^\nu}{|u^\nu|}\right)\right|\right)^2 - (-r)\left(\frac{a}{2\xi} - \eta a^2\right)|\zeta|^2|u^\tau|^2 - (-r)\frac{qa}{2}|u^\tau|^2.$$

Dividing this inequality through by $|u^\tau|^2$ results in

$$L_q^r\left(\frac{u^\tau}{|u^\tau|}, \frac{u^\tau}{|u^\tau|}\right) > (1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right| + \left|L_q^r\left(\frac{u^\tau}{|u^\tau|}, \frac{u^\nu}{|u^\nu|}\right)\right|\right)^2 - (-r)\left(\frac{a}{2\xi} - \eta a^2\right)|\zeta|^2 - (-r)\frac{qa}{2}. \tag{23}$$

Rearranging, we have

$$\left(L_q^r\left(\frac{u^\tau}{|u^\tau|}, \frac{u^\tau}{|u^\tau|}\right) - r\left(\frac{a}{2\xi} - \eta a^2\right)|\zeta|^2 - r\frac{qa}{2}\right)\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right| + \left|L_q^r\left(\frac{u^\tau}{|u^\tau|}, \frac{u^\nu}{|u^\nu|}\right)\right|\right)^{-1} > (1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right| + \left|L_q^r\left(\frac{u^\tau}{|u^\tau|}, \frac{u^\nu}{|u^\nu|}\right)\right|\right).$$

From this inequality, there exists a constant $d > 0$ such that we can write the following two inequalities

$$\frac{1}{2}\left(L_q^r\left(\frac{u^\tau}{|u^\tau|}, \frac{u^\tau}{|u^\tau|}\right) - r\left(\frac{a}{2\xi} - \eta a^2\right)|\zeta|^2 - r\frac{qa}{2}\right)\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right| + \left|L_q^r\left(\frac{u^\tau}{|u^\tau|}, \frac{u^\nu}{|u^\nu|}\right)\right|\right)^{-1} > d \tag{24}$$

and

$$d > \frac{1}{2}(1-\eta)^{-1}(-r)\left|\frac{\partial r}{\partial z_n}\right|^{-2}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right| + \left|L_q^r\left(\frac{u^\tau}{|u^\tau|}, \frac{u^\nu}{|u^\nu|}\right)\right|\right). \tag{25}$$

We can rearrange (24) and multiply through by $\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}$ to get

$$\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\tau}{|u^\tau|}\right)+\eta\left(\frac{a}{2\xi}-\eta a^2\right)(-r)^\eta e^{-\eta a\varphi(z)}|\zeta|^2+\eta(-r)^\eta e^{-\eta a\varphi(z)}\frac{qa}{2}$$
$$-2\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\right)\right|\right)d>0 \quad (26)$$

We can rearrange (25) and multiply through by $\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}$ to get

$$0<-\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\right)\right|\right)\frac{1}{2d}$$
$$+\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left|\frac{\partial r}{\partial z_n}\right|^2. \quad (27)$$

Next we multiply (26) through by $|u^\tau|^2$ to get

$$\Bigg(\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\tau}{|u^\tau|}\right)$$
$$+\eta\left(\frac{a}{2\xi}-\eta a^2\right)(-r)^\eta e^{-\eta a\varphi(z)}|\zeta|^2+\eta(-r)^\eta e^{-\eta a\varphi(z)}\frac{qa}{2}\Bigg)|u^\tau|^2$$
$$-2\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\right)\right|\right)d|u^\tau|^2>0. \quad (28)$$

Similarly, we multiply (27) through by $|u^\nu|^2$ to get

$$0<-\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\right)\right|\right)\frac{1}{2d}|u^\nu|^2$$
$$+\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left|\frac{\partial r}{\partial z_n}\right|^2|u^\nu|^2. \quad (29)$$

Next we add inequalities (28) and (29) together to get

$$\begin{aligned}
\Bigg(\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\tau}{|u^\tau|}\right)& \\
+\eta\left(\frac{a}{2\xi}-\eta a^2\right)(-r)^\eta e^{-\eta a\varphi(z)}|\zeta|^2+\eta(-r)^\eta e^{-\eta a\varphi(z)}\frac{qa}{2}\Bigg)\,|u^\tau|^2& \\
-2\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\right)\right|\right)d\big|u^\tau\big|^2& \\
-\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\right)\right|\right)\frac{1}{2d}|u^\nu|^2& \\
+\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left|\frac{\partial r}{\partial z_n}\right|^2\big|u^\nu\big|^2>0.& \quad (30)
\end{aligned}$$

By Cauchy's inequality $ab\le da^2+\frac{b^2}{4d}$ for $a,b>0$. Using this in inequality (30), we have

$$\begin{aligned}
\Bigg(\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\tau}{|u^\tau|}\right)& \\
+\eta\left(\frac{a}{2\xi}-\eta a^2\right)(-r)^\eta e^{-\eta a\varphi(z)}|\zeta|^2+\eta(-r)^\eta e^{-\eta a\varphi(z)}\frac{qa}{2}\Bigg)\,|u^\tau|^2& \\
-2\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\right)\right|\right)|u^\tau||u^\nu|& \\
+\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left|\frac{\partial r}{\partial z_n}\right|^2\big|u^\nu\big|^2>0.& \quad (31)
\end{aligned}$$

The next goal is to show that the left side of this inequality is less than or equal to $\partial\bar\partial\left(-\left(-\rho\right)^\eta\right)(u,u)$. To work to this goal, we will consider the $|u^\tau|^2$, $|u^\tau||u^\nu|$, and $|u^\nu|^2$ terms from the left side of this inequality individually. After we have examined each of these individually we will bring the results back together.

We first compute $\partial\bar{\partial}\left(-\left(-\rho\right)^{\eta}\right)(u,u)$:

$$\partial\bar{\partial}\left(-\left(-\rho\right)^{\eta}\right)(u,u) = \sum_{j,k=1}^{n}{\sum_{H}}' \left(\eta(1-\eta)(-\rho)^{\eta-2}\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial\overline{z_k}} + \eta(-\rho)^{\eta-1}\frac{\partial^2\rho}{\partial z_j\partial\overline{z_k}}\right) u_{jH}\overline{u_{kH}}$$

$$= \sum_{j,k=1}^{n}{\sum_{H}}' \left(\eta(1-\eta)\left(-re^{-a\varphi(z)}\right)^{\eta-2}\left(\frac{\partial r}{\partial z_j}e^{-a\varphi(z)} - ra\frac{\partial\varphi}{\partial z_j}e^{-a\varphi(z)}\right)\right.$$

$$\left(\frac{\partial r}{\partial\overline{z_k}}e^{-a\varphi(z)} - ra\frac{\partial\varphi}{\partial\overline{z_k}}e^{-a\varphi(z)}\right) + \eta\left(-re^{-a\varphi(z)}\right)^{\eta-1}\left(\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}e^{-a\varphi(z)} - a\frac{\partial r}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}e^{-a\varphi(z)}\right.$$

$$\left.\left.-ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}e^{-a\varphi(z)} - a\frac{\partial r}{\partial\overline{z_k}}\frac{\partial\varphi}{\partial z_j}e^{-a\varphi(z)} + ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}e^{-a\varphi(z)}\right)\right) u_{jH}\overline{u_{kH}}$$

$$= \sum_{j,k=1}^{n}{\sum_{H}}' \left(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left(\frac{\partial r}{\partial z_j} - ra\frac{\partial\varphi}{\partial z_j}\right)\left(\frac{\partial r}{\partial\overline{z_k}} - ra\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right.$$

$$\left.+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}} - a\frac{\partial r}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}} - ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}} - a\frac{\partial r}{\partial\overline{z_k}}\frac{\partial\varphi}{\partial z_j} + ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right) u_{jH}\overline{u_{kH}}$$

$$= \sum_{j,k=1}^{n}{\sum_{H}}' \left[\left(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}r^2a^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}} + \eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}\right.\right.$$

$$\left.-\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}} + \eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\right) u^{\tau}_{jH}\overline{u^{\tau}_{kH}}$$

$$+\left(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}r(-a)\frac{\partial\varphi}{\partial z_j}\left(\frac{\partial r}{\partial\overline{z_k}} - ra\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right.$$

$$\left.+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}} - ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}} - a\frac{\partial r}{\partial\overline{z_k}}\frac{\partial\varphi}{\partial z_j} + ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right) u^{\tau}_{jH}\overline{u^{\nu}_{kH}}$$

$$+\left(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}r(-a)\frac{\partial\varphi}{\partial\overline{z_k}}\left(\frac{\partial r}{\partial z_j} - ra\frac{\partial\varphi}{\partial z_j}\right)\right.$$

$$\left.+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}} - ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}} - a\frac{\partial r}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}} + ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right) u^{\nu}_{jH}\overline{u^{\tau}_{kH}}$$

$$+\left(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left(\frac{\partial r}{\partial z_j} - ra\frac{\partial\varphi}{\partial z_j}\right)\left(\frac{\partial r}{\partial\overline{z_k}} - ra\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right.$$

$$+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}} - a\frac{\partial r}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}} - ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}} - a\frac{\partial r}{\partial\overline{z_k}}\frac{\partial\varphi}{\partial z_j}\right.$$

$$\left.\left.\left.+ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right) u^{\nu}_{jH}\overline{u^{\nu}_{kH}}\right]. \quad (32)$$

Now we estimate each term in (31) with the corresponding term in (32). From the $|u^{\tau}|^2$

term of (31) we use (3) and (4) to show

$$\begin{aligned}
\Bigg(\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\tau}{|u^\tau|}\right)& \\
+\eta\left(\frac{a}{2\xi}-\eta a^2\right)(-r)^\eta e^{-\eta a\varphi(z)}|\zeta|^2+\eta(-r)^\eta e^{-\eta a\varphi(z)}\frac{qa}{2}\Bigg)\,|u^\tau|^2& \\
\leq \sum_{j,k=1}^{n}\sum_H{}'\Bigg(\eta(-r)^{(\eta-1)}e^{-\eta a\varphi(z)}\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}& \\
+\eta(-r)^\eta e^{-\eta a\varphi(z)}\left(\left(\frac{a}{2\xi}-\eta a^2\right)\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}+\frac{a}{2}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}\right)\Bigg)\,u_{jH}^\tau\overline{u_{kH}^\tau}& \\
\leq \sum_{j,k=1}^{n}\sum_H{}'\Bigg(\eta(-r)^{(\eta-1)}e^{-\eta a\varphi(z)}\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}& \\
+\eta(-r)^\eta e^{-\eta a\varphi(z)}\left(-\eta a^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}+a\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}\right)\Bigg)\,u_{jH}^\tau\overline{u_{kH}^\tau}.& \quad (33)
\end{aligned}$$

Rearranging terms results in

$$\begin{aligned}
\sum_{j,k=1}^{n}\sum_H{}'\Bigg(\eta(-r)^{(\eta-1)}e^{-\eta a\varphi(z)}\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}& \\
+\eta(-r)^\eta e^{-\eta a\varphi(z)}\left(-\eta a^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}+a\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}\right)\Bigg)\,u_{jH}^\tau\overline{u_{kH}^\tau}& \\
=\sum_{j,k=1}^{n}\sum_H{}'\Bigg(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}r^2a^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}& \\
-\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\Bigg)\,u_{jH}^\tau\overline{u_{kH}^\tau}.& \quad (34)
\end{aligned}$$

Thus we have

$$
\begin{aligned}
&\Bigg(\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}L^r_q\Big(\frac{u^\tau}{|u^\tau|},\frac{u^\tau}{|u^\tau|}\Big)\\
&\qquad\qquad +\eta\Big(\frac{a}{2\xi}-\eta a^2\Big)(-r)^\eta e^{-\eta a\varphi(z)}|\zeta|^2+\eta(-r)^\eta e^{-\eta a\varphi(z)}\frac{qa}{2}\Bigg)|u^\tau|^2\\
&\quad\le\sum_{j,k=1}^{n}{\sum_H}'\Bigg(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}r^2a^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}\\
&\qquad\qquad -\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\Bigg)u^\tau_{jH}\overline{u^\tau_{kH}}. \qquad (35)
\end{aligned}
$$

This provides us with an upper bound for the $|u^\tau|^2$ term.

Next, we consider the $|u^\tau||u^\nu|$ expression from the left side of (31). Due to our choice of coordinates, $u^\tau_{jH}\overline{u^\nu_{kH}}\neq 0$ only if $n\notin H$, $j\neq n$ and $k=n$. So here we have

$$
\begin{aligned}
&\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L^r_q\Big(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\Big)\right|\right)|u^\tau||u^\nu|\\
&\qquad\ge\eta^2(-r)^{\eta-1}e^{-\eta a\varphi(z)}a|\zeta|\left|\frac{\partial r}{\partial z_n}\right||u^\tau|\Bigg({\sum_{H,n\notin H}}'|\overline{u^\nu_{nH}}|^2\Bigg)^{\frac{1}{2}}\\
&\qquad\quad+\left|\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}L^r_q\Big(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\Big)\right||u^\tau||u^\nu|\\
&\quad\ge\left|\sum_{j=1}^{n-1}{\sum_{H,n\notin H}}'\eta^2(-r)^{\eta-1}e^{-\eta a\varphi(z)}a\Big(ar\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_n}}-\frac{\partial r}{\partial\overline{z_n}}\frac{\partial\varphi}{\partial z_j}\Big)u^\tau_{jH}\overline{u^\nu_{nH}}\right|\\
&\qquad\quad+\left|\sum_{j,k=1}^{n}{\sum_H}'\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\frac{\partial^2 r}{\partial z_j\partial\overline{z_n}}u^\tau_{jH}\overline{u^\nu_{nH}}\right|\\
&\qquad\qquad+\left|\sum_{j=1}^{n-1}{\sum_{H,n\notin H}}'\eta(-r)^\eta e^{-\eta a\varphi(z)}a\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_n}}u^\tau_{jH}\overline{u^\nu_{nH}}\right|-O((-r)^\eta).
\end{aligned}
$$

Then rearranging and using the triangle inequality, we have

$$\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\right)\right|\right)|u^\tau||u^\nu|$$
$$\geq\left|\sum_{j,k=1}^{n}{\sum_H}'\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}u^\tau_{jH}\overline{u^\nu_{nH}}\right|$$
$$+\left|\sum_{j=1}^{n-1}{\sum_{H,n\notin H}}'\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(-\eta a\frac{\partial\varphi}{\partial z_j}\frac{\partial r}{\partial\overline{z_n}}+\eta a^2 r\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_n}}-ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_n}}\right)u^\tau_{jH}\overline{u^\nu_{nH}}\right|-O((-r)^\eta)$$
$$\geq\left|\sum_{j,k=1}^{n}{\sum_H}'\left(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}r(-a)\frac{\partial\varphi}{\partial z_j}\left(\frac{\partial r}{\partial\overline{z_k}}-ra\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right.\right.$$
$$\left.\left.+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}-ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}-a\frac{\partial r}{\partial\overline{z_k}}\frac{\partial\varphi}{\partial z_j}+ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right)u^\tau_{jH}\overline{u^\nu_{kH}}\right|-O((-r)^\eta).\quad(36)$$

Finally, we consider the $|u^\nu|^2$ expression from (31). Here we note that $u^\nu_{jH}\overline{u^\nu_{kH}}\neq 0$ only if $n\in H$ or $j=k=n$. We then have the following

$$\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left|\frac{\partial r}{\partial z_n}\right|^2\left|u^\nu\right|^2\ \leq\ {\sum_{H,n\notin H}}'\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left|\frac{\partial r}{\partial z_n}\right|^2u^\nu_{nH}\overline{u^\nu_{nH}}.\quad(37)$$

Note that we can rewrite the right-hand side of (37)

$${\sum_{H,n\notin H}}'\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left|\frac{\partial r}{\partial z_n}\right|^2u^\nu_{nH}\overline{u^\nu_{nH}}$$
$$=\sum_{j,k=1}^{n}{\sum_H}'\left(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left(\frac{\partial r}{\partial z_j}-ra\frac{\partial\varphi}{\partial z_j}\right)\left(\frac{\partial r}{\partial\overline{z_k}}-ra\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right.$$
$$\left.+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}-a\frac{\partial r}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}-ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}-a\frac{\partial r}{\partial\overline{z_k}}\frac{\partial\varphi}{\partial z_j}+ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\right)\right)u^\nu_{jH}\overline{u^\nu_{kH}}$$
$$+O((-r)^\eta).\quad(38)$$

Note, the terms of the form $\frac{\partial r}{\partial z_j}\frac{\partial r}{\partial\overline{z_k}}$ can be ignored since in our coordinates, $\left|\frac{\partial r}{\partial z_j}\right|\leq O(|r|)$

when $1 \leq j \leq n-1$. Thus

$$\begin{aligned}
&\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left|\frac{\partial r}{\partial z_n}\right|^2\left|u^\nu\right|^2 \\
&\qquad \leq \sum_{j,k=1}^{n}\sum_H{}' \left(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left(\frac{\partial r}{\partial z_j}-ra\frac{\partial \varphi}{\partial z_j}\right)\left(\frac{\partial r}{\partial \overline{z_k}}-ra\frac{\partial \varphi}{\partial \overline{z_k}}\right)\right. \\
&+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left.\left(\frac{\partial^2 r}{\partial z_j\partial \overline{z_k}}-a\frac{\partial r}{\partial z_j}\frac{\partial \varphi}{\partial \overline{z_k}}-ra\frac{\partial^2 \varphi}{\partial z_j\partial \overline{z_k}}-a\frac{\partial r}{\partial \overline{z_k}}\frac{\partial \varphi}{\partial z_j}+ra^2\frac{\partial \varphi}{\partial z_j}\frac{\partial \varphi}{\partial \overline{z_k}}\right)\right)u^\nu_{jH}\overline{u^\nu_{kH}} \\
&\qquad\qquad + O((-r)^\eta). \quad (39)
\end{aligned}$$

Now bringing (35), (36), and (39) back to (31), we have

$$
\begin{aligned}
&\left(\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\tau}{|u^\tau|}\right)-\eta^2(-r)^\eta e^{-\eta a\varphi(z)}a^2|\zeta|^2+\eta(-r)^\eta e^{-\eta a\varphi(z)}qa\right)|u^\tau|^2\\
&\qquad -2\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\eta a|\zeta|\left|\frac{\partial r}{\partial z_n}\right|+\left|L_q^r\left(\frac{u^\tau}{|u^\tau|},\frac{u^\nu}{|u^\nu|}\right)\right|\right)|u^\tau||u^\nu|\\
&\qquad\qquad +\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left|\frac{\partial r}{\partial z_n}\right|^2|u^\nu|^2\\
&\leq \sum_{j,k=1}^n{\sum_H}'\Bigg[\bigg(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}r^2a^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}\\
&\qquad -\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}+\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\bigg)u^\tau_{jH}\overline{u^\tau_{kH}}\\
&\qquad +\bigg(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}r(-a)\frac{\partial\varphi}{\partial z_j}\left(\frac{\partial r}{\partial\overline{z_k}}-ra\frac{\partial\varphi}{\partial\overline{z_k}}\right)\\
&\quad +\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}-ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}-a\frac{\partial r}{\partial\overline{z_k}}\frac{\partial\varphi}{\partial z_j}+ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\right)\bigg)u^\tau_{jH}\overline{u^\nu_{kH}}\\
&\qquad +\bigg(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}r(-a)\frac{\partial\varphi}{\partial\overline{z_k}}\left(\frac{\partial r}{\partial z_j}-ra\frac{\partial\varphi}{\partial z_j}\right)\\
&\quad +\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\left(\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}-ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}-a\frac{\partial r}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}+ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\right)\bigg)u^\nu_{jH}\overline{u^\tau_{kH}}\\
&\qquad +\bigg(\eta(1-\eta)(-r)^{\eta-2}e^{-\eta a\varphi(z)}\left(\frac{\partial r}{\partial z_j}-ra\frac{\partial\varphi}{\partial z_j}\right)\left(\frac{\partial r}{\partial\overline{z_k}}-ra\frac{\partial\varphi}{\partial\overline{z_k}}\right)\\
&\qquad +\eta(-r)^{\eta-1}e^{-\eta a\varphi(z)}\bigg(\frac{\partial^2 r}{\partial z_j\partial\overline{z_k}}-a\frac{\partial r}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}-ra\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}-a\frac{\partial r}{\partial\overline{z_k}}\frac{\partial\varphi}{\partial z_j}\\
&\qquad\qquad +ra^2\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial\overline{z_k}}\bigg)\bigg)u^\nu_{jH}\overline{u^\nu_{kH}}\Bigg]+O((-r)^\eta). \qquad (40)
\end{aligned}
$$

From (32), we see that the left-side of (40) is equal to $\partial\bar{\partial}\left(-(-\rho)^\eta\right)(u,u)$.

From Theorem 4.18 [13], the boundary of every C^2 domain has positive reach. Thus for every point $z\in\Omega$ sufficiently close to $\partial\Omega$ there exists a unique closest $p\in\partial\Omega$ and so our proof holds for every $z\in\Omega$ on a neighborhood of the boundary. By Step 2 in the proof of Theorem 1 in [11], we can extend to all of Ω. We have shown (i) from Lemma 4.7 in [29] to be true. This lemma states that what we have shown is equivalent to the sum of the q smallest eigenvalues being nonnegative. Hence, for the conditions of the theorem and when

$a = \frac{\xi(1-\eta)}{2\eta} - \left(\frac{\xi}{q}\right)^{\frac{1}{2}} \mathcal{A}$, we have that $-(-\rho)^\eta$ is q-subharmonic on Ω. □

6 Proofs of Corollaries

In this section we will be proving the three corollaries. We start with the proof of Corollary 3.2.

Proof. Let $\varphi(z) = |z|^2$. We have that (4) holds with equality. If $\xi \geq \sup_\Omega |z|^2$ then we also have (3). This may mean that ξ is very large. However, for every ξ there exists an η so that (7) holds since $\mathcal{A} \geq 0$ and (8) also holds. Hence, for η sufficiently small, by Theorem 3.1 there exists a nonnegative real valued number a and a C^3 defining function ρ such that $\rho = re^{-a\varphi(z)}$ on a neighborhood of $\partial\Omega$ in Ω and $-(-\rho)^\eta$ is q-subharmonic on Ω. □

Next, we will prove Corollary 3.3.

Proof. Let $\varphi(z) = |z|^2$. We have that (4) holds with equality. If $\xi \geq \sup_\Omega |z|^2$ then we also have (3). This may mean that ξ is very large. However, for every ξ all $0 < \eta < 1$ satisfy (7) since $\mathcal{A} = 0$. Now, since $\mathcal{B} \leq 0$, $\mathcal{A} = 0$, and $0 \leq \frac{q(1-\eta)}{4\xi\eta}$ we have that (8) holds. Hence for all $0 < \eta < 1$ by Theorem 3.1 there exists a nonnegative real valued number a and a C^3 defining function ρ such that $\rho = re^{-a\varphi(z)}$ on a neighborhood of $\partial\Omega$ in Ω and $-(-\rho)^\eta$ is q-subharmonic on Ω. □

Finally, we will prove Corollary 3.4.

Proof. Let r be the signed distance function. Then by Lemma 4.4 we have that $\mathcal{B} \leq 0$. Let f be given by Definition 2.12 for some $M > 0$. Let $\varphi = \frac{q}{M} f$. Then $\xi = \frac{Cq}{M}$ for some positive constant C, so we can make ξ as small as we want. $\mathcal{A}$ may be very large, but for

$$\xi < \frac{q(1-\eta)^2}{4\mathcal{A}^2\eta^2}$$

we will satisfy (7) for any $0 < \eta < 1$. Since we can make ξ as small as we want, pick a ξ so that

$$\xi < \frac{q\left(\eta^{-\frac{1}{2}} - 1\right)^2}{4\mathcal{A}^2}.$$

Then, we have that $0 < \frac{q(1-\eta)}{4\xi\eta} - \frac{q^{\frac{1}{2}}}{\xi^{\frac{1}{2}}}\mathcal{A} - \mathcal{A}^2$. Since we also have that $\mathcal{B} \leq 0$, (8) holds. Hence, for all $0 < \eta < 1$ by Theorem 3.1 there exists a nonnegative real valued number a and a C^3 defining function ρ such that $\rho = re^{-a\varphi(z)}$ on a neighborhood of $\partial\Omega$ in Ω and $-(-\rho)^\eta$ is q-subharmonic on Ω. □

7 Local Examples

In this section, we will examine a couple of examples. Each of the examples below are only constructed locally but still demonstrate important differences. They are unbounded domains but since we are only considering a local construction they could be pieces of a bounded domain. By a similar process used in Proposition 6.6 of [19], we can construct a bounded domain $\widetilde{\Omega}$ with the same properties as our domain Ω near zero. By Corollary 3.3, for all $0 < \eta < 1$ there exists a nonnegative real valued number a for every C^3 defining function ρ such that $\rho = re^{-a\varphi(z)}$ on a neighborhood of $\partial\widetilde{\Omega}$ in $\widetilde{\Omega}$.

Proposition 7.1 will present a diagonalizable Levi form while the Levi form in Proposition 7.2 is non-diagonalizable. In many applications, it is helpful to know whether the Levi form is diagonalizable or non-diagonalizable, see Derridj in [10] for examples. Ben Moussa in [2] considered Derridj's ideas when $q > 1$ on pseudoconvex domains and Harrington and Raich in [18] consider Derridj's ideas when $q > 1$ on non-pseudoconvex domains. In both example provided in this section, we note that they are on weakly q-convex domains that are not always hyper q-convex domains, as in Definition 2.4.

We begin by computing the trace of the Levi form in general. Let ρ be a C^2 defining function for Ω. Let $\{L_j\}_{j=1}^{n-1}$ be an orthonormal basis for $T^{1,0}(\partial\Omega)$ with $L_n = |\partial\rho|^{-1}\sum_{j=1}^{n-1} \frac{\partial\rho}{\partial\bar{z}_j}\frac{\partial}{\partial z_j}$. Let a_{jk} be C^1 functions so that $L_j = \sum_{k=1}^n a_{jk}\frac{\partial}{\partial z_k}$. Then $\{L_j\}_{j=1}^n$ and $\{\frac{\partial}{\partial z_k}\}_{k=1}^n$ are both orthonormal.

The characteristic equation for a matrix is independent of basis [21]. The trace, as part of the characteristic equation, is then independent of the orthonormal basis. The trace of the Levi form is then

$$\begin{aligned}\sum_{j=1}^{n-1}\partial\overline{\partial}\rho\big(L_j\wedge\overline{L_j}\big) &= \sum_{j=1}^{n}\partial\overline{\partial}\rho\big(L_j\wedge\overline{L_j}\big)-\partial\overline{\partial}\rho\big(L_n\wedge\overline{L_n}\big)\\ &= \sum_{k=1}^{n}\partial\overline{\partial}\rho\left(\frac{\partial}{\partial z_k}\wedge\frac{\partial}{\partial\overline{z_k}}\right)-\partial\overline{\partial}\rho\big(L_n\wedge\overline{L_n}\big)\\ &= \sum_{j=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z_j}}-|\partial\rho|^{-2}\sum_{j,k=1}^{n}\frac{\partial\rho}{\partial z_k}\frac{\partial^2\rho}{\partial z_j\partial\overline{z_k}}\frac{\partial\rho}{\partial\overline{z_j}}.\end{aligned}\tag{41}$$

Proposition 7.1. *Let $n=3$, $q=2$, and $u\in\Lambda^{(0,2)}$. Let*

$$f(z_1,z_2)=\lambda_1|z_1|^2+\lambda_2|z_2|^2+\lambda_3|z_1|^4+\lambda_4|z_2|^4,$$

$$r(z)=\lambda_1|z_1|^2+\lambda_2|z_2|^2+\lambda_3|z_1|^4+\lambda_4|z_2|^4-\operatorname{Im}z_3,$$

and $\Omega=\{z\in\mathbb{C}^3:\operatorname{Im}z_3>f(z_1,z_2)\}$. If either of the following are true

- $\lambda_1+\lambda_2>0$ *or*
- $\lambda_1+\lambda_2=0$, $\lambda_3-\lambda_1^3>0$, *and* $\lambda_4-\lambda_2^3>0$,

then $\partial\Omega$ is weakly 2-convex in a neighborhood of zero. Additionally, note that

- *if $\lambda_1+\lambda_2>0$, then $\mathcal{N}_0^2$ is trivial and*
- *if $\lambda_1+\lambda_2=0$, $\lambda_3-\lambda_1^3>0$, and $\lambda_4-\lambda_2^3>0$, then $\mathcal{N}_0^2=T_0^{0,2}(\partial\Omega)$.*

In either case, $\mathcal{B}=0$ and $\mathcal{A}=0$ where $\mathcal{B}$ is defined by (5) and $\mathcal{A}$ is defined by (6) from the statement of Theorem 3.1.

Proof. We start by using our defining function to compute some necessary derivatives. Let $r(z)$ be as in the statement and compute

$$\partial r = \big(\lambda_1 \overline{z_1} + 2\lambda_3 z_1 \overline{z_1}^2\big) dz_1 + \big(\lambda_2 \overline{z_2} + 2\lambda_4 z_2 \overline{z_2}^2\big) dz_2 - \frac{1}{2i} dz_3 \tag{42}$$

and

$$\partial\overline{\partial} r = \big(\lambda_1 + 4\lambda_3 |z_1|^2\big) dz_1 \wedge d\overline{z_1} + \big(\lambda_2 + 4\lambda_4 |z_2|^2\big) dz_2 \wedge d\overline{z_2}. \tag{43}$$

At 0, $\partial r = -\frac{1}{2i} dz_3$ and $T_0^{1,0}(\partial\Omega) = span\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\}$. In a neighborhood of zero, the trace of the Levi form using (41) is

$$\begin{aligned}
\sum_{j=1}^{n-1} \partial\overline{\partial}\rho\big(L_j \wedge \overline{L_j}\big) &= \lambda_1 + 4\lambda_3 |z_1|^2 + \lambda_2 + 4\lambda_4 |z_2|^2 \\
&\quad - \big(4 + O\big(|z_1|^2 + |z_2|^2\big)\big) \Big(\big(\lambda_1 \overline{z_1} + 2\lambda_3 z_1 \overline{z_1}^2\big)\big(\lambda_1 + 4\lambda_3 |z_1|^2\big)\big(\lambda_1 z_1 + 2\lambda_3 z_1^2 \overline{z_1}\big) \\
&\quad + \big(\lambda_2 \overline{z_2} + 2\lambda_4 z_2 \overline{z_2}^2\big)\big(\lambda_2 + 4\lambda_4 |z_2|^2\big)\big(\lambda_2 z_2 + 2\lambda_4 z_2^2 \overline{z_2}\big)\Big) \\
&= \lambda_1 + 4\big(\lambda_3 - \lambda_1^3\big)|z_1|^2 + \lambda_2 + 4\big(\lambda_4 - \lambda_2^3\big)|z_2|^2 + O\big(|z_1|^4 + |z_2|^4\big)
\end{aligned}$$

Hence, $\partial\Omega$ is weakly 2-convex in a neighborhood of zero when $\lambda_1 + \lambda_2 > 0$ or if $\lambda_1 + \lambda_2 = 0$, $\lambda_3 - \lambda_1^3 > 0$, and $\lambda_4 - \lambda_2^3 > 0$. One can also check that $T_0^{0,2}(\partial\Omega) = \{u = u_{12} d\overline{z_1} \wedge d\overline{z_2}\}$. Thus, when $\lambda_1 + \lambda_2 > 0$ we have that $\mathcal{N}_0^2$ is trivial and when $\lambda_1 + \lambda_2 = 0$, $\lambda_3 - \lambda_1^3 > 0$, and $\lambda_4 - \lambda_2^3 > 0$, we have that $\mathcal{N}_0^2 = T_0^{0,2}(\partial\Omega)$.

When $\lambda_1 + \lambda_2 > 0$ we have that $\mathcal{N}_0^2$ is trivial. The set of $v \in \mathcal{N}_p^q$ such that $v \neq 0$ is then an empty set. Thus $\mathcal{B} = 0$ and $\mathcal{A} = 0$.

When $\lambda_1 + \lambda_2 = 0$, $\lambda_3 - \lambda_1^3 > 0$, and $\lambda_4 - \lambda_2^3 > 0$, $\mathcal{N}_0^2 = T_0^{0,2}(\partial\Omega)$. So the set of $v \in \mathcal{N}_0^2$ such that $v \neq 0$ is no longer an empty set. We start by considering $\mathcal{B}$. We have from Definition 2.7 and (43) that

$$L_2^r(v, v) = \big(\lambda_1 + 4\lambda_3 |z_1|^2 + \lambda_2 + 4\lambda_4 |z_2|^2\big) v_{12} \overline{v_{12}}. \tag{44}$$

In $U_{p,\mu}$, we have that $z_1 = z_2 = 0$. Now since $\lambda_1 + \lambda_2 = 0$, we have that $L_2^r(v, v) = 0$ for all $z \in U_{p,\mu}$. Therefore, $\mathcal{B} = 0$.

Now we will consider $\mathcal{A}$. We have from Definition 2.7 and (43) for $z \in U_{p,\mu}$ that

$$L_q^r\left(\frac{v}{|v|}, \frac{\partial r \wedge w}{|\partial r \wedge w|}\right) = \left(\lambda_1 + 4\lambda_3|z_1|^2 + \lambda_2 + 4\lambda_4|z_2|^2\right)\left(\frac{v}{|v|}\right)_{12}\overline{\left(\frac{\partial r \wedge w}{|\partial r \wedge w|}\right)_{12}}. \tag{45}$$

At $0 \in \partial\Omega$, we have that $(\partial r \wedge w)$ is only a linear combination of $dz_3 \wedge dz_1$ and $dz_3 \wedge dz_2$. So $(\partial r \wedge w)_{12} = 0$. Hence $L_q^r\left(\frac{v}{|v|}, \frac{\partial r \wedge w}{|\partial r \wedge w|}\right) = 0$. Therefore, $\mathcal{A} = 0$. Thus the only possibilities for $\mathcal{B}$ and $\mathcal{A}$ for this example are $\mathcal{B} = 0$ and $\mathcal{A} = 0$. □

Remark 1. It is important to note why there is a problem with $\lambda_1 + \lambda_2 = 0$. In the case where $\lambda_1 = 0 = \lambda_2$, if $z_1 \neq 0$, $z_2 = 0$, and $\lambda_3 < 0$ the trace is negative. For the case where $\lambda_1 = 0 = \lambda_2$, if $z_2 \neq 0$, $z_1 = 0$, and $\lambda_4 < 0$ the trace is negative. Otherwise, if $\lambda_1 + \lambda_2 = 0$ either λ_1 or λ_2 will be positive. If λ_1 is positive then when $z_1 \neq 0$, $z_2 = 0$, and $\lambda_3 - \lambda_1^3$ is small enough the trace is negative. Similarly, if λ_2 is positive then when $z_1 = 0$, $z_2 \neq 0$, and $\lambda_4 - \lambda_2^3$ is small enough the trace is negative. So in these cases we cannot construct a ball about the origin to make the trace nonnegative.

Proposition 7.2. *Let $n = 3$, $q = 2$, and $u \in \Lambda^{(0,2)}$. Let*

$$f(z) = \lambda_1|z_1|^2 + \lambda_2|z_2|^2 + \lambda_3|z_1|^4 + \lambda_4|z_2|^4 + \lambda_5|z_1|^2|z_2|^2,$$

$$r(z) = \lambda_1|z_1|^2 + \lambda_2|z_2|^2 + \lambda_3|z_1|^4 + \lambda_4|z_2|^4 + \lambda_5|z_1|^2|z_2|^2 - \operatorname{Im} z_3,$$

and $\Omega = \{z \in \mathbb{C}^3 : \operatorname{Im} z_3 > f(z_1, z_2)\}$. If either of the following are true

- $\lambda_1 + \lambda_2 > 0$ *or*
- $\lambda_1 + \lambda_2 = 0$, $4\lambda_4 + \lambda_5 - 4\lambda_1^3 > 0$, *and* $4\lambda_3 + \lambda_5 - 4\lambda_2^3 > 0$,

then $\partial\Omega$ is weakly 2-convex in a neighborhood of zero. Additionally, note that

- *if $\lambda_1 + \lambda_2 > 0$, then $\mathcal{N}_0^2$ is trivial and*

- *if* $\lambda_1 + \lambda_2 = 0$*, then* $\mathcal{N}_0^2 = T_0^{0,2}(\partial\Omega)$.

In either case, $\mathcal{B} = 0$ *and* $\mathcal{A} = 0$ *where* $\mathcal{B}$ *is defined by (5) and* $\mathcal{A}$ *is defined by (6) from the statement of Theorem 3.1. Finally, in the case where* $\lambda_1 = \lambda_2 = 0$*, if the following hold*

- $\lambda_5 \neq 0$ *and*

- $\lambda_4 \neq \lambda_3$,

then the Levi form is not locally diagonalizable in a neighborhood of zero.

Proof. For this example, we again start computing some necessary derivatives. Let $r(z)$ be as assumed in the statement and we compute

$$\partial r = \big(\lambda_1 \overline{z_1} + 2\lambda_3 z_1 \overline{z_1}^2 + \lambda_5 \overline{z_1} z_2 \overline{z_2}\big) dz_1 + \big(\lambda_2 \overline{z_2} + 2\lambda_4 z_2 \overline{z_2}^2 + \lambda_5 z_1 \overline{z_1 z_2}\big) dz_2 - \frac{1}{2i} dz_3 \tag{46}$$

and

$$\partial\overline{\partial} r = \big(\lambda_1 + 4\lambda_3 |z_1|^2 + \lambda_5 |z_2|^2\big) dz_1 \wedge d\overline{z_1} + \lambda_5 \overline{z_1} z_2 dz_1 \wedge d\overline{z_2}$$
$$+ \lambda_5 z_1 \overline{z_2} dz_2 \wedge d\overline{z_1} + \big(\lambda_2 + 4\lambda_4 |z_2|^2 + \lambda_5 |z_1|^2\big) dz_2 \wedge d\overline{z_2}. \tag{47}$$

At 0, $\partial r = -\frac{1}{2i} dz_3$ and $T_0^{1,0}(\partial\Omega) = span\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\}$. In a neighborhood of zero, the trace of the Levi form using (41) is

$$\sum_{j=1}^{n-1}\partial\overline{\partial}\rho\big(L_j\wedge\overline{L_j}\big)=\lambda_1+4\lambda_3|z_1|^2+\lambda_5|z_2|^2+\lambda_2+4\lambda_4|z_2|^2+\lambda_5|z_1|^2$$
$$-\Bigg(\big(\lambda_1\overline{z_1}+2\lambda_3 z_1\overline{z_1}^2+\lambda_5\overline{z_1}|z_2|^2\big)\big(\lambda_1 z_1+2\lambda_3 z_1^2\overline{z_1}+\lambda_5 z_1|z_2|^2\big)$$
$$+\big(\lambda_2\overline{z_2}+2\lambda_4 z_2\overline{z_2}^2+\lambda_5|z_1|^2\overline{z_2}\big)\big(\lambda_2 z_2+2\lambda_4 z_2^2\overline{z_2}+\lambda_5|z_1|^2 z_2\big)+\frac{1}{4}\Bigg)^{-1}$$
$$\cdot\Big(\big(\lambda_1\overline{z_1}+2\lambda_3 z_1\overline{z_1}^2+\lambda_5\overline{z_1}|z_2|^2\big)\big(\lambda_1+4\lambda_3|z_1|^2+\lambda_5|z_2|^2\big)\big(\lambda_1 z_1+2\lambda_3 z_1^2\overline{z_1}+\lambda_5 z_1|z_2|^2\big)$$
$$+\big(\lambda_2\overline{z_2}+2\lambda_4 z_2\overline{z_2}^2+\lambda_5|z_1|^2\overline{z_2}\big)(\lambda_5\overline{z_1}z_2)\big(\lambda_1 z_1+2\lambda_3 z_1^2\overline{z_1}+\lambda_5 z_1|z_2|^2\big)$$
$$+\big(\lambda_1\overline{z_1}+2\lambda_3 z_1\overline{z_1}^2+\lambda_5\overline{z_1}|z_2|^2\big)(\lambda_5 z_1\overline{z_2})\big(\lambda_2 z_2+2\lambda_4 z_2^2\overline{z_2}+\lambda_5|z_1|^2 z_2\big)$$
$$+\big(\lambda_2\overline{z_2}+2\lambda_4 z_2\overline{z_2}^2+\lambda_5|z_1|^2\overline{z_2}\big)\big(\lambda_2+4\lambda_4|z_2|^2+\lambda_5|z_1|^2\big)\big(\lambda_2 z_2+2\lambda_4 z_2^2\overline{z_2}+\lambda_5|z_1|^2 z_2\big)\Big)\,.$$

Simplifying this we have

$$\sum_{j=1}^{n-1}\partial\overline{\partial}\rho\big(L_j\wedge\overline{L_j}\big)=\lambda_1+\lambda_2+\big(4\lambda_3+\lambda_5-4\lambda_1^3\big)|z_1|^2+\big(\lambda_5+4\lambda_4-4\lambda_2^3\big)|z_2|^2+O\big(|z|^3\big).\quad(48)$$

Hence, $\partial\Omega$ is weakly 2-convex in a neighborhood of zero if $\lambda_1+\lambda_2>0$ or if $\lambda_1+\lambda_2=0$, $4\lambda_4+\lambda_5-4\lambda_1^3>0$, and $4\lambda_3+\lambda_5-4\lambda_2^3>0$.

Similar to Proposition 7.1, when $\lambda_1+\lambda_2>0$ we have that $\mathcal{N}_0^2$ is trivial. Also, when $\lambda_1+\lambda_2=0$ we again have that $\mathcal{N}_0^2=T_0^{0,2}(\partial\Omega)$.

When $\lambda_1+\lambda_2>0$ we have that $\mathcal{N}_0^2=\{0\}$. Similar to the previous example, this means that the set of $v\in\mathcal{N}_0^2$ such that $v\neq 0$ is an empty set. Thus $\mathcal{B}=0$ and $\mathcal{A}=0$.

When $\lambda_1+\lambda_2=0$, $\mathcal{N}_0^2=T_0^{0,2}(\partial\Omega)$. So the set of $v\in\mathcal{N}_0^2$ such that $v\neq 0$ is no longer an empty set.

We start by considering $\mathcal{B}$. We have from Definition 2.7 and (47) that

$$L_2^r(v,v)=\big(\lambda_1+4\lambda_3|z_1|^2+\lambda_5|z_2|^2+\lambda_2+4\lambda_4|z_2|^2+\lambda_5|z_1|^2\big)v_{12}\overline{v_{12}}.\quad(49)$$

In $U_{p,\mu}$, we have that $z_1 = z_2 = 0$. Now since $\lambda_1 + \lambda_2 = 0$, we have that $L_2^r(v, v) = 0$ for all $z \in U_{p,\mu}$. Therefore, $\mathcal{B} = 0$.

Now we will consider $\mathcal{A}$. We have from Definition 2.7 and (47) for $z \in U_{p,\mu}$ that

$$L_q^r\left(\frac{v}{|v|}, \frac{\partial r \wedge w}{|\partial r \wedge w|}\right) = \big(\lambda_1 + 4\lambda_3|z_1|^2 + \lambda_5|z_2|^2 + \lambda_2 + 4\lambda_4|z_2|^2 + \lambda_5|z_1|^2\big) \left(\frac{v}{|v|}\right)_{12} \overline{\left(\frac{\partial r \wedge w}{|\partial r \wedge w|}\right)_{12}}. \quad (50)$$

At $0 \in \partial\Omega$, we have that $(\partial r \wedge w)$ is only a linear combination of $dz_3 \wedge dz_1$ and $dz_3 \wedge dz_2$. So $(\partial r \wedge w)_{12} = 0$. Hence $L_q^r\left(\frac{v}{|v|}, \frac{\partial r \wedge w}{|\partial r \wedge w|}\right) = 0$. Therefore, $\mathcal{A} = 0$. Thus, similar to the previous example, the only possibilities for $\mathcal{B}$ and $\mathcal{A}$ are $\mathcal{B} = 0$ and $\mathcal{A} = 0$.

Now we assume that $\lambda_1 = \lambda_2 = 0$, $\lambda_5 \neq 0$, and $\lambda_4 \neq \lambda_3$ and we will show that the Levi form is not diagonalizable in this case. First note that

$$\widetilde{L_1} = \frac{\partial}{\partial z_1} + 2i\big(2\lambda_3|z_1|^2 + \lambda_5|z_2|^2\big)\overline{z_1}\frac{\partial}{\partial z_3}$$

and

$$\widetilde{L_2} = \frac{\partial}{\partial z_2} + 2i\big(2\lambda_4|z_1|^2 + \lambda_5|z_2|^2\big)\overline{z_2}\frac{\partial}{\partial z_3}$$

form a basis for $T^{1,0}(\partial\Omega)$ but not an orthonormal basis. The Levi form with respect to the non-orthonormal basis $\widetilde{L_1}$ and $\widetilde{L_2}$ is

$$\begin{pmatrix} 4\lambda_3|z_1|^2 + \lambda_5|z_2|^2 & \lambda_5\overline{z_1}z_2 \\ \lambda_5 z_1\overline{z_2} & 4\lambda_4|z_2|^2 + \lambda_5|z_1|^2 \end{pmatrix}. \quad (51)$$

We will look at the three situations where

- $z_1 = 0$ and $z_2 \neq 0$,
- $z_1 \neq 0$ and $z_2 = 0$, and
- $z_1 = z_2 \neq 0$.

Case 1: Let $z_1 = 0$ and $z_2 \neq 0$.

The eigenvalues can be found using the characteristic equation $0 = \det(A - \nu I)$. So, the eigenvalues from (51) can be found from

$$0 = \nu^2 - (\lambda_5 + 4\lambda_4)|z_2|^2\nu + 4\lambda_4\lambda_5|z_2|^4.$$

Solving this quadratic gives eigenvalues of $\lambda_5|z_2|^2$ or $4\lambda_4|z_2|^2$. Using these we can show that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are normalized eigenvectors.

Case 2: Let $z_1 \neq 0$ and $z_2 = 0$.

The eigenvalues in this case can be found from

$$0 = \nu^2 - (\lambda_5 + 4\lambda_3)|z_1|^2\nu + 4\lambda_3\lambda_5|z_1|^4.$$

Solving this quadratic gives eigenvalues of $\lambda_5|z_1|^2$ or $4\lambda_3|z_1|^2$. Using these we can show that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are normalized eigenvectors.

Case 3: Let $z_1 = z_2 \neq 0$.

The eigenvalues in this case can be found from

$$0 = \nu^2 - (2\lambda_5 + 4\lambda_3 + 4\lambda_4)|z_1|^2\nu + (4\lambda_3 + \lambda_5)(4\lambda_4 + \lambda_5)|z_1|^4 - \lambda_5^2|z_1|^4.$$

Solving this quadratic gives eigenvalues of $\nu = \left(2\lambda_3 + 2\lambda_4 + \lambda_5 \pm \left(4(\lambda_3 - \lambda_4)^2 + \lambda_5^2\right)^{\frac{1}{2}}\right)|z_1|^2$. Using these, we can show that $\begin{pmatrix} \frac{4\lambda_4|z_1|^2 - \nu}{\left((4\lambda_4|z_1|^2-\nu)^2 + (4\lambda_3|z_1|^2-\nu)^2\right)^{\frac{1}{2}}} \\ \frac{4\lambda_3|z_1|^2 - \nu}{\left((4\lambda_4|z_1|^2-\nu)^2 + (4\lambda_3|z_1|^2-\nu)^2\right)^{\frac{1}{2}}} \end{pmatrix}$ are normalized eigenvectors. So as long as $\lambda_5 \neq 0$ and $\lambda_3 \neq \lambda_4$, the eigenvalues are distinct, non-zero, and more importantly the eigenvectors are not the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Now that it is known that there are unique, nonzero eigenvalues in each of our cases, we will compare our Levi form with respect to our non-orthonormal coordinates to the Levi form with respect to orthonormal coordinates.

To find an orthonormal basis we will use the Gram-Schmidt Process. Let $L_1 = \frac{\widetilde{L_1}}{|\widetilde{L_1}|}$. From the next step of the Gram-Schmidt Process, we have that $L_2 = \frac{\widetilde{L_2} - \langle \widetilde{L_2}, L_1 \rangle L_1}{\left(|\widetilde{L_2}|^2 - |\langle \widetilde{L_2}, L_1 \rangle|^2\right)^{\frac{1}{2}}}$. Now L_1 and L_2 form an orthonormal basis for $T^{1,0}(\partial\Omega)$. Next we will compare L_1 with $\widetilde{L_1}$ and L_2 with $\widetilde{L_2}$.

Note that $1 \leq |\widetilde{L_1}|^2 \leq 1 + O\Big((|z_1|^2 + |z_2|^2)^2 |z_1|^2\Big)$. From this, we have that

$$1 \geq \frac{1}{|\widetilde{L_1}|} \geq 1 - O\Big((|z_1|^2 + |z_2|^2)^2 |z_1|^2\Big).$$

Thus when comparing L_1 and $\widetilde{L_1}$ we find that

$$\Big|L_1 - \widetilde{L_1}\Big| \leq |\widetilde{L_1}| \Big|1 - \frac{1}{|\widetilde{L_1}|}\Big| \leq O\Big((|z_1|^2 + |z_2|^2)^2 |z_1|^2\Big).$$

Next we will compare L_2 with $\widetilde{L_2}$. First note that

$$\big|\langle \widetilde{L_1}, \widetilde{L_2} \rangle\big| \leq O\Big((|z_1|^2 + |z_2|^2)^2 |z_1||z_2|\Big).$$

Then

$$\big|\langle \widetilde{L_2}, L_1 \rangle\big| = \frac{|\langle \widetilde{L_1}, \widetilde{L_2} \rangle|}{|\widetilde{L_1}|} \leq O\Big((|z_1|^2 + |z_2|^2)^2 |z_1||z_2|\Big).$$

Thus

$$\Big|\widetilde{L_2} - \Big(\widetilde{L_2} - \langle \widetilde{L_2}, L_1 \rangle L_1\Big)\Big| = \Big|\langle \widetilde{L_2}, L_1 \rangle L_1\Big| \leq O\Big((|z_1|^2 + |z_2|^2)^2 |z_1||z_2|\Big)$$

and

$$\begin{aligned}\left|L_2-\left(\widetilde{L_2}-\langle\widetilde{L_2},L_1\rangle L_1\right)\right| &= \left|\widetilde{L_2}-\langle\widetilde{L_2},L_1\rangle L_1\right|\left|\frac{1}{|\widetilde{L_2}-\langle\widetilde{L_2},L_1\rangle L_1|}-1\right| \\ &\leq O\Big((|z_1|^2+|z_2|^2)^2|z_2|^2\Big).\end{aligned}$$

Hence, $\left|L_2-\widetilde{L_2}\right|\leq O\Big((|z_1|^2+|z_2|^2)^2|z_2|^2\Big)$.

Let $\mathcal{L}_1$ be the Levi form with respect to $\{L_j\}_{j=1}^2$ and $\mathcal{L}_0$ be the Levi form with respect to $\{\widetilde{L_j}\}_{j=1}^2$. Now we estimate $|\mathcal{L}_1-\mathcal{L}_0|$ with

$$\begin{aligned}&\left|\partial\overline{\partial}\rho(L_j\wedge L_k)-\partial\overline{\partial}\rho\Big(\widetilde{L_j}\wedge\widetilde{L_k}\Big)\right| \\ &\quad=\left|\partial\overline{\partial}\rho\Big(\Big(\widetilde{L_j}+\Big(L_j-\widetilde{L_j}\Big)\Big)\wedge\Big(\widetilde{L_k}+\Big(L_k-\widetilde{L_k}\Big)\Big)\Big)-\partial\overline{\partial}\rho\Big(\widetilde{L_j}\wedge\widetilde{L_k}\Big)\right| \\ &\quad=\Big|\partial\overline{\partial}\rho\Big(\widetilde{L_j}\wedge\widetilde{L_k}\Big)+\partial\overline{\partial}\rho\Big(\widetilde{L_j}\wedge\Big(L_k-\widetilde{L_k}\Big)\Big)+\partial\overline{\partial}\rho\Big(\Big(L_j-\widetilde{L_j}\Big)\wedge\widetilde{L_k}\Big) \\ &\qquad+\partial\overline{\partial}\rho\Big(\Big(L_j-\widetilde{L_j}\Big)\wedge\Big(L_k-\widetilde{L_k}\Big)\Big)-\partial\overline{\partial}\rho\Big(\widetilde{L_j}\wedge\widetilde{L_k}\Big)\Big| \\ &=\left|\partial\overline{\partial}\rho\Big(\widetilde{L_j}\wedge\Big(L_k-\widetilde{L_k}\Big)\Big)+\partial\overline{\partial}\rho\Big(\Big(L_j-\widetilde{L_j}\Big)\wedge\widetilde{L_k}\Big)+\partial\overline{\partial}\rho\Big(\Big(L_j-\widetilde{L_j}\Big)\wedge\Big(L_k-\widetilde{L_k}\Big)\Big)\right|.\end{aligned}$$

Thus

$$|\mathcal{L}_1-\mathcal{L}_0|\leq O\Big(\big(|z_1|^2+|z_2|^2\big)^4\Big). \tag{52}$$

Now, consider the cases above again.

Case 1: Let $z_1=0$, $z_2\neq 0$.

From (52), we have $|\mathcal{L}_1-\mathcal{L}_0|\leq O\Big(\big(|z_2|^2\big)^4\Big)$. Let $\mathcal{L}_t=t\mathcal{L}_1+(1-t)\mathcal{L}_0$ be a differentiable family of matrices connecting $\mathcal{L}_1$ and $\mathcal{L}_0$. Then $\frac{d}{dt}\mathcal{L}_t=\mathcal{L}_1-\mathcal{L}_0$. Let $\nu_1(0)$ and $\nu_2(0)$ be the eigenvalues for $\mathcal{L}_0$ and $\nu_1(1)$ and $\nu_2(1)$ be the eigenvalues for $\mathcal{L}_1$. Now by A1 in [17],

$$|\nu_1(0)-\nu_1(1)|\leq O\Big(\big(|z_1|^2+|z_2|^2\big)^4\Big)|1-0|.$$

Note that

$$||\mathcal{L}_t||_{C^1} = ||\mathcal{L}_t|| + \left|\left|\frac{d}{dt}\mathcal{L}_t\right|\right|.$$

In [17], the C^1 norm is used in the proof but the part of their proof that we rely on only requires the norm of the derivative. We have that $||\frac{d}{dt}\mathcal{L}_t|| \leq O\Big((|z_1|^2 + |z_2|^2)^4\Big)$. From our previous work for case 1, we have that $|\nu_1(0) - \nu_2(0)| = |\lambda_5 - 4\lambda_4||z_2|^2$. Then

$$|\nu_1(1) - \nu_2(1)| \geq |\lambda_5 - 4\lambda_4||z_2|^2 - O\big(|z_2|^8\big).$$

So on a neighborhood close to the origin there are distinct eigenvalues that are bounded away from zero when $\lambda_5 - 4\lambda_4 \neq 0$. Now let $P_1(1)$ be the orthogonal projection onto the span of eigenvectors corresponding to ν_1 for $\mathcal{L}_1$ and let $P_1(0)$ be the orthogonal projection onto the span of eigenvectors corresponding to ν_1 for $\mathcal{L}_0$. By A2 in [17],

$$|P_1(1) - P_1(0)| \leq O\bigg(\frac{1}{|z_2|^2}|z_2|^8\bigg) = O\big(|z_2|^6\big).$$

So the eigenvectors for $\mathcal{L}_1$ in this case are arbitrarily close to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ on some neighborhood of the origin.

Case 2: Let $z_2 = 0$, $z_1 \neq 0$.

We have a similar setup for the differentiable family of matrices $\mathcal{L}_t$ as we had in case 1 where $\mathcal{L}_t = t\mathcal{L}_1 + (1-t)\mathcal{L}_0$ and $\frac{d}{dt}\mathcal{L}_t = \mathcal{L}_1 - \mathcal{L}_0$. Also similar, let $\nu_1(0)$ and $\nu_2(0)$ be the eigenvalues for $\mathcal{L}_0$ and $\nu_1(1)$ and $\nu_2(1)$ be the eigenvalues for $\mathcal{L}_1$. So by A1 in [17],

$$|\nu_1(0) - \nu_1(1)| \leq O\Big(\big(|z_1|^2 + |z_2|^2\big)^4\Big)|1 - 0|.$$

From our previous work for case 2, we have that $|\nu_1(0) - \nu_2(0)| = |\lambda_5 - 4\lambda_3||z_1|^2$. Then $|\nu_1(1) - \nu_2(1)| \geq |\lambda_5 - 4\lambda_3||z_1|^2 - O(|z_1|^8)$. So on a neighborhood close to the origin there are distinct eigenvalues that are bounded away from zero when $\lambda_5 - 4\lambda_3 \neq 0$. By A2 in [17],

$$|P_1(1) - P_1(0)| \leq O\Big(\frac{1}{|z_1|^2}|z_1|^8\Big) = O\big(|z_1|^6\big).$$

So the eigenvectors for $\mathcal{L}_1$ in this case are arbitrarily close to $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ on some neighborhood of the origin.

By hypothesis, $\lambda_5 \neq 0$ and $\lambda_4 \neq \lambda_3$. Thus, at least one of $\lambda_5 - 4\lambda_4 \neq 0$ or $\lambda_5 - 4\lambda_3 \neq 0$ is true. Therefore in at least one of case 1 or case 2 there are eigenvectors for $\mathcal{L}_1$ that are arbitrarily close to $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ on some neighborhood of the origin.

Case 3: Let $z_1 = z_2 \neq 0$.

We have a similar setup for the differentiable family of matrices $\mathcal{L}_t$ and definitions for $\nu_1(0)$, $\nu_2(0)$, $\nu_1(1)$, and $\nu_2(1)$ as we had in case 1. So by A1 in [17],

$$|\nu_1(0) - \nu_1(1)| \leq O\big(|z_1|^8\big)|1 - 0|.$$

From our previous work for case 3, we have that $|\nu_1(0) - \nu_2(0)| = |2\big(4(\lambda_3 - \lambda_4)^2 + \lambda_5^2\big)^{\frac{1}{2}}||z_1|^2$. Then

$$|\nu_1(1) - \nu_2(1)| \geq |2\big(4(\lambda_3 - \lambda_4)^2 + \lambda_5^2\big)^{\frac{1}{2}}||z_1|^2 - O\big(|z_1|^8\big).$$

So on a neighborhood close to the origin there are unique eigenvalues that are bounded away from zero. By A2 in [17],

$$|P_1(1) - P_1(0)| \leq O\Big(\frac{1}{|z_1|^2}|z_1|^8\Big) = O\big(|z_1|^6\big).$$

So the eigenvectors for $\mathcal{L}_1$ are bounded away from $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ on some neighborhood of the origin. Hence, our $\mathcal{L}_1$ is non-diagonalizable.

$\square$

8 Future Research

There are a few ideas that the local examples presented here motivate. In the examples presented both $\mathcal{A}$ and $\mathcal{B}$ are found to be zero. Finding a domain where one or both of these cannot equal zero is an interesting direction to consider. In an initial look into this, we found that finding a domain where $\mathcal{B} \neq 0$ or $\mathcal{A} \neq 0$ will likely require an understanding about higher dimensional cohomology. When $q = 1$, it is know that $\mathcal{A}$ is bounded away from zero on the Diederich-Fornæss worm domain (in fact, Liu [24] has computed the precise value) because of the annulus in the boundary.

Another possible future project to be to generalize the methods of Berndtsson and Charpentier in [3] or Harrington in [15] to weakly q-convex domains. These both use bounded plurisubharmonic exhaustion functions to prove that the Bergman Projection is continuous in certain Sobolev spaces. Pinton and Zampieri made significant progress in this direction in [26].

9 References

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