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Lyubeznik Numbers of Unmixed Edge Ideals

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Abstract

Lyubeznik numbers, defined in terms of local cohomology, are invariants of local rings that are able to detect many algebraic and geometric properties. Notably they recognize topological behaviors of various structures associated to their rings. We will discuss computations of these numbers for unmixed edge ideals by giving a completely combinatorial construction which realizes the connectedness information captured by these numbers.

Acknowledgements

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1 Introduction

Lyubeznik numbers are invariants of local rings originally introduced by Lyubenzik in [Lyu93]. These numbers characterize surprising topological connectedness information. They are arranged in a square integer matrix $\Lambda = (\lambda_{i,j})$. On the one hand, the bottom rightmost entry of this matrix, the so called highest Lyubeznik number, gives information about the connected components of the Hochster-Huneke graph, [Lyu06, Zha07]. Its connectivity is inherently algebraic. On the other hand, Lyubeznik numbers are subtle enough to detect geometric connectivity. For example, they are related in certain cases to the number of connected components of the punctured spectrum, as a consequence of the second vanishing theorem of Ogus, Hartshorne-Speiser, or Huneke-Lyubeznik, [HL90]. For more detailed results see [NBWZ16, NBSW19] and the references therein, or Section 2.1.1 for a summary.

Lyubeznik numbers in general are difficult to compute. However, for square-free monomial ideals, Lyubenzik numbers can be tractable to study. There is a specific computation due to Álvarez Montaner and Vahidi, [MV14], which builds on earlier work of Mustaţă, [Mus00], and Yanagawa, [Yan00, Yan01], that essentially breaks the difficulty down to computing homology of the free resolution of the associated Alexander dual. While this is more tractable in specific cases, it does not yield a complete characterization of all Lyubeznik numbers of such ideals. We seek to build upon the information this theorem offers in the case of edge ideals, a commonly studied class of square-free monomial ideals.

Question 1.0.1. *Given an edge ideal I in a polynomial ring R arising as an edge ideal, is there a purely combinatorial interpretation of the Lyubeznik numbers of the localization of R/I at the homogeneous maximal ideal?*

The main aim of this thesis is to address this question by building on the techniques present and providing supporting tools towards the computation of these invariants. We summarize now the main theorems, but leave the precise definition of some terms to the relevant chapters of the thesis.

The first result we obtain is a complete classification for a restricted class of edge ideals. Specifically, we introduce a finite family of graphs $\mathrm{Clq}_i(G)$ and $\mathrm{Clq}_{\geq i}(G)$ from a given graph G. These graphs are defined and studied in detail in Chapter 3, but were independently introduced in [DDDGHL19] as graphs associated to higher nerve complexes and arise isomorphically in [NBSW19]. See Chapter 3.4.1 for explicit translations and connections.

The class of graphs to which we can give a complete understanding are those defining unmixed edge ideals of dimension at least one and for which both the graph complement G^C and each $\mathrm{Clq}_{\geq i}(G^C)$ is chordal; a condition we call clique-cochordal. We have the following combinatorial description of the Lyubeznik numbers in this setting, see Theorem 4.1.3 whose statement follows. We denote by $\pi_0(G)$ the set of connected components of G, and by $I(G)$ the associated edge ideal.

Theorem 1.0.2. *Let k be a field and $I = I(G)$ an unmixed edge ideal of dimension $d \geq 1$ in the polynomial ring $R = k[x_1, \dots, x_n]$ where G is a clique-cochordal graph. The Lyubeznik table of R/I is given by*

$$\Lambda(R/I) = \begin{bmatrix} 0 & C_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & C_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & C_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & C_{d-2} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & J \end{bmatrix}$$

where $J = \#\pi_0(\mathrm{Clq}_{d-1}(G^C))$ and $C_i = \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C)) - \#\pi_0(\mathrm{Clq}_{\geq i}(G^C))$.

The previous theorem shows, in a new way, connectivity information encoded in the Lyubeznik table; this time concerning the complement of the graph associated to the edge ideal. The clique-cochordality already includes a wide class of graphs, however, this condition can be changed to an even weaker chordality condition, which we explain further in Section 4.2, specifically in Definition 4.2.2, while keeping the last column of the Lyubeznik table given in Theorem 1.0.2 intact. We see this in Theorem 4.2.15
Denote by $I^\vee$ the Alexander dual of I, which we discuss in more detail in Section 2.2.

Theorem 1.0.3. *Let $I = I(G)$ be an edge ideal of dimension d in the polynomial ring $R = k[x_1, \ldots, x_n]$ wth $I^\vee$ having projective dimension at most two. If G is a $(d-1)-$chordal graph, then $\lambda_{d,d}(R/I) = \#(\pi_0(\mathrm{Clq}_{d-1}(G^C)))$ and $\lambda_{j,d}(R/I) = 0$ for all $j \leq d-1$.*

Another natural question then is if the Lyubeznik table of an edge ideal associated to a graph G can be decomposed as a sum with each summand the Lyubeznik table associated to a connected component G^C. We see an answer to this question in Theorem 5.1.11 whose statement is given below. We study this topic in more detail in Chapter 5. There we will explore a technical condition for when G^C is a union of graphs $G^C|_S$ and $G^C|_T$ that is sufficient to split the Lyubeznik table. We call this a $(G^C, \ell)-$splitting where ℓ is an integer. See Definition 5.1.2 for more.

Let $C_{i,j}$ denote the matrix with 1 in the i^{th} row and j^{th} column and 0 for all other entries.

Theorem 1.0.4. *Let k be a field and $I = I(G)$ an unmixed edge ideal of dimension $d \geq 1$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ where G is a clique-cochordal graph with $G^C|_S \cup G^C|_T$ a $(G^C, \ell)-$splitting for $\ell \geq 0$. Set $K = I(G|_S)$ and $L = I(G|_T)$.*

If $\ell = d - 1$, then

$$\Lambda(R/I) = \Lambda(R/KR) + \Lambda(R/LR) - C_{d,d}.$$

If $\ell < d-1$, then

$$\Lambda(R/I) = \Lambda(R/KR) + \Lambda(R/LR) + C_{\ell,\ell+1}.$$

We will see in Chapter 4 that the Eagon-Reiner formula for Betti numbers is the tool that bridges the gap from graph combinatorics to Lyubeznik number calculations. Therefore, a Betti splitting, discussed further in Chapter 5, is used to provide the splitting in the above theorem. A Betti splitting is a decomposition of an ideal in a way that preserves Betti numbers under addition. That the Betti splittings of $I^\vee$ result from a $(G^C, 0)-$splitting answers a question posed by Van Tuyl in [Van13, Qu. 68]. Francisco, Há, and Van Tuyl found that there is a Betti splitting of a cover ideal, which is the Alexander dual of an edge ideal, in the case the G is a Cohen-Macaulay bipartite graph, [FHVT, Thm. 3.8], but sought other ways to naturally split the cover ideal. A consequence of Theorem 5.1.8 provides an answer to this question.

Theorem 1.0.5. *Let $I = I(G)$ be an edge ideal. If $G^C = G^C|_S \cup G^C|_T$ is a $(G^C, 0)$-splitting, then the Alexander dual of I has a Betti splitting.*

2 Preliminaries

In this section, we review formal definitions for Lyubeznik numbers, their basic properties, as well as combinatorial concepts we consider. Finally, we describe explicit methods of computing Lybenznik numbers for monomial ideals which will be tacitly used throughout the thesis. Our review in this background is unfortunately not comprehensive. See [NBWZ16, MY16, MV14] for more details. We utilize the following notational conventions and assumptions which we outline below.

Notation: Generally, throughout k will denote a field, though at times also an integer, the context will make it clear. Local rings with residue field k will be denoted $(R, \mathfrak{m}, k)$ a local ring R with maximal ideal $\mathfrak{m}$. We rely on common constructions in commutative and homological algebra. Specifically, we assume the reader has knowledge of free resolutions and Ext modules as well as basic notions such as height and associated primes, denoted $\mathrm{ht}(I)$ and $\mathrm{Ass}(I)$ respectively, for ideal I or module M.

To set notation, for a local ring R and finitely generated R-module M, we use $F_{\bullet} \to M$ to denote a free resolution or when the module is clear simply by $F_{\bullet}$. Specifically, this is an exact complex $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ where F_i is a free R-module. When $F_{\bullet}$ is minimal, i.e., each F_i has finite rank and the matrices defining the maps between adjacent free modules contain only non-units. Set β_i the rank of F_i, the i-th Betti number. If the module is in question, these will be denoted $F_{\bullet}(M)$ and $\beta_i(M)$ respectively. Often, R will be a localization of a polynomial ring at its ideal of variables and M finitely generated, whence by the Hilbert syzygy theorem, minimal free resolutions are finite. We denote the projective dimension, i.e., minimal length of a minimal free resolution of M by $\mathrm{pdim}(M)$.

When R is a polynomial ring, it is also naturally a graded k-algebra and so we keep track of graded shifts. We will not use any exotic gradings, so any polynomial ring will be standard graded when considered as a graded algebra. Set $[R]_n$ for the degree n component of R and for an integer j, set $R(-j)$ for the graded shift with $[R(-j)]_n = [R]_{n-j}$. For M a graded R-module, and $F_\bullet$ a minimal graded free resolution, $F_i \cong \bigoplus_j R(-j)^{\oplus \beta_{i,j}}$ a finite direct sum of graded shifts, i.e., all but finitely many $\beta_{i,j}$ are 0 for fixed i. It's common to denote F_i by $\bigoplus_j R(-j)^{\beta_{i,j}}$ where the direct sum is understood. The numbers $\beta_{i,j}$ are the graded Betti numbers. We denote by $\operatorname{reg} M$ the Castelnuovo-Mumford regularity, i.e., the maximal j with $\beta_{i,i+j} \neq 0$ occurring in a minimial graded free resolution.

We have need to also work with homologically indexed but possibly non-exact complexes $M_\bullet$ as well. We denote by $H_i(M_\bullet) = \ker(M_i \to M_{i-1}) / \operatorname{im}(M_{i+1} \to M_i)$ the i-th homology.

Finally, we review a bit of combinatorial basics and notation with a more detailed review in Section 2.3. Throughout, for a finite set X, we denote by $\mathcal{P}(X)$ the power set, i.e,. the set of all subsets of X. A graph will be denoted by G and we denote by $V(G)$ its vertex set and $E(G)$ its edge set. All graphs for us will be finite and simple, i.e., having no loops nor multiple edges.

A subgraph of a graph G is a graph H for which both $V(H) \subset V(G)$ and $E(H) \subset E(G)$. However, special subgraphs will be useful. Specifically, for a fixed graph G and $H \subset V(G)$, we denote by $G|_H$ the induced subgraph on those vertices. In particular, there is an edge in $G|_H$ between vertex x and vertex y in H if and only if there is an edge between x and y in G. It will be sometimes useful to denote this as $G[H]$ as well. It is unfortunately useful to conflate a symbol H with the subgraph it induces. Specifically, at times H can be either a subset of $V(G)$ or the subgraph $G|_H$; however this should cause no confusion by context. We throughout reserve the notation K_n for the complete graph on n-vertices, that is the unique up to isomorphism graph with n-vertices and all possible edges.

2.1 Local cohomology and Lyubeznik numbers

We now review the primary invariant of interest. For a general reference for local cohomology, see [ILL+07] and for specific references for Lyubeznik numbers, see [NBWZ16].

Throughout this thesis we consider local rings essentially of finite type over a field k. Specifically, local rings $(S, \mathfrak{m}_S, k)$ considered always admit as surjection $R \twoheadrightarrow S$ where $(R, \mathfrak{m}_R, k)$ is a regular local ring with the same residue field. Concretely, throughout this section R will be the localization at the unique homogeneous maximal ideal of a polynomial ring over k in finitely many variables, and $S \cong R/I$ for an ideal I in R. Note that in this case $\mathfrak{m}_R S = \mathfrak{m}_S$.

For an ideal I in R, consider the I-torsion functor sending an R-module M to $\Gamma_I(M) := \{m \in M \colon I^t m = 0 \text{ for some } t\}$. It is routine to verify that this is left exact. One may define local cohomology efficiently as the derived functors of this functor; the i-th local cohomology denoted $H^i_I(M)$ for integer i. More explicitly, $H^i_I(M)$ can also be recognized as the colimit $\varinjlim_{t\to\infty} \operatorname{Ext}^i_R(R/I^t, M)$, [ILL+07, Thm. 7.8]. We are most interested in the case where $M = R$.

Suppose $n = \dim R$ and set $\varphi \colon R \to S$ a surjection with kernel I, so $S \cong R/I$. The dimensions $\dim_k \operatorname{Ext}^i_R(k, H^{n-j}_I(R))$ only depend on S, i, and j. They are independent of R and φ. Even finiteness of these numbers is an important and surprising result, [HS93, Lyu93]. To resolve independence, in the simplified case when R and R' are both localizations of polynomial rings surjecting onto S, one consider $R'' := R \otimes_k R'$ and the natural surjection $\varphi'' \colon R'' \to S$ extending the given surjections and relating the local cohomology in question, see [NBWZ16, Thm. 3.1]. This leads to the well-defined collection of integers which are at the center of our investigation.

Definition 2.1.1 (Lyubeznik Numbers)**.** Fix $(R, \mathfrak{m}_R, k)$ a regular local ring and $(S, \mathfrak{m}_S, k)$

a local ring with surjection $R \to S$ with kernel I. For each i and j non-negative integers, set

$$\lambda_{i,j} := \dim_k \operatorname{Ext}^i_R(k, H^{n-j}_I(R)).$$

The integers $\lambda_{i,j}$ are called the **Lyubeznik numbers**.

When we need to be specific, we write $\lambda_{i,j}(S)$ to specify the ring in question. Many of these invariants are automatically 0, [Lyu93, pg. 54]. In particular, recall $n = \dim R$ and set $d = \dim S$. When $n - j$ exceeds the cohomological dimension of I, $\lambda_{i,j} = 0$. However, as cohomological dimension bounds the height which is at least $n - d$, we have that $\lambda_{i,j} = 0$ for $j > d$. Similarly, the dimension of $H^{n-j}_I(S)$ is at most j and as this bounds the injective dimension of the module in question, $\lambda_{i,j} = 0$ when $i > j$. It is however an application of a reformulation of the definition above and the Grothendieck spectral sequence of iterated local cohomology which allows one to conclude that $\lambda_{d,d} \neq 0$, [Lyu93, pg. 54]. This means we can describe the Lyubeznik numbers as an upper triangular $(d+1) \times (d+1)$-matrix

$$\Lambda(S) = (\lambda_{i,j}(S)) = \begin{pmatrix} \lambda_{0,0} & \lambda_{0,1} & \cdots & \lambda_{0,d} \\ 0 & \lambda_{1,1} & \cdots & \lambda_{1,d} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{d,d} \end{pmatrix}.$$

When $\lambda_{i,j} = 0$ whenever $i \neq d$ or $j \neq d$, that is when $\Lambda(S) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$, $\Lambda(S)$ is called a trivial Lyubeznik table. Notably this happens when S is defined by a square-free Cohen-Macaulay monomial ideal. Weaker notions than Cohen-Macaulay also give trivial Lyubeznik tables, [Mon14, Cor. 3.4]. Another consequence of the Grothendieck spectral sequence is the following useful relationship, [Mon14, Prop. 1.1].

Theorem 2.1.2 (Euler Characteristic Formula)**.** *For a local ring $(S, \mathfrak{m}, k)$ with $d = \dim S$,*

$$\sum_{0 \leq i,j \leq d} (-1)^{i-j} \lambda_{i,j} = 1.$$

2.1.1 Lyubeznik numbers and connectedness

This section articulates some known relationships between Lyubeznik numbers and both algebraic and geometric connectedness results in more detail, as they are very much so complementary to our results. Throughout, we fix S an equidimensional local ring of dimension $d \geq 2$ with separably closed residue field.

The primary algebraic connectedness considered is in terms of Hochster-Huneke graphs, initiated by Zhang in [Zha07]. This interpretation is expanded upon in [NBSW19]. Specifically, there the following family of graphs is considered, [NBSW19, Def. 2.2].

Definition 2.1.3. For an integer $1 \leq t \leq d-1$, set $\Gamma_t(S)$ to be the graph with:

1. The vertices indexed by the minimal primes of S
2. An edge between minimal primes $\mathfrak{p}$ and $\mathfrak{q}$ if and only if $\text{ht}(\mathfrak{p}+\mathfrak{q}) \leq t$.

The graph $\Gamma_1(S)$ is the famous Hochster-Huneke graph. Zhang showed that $\lambda_{d,d}(S)$ counts the number of connected components of $\Gamma_1(S)$, [Zha07, Main Theorem]. However, this connection is made deeper in the following fashion. Denote by $\pi_0(\Gamma_t(S))$ the connected components of $\Gamma_t(S)$.

Theorem 2.1.4 ([NBSW19], Thm.5.4)**.** *One has* $\lambda_{1,2}(S) = \#\pi_0\Gamma_{d-2}(S) - \#\pi_0\Gamma_{d-1}(S)$ *and*

$$\lambda_{i,i+1}(S) \geq \#\pi_0\Gamma_{d-i-1}(S) - \#\pi_0\Gamma_{d-i}(S)$$

for $1 \leq i \leq d-2$.

Later, these graphs will be reinterpreted and in some cases, the inequality in this theorem is strengthened to equality. See Theorem 4.1.3

Also, there are geometric connectivity relationships, one rooted in the second vanishing theorem of local cohomology, which is used in [NBSW19] to show $\lambda_{0,1}(S) = 0$ if and only if $c(S) \geq 1$. Here $c(S)$ is the *connectedness dimension of S*, the minimal dimension of a closed subset Z of Spec(S) for which Spec$(S)\backslash Z$ is a disconnected topological space. However, the authors go further, [NBSW19, Thm. B].

Theorem 2.1.5. *One has $c(S) \geq i$ if and only if $\lambda_{0,1}(S) = \lambda_{1,2}(S) = \cdots = \lambda_{i,i+1}(S) = 0$.*

The point is that the entries of the superdiagonal of the Lyubeznik table determine a bound on the maximal number of connected components of open subsets of Spec A of bounded dimension. An even more precise estimate of the number of such components is given in [NBSW19, Prop. 6.10].

The take away here is that by incorporating all the Lyubeznik numbers, one gets deeper connectivity results. The thrust of this thesis is to expand on this combinatorially for the case of edge ideals.

2.2 Lyubeznik numbers of monomial ideals

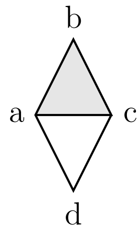
Throughout this section, R denotes a standard graded polynomial ring. The main ideals of interest for this thesis are monomial ideals $I \subset R$. Note these are automatically homogeneous. For consistency of notation, suppose R is $k[\underline{x}]$ where k is a field and $\underline{x} = x_1, \ldots, x_n$ is a set of variables. We denote by $\underline{x}^{\underline{a}}$ for a tuple of non-negative integers $\underline{a} = (a_1, \ldots, a_n)$ the monomial $x_1^{a_1} \cdots x_n^{a_n}$.

There are various constructions of importance for us which are efficiently explained in terms of simplicial complexes. For a finite set $X = \{x_1, \ldots, x_n\}$ a simplicial complex Δ on X is a subset of the power set $\mathcal{P}(X)$ that is closed under taking subsets. That is, if $A \subset B$ for $B \in \Delta$, then $A \in \Delta$. Elements of Δ are called simplicies or faces, and a simplex $A \in \Delta$ not properly contained in any other simplex is called a facet.

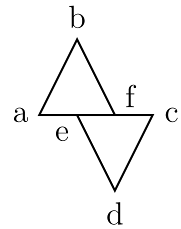
Example 2.2.1. Let $X = \{a, b, c, d, e, f\}$ and let P_1 and P_2 be subsets of $\mathcal{P}(X)$ where

$P_1 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}\}$, and

$P_2 = \{\{a\}, \{b\}, \{c\}, \{d\}\}, \{a, b\}, \{b, f\}, \{a, f\}, \{e, c\}, \{c, d\}, \{d, e\}, \{a, b, f\}, \{e, c, d\}\}$



P_1: Simplicial Complex



P_2: Not a Simplicial Complex

Note in P_1 the simplicies, or faces, consist of all vertices, $\{a\}, \{b\}, \{c\}$, and $\{d\}$, all edges $\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}$, and $\{a, c\}$, but only the shaded triangle $\{a, b, c\}$. The triangle is a facet.

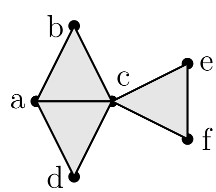
However, the collection of subsets on the right, P_2 is not a simplicial complex because triangles $\{a, b, f\}$ and $\{d, e, c\}$ are in P_2, however their intersection is not in P_2.

Via Stanley-Reisner theory, simplicial complexes determine square-free monomial ideals, i.e., those monomial ideals with generators $\underline{x}^{\underline{a}}$ with $a_i \leq 1$ for all $1 \leq i \leq n$. For such a square-free monomial, one may pick $r \in \mathbb{N}$ and indicies i_j for $1 \leq j \leq r$ so that $\underline{x}^{\underline{a}} = x_{i_1} \cdots x_{i_r}$ where $a_i = 1$ if and only if $i = i_j$ for some j.

In particular, if $R = k[\underline{x}]_{(\underline{x})}$ with $\underline{x} = x_1, \ldots, x_n$, and Δ is a simplicial complex on $X := \{x_1, \ldots, x_n\}$ then the Stanley-Reisner ideal is

$$I_\Delta := (\underline{x}^{\underline{a}} = x_{i_1} \cdots x_{i_r} : \{i_1, \ldots, i_r\} \notin \Delta\}).$$

Example 2.2.2. Consider the following simplicial complex Δ:

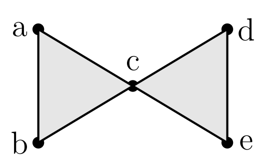


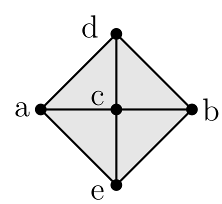
We need only consider the minimal non-faces of Δ, which in this case are the edges not in Δ. These are $\{a, e\}, \{a, f\}, \{b, e\}, \{b, f\}, \{b, d\}, \{d, e\}$, and $\{d, f\}$. Therefore, I_Δ is the monomial ideal $(ae, af, bd, be, bf, de, df)$ in $R = k[a, b, c, d, e, f]$.

Two constructions will be of central importance to our work. The first is a duality. The key fact is that when Δ is a simplicial complex on $X = \{x_1, \ldots, x_n\}$, then we get an associated simplicial complex by flipping the notion of face. Namely, set $\Delta^\vee := \{X \setminus A \colon A \notin \Delta\}$, so faces of $\Delta^\vee$ are precisely the complements of non-faces of Δ and the facets of $\Delta^\vee$ are the complements of the minimal non-faces of Δ under the inclusion order. It is an easy exercise to verify that $\Delta^\vee$ is again a simplicial complex.

Definition 2.2.3. For $I \subset R$ a square-free monomial ideal, set Δ its associated simplicial complex. The Alexander dual is $I_{\Delta^\vee}$, which we sometimes denote by $I^\vee$.

Example 2.2.4. Let $X = \{a, b, c, d, e\}$. We will consider the simplicial complex Δ, where $\Delta = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, b\}, \{b, c\}, \{a, c\}, \{c, d\}, \{d, e\}, \{c, e\}, \{a, b, c\}, \{c, d, e\}\}$, and will see that $\Delta^\vee$ is given by the simplicial complex $\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, d\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, c\}, \{b, e\}, \{c, d\}, \{c, e\}, \{b, c, e\}, \{a, c, d\}, \{b, c, d\}, \{a, c, e\}\}$.

Δ: 

$\Delta^\vee$: 

Notice in Δ that the minimal non-faces are the edges $\{a, d\}, \{b, e\}, \{a, e\}$ and $\{b, d\}$. So the facets of $\Delta^\vee$ are the complements of those edges, i.e. $\{b, c, e\}, \{a, c, d\}, \{b, c, d\}$ and $\{a, c, e\}$.

Since the minimal non-faces of Δ are $\{a, d\}, \{b, e\}, \{a, e\}$ and $\{b, d\}$, then I_Δ is given by (ad, be, ae, bd). Since the minimal non-faces of $\Delta^\vee$ are $\{d, e\}$ and $\{a, b\}$, then $I^\vee$, the Alexander dual of I_Δ, is the monomial ideal (ab, de).

2.2.1 Lyubeznik numbers via linear strands

We now describe a computational interpretation of Lyubeznik numbers in terms of free resolutions. For more, see [MV14] or [Mon13, MY16]. This approach specifically identifies the Lyubeznik numbers in terms of homology of linear strands which we now review.

Definition 2.2.5. Fix $I \subset R$ a homogeneous ideal and set set $F_\bullet \to I \to 0$ a graded minimal free resolution. Fixing an integer r, one has a subcomplex of $k-$vector spaces

$$\mathbb{L}_\bullet^{\langle r \rangle} : 0 \to \mathbb{L}_m^{\langle r \rangle} \to \mathbb{L}_{m-1}^{\langle r \rangle} \to \cdots \to \mathbb{L}_1^{\langle r \rangle} \to \mathbb{L}_0^{\langle r \rangle} \to 0,$$

where $\mathbb{L}_j^{\langle r \rangle} = \bigoplus_{a=j+r} R(-a)^{\beta_{j,a}}$. The differentials are constructed via the corresponding differentials of the original resolution. This complex is called the ***r*-th linear strand of $F_\bullet$**.

In cases where the ideal in question is unclear, we denote this by $\mathbb{L}_\bullet^{\langle r \rangle}(I)$ to clarify.

Example 2.2.6. Suppose I is generated by homogeneous polynomials of degree d. If all the non-zero terms of the maps in the free resolution $F_\bullet$ are linear forms, then the resolution has the shape $F_i \cong \bigoplus R(-d+i-1)^{\beta_{i,-d+i-1}}$ for all $i > 0$. In particular, $F_0 \cong R$, $F_1 \cong R(-d)^\mu$ where μ is the minimal number of generators of I. In this case, the only linear strand which is not zero is $r = d$, and such resolutions are called linear.

Example 2.2.7. Let I be the ideal (xy, zw) in $R = k[x, y, z, w]$. The minimal graded free resolution of I is given by:

$$0 \longrightarrow R(-4) \xrightarrow{\begin{bmatrix} -zw \\ xy \end{bmatrix}} R^2(-2) \xrightarrow{\begin{bmatrix} xy & zw \end{bmatrix}} I \longrightarrow 0.$$

So $\beta_{1,4}(I) = 1$ and $\beta_{0,2}(I) = 2$. Therefore we will need to calcuate $\mathbb{L}_\bullet^{\langle 2 \rangle}$ and $\mathbb{L}_\bullet^{\langle 3 \rangle}$.

$\mathbb{L}_0^{\langle 2\rangle} = R(-0-2)^{\beta_{0,0+2}(I)} = R^2(-2)$ and $\mathbb{L}_i^{\langle 2\rangle} = 0$ for all $i \neq 0$. So we have:

$$\mathbb{L}_\bullet^{\langle 2\rangle} \colon 0 \longrightarrow R^2(-2) \longrightarrow 0.$$

$\mathbb{L}_1^{\langle 3\rangle} = R(-1-3)^{\beta_{1,1+3}(I)} = R(-4)^1$ and $\mathbb{L}_i^{\langle 3\rangle} = 0$ for all $i \neq 1$. So we have:

$$\mathbb{L}_\bullet^{\langle 3\rangle} \colon 0 \longrightarrow R(-4) \longrightarrow 0 \longrightarrow 0.$$

The relationship between linear strands and Lyubeznik numbers is not direct. That is, fixing a square-free monomial ideal $I \subset R$, the Lyubeznik numbers of $S \cong R/I$ are not directly computed in terms of the linear strands of a graded free resolution of S as an R-module. Instead, one has to replace I by its Alexander dual $I^\vee$ and rationalize the homology, [MV14, Cor. 4.2].

Theorem 2.2.8 (Montaner-Yanagawa)**.** *Suppose $I \subset R = k[x_1, \ldots, x_n]$ is a square-free monomial ideal. Set K to be the fraction field of R. One has*

$$\lambda_{i,j}(R/I) = \dim_K H_{j-i}(\mathbb{L}_\bullet^{\langle n-j\rangle}(I^\vee) \otimes_R K).$$

Example 2.2.9. Let $R = k[x, y, z, w]_{(x,y,z,w)}$ and I be the ideal (xz, zw, yz, yw). One has $I^\vee = (xy, zw)$ and the graded free resolution of $I^\vee$ is

$$0 \longrightarrow R(-4) \xrightarrow{\begin{bmatrix} -zw \\ xy \end{bmatrix}} R^2(-2) \xrightarrow{\begin{bmatrix} xy & zw \end{bmatrix}} I^\vee \longrightarrow 0.$$

So the non-zero graded Betti numbers are $\beta_{1,4} = 1$ and $\beta_{0,2} = 2$. We will calculate the 2-linear strand and 3-linear strand.

Directly, one sees $\mathbb{L}_0^{\langle 2\rangle} = R(-0-2)^{\beta_{0,0+2}} = R(-2)^2$ and $\mathbb{L}_i^{\langle 2\rangle} = 0$ for all $i \neq 0$. So

$\mathbb{L}_\bullet^{\langle 2\rangle}(I^\vee)$ is the complex

$$\mathbb{L}_\bullet^{\langle 2\rangle}(I^\vee)\colon 0 \longrightarrow R^2(-2) \longrightarrow 0,$$

which gives us that $\mathbb{L}_\bullet^{\langle 2\rangle}(I^\vee) \otimes_R K$ is $0 \longrightarrow K^2 \longrightarrow 0$.

For the 3-linear strand, $\mathbb{L}_1^{\langle 3\rangle} = R(-1-3)^{\beta_{1,1+3}} = R(-4)^1$ and $\mathbb{L}_i^{\langle 3\rangle} = 0$ for all $i \neq 1$. So $\mathbb{L}_\bullet^{\langle 3\rangle}$ is

$$\mathbb{L}_\bullet^{\langle 3\rangle}\colon 0 \longrightarrow R(-4) \longrightarrow 0 \longrightarrow 0,$$

which gives us that $\mathbb{L}_\bullet^{\langle 3\rangle} \otimes_R K$ is $0 \longrightarrow K \longrightarrow 0 \longrightarrow 0$.

Using Theorem 2.2.8, we see that

$$\lambda_{0,1}(R/I) = \dim_K H_1(\mathbb{L}_\bullet^{\langle 3\rangle} \otimes_R K) = \dim_K(K) = 1,$$

$$\lambda_{2,2}(R/I) = \dim_K H_0(\mathbb{L}_\bullet^{\langle 2\rangle} \otimes_R K) = \dim_K(K^2) = 2,$$

and $\lambda_{i,j}(R/I) = \dim_K H_{j-i}(\mathbb{L}_\bullet^{\langle 4-j\rangle} \otimes_R K) = 0$ for all other i, j. Therefore, the Lyubeznik table for R/I is given by:

$$\Lambda(R/I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

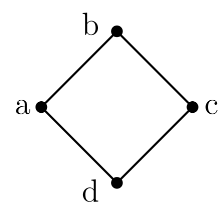
2.3 Graph theory

In this section, we review and introduce concepts from graph theory and combinatorial commutative algebra needed to calculate the Lyubeznik numbers of certain monomial ideals.

Throughout this section, G denotes a graph with $V(G) = \{x_1, \ldots, x_n\}$ and R denotes the polynomial ring $k[\underline{x}]$ where k is a field and $\underline{x} = x_1, \ldots, x_n$.

Definition 2.3.1. Fixing a graph G, the **edge ideal** of G is the R-ideal $I(G)$ generated by monomials x_ix_j when $\{x_i, x_j\}$ forms an edge in G.

Example 2.3.2. Let G be the following graph:



Set $R = k[a, b, c, d]$. The edge ideal of G, $I(G)$, is the monomial ideal (ab, bc, cd, ad).

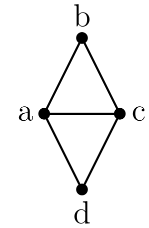
As G is simple, each edge ideal is of course a square-free monomial ideal, and as such it is recognized as the Stanley-Reisner ideal of a simplicial complex. This simplicial complex can be described in terms of the graph, or more importantly by the cliques of the graph.

Definition 2.3.3. Fix a graph G. A **clique** is a subgraph $G|_F$ for $F \subset V(G)$ which is isomorphic to a complete graph K_m for some m. We call m the **size of the clique**.

As per our abusive notational conventions, a clique can simply be denoted $F \subset V(G)$, where interpreting as a graph we mean $G|_F$. We will denote by $\#F$ for its size. A clique H in G is **maximal** if there is no other clique of G which contains H.

Example 2.3.4. Consider the graph:

G:



Notice that all vertices, $\{a\}, \{b\}, \{c\}$ and $\{d\}$, all edges $\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}$, and $\{a, c\}$, and the two triangles $\{a, b, c\}$ and $\{a, c, d\}$ are all cliques. However, only the two triangles are maximal cliques since all other cliques are contained in at least one of the two triangles.

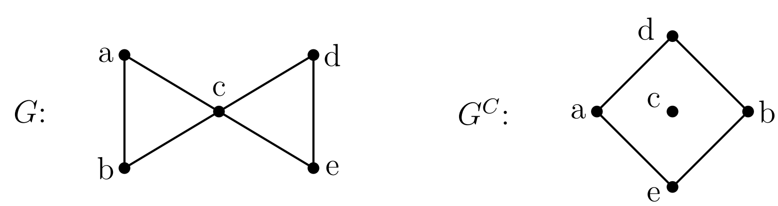
Note that the set of cliques of G, considered as subsets of $V(G)$, satisfies naturally the definition of a simplicial complex. In particular, it is immediate to verify that the intersection of two cliques in G is again a clique.

Definition 2.3.5. Given a graph G. The simplicial complex $\Delta(G)$ whose faces are the cliques of G is called the **clique complex**.

Example 2.3.6. Consider Example 2.2.1 and Example 2.3.4. We see that the set of faces in Δ is the same as the set of cliques in G and similarly the facets of Δ are the maximal clques in G. So the simplicial complex Δ in Example 2.2.1 is actually $\Delta(G)$, the clique complex of the graph G in Example 2.3.4.

Finally, given a graph G, its complement G^C is simply the graph with the same vertex set, so $V(G) = V(G^C)$, however there is an edge between vertices x and y in G^C if and only if there is no edge between x and y in G.

Example 2.3.7. Consider the graphs:



Here the graphs labeled G and G^C have the same vertex set, but G^C contains all the edges not in G and does not contain any edges found in G. So G^C is indeed the graph complement of G.

One can now proceed to calculate the Lyubeznik numbers of an edge ideal via the general method of linear strands reviewed in Chapter 2.2.1. A critical feature of the duality used there is that it admits a familiar purely graph theoretic interpretation. The following result has a convenient easy proof which can be found at [Fer06, Lem. 3.2].

Lemma 2.3.8. *For G a graph with edge ideal $I := I(G)$, then $I = I_{\Delta(G^C)}$ where $\Delta(G^C)$ is the clique complex of G^C.*

Since an edge ideal I can be interpreted using a graph or simplicial complex, we often borrow topological terminology to describe graphs. For example, a graph is connected if there is a path between any two vertices. A connected component of a graph G then is a connected subgraph H of G with the property that no vertex in H shares an edge with any vertex in G not in H.

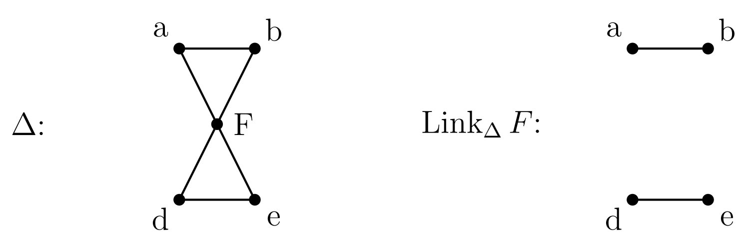
2.4 Calculating graded Betti numbers via Eagon-Reiner

We now need a critical famous reinterpretation of graded Betti numbers which will play a central role in our calculations. This is done in terms of topological restricted homology of a combinatorial deleted neighborhood called a link.

Definition 2.4.1. Fix Δ a simplicial complex and F a face in Δ, the **link of** F is the simplicial complex

$$\operatorname{Link}_\Delta F := \{G \in \Delta \colon G \cup F \in \Delta \text{ and } G \cap F = \emptyset\}.$$

Example 2.4.2. Consider the following simplicial complex Δ with face F:



Notice that to find $\operatorname{Link}_\Delta F$, we are basically looking for the facets in Δ containing F, which in this case are the triangles $\{a, b, F\}$ and $\{d, e, F\}$. We see then that $\operatorname{Link}_\Delta F$ consists of those facets excluding F, i.e. the edges $\{a, b\}$ and $\{d, e\}$.

Recall, that under usual homology of topological spaces, or simplicial complexes more specifically, H_0 calculates connected components. Working generally with $\mathbb{Z}$-coefficients, if X is a one point space, $H_0(X) \cong \mathbb{Z}$, where as asymmetrically $H_i(X) = 0$ for all $i > 0$. This asymmetry causes issues in uniform statements, for example the homology of the wedge sum of two spaces is not simply a direct sum of their homologies. To correct this, one augments the usual chain complex, [Hat02, pg. 110]. Denote by $\tilde{H}_i(X)$ for the homology of this augmented complex. This has the effect of changing no homology in positive degree, that is $\tilde{H}_i(X) \cong H_i(X)$ when $i > 0$. However, $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$ and so $\operatorname{rank} \tilde{H}_0(X) = \operatorname{rank} H_0(X) - 1$ giving the right correction. This does introduce now a valid $\tilde{H}_{-1}(X)$ however, this group is 0 unless X is empty in which case $\tilde{H}_{-1}(X) \cong \mathbb{Z}$.

Remark 2.4.3. *Reduced homology will be applied to simplicial complexes hence there is a non-canonical choice of coefficients. When considering for a simplicial complex Δ its associated Stanley-Reisner ideal I_Δ in a polynomial ring over a given field k, we always take k-coefficients for reduced homology. That is, $\tilde{H}_i(X)$ will be a k-vector space even though k won't appear in the notation.*

The famous formula now relating reduced homology of links with graded Betti numbers will be referred to throughout this work as the Eagon-Reiner formula, which is a reformulation of a result of Hochster, [Hoc77], to the case of simplical complexes, [ER98, Prop. 1].

We again let R denote the localization at the homogeneous maximal ideal of the polynomial ring $k[\underline{x}]$ where k is a field and $\underline{x} = x_1, \ldots, x_n$.

Theorem 2.4.4 (Eagon-Reiner)**.** *For Δ a simplicial complex denote by I_Δ its*

Stanley-Reisner ideal in R . The graded Betti numbers of R/I are given by

$$\beta_{i,j} := \sum_{F \in \Delta^\vee, \#F = n-j} \dim_k \tilde{H}_{i-1}(\mathrm{Link}_{\Delta^\vee} F).$$

2.4.1 Summary of the approach

We summarize our approach to address our main question on Lyubeznik numbers on edge ideals. First, one must wrestle with the fact that edge ideals, or more specifically their duals, can have complicated free resolutions and hence can also have complicated linear strands. Outside of special cases, there are no *simple* effective ways to calculate a specifically minimal graded free resolution in a uniform way. In particular, while one can consider non-minimal resolutions, such as the Taylor or Lyubenzik resolutions, they are not canonically pruned to a minimum one. Surprisingly, deterministic and canonical algorithms have recently been shown to exist, for example [EMO], however their practical implementations are fraught with complications to apply in our specific case. To summarize, it seems out of reach to give a complete answer for all edge ideals via purely computational tools due to the wild variability of resolutions of edge ideals.

As such, we prefer to take an approach which produces the most detailed answer first at the expensive of limiting the graphs we consider. These limitations will force the free resolutions to be simple enough that the homology of their linear strands will be effectively computed by computing graded Betti numbers. The Eagon-Reiner formula then allows us to translate the problem to a combinatorial one in terms of connected components of links in an associated clique complex. To achieve an optimal answer, we then reinterpret those graded Betti numbers in terms of purely combinatorial data from the graph using a refined family of graphs which essentially count 'clique connectivity' which we call clique graphs.

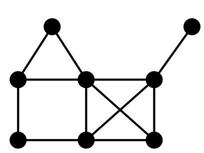
3 Clique graphs and the Clique-Link correspondence

This chapter introduces clique graphs, the main technical tool in the thesis as well as many supporting detailed combinatorial results. These calculations can be understood independently of their applications to Lyubeznik numbers and might be of independent combinatorial interest. The reader interested only in those applications can, upon accepting the results Corollary 3.2.10 and Theorem 3.3.14 as valid calculations, proceed directly to the topics of interest in Chapter 4. While clique graphs were discovered while working on this thesis, they appear independently in other sources. While the graphs in these sources are isomorphic, our techniques are inherently different. We draw connectections between clique graphs and the higher nerve complexes in [DDDGHL19] in Chapter 3.4.1 as well as higher Hochster-Huneke graphs in [NBSW19] in Section 3.4.2.

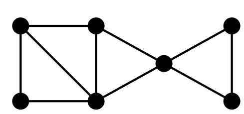
3.1 Clique graphs

Clique graphs will be graphs built from the set of maximal cliques of a given graph G. We will most often only consider graphs G for which all maximal cliques have the same size. Such graphs are called **pure**. If G is pure with the common size of all maximal cliques d one says that G is a $(d-1)$**-dimensional graph**. The reason for the shift by 1 is to allow the ring defined by the edge ideal of G to have Krull dimension d.

Example 3.1.1. We observe here an example of a pure and non-pure graph. Notice G_1 has a maximal clique of size 4, a maximal clique of size 3, and 3 maximal cliques of size 2, whereas all maximal cliques in G_2 are size 3.

G_1: 

non-pure graph

G_2: 

pure graph

Since we will reference maximal cliques often, denote by $\mathrm{MaxClq}(G)$ the set of all maximal cliques in G, denote by $\text{MaxClq}_i(G)$ the set of all maximal cliques cliques in G of size i, and for fixed clique F in G, set

$$\text{MaxClq}(G/F) = \{H \in \text{MaxClq}(G) : F \subset H\}.$$

Definition 3.1.2. Let G be a pure graph. For each i, we will define two kinds of **clique graphs** as follows. The graph $\text{Clq}_{\geq i}(G)$ is the graph with:

1. vertex set $V(\text{Clq}_{\geq i}(G)) = \text{MaxClq}(G)$, and

2. edge set $E(\text{Clq}_{\geq i}(G)) = \{\{U, V\} : \#(U \cap V) \geq i \text{ in } G\}$.

The graph $\text{Clq}_i(G)$ is the graph with:

1. vertex set $V(\text{Clq}_i(G)) = \text{MaxClq}(G)$, and

2. edge set $E(\text{Clq}_i(G)) = \{\{U, V\} : \#(U \cap V) = i \text{ in } G\}$.

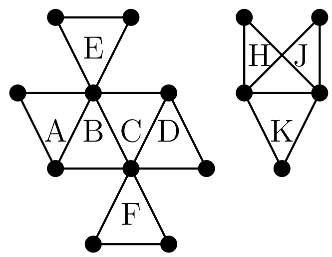
Note, all clique graphs have the same vertex set $\text{MaxClq}(G)$. The differences only lie in the edges based on equality or inequality of sizes of intersections.

Remark 3.1.3. *Let G be a $(d-1)$-dimensional pure graph. First notice that $\#(U \cap V) < d$ for all U and V in $\text{MaxClq}(G)$ since $\#U = d$ for all $U \in \text{MaxClq}(G)$. Also, $\#F \geq 0$ for all $F \in \Delta(G)$, so $\#(U \cap V) \geq 0$ for all U and V in $\text{MaxClq}(G)$. Therefore, $\text{Clq}_i(G)$ is the graph of isolated points for $i \geq d$ or $i < 0$, $\text{Clq}_{\geq i}(G)$ is the graph of isolated points for $i \geq d$, and $\text{Clq}_{\geq i}(G)$ is a complete graph for $i \leq 0$. For this reason, the most interesting clique graphs are $\text{Clq}_{\geq i}(G)$ for $0 < i \leq d-1$ and $\text{Clq}_i(G)$ for $0 \leq i \leq d-1$.*

We also remark here that $\text{Clq}_{\geq d-1}(G) \cong \text{Clq}_{d-1}(G)$ since, as we noted above, no pair of maximal cliques in G can intersect at a clique of size d or greater.

We look now an example which will be recurring and through which we will develop and explain the ideas that lead one to establish the Clique-Link correspondence which will be stated formally as Theorem 3.2.8.

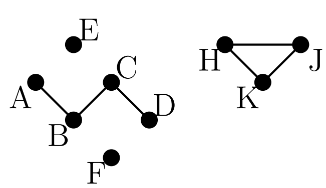
Example 3.1.4. Consider the graph G given by:



Note that G is a pure graph, where all maximal cliques have size three. We compute the clique graphs, $\mathrm{Clq}_{\geq 2}(G) \cong \mathrm{Clq}_2(G)$, $\mathrm{Clq}_{\geq 1}(G)$, $\mathrm{Clq}_1(G)$, and $\mathrm{Clq}_0(G)$. The set $\mathrm{MaxClq}(G) = \{A, B, C, D, E, F, G, H, J, K\}$ will serve as the set of vertices for all clique graphs.

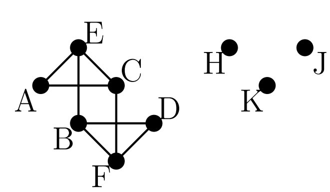
We start by computing $\mathrm{Clq}_2(G) \cong \mathrm{Clq}_{\geq 2}(G)$. This will have an edge for every pair of maximal cliques in G whose intersection is an edge, i.e.

$\mathrm{Clq}_{\geq 2}(G) \cong \mathrm{Clq}_2(G)$:



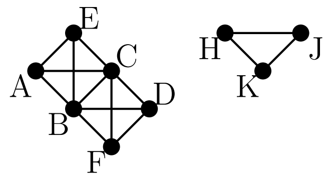
The clique graph $\mathrm{Clq}_1(G)$ will have an edge for every pair of maximal cliques in G whose intersection is a point. Notice that $E \cap A, E \cap B, E \cap C$ and $F \cap B, F \cap C, F \cap D$ are all points, but also $A \cap C$ and $B \cap D$ are points. So we have:

$\mathrm{Clq}_1(G)$:



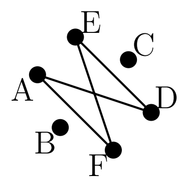
We see that $\mathrm{Clq}_{\geq 1}(G)$ will have an edge for every pair of maximal cliques in G that intersect at a point or an edge. So we have all the edges in $\mathrm{Clq}_2(G)$ and $\mathrm{Clq}_1(G)$

$\mathrm{Clq}_{\geq 1}(G)$:



The clique graph $\mathrm{Clq}_0(G)$ has an edge for every pair of maximal cliques in G that do not intersect. Obviously there would be an edge from H to each of A, B, C, D, E, F, from J to each of A, B, C, D, E, F, and from K to each of A, B, C, D, E, F. So to simplify our graph, we will only consider restricted portion of the clique graph, $\mathrm{Clq}_0(G)|_{\{A,B,C,D,E,F\}}$.

$\mathrm{Clq}_0(G)|_{\{A,B,C,D,E,F\}}$:



After examining many examples, certain patterns emerge. In particular, we notice a few correspondences between the graph G and the clique graphs which we explain in this example. One such correspondence is seen in $\mathrm{Clq}_{\geq 1}(G)$, where the number of connected components of $\mathrm{Clq}_{\geq 1}(G)$ is equal to the number of connected components in G, i.e. $\pi_0(\mathrm{Clq}_{\geq 1}(G))$ is in bijection with $\pi_0(G)$. Upon closer examination, we also notice a correspondence between maximal cliques in $\mathrm{Clq}_{\geq i}(G)$ and maximal cliques in G that have an intersection of size i.

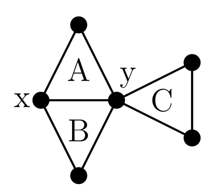
In the following section, we will provide general proof of these and other correspondences which will ultimately relate links of cliques in a graph and cliques in the clique graph, hence allowing a relationship between graded Betti numbers of the ring defined by an edge ideal and purely graph theoretic combinatorial data via Eagon-Reiner.

3.2 Clique-Link Correspondence

Here we describe a sequence of correspondences that will be useful in calculating the Betti numbers of Alexander duals of edge ideals using clique graphs (see Chapter 4). We refer to these as the **Clique-Link correspondences**, and will see they come about quite naturally.

Let us first notice that there is a natural map $\psi : \Delta(G) \longrightarrow \mathcal{P}(\mathrm{MaxClq}(G))$, sending F to $\mathrm{MaxClq}(G/F)$. There is also a natural map $\phi : \mathcal{P}(\mathrm{MaxClq}(G)) \longrightarrow \Delta(G)$ sending $H \subseteq \mathrm{MaxClq}(G)$ to the intersection $\bigcap_{Q \in H} Q$. It is clear that $F \subset \phi(\psi(F))$ and $H \subset \psi(\phi(H))$. However we will see in the next example, that in general, these containments can be strict.

Example 3.2.1. Let G be the graph:



Notice that $\psi(\{x\}) = \{A, B\}$. Then $\phi(\{A, B\}) = A \cap B = \{x, y\}$. So we see that $\{x\} \subsetneq \{x, y\}$ which shows the containment $F \subset \psi(\phi(F))$ can be strict.

Also notice that $\phi(\{A, C\}) = \{y\}$, but $\psi(\{y\}) = \{A, B, C\}$. So again, we have an $H \subsetneq \psi(\phi(H))$.

The natural question to ask is what restrictions can be made on the domains and codomains of ψ and ϕ to give a bijection. Before we address this question, recall that $\mathrm{MaxClq}(G)$ serves as the vertex set for $\mathrm{Clq}_{\geq i}(G)$ for any i. Therefore, for each subset, S, of $\mathrm{MaxClq}(G)$, we can consider S as the subset of $V(\mathrm{Clq}_{\geq i}(G))$ for any i, and from this, we can consider the induced subgraph of $\mathrm{Clq}_{\geq i}(G)$ on the vertices in S, i.e $\mathrm{Clq}_{\geq i}(G)|_S$. So, for $F \in \Delta(G)$, we can consider the map

$$F \xrightarrow{\psi} \psi(F) \subseteq \mathrm{MaxClq}(G) = \mathrm{Clq}_{\geq i}(G)|_{\psi(F)}.$$

Also, for $H \subseteq V((\mathrm{Clq}_{\geq i}(G)))$ for some i, recalling the abuse conflating H for $\mathrm{Clq}_{\geq i}(G)|_H$, we can consider the map

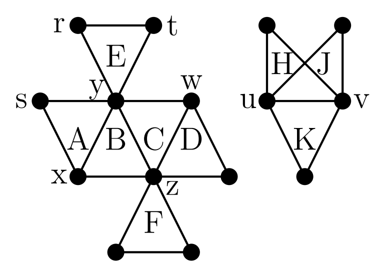
$$H \subseteq \mathrm{MaxClq}(G) \xrightarrow{\phi} \bigcap_{Q \in H} Q.$$

With this in mind, we will abuse notation in the following way when the situation is clear. We let ψ denote the map $\psi : \Delta(G) \longrightarrow \Delta(\mathrm{Clq}_{\geq i}(G))$ for any i, defined by $\psi(F) = \mathrm{Clq}_{\geq i}(G)[\mathrm{MaxClq}(G/F)]$. Recall $\mathrm{Clq}_{\geq i}(G)[\mathrm{MaxClq}(G/F)]$ denotes the induced subgraph of $\mathrm{Clq}_{\geq i}(G)$ on the set of vertices $\mathrm{MaxClq}(G/F)$. We let ϕ denote the map $\phi : \Delta(\mathrm{Clq}_{\geq i}(G)) \longrightarrow \Delta(G)$ for any i, defined by $\phi(H) = \bigcap_j A_j$ where $\{A_1, \ldots, A_k\}$ is the set of maximal cliques in G represented by $V(H)$.

Returning to our question of when ψ is a bijection, we now see it would be useful to consider the subgraph $\psi(F)$. We return to the graph we used in Example 3.1.4 to explore this concretely.

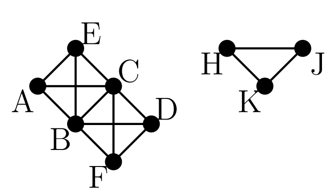
Example 3.2.2. Consider the graph G as given in Example 3.1.4.

G:



Recall that $\mathrm{Clq}_{\geq 1}(G)$ was given by:

$\mathrm{Clq}_{\geq 1}(G)$:



The calculation summarized in Figure 3.2.2 below suggests that when Link F is disconnected, we will have a bijection between F and $\phi(\psi(F)) \in \Delta(\mathrm{Clq}_{\geq i}(G))$, and when Link F is connected this correspondence is obstructed. This observation leads to one restriction we need to make on the domain of ψ.

$F \in \Delta(G)$	$V(\psi(F))$	$\phi(\psi(F))$	Link F connected?
{y}	{A,B,C,E}	{y}	No
{u}	{H,J,K}	{u,v}	Yes

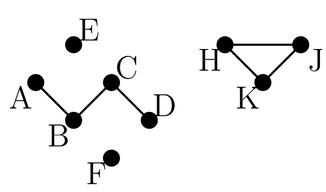
Figure 3.1: Correspondence between connected components of links

Also, notice that in G, $\#(A \cap B \cap C \cap E) = 1$ and $\#(H \cap J \cap K) = 2$, and recall we're considering $\mathrm{Clq}_{\geq 1}(G)$. In the general case, this would be saying we need to consider only the faces F of size i in $\Delta(G)$ to recognize a bijection on a restriction of $\psi : \Delta(G) \longrightarrow \mathrm{Clq}_{\geq i}(G)$.

Another observation we can make here is that $\psi(\{y\})$ is a maximal clique in $\mathrm{Clq}_{\geq 1}(G)$. Restricting to maximal cliques in $\mathrm{Clq}_{\geq 1}(G)$ is a restriction we will place on the codomain of $\psi : \Delta(G) \longrightarrow \mathrm{Clq}_{\geq 1}(G)$ to force a bijection.

To see another restriction we need to make on the codomain of ψ, we consider $\mathrm{Clq}_{\geq 2}(G)$ which we recall is the following graph.

$\mathrm{Clq}_{\geq 2}(G)$:



Note that $\{y\}$ in G is equal to $\phi(\{A, B, C, E\})$ and $\mathrm{Clq}_{\geq 2}(G)[\{A, B, C, E\}]$ is disconnected, whereas $\{u\}$ was not equal to $\phi(\{H, J, K\})$, and $\mathrm{Clq}_{\geq 2}(G)[\{H, J, K\}]$ is connected. So we will restrict the codomain of ψ further to require $\psi(F)$ in $\mathrm{Clq}_{\geq i}(G)$ to be such that $\mathrm{Clq}_{\geq i+1}(G)|_{\psi(F)}$ is disconnected.

Armed with these observations, we redefine the domains and codomains for ψ and ϕ to force a bijection. This bijection is a critical conduit for our calculations to relate Lyubeznik numbers with the combinatorial data from G.

Definition 3.2.3. Let G be a pure $(d-1)$-dimensional graph. Fix $0 \leq i \leq d$. We define the sets $\mathcal{F}_i$ and $\mathcal{H}_i$ as follows:

$$\mathcal{F}_i = \{F \in \Delta(G) : \#F = i \text{ and } \operatorname{Link}(F) \text{ is disconnected}\},$$

$$\mathcal{H}_i = \{H \in \operatorname{MaxClq}(\mathrm{Clq}_{\geq i}(G)) : \mathrm{Clq}_{\geq i+1}(G)|_H \text{ is disconnected}\}.$$

To see the adjustments, we look again at the graphs from Example 3.1.4

Example 3.2.4. In Example 3.1.4, we see that $\mathcal{F}_1$ is equal to $\{y, z\}$ and $\mathcal{H}_1 = \{\{A, B, C, E\}, \{B, C, D, F\}\}$, where $\{y\}$ corresponds to $\{A, B, C, E\}$ and $\{z\}$ corresponds to $\{B, C, D, F\}$. Also, $\mathcal{F}_2$ is equal to $\{\{x, y\}, \{y, z\}, \{z, w\}, \{u, v\}\}$ and $\mathcal{H}_2$ is equal to $\{\{A, B\}, \{B, C\}, \{C, D\}, \{H, J, K\}\}$.

We now state and prove the main form of the Clique-Link correspondence and necessary lemmas.

Remark 3.2.5. *Throughout these proofs and the proofs of the rest of the results of this chapter, we write* $\mathrm{Clq}_{\geq i}$ *for* $\mathrm{Clq}_{\geq i}(G)$*, as* G *is fixed throughout, and revert to the original notation when necessary. We also introduce a couple convenient notations for this proof. First if* A *is an induced subgraph of* G *on a given set of vertices and* F *is a subgraph of* A*, then we use the notation* $A \backslash F$ *to be the induced subgraph of* G *on* $V(A)\backslash V(F)$*. Next, for* $H = \{A_1, \ldots, A_k\}$*, a subset of* $V(G)$ *for graph* G*, denote by* $G[A_1, \ldots, A_k]$ *for* $G|_H$*. We will apply this to* $\mathrm{Clq}_{\geq i}$*, so* $A_1, \ldots, A_k$ *will be maximal cliques in* G*, and* $\mathrm{Clq}_{\geq i}[A_1, \ldots, A_k]$ *is the subgraph induced by these.*

Lemma 3.2.6. *Let G be a pure $(d-1)$-dimensional graph and let $\mathcal{H}_i$ be as defined in 3.2.3. Let H be in $\mathcal{H}_i$ where $V(H) = \{A_1, \ldots, A_k\}$, then in G, $\#(\bigcap_j A_j) = i$.*

Proof. We first note that $\text{Clq}_{\geq i+1}(G)|_H$ is disconnected, and so there exists p, q such that A_p and A_q are in differenct connected components in $\text{Clq}_{\geq i+1}(G)|_H$. Therefore, $\{A_p, A_q\}$ is not an edge in $\text{Clq}_{\geq i+1}(G)|_H$. This gives us that $\#(A_p \cap A_q) = i$ in G.

Claim: $(A_p \cap A_q) \subset A_j$ in G for each $j = 1, \ldots, k$.
Assume by contradiction $A_p \cap A_q$ is not in A_m for some $m = 1, \ldots, k$.

By elementary set theory, $Z = (Z \cap W) \cup (Z \backslash W)$. Taking $Z = A_p \cap A_m$ and $W = A_q$, this gives us

$$A_p \cap A_q = (A_p \cap A_m \cap A_q) \cup ((A_p \cap A_m) \backslash A_q).$$

By assumption, since $\#(A_p \cap A_q) = i$ and $A_p \cap A_q \nsubseteq A_m$, we see that $\#(A_p \cap A_m \cap A_q) < i$. Therefore, $(A_p \cap A_m) \backslash A_q \neq \varnothing$ and so there exists $y \in (A_p \cap A_m) \backslash A_q$.
Similarly, there exists $v \in (A_q \cap A_m) \backslash A_p$.
By clique definitions, $A_p \cap A_q \cup \{y\} \cup \{v\}$ is a clique in G and so $A_p \cap A_q \cup \{y\} \cup \{v\} \subseteq A_c$ for some c. But then $\#(A_q \cap A_c) \geq \#(A_p \cap A_q \cup \{v\}) = i + 1$ and $\#(A_p \cap A_c) \geq \#(A_p \cap A_q \cup \{y\}) = i + 1$. So there exist edges $\{A_q, A_c\}$ and $\{A_c, A_p\}$ in $\text{Clq}_{\geq i+1}(G)$, which means there is a path from vertex A_q to A_p in $\text{Clq}_{\geq i+1}(G)$. This is a contradiction since A_p and A_q are in different connected components. So the claim holds.

This gives us that

$$A_p \cap A_q \subseteq \bigcap_{j=1}^{k} A_j \subseteq A_p \cap A_q \text{ and so } A_p \cap A_q = \bigcap_{j=1}^{k} A_j.$$

Clearly then,

$$\#(\bigcap_{j=1}^{k} A_j) = \#(A_p \cap A_q) = i.$$

□

Lemma 3.2.7. *Let G be a pure $(d-1)$-dimensional graph and let $\mathcal{H}_i$ be as defined in 3.2.3, and consider the restrictions $\psi : \mathcal{F}_i \longrightarrow \mathcal{H}_i$ and $\phi : \mathcal{H}_i \longrightarrow \mathcal{F}_i$. If $F \in \mathcal{F}_i$ and $H \in \mathcal{H}_i$, then $\mathrm{Clq}_{\geq i+1}|_{\psi(F)}$ is disconnected and $\mathrm{Link}(\bigcap_{Q \in H} Q)$ is disconnected in G.*

Proof. We first verify that $\mathrm{Clq}_{\geq i+1}|_{\psi(F)}$ is disconnected by contradiction. So suppose that $\mathrm{Clq}_{\geq i+1}|_{\psi(F)}$ is connected. Let x, y be vertices in $\mathrm{Link}(F)$. Let $A, B \in \mathrm{MaxClq}(G/F)$ such that $x \in A, y \in B$. Note that A, B are vertices in H. Since $\mathrm{Clq}_{\geq i+1}|_{\psi(F)}$ is connected, there exists vertices $V_1, \ldots, V_r$ such that $\{A, V_1\}, \{V_1, V_2\}, \ldots, \{V_{r-1}, V_r\}, \{V_r, B\}$ are edges in $\mathrm{Clq}_{\geq i+1}|_{\psi(F)}$. Therefore, in G, we have that $\#(A \cap V_1) \geq i+1, \#(V_1 \cap V_2) \geq i+1, \ldots, \#(V_{r-1}, V_r) \geq i+1, \#(V_r \cap B) \geq i+1$. Since $i \geq 0$, we see that $i+1 \geq 1$. So let u_1 be in $A \cap V_1, u_2$ be in $V_1 \cap V_2, \ldots, u_r$ be in $V_{r-1} \cap V_r$, and u_{r+1} be in $V_r \cap B$. Since x and u_1 are both in A, there is an edge in G between x and u_1. Since u_1, u_2 are both in V_1, there is an edge in G between u_1 and $u_2, \ldots,$ and since u_r, u_{r+1} are both in V_{r+1}, there is an edge i G between u_r and u_{r+1}. Lastly, since u_{r+1}, y are both in B, there is an edge in G between u_{r+1} and y. Therefore, there is a path between x and y and therefore $\mathrm{Link}(F)$ is connected, which is a contradiction. This is enough to show that $\mathrm{Clq}_{\geq i+1}|_{\psi(F)}$ is not connected.

To show that $\mathrm{Link}(\bigcap_{Q \in H} Q)$ is disconnected, first set $F = \bigcap_{Q \in H} Q$. Now suppose that $\mathrm{Link}(F)$ is connected. Let $A, B \in V(H)$. So $A, B \in \mathrm{MaxClq}(G/F)$, and so $A \backslash F$ and $B \backslash F$ are contained in $\mathrm{Link}(F)$. Let x be in $A \backslash F$ and y be in $B \backslash F$. Since $\mathrm{Link}(F)$ is connected, there exists a path from x to y, i.e. there exists vertices $v_1, \ldots, v_{r-1}$ such that $\{x, v_1\}, \{v_1, v_2\}, \ldots, \{v_{r-2}, v_{r-1}\}$, and $\{v_{r-1}, y\}$ are edges in $\mathrm{Link}(F)$. Set $x = v_0$ and $y = v_r$. Let $U_j \in \mathrm{MaxClq}(G/F)$ such that $\{v_{j-1}, v_j\}$ is in U_j for all $1 \leq j \leq r$.

Now set $A = U_0, B = U_{r+1}$. For all $0 \leq j \leq r$, $v_j \in U_{j+1}$, and so $V(U_j \cap U_{j+1}) \supseteq V(F) \cup v_j$, which gives us that $\#(U_j \cap U_{j+1}) \geq i+1$. Therefore, there is an edge between U_j and U_{j+1} in Clq_{i+1} for all $0 \leq j \leq r$. So there is a path from U_0 to U_{r+1}, i.e. from A to B, in $\mathrm{Clq}_{\geq i+1}|_H$, and therefore $\mathrm{Clq}_{\geq i+1}|_H$ is connected. This is a contradiction, which gives us that $\mathrm{Link}(F)$ is not connected. □

Theorem 3.2.8. *Let G be a pure $(d-1)$-dimensional graph and let $\mathcal{F}_i$ and $\mathcal{H}_i$ be as defined in 3.2.3. The restrictions $\psi : \mathcal{F}_i \longrightarrow \mathcal{H}_i$ and $\phi : \mathcal{H}_i \longrightarrow \mathcal{F}_i$ form a bijection. In particular, there is a one to one correspondence between the sets $\mathrm{MaxClq}(G/F)$ and $V(H)$ for all $F \in \mathcal{F}_i$ corresponding to $H \in \mathcal{H}_i$.*

Proof. We show that $\psi(\mathcal{F}_i) \subseteq \mathcal{H}_i$, that $\phi(\mathcal{H}_i) \subseteq \mathcal{F}_i$, and that ψ and ϕ are inverses of each other.

1. $\psi(\mathcal{F}_i) \subseteq \mathcal{H}_i$.

   Let $F \in \mathcal{F}_i$ and set $\mathrm{MaxClq}(G/F) = \{A_1, \ldots, A_k\}$. Since $\mathrm{Link}(F)$ is not connected, we see that $k > 1$. Since $F \subseteq A_j$ for all j, which means that $F \subseteq A_l \cap A_j$ for all l, j, we have that $\#(A_l \cap A_j) \geq \#F = i$. So there is an edge between A_l and A_j for all $1 \leq l, j \leq k$ in $\mathrm{Clq}_{\geq i}$. Therefore, $\mathrm{Clq}_{\geq i}[\{A_1, \ldots, A_k\}]$ is a clique.

   To see that $\mathrm{Clq}_{\geq i}[\{A_1, \ldots, A_k\}]$ is a maximal clique in $\mathrm{Clq}_{\geq i}$, let V be a vertex in $\mathrm{Clq}_{\geq i}$ such that $\mathrm{Clq}_{\geq i}[\{A_1, \ldots, A_k, V\}]$ is a clique in $\mathrm{Clq}_{\geq i}$. One has $\#(V \cap A_j) \geq i$ in G for all j.

   Assume, for the sake of contradiction, that $F \not\subseteq V$ in G. For all $1 \leq j \leq k$, there exists at least one vertex v_j in $\#(V \cap A_j)$, such that v_j is not in F. Since v_j is in A_j and v_j is not in F for all j, we see that v_j is in $\mathrm{Link}(F)$ for all j.

   Let a, b be faces of size one in $\mathrm{Link}(F)$ which we at times will abuse notation and refer to as vertices since we are looking at things from a graph perspective, and let A_l and A_m be maximal cliques in G containing a and b respectively. Both a and v_l are in A_l, so there is an edge in G between a and v_l. Since v_l and v_m are both in V, there is an edge in G between v_l and v_m and since v_m and b are both in A_m, there is an edge in G between v_m and b. Therefore, there is a path in $\mathrm{Link}(F)$ connecting a and b. Since this is true for any two vertices in $\operatorname{Link}(F)$, we see that $\operatorname{Link}(F)$ is connected, which is a contradiction. Therefore, $F \subseteq V$ which means $V = A_j$ for some j. This gives us that $\psi(F) = \mathrm{Clq}_{\geq i}[\{A_1, \ldots, A_k\}]$ is indeed a maximal clique of size $k > 1$ in $\mathrm{Clq}_{\geq i}$.

We must also show that $\mathrm{Clq}_{\geq i+1}|_{\psi(F)}$ is disconnected, but for this, we refer to Lemma 3.2.7. Therefore, for any $F \in \mathcal{F}_i$, we see that $\psi(F) \in \mathcal{H}_i$.

2. $\phi(\mathcal{H}_i) \subseteq \mathcal{F}_i$, that is, if H is in $\mathcal{H}_i$, then $\phi(H)$ is in $\mathcal{F}_i$.

   As H is a clique in $\mathrm{Clq}_{\geq i}$, its vertices are a set of maximal cliques in G. Denote this set as $\{A_1, \ldots, A_k\}$ and set $F = \bigcap A_j$. Note that F itself is a clique in G, so we need to show that it has size i and that $\operatorname{Link} F$ is disconnected. We apply Lemma 3.2.6 for the former and Lemma 3.2.7 for the latter.

3. ψ and ϕ are inverses of each other. Let $H \in \mathcal{H}$ and $F = \phi(H)$. By definition, $H \subseteq \psi(\phi(H))$. Since H is a maximal clique and by (1), $\psi(\phi(H))$ is a maximal clique, $H = \psi(\phi(H))$.

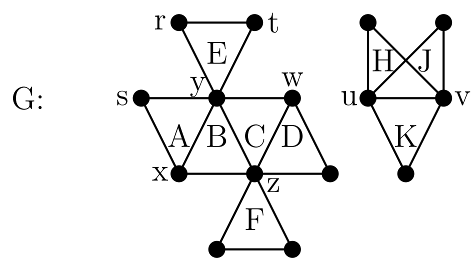
   Let $F \in \mathcal{F}_i$ with $\mathrm{MaxClq}(G/F) = \{A_1, \ldots, A_k\}, k > 1$. Consider $(\phi \circ \psi)(F)$. First, we know $\psi(F) = \mathrm{Clq}_{\geq i}[\{A_1, \ldots, A_k\}]$, which by (1) is a maximal clique in $\mathrm{Clq}_{\geq i+1}$. Call this maximal clique H. Then $\phi(H) = \bigcap_j A_j$, which by (2) is a clique of size i in G. Since $F \subset A_j$ for all j, we have that $F \subset \bigcap_j A_j$. Since F is also a clique of size i, $F = \bigcap_j A_j = \phi(H) = \phi(\psi(F))$.

   So we have showed there is a bijection between $\mathcal{F}_i$ and $\mathcal{H}_i$. We also see that if $F \in \mathcal{F}_i$ corresponds to $H \in \mathcal{H}_i$, then $\mathrm{MaxClq}(G/F) = \{A_1, \ldots, A_k\}$ in G and $V(H) = \{A_1, \ldots, A_k\}$ in $\mathrm{Clq}_{\geq i}$, and so there is a one to one correspondence between $\mathrm{MaxClq}(G/F)$ and $V(H)$.

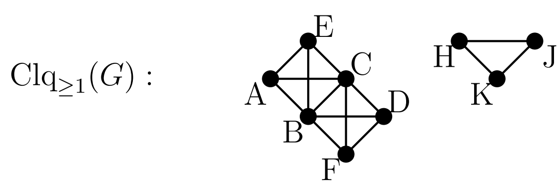
□

Now that we have answered our question about what restriction of the domain of codomain of ψ gives a bijection, we will see there is also a correspondence between connected components of Link(F) and connected components in $\mathrm{Clq}_{\geq i}(G)|_{\psi(F)}$, which follows in similar manner as Theorem 3.2.8. We will show this behavior in a concrete example before examining the general case.

Example 3.2.9. We will return to Example 3.1.4, where G is given as follows.

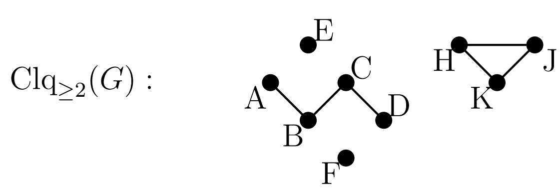


and $\mathrm{Clq}_{\geq 1}(G)$ is given by:



Recall $\{y\}$ is in one to one correspondence with $\mathrm{Clq}_{\geq 1}(G)[\{A, B, C, E\}\}]$, where $\#(\{y\}) = 1$. Notice Link($\{y\}$) has 2 connected components, namely the component that consists of $A\backslash\{y\}$ and the component that consists of $B\backslash\{y\}, C\backslash\{y\}$, and $E\backslash\{y\}$.

Also, recall that $\mathrm{Clq}_{\geq 2}(G)$ is given by the following.



Also, recall that $\mathrm{Clq}_{\geq 2}(G)|_{\psi(\{y\})} = \mathrm{Clq}_{\geq 2}(G)[\{A, B, C, D\}]$ has 2 connected components, namely the component consisting of $\{A, B, C\}$ and the component consisting of $\{E\}$.

Finally, we record a version of the Clique-Link correspondence in more explicit combinatorial terms in particular as an bijection between sets of connected components. Throughout, when X is either a graph or a simplicial complex, we denote its set of connected components simply by $\pi_0(X)$.

Corollary 3.2.10. *Let G be a pure graph. Let F in $\mathcal{F}_i$ correspond to H in $\mathcal{H}_i$ as described in Theorem 3.2.8. There is a one-to-one correspondence between $\pi_0(\mathrm{Link}(F))$ and $\pi_0(\mathrm{Clq}_{\geq i+1}(G)|_H)$.*

Proof. As G is fixed throughout, we write $\mathrm{Clq}_{\geq i}$ for $\mathrm{Clq}_{\geq i}(G)$ and revert to the original notation when necessary. Let $\{A_1, \ldots, A_k\}$ be a subset of $V(H)$ such that $\mathrm{Clq}_{\geq i+1}|_{\{A_1,\ldots,A_k\}} \in \pi_0(\mathrm{Clq}_{\geq i+1}|_H)$. The result will follow by a series of biconditional statements:

By definition of a connected component, the statement that $\{A_1, \ldots, A_k\}$ is a subset of $V(H)$ such that $\mathrm{Clq}_{\geq i+1}|_{\{A_1,\ldots,A_k\}} \in \pi_0(\mathrm{Clq}_{\geq i+1}|_H)$ is equivalent to saying $\{A_1, \ldots, A_k\}$ is a subset of $V(H)$ such that there is a path between any two vertices in $\mathrm{Clq}_{\geq i+1}|_{\{A_1,\ldots,A_k\}}$ and no vertex in $\mathrm{Clq}_{\geq i+1}|_{\{A_1,\ldots,A_k\}}$ is connected to any vertex not in $\mathrm{Clq}_{\geq i+1}|_{\{A_1,\ldots,A_k\}}$. Our proof will require us to label the path between the two vertices in $\mathrm{Clq}_{\geq i+1}|_{\{A_1,\ldots,A_k\}}$, so we will rewrite the previous statement symbolically: $\{A_1, \ldots, A_k\}$ is a subset of $V(H)$ such that for any A_j, A_l in $\{A_1, \ldots, A_k\}$, there exist vertices $A_{i_1}, \ldots, A_{i_r} \in \{A_1, \ldots, A_k\}$ such that $\{A_j, A_{i_1}\}, \{A_{i_1}, A_{i_2}\}, \ldots, \{A_{i_r}, A_l\}$ are edges in $\mathrm{Clq}_{\geq i+1}|_{\{A_1,\ldots,A_k\}}$ and for any $P \in V(H)\backslash\{A_1, \ldots, A_k\}$, then $\{P, A_j\}$ is not an edge in $\mathrm{Clq}_{\geq i+1}|_{V(H)}$ for any j.

Now, by definition of $\mathrm{Clq}_{\geq i+1}$, recalling that vertices in $\mathrm{Clq}_{\geq i+1}$ represent maximal cliques in G and edges in $\mathrm{Clq}_{\geq i+1}$ represent intersections of maximal cliques in G, we see this is equivalent to saying: $\{A_1, \ldots, A_k\}$ is a subset of $\mathrm{MaxClq}(G/F)$ such that for any $A_j, A_l \in \{A_1, \ldots, A_k\}$, there exists $A_{i_1}, \ldots, A_{i_r} \in \{A_1, \ldots, A_k\}$ such that $\#(A_j \cap A_{i_1}) \geq i+1, \#(A_{i_1} \cap A_{i_2}) \geq i+1, \ldots,$ and $\#(A_{i_r} \cap A_l) \geq i+1$ and for any P in $\mathrm{MaxClq}(G/F)\backslash\{A_1, \ldots, A_k\}$, then $\#(P \cap A_m) < i+1$ for all m.

Since F is a size i clique and is contained in $A_j, A_{i_1}, \ldots A_{i_1}$, and A_l, we see the next equivalent statement: $\{A_1, \ldots, A_k\}$ is a subset of $\mathrm{MaxClq}(G/F)$ such that for any

$A_j, A_l \in \{A_1, \ldots, A_k\}$, there exists $A_{i_1}, \ldots, A_{i_r} \in \{A_1, \ldots, A_k\}$ such that $\#(A_j \backslash F \cap A_{i_1} \backslash F) \geq 1, \#(A_{i_1} \backslash F \cap A_{i_2} \backslash F) \geq 1, \ldots,$ and $\#(A_{i_r} \backslash F \cap A_l \backslash F) \geq 1$ and for any $P \in \mathrm{MaxClq}(G/F) \backslash \{A_1, \ldots, A_k\}$, then $\#(P \backslash F \cap A_m \backslash F) = \emptyset$ for all m

Again noting the definition of a connected component of $\mathrm{Link}(F)$, we obtain the last equivalent statement: $\{A_1, \ldots, A_k\}$ is a subset of $\mathrm{MaxClq}(G/F)$ such that $G[A_1 \backslash F, \ldots, A_k \backslash F] \in \pi_0(\mathrm{Link}(F))$.

This shows the correspondence between $\pi_0(\mathrm{Link}(F))$ and $\pi_0(\mathrm{Clq}_{\geq i+1}|_H)$ for $F \in \mathcal{F}_i$ corresponding to $H \in \mathcal{H}_i$, and also the equality

$$\#(\pi_0(\mathrm{Link}(F))) = \#(\pi_0(\mathrm{Clq}_{\geq i+1}|_H)).$$

□

Corollary 3.2.11. *Let G be a pure, $(d-1)$-dimensional graph. There is a bijection between $\pi_0(\mathrm{Clq}_{\geq 1}(G))$ and $\pi_0(G)$.*

Proof. From Corollary 3.2.10, we see that there is a bijection between $\pi_0(\mathrm{Link}(F))$ and $\pi_0(\mathrm{Clq}_{\geq 1}(G)|_H)$ where $F \in \mathcal{F}_0$ corresponds to $H \in \mathcal{H}_0$. Notice though that since $F \in \mathcal{F}_0$, then $\#F = 0$ and so $F = \emptyset$, which tells us that $\mathrm{Link}(F) = G$. Also, since $H \in \mathcal{H}_0$, we see that $H \in \mathrm{MaxClq}(\mathrm{Clq}_{\geq 0}(G))$. However, we observed in Remark 3.1.3 that $\mathrm{Clq}_{\geq 0}(G)$ is the complete graph and so $H = \mathrm{MaxClq}(\mathrm{Clq}_{\geq 0}(G)) = \mathrm{Clq}_{\geq 0}(G)$. Therefore, $\mathrm{Clq}_{\geq 0}(G)|_H = \mathrm{Clq}_{\geq 0}(G)$. This is enough to show our result. □

Now we seek to generalize Corollary 3.2.10. In particular, we know the number of connected components of a simplicial complex is equal to the 0^{th} homology of the complex. Therefore, Corollary 3.2.10 showed that $H_0(\mathrm{Link}(F)) = H_0(\Delta(\mathrm{Clq}_{\geq i+1}(G)|_H)$ for $F \in \mathcal{F}_i$ corresponding to $H \in \mathcal{H}_i$. Recall $\Delta(\mathrm{Clq}_{\geq i+1}(G)|_H)$ is the clique complex of $\mathrm{Clq}_{\geq i+1}(G)|_H$, i.e. the simplicial complex whose faces are the cliques in $\mathrm{Clq}_{\geq i+1}(G)|_H$. With this in mind, we pose the following question.

Question 3.2.12. *Let G be a pure graph. Let F in $\mathcal{F}_i$ correspond to H in $\mathcal{H}_i$ as described in Theorem 3.2.8. Is it true that $H_j(\mathrm{Link}(F)) \cong H_j(\Delta(\mathrm{Clq}_{\geq i+1}(G)|_H))$ for any $j \geq 0$?*

We will address this question in detail in Chapter 3.4.1.

3.3 Clique chordal and clique-cochordal graphs

We introduce now a family of graphs to which our calculations can be applied. This is based on the familiar graph theoretic notion of chordality which we review.

Definition 3.3.1. A graph is **chordal** if every cycle of length at least three has a chord, i.e., an edge connecting two nonadjacent vertices. We say a graph is **cochordal** if the complement of the graph is chordal.

Example 3.3.2. We return to our graphs from Example 3.1.1 to better understand chordality.

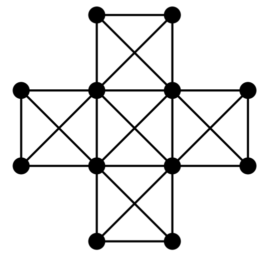
non-chordal graph chordal graph

Notice G_1 has a cycle of length 4, in particular the cycle with edges $\{a, b\}, \{b, c\}, \{c, d\}$, and $\{d, a\}$, but there is no chord, i.e. there is no edge connecting vertex a to c or b to d. However, in G_2, we also have the cycle of length 4 with edges $\{a, b\}, \{b, c\}, \{c, d\}$, and $\{d, a\}$, but now we have a chord, i.e. an edge connecting vertex b to d.

It would seem reasonable to ask if chordality translates from a graph G to its clique graphs. Indeed there are classes of graphs for which this occurs, see Theorem 3.3.10 for such a class. However, not all graphs enjoy this property.

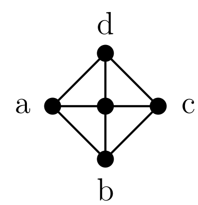
Example 3.3.3. Consider the graph:

G:



Notice we have the following clique graph:

$\text{Clq}_{\geq 1}(G)$:



In graph G, every cycle of length greater than 3 has a chord, i.e. G is chordal. However, in the clique graph, there is a cycle of length 4, namely the cycle with edges $\{a, b\}, \{b, c\}, \{c, d\}$, and $\{d, a\}$, that has no chord, i.e. $\text{Clq}_{\geq 1}(G)$ is not chordal.

We impose this condition however as a class of graphs to consider for our analysis in Chapter 4.

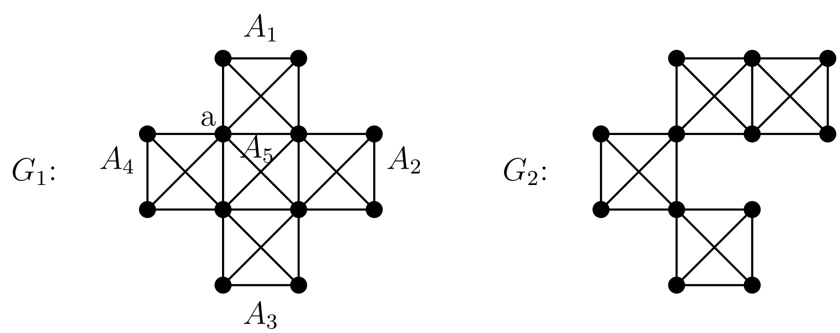
Definition 3.3.4. We call a chordal graph G **clique-chordal** if $\text{Clq}_{\geq i}(G)$ is chordal for all i.

Remark 3.3.5. *At this point it is tempting to assume that we will apply this condition to the input graph in Chapter 4 which contains our main results computing Lyubeznik numbers. Instead, owing to the duality necessary in computing Lyubenzik numbers via linear strands, it will be convenient for us to apply this definition to the complement of G. We therefore call a graph G* **clique-cochordal** *provided G^C is clique-chordal. Note this also assumes that G is cochordal.*

To provide an ample resource of such graphs, consider the following relatively mild conditions that ensure clique chordality. We will see the clique chordal condition is not completely unnatural.

Definition 3.3.6. A graph G is a **clique-forest** if G is chordal and $\#(\text{MaxClq}(G/F)) \leq 2$ for all cliques F in G.

Example 3.3.7. We return to the graph given in Example 3.3.3, denoted here as G_1, where $\text{MaxClq}(G_1) = \{A_1, A_2, A_3, A_4, A_5\}$. Also consider the graph G_2 defined below.



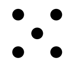
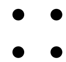
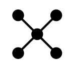
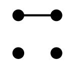
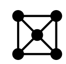
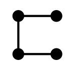
Notice that G_1 and G_2 are both chordal. However, in G_1, for $F = \{a\}$, $\text{MaxClq}(G_1/F) = \{A_1, A_4, A_5\}$ and so $\#(\text{MaxClq}(G_1/F)) = 3$. Therefore, G_1 is not clique-forest. On the other hand, in G_2, there are no cliques F such that $\#(\text{MaxClq}(G_2/F)) > 2$. Therefore, G_2 is a clique-forest.

In the next theorem, we will see a condition on the clique graphs that forces G to be a clique-forest. Theorem 3.3.10 will ensure all clique-forests are clique-chordal.

Recall that a connected graph is a **tree** if there are no cycles. A graph then is a **forest** if each connected component of the graph is a tree. We will see a graph is a clique-forest, precisely when all its clique graphs, except $\text{Clq}_{\geq 0}(G)$, are forests.

Recall that $\text{Clq}_{\geq 0}(G)$ is a complete graph for a graph G. Therefore, $\text{Clq}_{\geq 0}(G)$ is never a forest when G has more than two maximal cliques.

Example 3.3.8. We return to graphs G_1 and G_2 in Example 3.3.7. Consider their clique graphs below.

i	$\mathrm{Clq}_{\geq i}(G_1)$	$\mathrm{Clq}_{\geq i}(G_2)$
3		
2		
1		

Notice that all clique graphs for G_2 are forests and recall from Example 3.3.7 that G_2 is a clique-forest. Also, notice $\mathrm{Clq}_{\geq 1}(G_1)$ is not a forest, and recall from Example 3.3.7 that G_1 is not a clique-forest. We see in the next theorem this is not a coincidence.

Lemma 3.3.9. *If* $\mathrm{Clq}_{\geq 1}(G)$ *is a forest, then* $\mathrm{Clq}_{\geq i}(G)$ *is a forest for all* $i \geq 1$*.*

Proof. We work by contrapositive. If $\mathrm{Clq}_{\geq i}(G)$ is not a forest for some $i \geq 1$, then there exists a cycle $\mathcal{C}$ in $\mathrm{Clq}_{\geq i}(G)$.

Since $V(\mathrm{Clq}_{\geq i}(G)) = V(\mathrm{Clq}_{\geq 1}(G))$ and since $E(\mathrm{Clq}_{\geq i}(G)) = \{\{U, V\} : \#(U \cap V) \geq i$ in $G\}$ where $i > 1$, we see that $\mathrm{Clq}_{\geq i}(G)$ is a subgraph of $\mathrm{Clq}_{\geq 1}(G)$. Therefore, there exists a cycle $\mathcal{C}$ in $\mathrm{Clq}_{\geq 1}(G)$, and so $\mathrm{Clq}_{\geq 1}(G)$ is not a forest.

□

Theorem 3.3.10. *If* G *is a clique-forest, then* $\mathrm{Clq}_{\geq i}(G)$ *is a forest for all* $i > 0$*.*

Proof. If $\mathrm{Clq}_{\geq i}(G)$ is not a clique-forest for some i, then $\mathrm{Clq}_{\geq 1}(G)$ is not a forest. So there exists a cycle $\mathcal{C}$ of length at least three in $\mathrm{Clq}_{\geq 1}(G)$. Either $\mathcal{C}$ contains a maximal clique in $\mathrm{Clq}_{\geq 1}(G)$ of size at least three, or $\mathcal{C}$ does not contain a maximal clique in $\mathrm{Clq}_{\geq 1}(G)$ of size at least three.

Case 1: Suppose $\mathcal{C}$ contains a maximal clique H of size at least three.

By the Clique-link correspondence, Theorem 3.2.8, H corresponds to $F \in \Delta(G)$, where $\#V(H) = \#\,\mathrm{MaxClq}(G/F)$. Since $\#V(H) \geq 3$, this shows that $\#\,\mathrm{MaxClq}(G/F) \geq 3$, which by definition shows G is not a clique-forest. This is a contradiction.

Case 2: Instead suppose $\mathcal{C}$ does not contain a maximal clique of size three or greater.

Therefore, there exists a chordless subcycle of $\mathcal{C}$, say $\{P_1, \ldots, P_r\}$, contained in a connected component.

So $P_1, \ldots, P_r \in \text{MaxClq}(G)$. Since $\{P_r, P_1\}$ and $\{P_s, P_{s+1}\}$ are edges in $\text{Clq}_{\geq 1}(G)$ for all s, they are therefore maximal cliques in $\text{Clq}_{\geq 1}(G)$ since there are no maximal cliques of size more than two. Now by the Clique-link correspondence, for each s, $\{P_s, P_{s+1}\}$ in $\text{Clq}_{\geq 1}(G)$ corresponds to a clique of size one, i.e. a vertex, F_s in G, where $\#(\text{MaxClq}(G/F_s)) = 2$ for all s. Similarly, $\{P_r, P_1\}$ in $\text{Clq}_{\geq 1}(G)$ corresponds to vertex F_r in G. .

Now, since F_s is in maximal clique P_{s+1} in G and F_{s+1} is in maximal clique P_{s+1} in G, we see that $\{F_s, F_{s+1}\}$ is an edge in G for all s. Similarly, $\{F_r, F_1\}$ is an edge in G.

Also, since F_s is not in maximal clique P_j in G for any $j \neq s, s+1$, we see that $F_s \neq F_j$ for any $j \neq s$. Therefore, $F_1, \ldots, F_r$ is a cycle.

Now again, since F_s is not in maximal clique P_j in G for any $j \neq s, s+1$, if there exists a chord $\{F_s, F_j\}$ for $|s - j| \neq 1$, then $\{F_s, F_j\}$ is in a maximal clique M in G where M is not equal to P_l for any l. But then $M, P_s, P_{s+1} \in \text{MaxClq}(G/F_s)$, and so G is not a clique forest, which is a contradiction. Therefore, $F_1, \ldots, F_r$ is a chordless cycle with $r > 3$, and therefore G is not chordal, and in such is not a clique-forest, which again is a contradiction. □

An immediate corollary shows that the clique-forest condition is enough to force clique-chordality, giving us a condition on the graph G itself that implies clique-cochordal, and also providing a wide class of graphs that meets this condition.

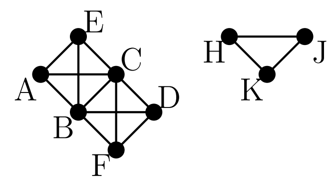
Corollary 3.3.11. *If G is a clique-forest, then G is clique-chordal.*

Proof. From Theorem 3.3.10, we see that $\text{Clq}_{\geq i}(G)$ is a forest for all i, and therefore contains no cycle. The result follows. □

The main reason for us to impose the clique-chordal condition is the following theorem which greatly expands the effectivity of the combinatorics of the Clique-Link correspondence. Before we state the following theorem though, we again consider Example 3.1.4.

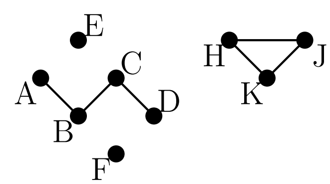
Example 3.3.12. Recall that $\mathrm{Clq}_{\geq 1}(G)$ in Example 3.1.4 is given by:

$\mathrm{Clq}_{\geq 1}(G)$:



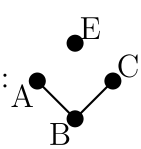
Let T be the maximal clique $\{A, B, C, E\}$ and S be the maximal clique $\{B, C, D, F\}$. Now consider $\mathrm{Clq}_{\geq 2}(G)|_T \cap \mathrm{Clq}_{\geq 2}(G)|_S$. Recall that $\mathrm{Clq}_{\geq 2}(G)$ is the graph:

$\mathrm{Clq}_{\geq 2}(G)$:

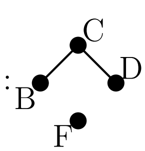


So $\mathrm{Clq}_{\geq 2}(G)|_T$ and $\mathrm{Clq}_{\geq 2}(G)|_S$ are given by:

$\mathrm{Clq}_{\geq 2}(G)|_T$:



$\mathrm{Clq}_{\geq 2}(G)|_S$:



Note that $\mathrm{Clq}_{\geq 2}(G)|_T \cap \mathrm{Clq}_{\geq 2}(G)|_S$ is the edge $\{B, C\}$ and in fact, it will always be the case that the intersection of clique graphs $\mathrm{Clq}_{\geq i}(G)$ restricted to maximal cliques in $\mathrm{Clq}_{\geq i-1}(G)$ will be connected, as we will see in Lemma 3.3.13.

Also notice that

$$(\#(\pi_0(\mathrm{Clq}_{\geq 2}(G)|_T)) - 1) + (\#(\pi_0(\mathrm{Clq}_{\geq 2}(G)|_S)) - 1) = 2 - 1 + 2 - 1 = 2.$$

This is equal to

$$\#(\pi_0(\mathrm{Clq}_{\geq 2}(G)|_R)) - 1,$$

where $R = T \cup S$. This demonstrates the property we will observe in Theorem 3.3.14 for a particular class of graphs.

Lemma 3.3.13. *Let G be a pure graph and $A \in \pi_0(\mathrm{Clq}_{\geq i-1}(G))$. Let $H, K \in \mathrm{MaxClq}(A)$. If $\mathrm{Clq}_{\geq i}(G)|_H \cap \mathrm{Clq}_{\geq i}(G)|_K \neq \emptyset$, then $\mathrm{Clq}_{\geq i}(G)|_H \cap \mathrm{Clq}_{\geq i}(G)|_K$ is connected.*

Proof. Assume by contradiction that $\mathrm{Clq}_{\geq i}|_H \cap \mathrm{Clq}_{\geq i}|_K$ is disconnected. So there exist vertices A, B in $\mathrm{Clq}_{\geq i}|_H \cap \mathrm{Clq}_{\geq i}|_K$ such that there is no path in $\mathrm{Clq}_{\geq i}|_H \cap \mathrm{Clq}_{\geq i}|_K$ from A to B. Therefore, A and B must be in different connected components of $\mathrm{Clq}_{\geq i}|_H$. Let H_1, H_2 be connected components of $\mathrm{Clq}_{\geq i}|_H$ so that $A \in H_1$ and $B \in H_2$. Also, A and B must be in different connected components of $\mathrm{Clq}_{\geq i}|_K$. Let K_1, K_2 be connected components of $\mathrm{Clq}_{\geq i}|_K$ so that $A \in K_1$ and $B \in K_2$.

First note that since $\mathrm{Clq}_{\geq i}|_H$ is disconnected, by Theorem 3.2.8, we have that H corresponds to a clique F of size $i-1$ in G, where $V(H)$ corresponds $\mathrm{MaxClq}(G/F)$. So $A, B \in \mathrm{MaxClq}(G/F)$. More specifically, since H is a clique in $\mathrm{Clq}_{\geq i-1}$ and A and B are in H, there is an edge between A and B in $\mathrm{Clq}_{\geq i-1}$, whereas there is not an edge between A and B in $\mathrm{Clq}_{\geq i}$. Therefore, we see that $\#(A \cap B) = i - 1$ and so $A \cap B = F$.

Similarly, $K \in \mathrm{MaxClq}(\mathrm{Clq}_{\geq i-1})$ such that $\mathrm{Clq}_{\geq i}|_K$ is disconnected, and so K corresponds to a clique L of size $i-1$ in G, where $L \neq F$. But A and B are also in K, and so we see that $A \cap B = L$. This is a contradiction since $A \cap B = F$ and $F \neq L$.

Therefore, $\mathrm{Clq}_{\geq i}|_H \cap \mathrm{Clq}_{\geq i}|_K$ is connected. □

Theorem 3.3.14. *Let G be a pure, $(d-1)$-dimensional graph where $\mathrm{Clq}_{\geq i-1}(G^C)$ is* *chordal for some i such that $1 \leq i \leq d$, and $A \in \pi_0(\mathrm{Clq}_{\geq i-1}(G))$. One has*

$$\sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(A)}} (\#(\pi_0(\mathrm{Clq}_{\geq i}(G)|_H)) - 1) = \#(\pi_0(\mathrm{Clq}_{\geq i}(G)|_A)) - 1.$$

Proof. We proceed by induction on r, where $r = \#(\mathrm{MaxClq}(A))$.

If $r = 1$, $H_1 = A$, which gives us

$$\sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(A)}} \#(\pi_0(\mathrm{Clq}_{\geq i}|_H)) - 1 = \#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_1})) - 1 = \#(\pi_0(\mathrm{Clq}_{\geq i}|_A)) - 1$$

If $r = 2$, $\mathrm{MaxClq}(A) = \{H_1, H_2\}$.

Notice that $\pi_0(\mathrm{Clq}_{\geq i}|_A)$ is the disjoint union of the sets $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ where

$$\mathcal{X} = \{\mathcal{C} \in \pi_0(\mathrm{Clq}_{\geq i}|_A) : \exists\, \mathcal{U} \in \pi_0(\mathrm{Clq}_{\geq i}|_{H_1}) \text{ s.t. } \mathcal{C} \supseteq \mathcal{U} \text{ and } \forall\, \mathcal{V} \in \pi_0(\mathrm{Clq}_{\geq i}|_{H_2}), \mathcal{C} \not\supseteq \mathcal{V}\}$$

$$\mathcal{Y} = \{\mathcal{C} \in \pi_0(\mathrm{Clq}_{\geq i}|_A) : \forall\, \mathcal{U} \in \pi_0(\mathrm{Clq}_{\geq i}|_{H_1}), \mathcal{C} \not\supseteq \mathcal{U} \text{ and } \exists\, \mathcal{V} \in \pi_0(\mathrm{Clq}_{\geq i}|_{H_2}) \text{ s.t. } \mathcal{C} \supseteq \mathcal{V}\}$$

$$\mathcal{Z} = \{\mathcal{C} \in \pi_0(\mathrm{Clq}_{\geq i}|_A) : \exists\, \mathcal{U} \in \pi_0(\mathrm{Clq}_{\geq i}|_{H_1}), \exists\, \mathcal{V} \in \pi_0(\mathrm{Clq}_{\geq i}|_{H_2}) \text{ s.t. } \mathcal{C} \supseteq \mathcal{U} \text{ and } \mathcal{C} \supseteq \mathcal{V}\}$$

Loosely, think about $\mathcal{X}$ as the set of all connected components in $\mathrm{Clq}_{\geq i}|_A$ that are strictly contained in $\mathrm{Clq}_{\geq i}|_{H_1}$, $\mathcal{Y}$ as the set of all connected components in $\mathrm{Clq}_{\geq i}|_A$ that are strictly contained in $\mathrm{Clq}_{\geq i}|_{H_2}$, and $\mathcal{Z}$ as the set of all connected components in $\mathrm{Clq}_{\geq i}|_A$ that nontrivially intersect $\mathrm{Clq}_{\geq i}|_{H_1}$ and $\mathrm{Clq}_{\geq i}|_{H_2}$.

Therefore, $\#(\pi_0(\mathrm{Clq}_{\geq i}|_A)) = \#\mathcal{X} + \#\mathcal{Y} + \#\mathcal{Z}$.

Since A is connected, we see that there exists a vertex p in $H_1 \cap H_2$, and therefore p is in $\mathrm{Clq}_{\geq i}|_{H_1} \cap \mathrm{Clq}_{\geq i}|_{H_2}$. So $\mathrm{Clq}_{\geq i}|_{H_1} \cap \mathrm{Clq}_{\geq i}|_{H_2} \neq \emptyset$, and therefore by Lemma 3.3.13, $\mathrm{Clq}_{\geq i}|_{H_1} \cap \mathrm{Clq}_{\geq i}|_{H_2}$ is connected. This shows that $\#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_1} \cap \mathrm{Clq}_{\geq i}|_{H_2})) = 1$. Therefore $\#\mathcal{Z} = 1$.

Also, $\#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_1})) = \#\mathcal{X} + \#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_1} \cap \mathrm{Clq}_{\geq i}|_{H_2}) = \#\mathcal{X} + 1$.

Similarly, $\#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_2})) = \#\mathcal{Y} + \#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_1} \cap \mathrm{Clq}_{\geq i}|_{H_2}) = \#\mathcal{Y} + 1$.

Therefore, $\#(\pi_0(\mathrm{Clq}_{\geq i}|_A)) = \#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_1})) - 1 + \#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_2})) - 1 + 1$, and so

$$\sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(A)}} (\#(\pi_0(\mathrm{Clq}_{\geq i}|_H)) - 1) = \#(\pi_0(\mathrm{Clq}_{\geq i}|_A)) - 1.$$

Now assume that if $\#(\mathrm{MaxClq}(A)) = r$, then

$$\sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(A)}} (\#(\pi_0(\mathrm{Clq}_{\geq i}|_H)) - 1) = \#(\pi_0(\mathrm{Clq}_{\geq i}|_A)) - 1$$

Consider the case where $\mathrm{MaxClq}(A) = \{H_1, \ldots, H_{r+1}\}$. Since $\mathrm{Clq}_{\geq i-1}$ is chordal, there exists an l such that $H_l \cap \mathrm{Clq}_{\geq i-1}[\bigcup_{j \neq l} H_j]$ is a clique. Therefore, $\mathrm{Clq}_{\geq i-1}[\bigcup_{j \neq l} H_j]$ is a connected component in $\mathrm{Clq}_{\geq i-1} \setminus A \cup \bigcup_{j \neq l} H_j$ with r maximal cliques. Without loss of generality, assume $l = r + 1$.

Notice that $\pi_0(\mathrm{Clq}_{\geq i}|_A)$ is the disjoint union of the sets $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ where

$\mathcal{X} = \{\mathcal{C} \in \pi_0(\mathrm{Clq}_{\geq i}|_A) : \exists\, \mathcal{U} \in \pi_0(\mathrm{Clq}_{\geq i}|_{\bigcup_{j \neq l} H_j}) \text{ s.t. } \mathcal{C} \supseteq \mathcal{U} \text{ and } \forall\, \mathcal{V} \in \pi_0(\mathrm{Clq}_{\geq i}|_{H_{r+1}}), \mathcal{C} \not\supseteq \mathcal{V}\}$

$\mathcal{Y} = \{\mathcal{C} \in \pi_0(\mathrm{Clq}_{\geq i}|_A) : \forall\, \mathcal{U} \in \pi_0(\mathrm{Clq}_{\geq i}|_{\bigcup_{j \neq l} H_j}), \mathcal{C} \not\supseteq \mathcal{U} \text{ and } \exists\, \mathcal{V} \in \pi_0(\mathrm{Clq}_{\geq i}|_{H_{r+1}}) \text{ s.t. } \mathcal{C} \supseteq \mathcal{V}\}$

$\mathcal{Z} = \{\mathcal{C} \in \pi_0(\mathrm{Clq}_{\geq i}|_A) : \exists\, \mathcal{U} \in \pi_0(\mathrm{Clq}_{\geq i}|_{\bigcup_{j \neq l} H_j}), \mathcal{V} \in \pi_0(\mathrm{Clq}_{\geq i}|_{H_{r+1}}) \text{ s.t. } \mathcal{C} \supseteq \mathcal{U} \text{ and } \mathcal{C} \supseteq \mathcal{V}\}$,

and therefore, $\#(\pi_0(\mathrm{Clq}_{\geq i}|_A)) = \#\mathcal{X} + \#\mathcal{Y} + \#\mathcal{Z}$.

Since $H_l \cap \mathrm{Clq}_{\geq i-1}[\bigcup_{j \neq l} H_j]$ is a clique, then $H_l \cap \mathrm{Clq}_{\geq i-1}[\bigcup_{j \neq l} H_j]$ is contained in some maximal clique H_m, with $m \neq l$. Therefore, we see that

$\#(\mathrm{Clq}_{\geq i}|_{\bigcup_{j \neq l} H_j} \cap \mathrm{Clq}_{\geq i}|_{H_{r+1}}) = \#(\mathrm{Clq}_{\geq i}|_{H_l} \cap \mathrm{Clq}_{\geq i}|_{H_m})$, and by Lemma 3.3.13,

$\#(\mathrm{Clq}_{\geq i}|_{H_l} \cap \mathrm{Clq}_{\geq i}|_{H_m}) = 1$. Therefore $\#\mathcal{Z} = 1$.

Also, $\#(\pi_0(\mathrm{Clq}_{\geq i}|_{\bigcup_{j\neq l} H_j})) = \#\mathcal{X} + \#(\mathrm{Clq}_{\geq i}|_{\bigcup_{j\neq l} H_j} \cap \mathrm{Clq}_{\geq i}|_{H_{r+1}}) = \#\mathcal{X} + 1$,

and similarly $\#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_{r+1}})) = \#\mathcal{Y} + \#(\mathrm{Clq}_{\geq i}|_{\bigcup_{j\neq l} H_j} \cap \mathrm{Clq}_{\geq i}|_{H_{r+1}}) = \#\mathcal{Y} + 1$

Therefore,

$$\#(\pi_0(\mathrm{Clq}_{\geq i}|_A) - 1 = \#(\pi_0(\mathrm{Clq}_{\geq i}|_{\bigcup_{j\neq l} H_j}) - 1 + \#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_{r+1}}) - 1 + 1 - 1$$

But by induction hypothesis,

$\#(\pi_0(\mathrm{Clq}_{\geq i}|_{\bigcup_{j\neq l} H_j})) - 1 = \sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(\bigcup_{j\neq l} H_j)}} (\#(\pi_0(\mathrm{Clq}_{\geq i}|_H)) - 1)$, and so

$$\begin{aligned} \#(\pi_0(\mathrm{Clq}_{\geq i}|_A)) - 1 &= \sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(\bigcup_{j\neq l} H_j)}} (\#(\pi_0(\mathrm{Clq}_{\geq i}|_H)) - 1) + \#(\pi_0(\mathrm{Clq}_{\geq i}|_{H_{r+1}})) - 1 \\ &= \sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(A)}} (\#(\pi_0(\mathrm{Clq}_{\geq i}|_H)) - 1) \end{aligned}$$

□

3.4 Interpretations of Clique Graphs

In this section, we explore two contructions that are isomorphic to the clique graphs, higher nerve complexes and higher Hochster-Huneke graphs.

3.4.1 Higher Nerve Complexes

We first remark that the results in this section are part of ongoing work with Justin Lyle. Throughout the section we will consider a simplical complex on a vertex set $[r] = \{1, 2, 3, \ldots, r\}$ to be a collection of subsets $F \subseteq [r]$ called faces, which is closed under inclusion. By abuse of notation, if $F = \{i_1, \ldots, i_s\} \subseteq [r]$, we will denote the face by $F = \{F_{i_1}, \ldots, F_{i_s}\}$.

Definition 3.4.1. Let $A = \{A_1, A_2, \ldots, A_r\}$ be the set of facets of a simplicial complex Δ. The **nerve complex of** Δ is defined to be

$$N(\Delta) = \{F \subseteq [r] : \bigcap_{i \in F} A_i \neq \varnothing\}.$$

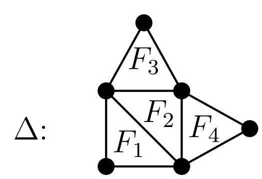
We note that Δ and $N(\Delta)$ are homotopic, [Bor48], due to a theorem we call the nerve theorem. What information about Δ can be captured by generalizing the notion of nerve complexes? In [DDDGHL19], this question was addressed. They generalized nerve complexes as follows.

Definition 3.4.2. Let $A = \{A_1, A_2, \ldots, A_r\}$ be the set of facets of a simplicial complex Δ. The i^{th} **nerve complex of** Δ is defined to be:

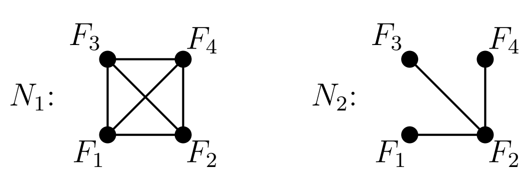
$$N_i(\Delta) = \{F \subseteq [r] : \#\left(\bigcap_{j \in F} A_j\right) \geq i\}.$$

Collectively, one refers to $N_i(\Delta)$ as the **higher nerve complexes.**

Example 3.4.3. Consider the simplicial complex where every possible face is realized:



The higher nerve complexes are given as follows:



We now give the aforementioned result relating higher nerve complexes with clique graphs to keep in mind throughout the rest of the section as we explore more information that can be drawn from higher nerve complexes.

Theorem 3.4.4. *For any graph G, the 1-skeleton of the higher nerve complex, $N_i(\Delta(G))$ is the clique graph, $\mathrm{Clq}_{\geq i}(G)$.*

Proof. Let $\{A_1, \ldots, A_r\}$ be the set of facets of $\Delta(G)$. Then a vertex P is in $N_i(\Delta(G))$ if and only if A_P is a facet of $\Delta(G)$ and $\#(A_P) \geq i$. Since it is clear that $\#(A_P) \geq i$ and by definition of the clique complex, $\Delta(G)$, we see that P is a vertex in $N_j(\Delta(G))$ if and only if A_P is a maximal clique in G. Lastly, by definition of the clique graph, P is a vertex in $N_j(\Delta(G))$ if and only if A_P is a vertex in $\mathrm{Clq}_{\geq i}(G)$.

Also, notice that $\{P, Q\}$ is an edge in $N_j(\Delta(G))$ if and only if A_P and A_Q are facets of $\Delta(G)$ and $\#(A_P \cap A_Q) \geq i$. By definition of $\Delta(G)$, this is equivalent to saying A_P and A_Q are maximal cliques in G and $\#(A_P \cap A_Q) \geq i$. Lastly, by definiton of the clique graph, we see that this is equivalent to saying (A_P, A_Q) is an edge in $\mathrm{Clq}_{\geq i}(G)$. □

This theorem shows the clique graphs are the $1-$skeleton of the higher nerve complexes. The next theorem however takes it a step further and shows that the clique complex of the clique graphs are actually isomorphic to the higher nerve complexes.

Recall if G is a graph, the clique complex G, denoted $\Delta(G)$ is the simplicial complex whose faces are the cliques in G, and therefore whose facets are maximal cliques in G.

Theorem 3.4.5. *For any graph G, the higher nerve complex, $N_m(\Delta(G))$ and clique complex $\Delta(\mathrm{Clq}_{\geq m}(G))$ are isomorphic.*

Proof. Note that J is a face in $\Delta(\mathrm{Clq}_{\geq m}(G))$ if and only if J is a clique in $\mathrm{Clq}_{\geq m}(G)$. Let $V(J) = \{J_1, \ldots, J_r\}$. Notice that J is contained in a maximal clique, H, in $\mathrm{Clq}_{\geq m}(G)$. By the Clique-Link correspondence, Theorem 3.2.8, H corresponds to a clique F of size m in G, and in particular, $V(H)$ corresponds to $\mathrm{MaxClq}(G/F)$. Since $\{J_1, \ldots, J_r\} \subseteq V(H)$ in $\mathrm{Clq}_{\geq m}(G)$, we see that $J_1, \ldots, J_r$ are all in $\mathrm{MaxClq}(G/F)$. Therefore, $F \subseteq J_i$ for all i where F has size m. Thus J being a clique in $\mathrm{Clq}_{\geq m}(G)$ is equivalent to saying $J_1, \ldots, J_r$ are maximal cliques in G such that $\bigcap_i J_i \geq m$ in G. As we see from Definition 3.4.2, this is also equivalent to saying J is a face in $N_m(\Delta(G))$.

$\square$

Homologies of Higher Nerves

Knowing the relationship between clique graphs and higher nerves, we focus now on gleaning homological data from the higher nerve complexes. The following theorem will outline some of the information about the simplicial complex the higher nerves are able to capture.

Theorem 3.4.6 ([DDDGHL19], Thm. 1.3)**.** *Let k be a field and Δ a simplicial complex of dimension $d-1$, and $k[\Delta]$ be the Stanley-Reisner ring. Letting $\tilde{H}_i$ denote the i^{th} reduced homology and χ denote the Euler characteristic, we have:*

1. $\tilde{H}_i(N_j(\Delta)) = 0$ *for all* $i + j > d$ *and* $1 \leq j \leq d$.

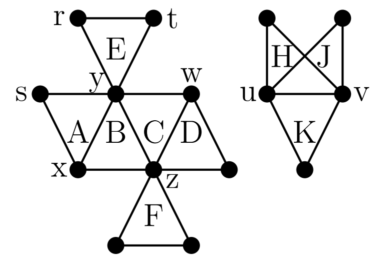
2. $\text{depth}(k[\Delta]) = \inf\{i + j : \tilde{H}_i(N_j(\Delta)) \neq 0\}$

3. *For $i \geq 0$, the number of $i-$dimensional faces of Δ is given by* $\sum_{j=i+1}^{d} \binom{j-1}{i} \chi(N_j(\Delta))$.

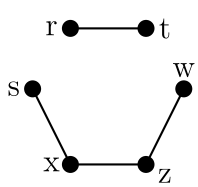
Notice that the third item in the previous theorem gives a way of relating the combinatorics of a simplicial complex to the higher nerves. Since our goal is to compute Lyubeznik numbers this idea is what we will focus in on. Specifically, we need information about the linear strands, including the maps and the graded Betti numbers, of the Alexander dual to calculate the Lyubeznik table. By the Eagon-Reiner formula, we know that we can find the graded Betti numbers by using the homologies of links of faces in the Stanley Reisner complex of the ideal. Therefore, if we can relate the homology of the link of a face to the homology of the higher nerves, this will help allow us to build a Lyubeznik table from combinatorial information in the higher nerve complexes. To this end, consider the following example.

Example 3.4.7. We will return to Example 3.1.4, where G is given as follows.

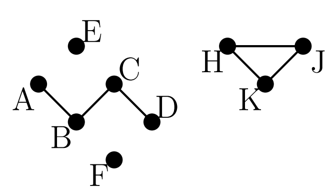
G:



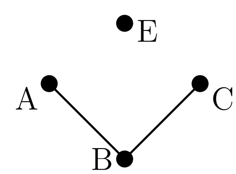
We will consider the link of the vertex y. Recall Link($\{y\}$) consists of all faces in $\Delta(G)$ that, when joined with y, produce a facet. We also give $N_1(\Delta(G))$ and $N_1(\Delta(G))|_{H_y}$, where H_y is the index set of all facets in $\Delta(G)$ containing $\{y\}$.



$\text{Link}_{\Delta(G)}(\{y\})$



$N_1(\Delta(G))$



$N_1(\Delta(G))|_{H_y}$

Notice that $\text{Link}_{\Delta(G)}(\{y\})$ and $N_1(\Delta(G))|_{H_y}$ are homotopic. We will show this homotopy in more generality in the following theorem.

Theorem 3.4.8. *Let $\{A_1, \ldots, A_r\}$ be the set of facets in a simplicial complex Δ, and let F be a face in Δ. If $H_F = \{i : F \subseteq A_i\} \subseteq [r]$, then* $\text{Link}_\Delta(F)$ *is homotopic to $N_{\#(F)+1}(\Delta)|_{H_F}$.*

Proof. First note that $\text{Link}_\Delta(F) = \{G \subseteq A_i \backslash F \text{ in } \Delta : i \in H_F\}$, and recall that $\text{Link}_\Delta(F)$ is homotopic to $N(\text{Link}_\Delta(F))$ by the nerve theorem. We now show that $N(\text{Link}_\Delta(F)) = N_{\#(F)+1}(\Delta)|_{H_F}$.

To do so, notice that $\sigma \subseteq [r]$ is a face in $N(\text{Link}_\Delta(F))$ if and only if $\sigma \subseteq H_F$ and $\bigcap_{i\in\sigma}(A_i \backslash F) \neq \varnothing$. Note though that $\bigcap_{i\in\sigma}(A_i \backslash F) = (\bigcap_{i\in\sigma} A_i)\backslash F$, and $(\bigcap_{i\in\sigma} A_i)\backslash F \neq \varnothing$ if and only if $\#(\bigcap_{i\in\sigma} A_i) \geq \#(F) + 1$. Therefore $\sigma \in N(\text{Link}_\Delta(F))$ if and only if $\sigma \subseteq H_F$ and

$\sigma \in N_{\#(F)+1}(\Delta)$, which occurs if and only if $\sigma \in N_{\#(F)+1}(\Delta)|_{H_T}$. □

Notice that Theorem 3.4.8 is able to relate the homologies of the higher nerves and the links of faces in a simplicial complex. However, we still use the Clique-link correspondence to give direct correspondences between the two, i.e. to show exactly which parts of the clique graphs are representing links of cliques in the graph. Working together, Theorem 3.4.8 and the Clique-link correspondence are able to recognize more information about a graph from combinatorial information in the clique graphs.

We will show in the following theorem that the higher nerves of the link of a face in the simplicial complex are equal to a particular higher nerve complex of the simplicial complex restricted again to the index set of facets containing the face. However, in doing so we lose the effect of the nerve theorem which allows us to related these higher nerves directly to the link, instead having to pass through a higher nerve of the link.

Theorem 3.4.9. *Let* $\{A_1, \ldots, A_r\}$ *be the set of facets in a simplicial complex* Δ*, and let* F *be a face in* Δ*. If* $H_F^j = \{i : F \subseteq A_i, \#(A_i) \geq j + \#(F)\} \subseteq [r]$*, then* $N_j(\mathrm{Link}_\Delta(F)) = N_{\#(F)+j}(\Delta)|_{H_F^j}$.

Proof. First note that $\mathrm{Link}_\Delta(F) = \{G \subseteq A_i \backslash F \text{ in } \Delta : i \in H_F\}$, and recall that $\mathrm{Link}_\Delta(F)$ is homotopic to $N(\mathrm{Link}_\Delta(F))$. We now show that $N_j(\mathrm{Link}_\Delta(F)) = N_{\#(F)+j}(\Delta)|_{H_F^j}$.

To do so, notice that $\sigma \subseteq [r]$ is a face in $N_j(\mathrm{Link}_\Delta(F))$ if and only if $\sigma \subseteq H_F^j$ and $\bigcap_{i\in\sigma}(A_i \backslash F) \geq j$. Note though that $\bigcap_{i\in\sigma}(A_i \backslash F) \geq j$ if and only if $\bigcap_{i\in\sigma}(A_i) \geq j + \#(F)$.

Therefore, $\sigma \in N_j(\mathrm{Link}_\Delta(F))$ if and only if $\sigma \subseteq H_F^j$ and $\sigma \in N_{\#(F)+j}(\Delta)$ which occurs if and only if $\sigma \in N_{\#(F)+j}(\Delta)|_{H_F^j}$. □

In Chapter 4, we will use these results to give a reinterpretation of the Eagon-Reiner formula for graded Betti numbers using clique graphs.

3.4.2 Higher Hochster-Huneke Graphs

Thus far, our constructions have stemmed from simplifying combinatorial information in the graph. In this section however, we study the higher Hochster-Huneke graphs, which are constructed from the ring. Recall the following definition, which again, can be found in [NBSW19, Def. 2.2].

Definition 3.4.10. Let S be an equidimensional local ring of dimension $d \geq 2$. For integer $1 \leq t \leq d-1$, set $\Gamma_t(S)$, the t^{th} higher Hochster-Huneke graph, to be the graph with:

1. The vertices indexed by the minimal primes of S,

2. An edge between minimal primes $\mathfrak{p}$ and $\mathfrak{q}$ if and only if $\operatorname{ht}(\mathfrak{p}+\mathfrak{q}) \leq t$.

Recall that $\Gamma_1(S)$ is the Hochster-Huneke graph of which $\lambda_{d,d}(S)$ counts the number of connected components, [Zha07, Main Theorem]. The higher Hochster-Hunke graphs provide information about other Lyubeznik numbers.

Theorem 3.4.11 ([NBSW19], Thm.5.4)**.** *One has* $\lambda_{1,2}(S) = \#\pi_0\Gamma_{d-2}(S) - \#\pi_0\Gamma_{d-1}(S)$ *and*

$$\lambda_{i,i+1}(S) \geq \#\pi_0\Gamma_{d-i-1}(S) - \#\pi_0\Gamma_{d-i}(S)$$

for $1 \leq i \leq d-2$.

We will see in Chapter 4 a setting where this result can be sharpened. However, we first need to establish the isomorphism between clique graphs and higher Hochster-Huneke graphs. Recall from Definition 3.1.2 that clique graphs are only defined for pure graphs. Since edge ideals of pure graphs are unmixed, we only show the isomorphsim in the case of unmixed edge ideals.

Let $X \backslash Y$ denote $X \cap Y^C$ when needed.

Theorem 3.4.12. *Let G be a graph with unmixed edge ideal $I = I(G)$ of dimension $d \geq 2$ in the polynomial ring $R = k[x_1, \ldots, x_n]$. For any $1 \leq t \leq d-1$, the graphs $\Gamma_t(I)$ and $\operatorname{Clq}_{\geq d-t}(G^C)$ are isomorphic.*

Proof. By definition, the vertices of $\Gamma_t(I)$ are indexed by the minimal primes P of I. However, since I is a square-free monomial ideal and therefore radical, we see that $\mathrm{Min}(I) = \mathrm{Ass}(I)$.

By [FMS14, Thm. 4.11], P is associated to I if and only if the complement of m is a facet of Δ_I, where m is a monomial such that $P = (x_i : x_i \in m)$ and Δ_I is the clique complex $\Delta(G^C)$. Therefore, the vertices of $\Gamma_t(I)$ are indexed by facets of $\Delta(G^C)$ which, by definition, also indexes the vertices of $\mathrm{Clq}_{\geq t}(G^C)$.

Now, there is an edge between two vertices P and Q in $\mathrm{Clq}_{\geq d-t}(G^C)$ if and only if $\#(P \cap Q) \geq d - t$ in G^C. Let $\tilde{P} = (x_i : x_i \in n \backslash P)$ and $\tilde{Q} = (x_i : x_i \in n \backslash Q)$. Since P and Q are facets in $\Delta(G^C)$, we know P and Q are complements of associated primes of I. Therefore, $\tilde{P}, \tilde{Q} \in \mathrm{Ass}(I)$. Let $S = R/I$. Note: $\mathrm{ht}(P + Q) = \dim(S) - \dim(S/(P + Q))$. Since $P \cap Q \geq d - t$ in G^C and since $\tilde{P} + \tilde{Q} = (x_i : x_i \in n \backslash P \text{ or } n \backslash Q) = (x_i : x_i \notin P \cap Q)$, we see that $\mathrm{ht}(P + Q) \leq t$. Therefore, there is an edge between $\tilde{P}$ and $\tilde{Q}$ in $\Gamma_t(I)$. □

4 Lyubeznik tables of unmixed edge ideals of clique co-chordal graphs

We now describe our main result on Lyubeznik tables of edge ideals. This involves applying Theorem 2.2.8 to compute these in terms of homology of linear strands, which often reduces simply to calculating graded Betti numbers via the Eagon-Reiner formula. One particular Lyubeznik number is easy to calculate in general, which we observe in Theorem 4.0.1.

Throughout, we fix a graph G, $I = I(G)$ its edge ideal which is assumed to be an unmixed ideal of dimension $d \geq 1$ in a polynomial ring R of dimension n over a field k. Note we sometimes will abusively conflate R with its localization at the ideal of variables when discussing results that require local rings. The fraction field of R will be denoted by K.

All references to graded Betti numbers in this section will be of the dual $I^\vee$, in particular $\beta_{i,j}$ will denote the (i,j)-th graded Betti number of the R-module $R/I^\vee$. This will carry over to linear strands $\mathbb{L}_\bullet^{\langle r\rangle}$ as well, i.e., this will be the linear strands of $I^\vee$. Also, as before, since G is fixed throughout, we write $\mathrm{Clq}_{\geq i}$ for $\mathrm{Clq}_{\geq i}(G^C)$ and revert to the original notation when necessary.

Theorem 4.0.1. *The Lyubeznik number $\lambda_{0,0}(R/I)$ is 0.*

Proof. Recall from Theorem 2.2.8 that $\lambda_{0,0}(R/I) = \dim_K[H_0(\mathbb{L}_\bullet^{\langle n\rangle} \otimes_R K)]$. The terms of the n-linear strand are given by $\mathbb{L}_i^{\langle n\rangle} = R(-i-n)^{\beta_{i,i+n}}$. We will use Eagon-Reiner to find the needed Betti numbers

$$\beta_{0,n} = \sum_{\substack{F \in \Delta(G^C) \\ \#F = 0}} \dim_k \tilde{H}_{-1}(\mathrm{Link}_{\Delta(G^C)} F).$$

The only F in $\Delta(G^C)$ such that $\#F = 0$ is $F = \emptyset$. Since $\mathrm{Link}_{\Delta(G^C)} \emptyset = \Delta(G^C) \neq \emptyset$, we have that $\tilde{H}_{-1}(\mathrm{Link}_{\Delta(G^C)} F, k) = 0$, and so $\beta_{0,n} = 0$. Now consider $\beta_{i,i+n}$ for $i \geq 1$. Again by Eagon-Reiner, we have $\beta_{i,i+n} = \sum_{\substack{F \in \Delta(G^C) \\ \#F = -i}} \dim_k \tilde{H}_{i-1}(\mathrm{Link}_{\Delta(G^C)} F)$. Since $i \geq 1$, we know

$-i \leq -1$, and there are no F in $\Delta(G^C)^\vee$ such that $\#F \leq -1$ since $\#F$ must be positive. Therefore, $\beta_{i,i+n} = 0$ for $i \geq 1$. This gives us that the n-linear strand of $I^\vee$ is 0, and so $\lambda_{0,0}(R/I) = 0$.

□

We can also articulate extremal Lyubeznik numbers of edge ideals of dimension at least one in a generality assuming only that the graph complement of G has no isolated points.

Lemma 4.0.2. *For all r, $\beta_{0,n-r}(I^\vee) = \#(\mathrm{MaxClq}_r(G^C))$*

Proof. By the Eagon-Reiner formula we have

$$\beta_{0,n-r} = \sum_{\substack{F \in \Delta(G^C) \\ \#F = r}} \dim_k \tilde{H}_{-1}(\mathrm{Link}_{\Delta(G^C)} F).$$

But $\tilde{H}_{-1}(\mathrm{Link}_{\Delta(G^C)} F))$ can be nonzero only when $\mathrm{Link}_{\Delta(G^C)} F = \emptyset$, in which case $\dim_k \tilde{H}_{-1}(\mathrm{Link}_{\Delta(G^C)} F) = 1$.

Notice that $\mathrm{Link}_{\Delta(G^C)} F = \emptyset$ implies that F is a facet in $\Delta(G^C)$ and therefore a maximal clique in G^C. This gives us that

$$\beta_{0,n-r} = \sum_{\substack{F \text{ in} \\ \mathrm{MaxClq}_r(G^C)}} (1)$$

and so $\beta_{0,n-r} = \#(\mathrm{MaxClq}_r(G^C))$. □

Theorem 4.0.3. *Suppose $d \geq 1$ with G^C having no isolated points. The Lyubeznik number $\lambda_{0,1}(R/I) = M - 1$ where $M = \#(\pi_0(G^C))$, and $\lambda_{1,1}(R/I) = 0$.*

Proof. From Theorem 2.2.8, the Lyubeznik numbers are given by $\lambda_{0,1}(R/I) = \dim_K[H_1(\mathbb{L}_\bullet^{\langle n-1\rangle} \otimes_R K)]$ and $\lambda_{1,1}(R/I) = \dim_K[H_0(\mathbb{L}_\bullet^{\langle n-1\rangle} \otimes_R K)]$. By definition the terms in the $(n-1)$-linear strand are given by $\mathbb{L}_i^{\langle n-1\rangle} = R(-i-n+1)^{\beta_{i,i+n-1}}$.

From Lemma 4.0.2, we see that $\beta_{0,n-1}(I^\vee) = \#(\mathrm{MaxClq}_1(G^C))$. However, since there are no isolated points in G^C, we have $\beta_{0,n-1} = 0$.

For $\beta_{1,n}$, we have:

$$\beta_{1,n} = \sum_{\substack{F \in \Delta(G^C) \\ \#F = 0}} \dim_k \tilde{H}_0(\operatorname{Link}_{\Delta(G^C)} F).$$

The only F in $\Delta(G^C)$ such that $\#F = 0$ is $F = \emptyset$. Since $\operatorname{Link}_{\Delta(G^C)} \emptyset = \Delta(G^C)$, we have that $\tilde{H}_0(\operatorname{Link}_{\Delta(G^C)} F) = k^M$ where $M = \#(\pi_0(\Delta(G^C))) = \#(\pi_0(G^C))$, and so $\tilde{H}_0(\operatorname{Link}_{\Delta(G^C)} F, k) = k^{M-1}$. Therefore $\beta_{1,n} = \dim_k(k^{M-1}) = M - 1$. Now consider $\beta_{i,i+n-1}$ for $i \geq 2$. Again by Eagon-Reiner, we have

$$\beta_{i,i+n-1} = \sum_{\substack{F \in \Delta(G^C) \\ \#F = -i+1}} \dim_k \tilde{H}_i(\operatorname{Link}_{\Delta(G^C)} F).$$

Since $i \geq 2$ we see that $-i + 1 \leq -1$, and there are no F in $\Delta(G^C)^\vee$ such that $\#F \leq 1$ since $\#F$ must be positive. Therefore, $\beta_{i,i+n-1}(I^\vee) = 0$ for $i \geq 2$. This gives us that the $(n-1)$-linear strand of $I^\vee$ is $0 \longrightarrow R(-n)^{M-1} \longrightarrow 0 \longrightarrow 0$, and so $\mathbb{L}_\bullet^{\langle n-1\rangle} \otimes_R K$ is given by $0 \longrightarrow K^{M-1} \longrightarrow 0 \longrightarrow 0$. Since $H_1(\mathbb{L}_\bullet^{\langle n-1\rangle} \otimes_R K) = K^{M-1}$, we have that $\lambda_{0,1}(R/I) = \dim_K(K^{M-1}) = M - 1$, and since $H_0(\mathbb{L}_\bullet^{\langle n-1\rangle} \otimes_R K) = 0$, we have that $\lambda_{1,1}(R/I) = \dim_K(0) = 0$.

□

To go further, note that the combinatorial condition of G^C having no isolated points has a simple algebraic interpretation.

Lemma 4.0.4. *If $d \geq 1$, then* $\operatorname{ht}(\mathfrak{p}) < n - 1$ *for all associated primes* $\mathfrak{p} \in \operatorname{Ass}(I)$ *if and only if G^C has no isolated points.*

Proof. First, note that G^C has no isolated points if and only if all facets in $\Delta(G^C)$ have size greater than one. Since there are n vertices in G^C, it easily follows that G^C has no isolated points if and only if all complements of facets in $\Delta(G^C)$ have size at most $n - 1$. We then observe that the prime ideal $P_m = (x_i : x_i \in m)$ is associated to $I(G)$ for m a square-free monomial if and only if m is the complement of a facet in $\Delta_{I(G)}$, [FMS14, Thm.

4.11]. Therefore, we see that G^C has no isolated points if and only if $\#(m) < n - 1$. Since the height of P_m in R is equal to $\#(m)$, the result follows.

□

We next impose a few conditions. First, we utilize an unmixedness assumption on the edge ideal in question. This ensures that the graph complement is pure, whence unmixed edge ideals of dimension at least one satisfy the conclusion of Theorem 4.0.3. Purity follows easily from a natural association of generators of minimal primes of edge ideals to cliques in the graph complement.

Lemma 4.0.5. *An edge ideal $I(G)$ is unmixed of dimension d if and only if all the maximal cliques of G^C have size d.*

Proof. As in Lemma 4.0.4, we again observe that the prime ideal $P_m = (x_i : x_i \in m)$ is associated to $I(G)$ for m a square-free monomial if and only if m is the complement of a facet in $\Delta_{I(G)}$, [FMS14, Thm. 4.11]. Therefore, $I(G)$ is unmixed if and only if all complements of facets of $\Delta_{I(G)}$ are the same dimension, which of course is true if and only if $\Delta_{I(G)}$ is pure. However, we see in Lemma 2.3.8 that $\Delta_{I(G)} = \Delta(G^C)$. So by definition of the clique complex, we see that $I(G)$ is unmixed if and only if all maximal cliques of G^C are the same size.

Also, notice maximal cliques have size $n - r_m$ for square-free monomial m, where r_m is the number of vertices in m, the complement of a facet, and n is the number of vertices in G^C. Since $\operatorname{ht}(P_m)$ is also equal to r_m, we see that the size of the maximal cliques in G^C have size $n - \operatorname{ht}(P_m) = \dim(R/I(G))$. □

So far we have *not* used any results in Chapter 3. In order to apply those results, in particular the consequence of the Clique-Link correspondence in Theorem 3.3.14, we need to assume the graphs in question are additionally clique-cochordal. Recall that when G is clique-cochordal, it is also cochordal. This imposes a particularly useful restriction to the resolutions from which we take linear strands. This is a consequence of a sequence of results.

Corollary 4.0.6. *The graph G is cochordal if and only if* $\text{pdim}(I^\vee) = 1$.

Proof. First, G is cochordal if and only if I has a linear resolution, [Fro90, Thm. 1]. However, since I is a homogeneous ideal generated in degree two, we know that I has a linear resolution if any only if $\text{reg}(I) = 2$. We then see that $\text{pdim}(S/I^\vee) = \text{reg}(I) = 2$ [Ter99, Cor. 0.3]. Since $\text{pdim}(I^\vee) = \text{pdim}(S/I^\vee) - 1$, we have our result. □

4.1 The unmixed case

We see from Corollary 4.0.6, that if G is cochordal, then the only nontrivial graded Betti numbers of $I^\vee$ are $\beta_{0,n-d}$ and $\beta_{1,n-i+1}$ for all $1 \leq i \leq d$. In this section, we will calculate these Betti numbers using the clique graphs described above.

Proposition 4.1.1. *For all $r \neq d$, $\beta_{0,n-r} = 0$ and*

$$\beta_{0,n-d} = \#V(\text{Clq}_{d-1}(G^C)) = \#(\text{MaxClq}(G^C)).$$

Proof. We saw in Lemma 4.0.2 that $\beta_{0,n-r}$ is equal to the number of maximal cliques of size r in G^C for any r. However, since I is unmixed of dimension d, by Lemma 4.0.5, all maximal cliques of G^C have size d. This tells us that $\beta_{0,n-r} = 0$ for all $r \neq d$ and $\beta_{0,n-r} = \#(\text{MaxClq}(G^C))$.

Since there is a vertex in $\text{Clq}_{d-1}(G^C)$ for every maximal clique in G^C, we also have that $\beta_{0,n-d} = \#(V(\text{Clq}_{d-1}(G^C)))$. □

Theorem 4.1.2. *Let G be a graph and $1 \leq i \leq d$ be such that $\text{Clq}_{\geq i-1}(G^C)$ is chordal, then*

$$\beta_{1,n-i+1} = \#(\pi_0(\text{Clq}_{\geq i}(G^C))) - \#(\pi_0(\text{Clq}_{\geq i-1}(G^C))).$$

Proof. By the Eagon-Reiner formula, we have that

$$\beta_{1,n-i+1} = \sum_{\#F=i-1} \dim_k \tilde{H}_0(\operatorname{Link}_{\Delta(G^C)} F)$$

and by definition of reduced homology, we see this is equal to

$$\sum_{\#F=i-1} (\#(\pi_0(\operatorname{Link}_{\Delta(G^C)} F)) - 1).$$

By Theorem 3.2.8, we see that there is a bijection between $\mathcal{F}_{i-1}$ and $\mathcal{H}_{i-1}$ where

$$\mathcal{F}_{i-1} = \{F \in \Delta(G^C) : \#F = i-1 \text{ and } \operatorname{Link}(F) \text{ is disconnected}\},$$

$$\mathcal{H}_{i-1} = \{H \in \operatorname{MaxClq}(\operatorname{Clq}_{\geq i}) : \operatorname{Clq}_{\geq i}|_H \text{ is disconnected}\}.$$

Let F be in $\mathcal{F}_{i-1}$. By Corollary 3.2.10, we have that $\#(\pi_0(\operatorname{Link}(F))) = \#(\pi_0(\operatorname{Clq}_{\geq i}|_H))$, where H is the maximal clique in $\mathcal{H}$ that corresponds to F. Notice that for $F \notin \mathcal{F}_{i-1}$, then $\#(\pi_0(\operatorname{Link}(F))) = 1$. Therefore, for $F \notin \mathcal{F}_{i-1}$, then $\#(\pi_0(\operatorname{Link}(F))) - 1 = 0$. Similarly, for $H \notin \mathcal{H}_{i-1}$, then $\operatorname{Clq}_{\geq i}$ is connected, and so $\#(\pi_0(\operatorname{Clq}_{\geq i}|_H)) - 1 = 0$. Putting these facts together, we see that

$$\begin{aligned}
\sum_{\#F=i-1} (\#(\pi_0(\operatorname{Link}_{\Delta(G^C)} F)) - 1) &= \sum_{F \in \mathcal{F}} (\#(\pi_0(\operatorname{Link}_{\Delta(G^C)} F)) - 1) \\
&= \sum_{H \in \mathcal{H}} (\#(\pi_0(\operatorname{Clq}_{\geq i-1}|_H)) - 1) \\
&= \sum_{\substack{H \text{ in} \\ \operatorname{MaxClq}(\operatorname{Clq}_{\geq i-1})}} (\#(\pi_0(\operatorname{Clq}_{\geq i}|_H)) - 1)
\end{aligned}$$

Now notice that

$$\sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(\mathrm{Clq}_{\geq i}(G^C))}} (\#(\pi_0(\mathrm{Clq}_{\geq i}|_H)) - 1) = \sum_{\substack{A \text{ in} \\ \pi_0(\mathrm{Clq}_{\geq i-1})}} \left(\sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(A)}} (\#(\pi_0(\mathrm{Clq}_{\geq i}|_H)) - 1) \right).$$

By Theorem 3.3.14, we have that for A in $\pi_0(\mathrm{Clq}_{\geq i-1})$,

$$\sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(A)}} (\#(\pi_0(\mathrm{Clq}_{\geq i}|_H)) - 1) = \#(\pi_0(\mathrm{Clq}_{\geq i}|_A)) - 1.$$

So we have that

$$\begin{aligned}
\sum_{\substack{A \text{ in} \\ \pi_0(\mathrm{Clq}_{\geq i-1})}} \left(\sum_{\substack{H \text{ in} \\ \mathrm{MaxClq}(A)}} (\#(\pi_0(\mathrm{Clq}_{\geq i}|_H)) - 1 \right) &= \sum_{\substack{A \text{ conn. comp.} \\ \text{in } \mathrm{Clq}_{\geq i-1}}} (\#(\pi_0(\mathrm{Clq}_{\geq i}|_A)) - 1) \\
&= \sum_{\substack{A \text{ conn. comp.} \\ \text{in } \mathrm{Clq}_{\geq i}}} \#(\pi_0(\mathrm{Clq}_{\geq i}|_A))) - \#(\pi_0(\mathrm{Clq}_{\geq i-1})) \\
&= \#(\pi_0(\mathrm{Clq}_{\geq i})) - \#(\pi_0(\mathrm{Clq}_{\geq i-1})).
\end{aligned}$$

□

Theorem 4.1.3. *Let $I = I(G)$ be an unmixed edge ideal of dimension $d \geq 1$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ where G is a clique-cochordal graph. The Lyubeznik table of R/I is given by*

$$\Lambda(R/I) = \begin{bmatrix}
0 & C_0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & C_1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & C_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
0 & 0 & 0 & \ldots & 0 & C_{d-2} & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & J
\end{bmatrix}$$

where $J = \#\pi_0(\mathrm{Clq}_{d-1}(G^C))$ *and* $C_i = \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C)) - \#\pi_0(\mathrm{Clq}_{\geq i}(G^C))$.

Proof. Recall from Proposition 4.1.1, we have $\beta_{0,n-r} = 0$ for all $r \neq d$ and $\beta_{0,n-d} = \#\,\mathrm{MaxClq}(G^C)$. Also, from Corollary 4.0.6, we see that $\beta_i = 0$ for all $i > 1$. This gives us the minimal graded free resolution

$$F_\bullet(I^\vee) : 0 \longrightarrow \bigoplus_{0\leq i\leq d} R^{\beta_{1,n-i+1}}(-n+i-1) \xrightarrow{\phi} R^{\beta_{0,n-d}(I^\vee)}(-n+d) \longrightarrow I^\vee \longrightarrow 0.$$

Consider $\phi_{d-1} : R^{\beta_{1,d-1}}(-d+1) \longrightarrow R^{\beta_{0,n-d}}(-n+d)$, a restriction of ϕ. Let $\mathbf{w} \in \mathrm{Ker}(\phi_{d-1})$. So $\phi_{d-1}(\mathbf{w}) = 0$ in $R^{\beta_{0,n-d}}(-n+d)$. However, since ϕ is injective, we know $\mathrm{Ker}\,\phi = 0$ and then it must be that case that $(\mathbf{v}, \mathbf{w}) = 0$. Therefore, $\mathbf{w} = 0$ and so $\mathrm{Ker}(\phi_{d-1}) = 0$ and so ϕ_{d-1} is injective. From the linear free resolution of $I^\vee$, we get the linear strands:

$$\mathbb{L}_\bullet^{\langle n-i\rangle} : 0 \longrightarrow R^{\beta_{1,n-i+1}}(-n+i-1) \longrightarrow 0 \longrightarrow 0 \text{ for all } 0 \leq i \leq d-1, \text{ and}$$

$$\mathbb{L}_\bullet^{\langle n-d\rangle} : 0 \longrightarrow R^{\beta_{1,n-d+1}}(-n+d-1) \xrightarrow{\phi_{d-1}} R^{\beta_{0,n-d}}(-n+d) \longrightarrow 0$$

From this we apply Theorem 2.2.8 to see

$$\lambda_{i,i}(R/I) = \dim_K[H_0(\mathbb{L}_\bullet^{\langle n-i\rangle} \otimes_R K)] = 0 \text{ for all } 0 \leq i \leq d-1$$

$$\lambda_{i,j}(R/I) = \dim_K[H_{j-i}(\mathbb{L}_\bullet^{\langle n-i\rangle} \otimes_R K)] = 0 \text{ for all } 0 \leq i \leq d-2, j \geq i+2$$

$$\lambda_{i,i+1}(R/I) = \dim_K[H_1(\mathbb{L}_\bullet^{\langle n-i-1\rangle} \otimes_R K)] = \beta_{1,n-i} \text{ for all } 0 \leq i \leq d-2$$

From Theorem 4.1.2, we see that

$$\beta_{1,n-i} = \#(\pi_0(\mathrm{Clq}_{\geq i+1})) - \#(\pi_0(\mathrm{Clq}_{\geq i})) = \dim_k H_0(\mathrm{Clq}_{\geq i+1}, k) - \dim_k H_0(\mathrm{Clq}_{\geq i}, k).$$

Lastly, since $\mathrm{Ker}(R^{\beta_{0,n-d}}(-n+d) \longrightarrow 0) = R^{\beta_{0,n-d}}(-n+d)$ and since ϕ_{d-1} is injective, we have that

$\mathrm{Ker}(\phi_{d-1}) = 0$ and $\mathrm{Im}(\phi_{d-1}) = R^{\beta_{1,n-d+1}}(-n+d-1)$, and so

$$\lambda_{d-1,d}(R/I) = \dim_K[H_1(\mathbb{L}_{\bullet}^{\langle n-d\rangle} \otimes_R K)] = 0 \text{ and}$$

$$\lambda_{d,d}(R/I) = \dim_K[H_0(\mathbb{L}_{\bullet}^{\langle n-d\rangle} \otimes_R K)] = \beta_{0,n-d} - \beta_{1,n-d+1}.$$

We need to show: $\beta_{0,n-d} - \beta_{1,n-d+1} = \#\pi_0((\mathrm{Clq}_{d-1}))$.

From Theorems 4.1.1 and 4.1.2, we see that

$$\beta_{0,n-d} - \beta_{1,n-d+1} = \#V(\mathrm{Clq}_{d-1}) - \#\pi_0(\mathrm{Clq}_{\geq d}) + \#(\pi_0(\mathrm{Clq}_{\geq d-1})).$$

But $\mathrm{Clq}_{\geq d}$ is a graph of isolated points and so $\#\pi_0(\mathrm{Clq}_{\geq d}) = \#V(\mathrm{Clq}_{\geq d}) = \#V(\mathrm{Clq}_{\geq d-1})$. Also, since $\mathrm{Clq}_{\geq d-1} \cong \mathrm{Clq}_{d-1}$, we see that $\lambda_{d,d}(R/I) = \#\pi_0(\mathrm{Clq}_{\geq d-1})$. This is enough to complete the Lyubeznik table of R/I. □

Recall Theorem 3.4.11 gives us the inequality

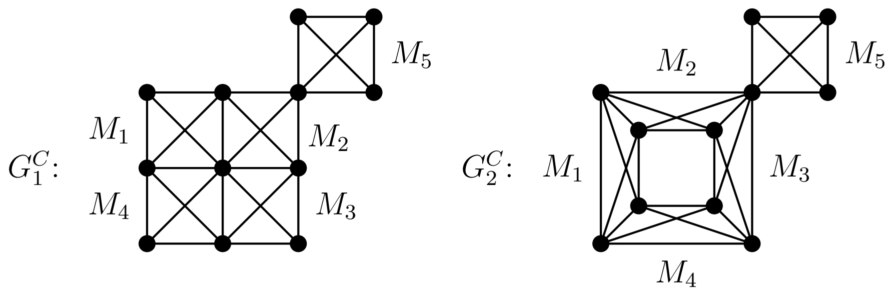
$$\lambda_{i,i+1}(S) \geq \#\pi_0\Gamma_{d-i-1}(S) - \#\pi_0\Gamma_{d-i}(S)$$

for $1 \leq i \leq d-2$ when S is an equidimensional local ring of dimension $d \geq 2$. Noting that the higher Hochster-Huneke graphs and clique graphs are isomorphic, we see this inequality is strengthed to an equality in Theorem 4.1.3 in the case of an unmixed edge ideal where the associated graph is clique-cochordal.

4.2 Results in Higher Projective Dimension

As we observed in Section 3.3, Theorem 4.1.3 holds for a wide class of edge ideals. However, one can ask about even more generality. The difficulty of the non-clique-chordal case, i.e. for higher projective dimension of the dual of the edge ideal, arises because the maps in the linear strands can have varied behavior. We see this demonstrated in the following example.

Example 4.2.1. Let G_1^C and G_2^C be the graphs given below, whose maximal cliques are each given by M_1, M_2, M_3, M_4 and M_5.



Let $I_1 = I(G_1)$ and $I_2 = I(G_2)$. The linear strands for $I_1^\vee$ are given by

$$\mathbb{L}_\bullet^{\langle 8 \rangle}(I_1^\vee)\colon 0 \longrightarrow R^5 \longrightarrow 0$$

$$\mathbb{L}_\bullet^{\langle 9 \rangle}(I_1^\vee)\colon 0 \longrightarrow R \xrightarrow{\phi_9} R^4 \longrightarrow 0 \longrightarrow 0$$

$$\mathbb{L}_\bullet^{\langle 10 \rangle}(I_1^\vee)\colon 0 \longrightarrow R \longrightarrow 0 \longrightarrow 0$$

whereas the linear strands for $I_2^\vee$ are given by

$$\mathbb{L}_\bullet^{\langle 7 \rangle}(I_2^\vee)\colon 0 \longrightarrow R^5 \longrightarrow 0$$

$$\mathbb{L}_\bullet^{\langle 8 \rangle}(I_2^\vee)\colon 0 \longrightarrow R^4 \longrightarrow 0 \longrightarrow 0$$

$$\mathbb{L}_\bullet^{\langle 9 \rangle}(I_2^\vee)\colon 0 \longrightarrow R \xrightarrow{\cdot 0} R \longrightarrow 0 \longrightarrow 0.$$

Now consider the Lyubeznik tables for R/I_1 and R/I_2:

$$\Lambda(R/I_1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \qquad \Lambda(R/I_2) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Notice for I_1, the only linear strand of $I_1^\vee$ with a potentially nontrivial map is $\mathbb{L}_{\bullet}^{\langle 9 \rangle}(I_1^\vee)$. This map, ϕ_9 is injective. Therefore, to compute $\lambda_{2,3}(R/I_1)$, we subtract $\beta_{2,11}(I_1^\vee)$ from $\beta_{1,10}(I_1^\vee)$.

However, for I_2, the only linear strand of $I_1^\vee$ with a potentially nontrivial map is $\mathbb{L}_{\bullet}^{\langle 9 \rangle}(I_1^\vee)$. However, this map, $\phi_9 : R \longrightarrow R$, is the zero map. Therefore, $\lambda_{0,2}(R/I_2)$ is given by $\beta_{1,10}(I_1^\vee)$ and $\lambda_{1,2}(R/I_2)$ is given by $\beta_{2,11}(I_1^\vee)$.

This tells us to calculate the Lyubeznik tables for edge ideals whose duals have projective dimension > 1, it is not always as simple as a Betti number calculation. Here information about the maps of the linear strands is also needed.

However, if we can find a condition that forces the maps of the linear strands to be injective, the maps become much more manageable. We explore a potential condition in the following section.

4.2.1 $\ell-$Chordality

We explore how to obtain a similar statement to Theorem 4.1.3 in more generality by studying what causes the difference between I_1 and I_2 in the example above. In this pursuit, we define the condition below which we call $\ell-$chordal.

Definition 4.2.2. If there are maximal cliques $M_1, \ldots, M_k$ in a graph G and ℓ is the maximal integer such that $\#(M_j \cap M_{j+1}) \geq \ell$ for all $1 \leq j < k$ and $\#(M_k \cap M_1) \geq \ell$, and if $\#(\bigcap_j M_j) = m$, then we say $\{M_1, \ldots, M_k\}$ is an $(\boldsymbol{\ell}, \mathbf{m}) -$ **cycle** in G.

Fix ℓ. We call G an $\boldsymbol{\ell}$**-chordal** graph if there are no $(\ell, m)-$cycles in G with $\ell - m > 1$.

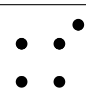
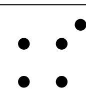
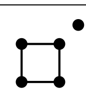
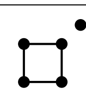
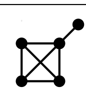
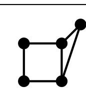
We return to edge ideals I_1 and I_2 from Example 4.2.1 to better understand the previous definitions.

Example 4.2.3. Notice in G_1^C in Example 4.2.1 that each pair of maximal cliques in the set $\{M_1, M_2, M_3, M_4\}$ intersect at a clique of size 2. Also, $M_1 \cap M_2 \cap M_3 \cap M_4$ is a clique of size 1. Therefore, $\{M_1, M_2, M_3, M_4\}$ is a $(2, 1)-$cycle in G_1^C. Since $\ell - m = 1$ and since there are no other $(\ell, m)-$cycles for any m, we see that G_1^C is $2-$chordal.

Now consider G_2^C in Example 4.2.1. Notice that each pair of maximal cliques in the set $\{M_1, M_2, M_3, M_4\}$ intersect at a clique of size 2. However $M_1 \cap M_2 \cap M_3 \cap M_4$ is a clique of size 0. Therefore, $\{M_1, M_2, M_3, M_4\}$ is a $(2, 0)-$cycle in G_2^C. Since $\ell - m = 2 > 1$, we see that G_2^C is not $2-$chordal.

We now see how the $\ell-$chordal condition on the graph effects the clique graphs, which also inspires the term $\ell-$chordal.

To do so, consider the clique graphs of G_1^C and G_2^C from Example 4.2.1.

i	$\text{Clq}_{\geq i}(G_1^C)$	$\text{Clq}_{\geq i}(G_2^C)$
3		
2		
1		

Recall that we found G_1^C to be 2-chordal. Now notice that there is a cycle in $\text{Clq}_{\geq 2}(G_1^C)$ that becomes a complete graph, i.e. a chordal subgraph, in $\text{Clq}_{\geq 1}(G_1^C)$.

Also recall that we found G_2^C to not be 2-chordal. Now notice there is a cycle in $\text{Clq}_{\geq 2}(G_2^C)$ that is does not become a complete subgraph in $\text{Clq}_{\geq 1}(G_2^C)$, and in fact is still a non-chordal subgraph.

We show a generalization of this behavior in the following theorem.

Theorem 4.2.4. *Let G be a graph. We have the following:*

1. *$\{M_1, \ldots, M_k\}$ is an $(\ell, m)-$cycle in G for some m if and only if ℓ is maximal such that $\{M_1, \ldots, M_k\}$ is a cycle in $\mathrm{Clq}_{\geq \ell}(G)$.*

2. *The following are equivalent*

   (a) *G is $\ell-$chordal*

   (b) *If ℓ is maximal such that $\{M_1, \ldots, M_k\}$ is a cycle in $\mathrm{Clq}_{\geq \ell}(G)$, then $\mathrm{Clq}_{\geq \ell-1}(G)[M_1, \ldots, M_k]$ is complete.*

Proof. Notice ℓ is maximal such that $\{M_1, \ldots, M_k\}$ is a cycle in $\mathrm{Clq}_{\geq \ell}(G)$ if and only if ℓ is maximal such that $M_j \cap M_{j+1} \geq \ell$ for all $1 \leq j \leq k-1$ and $M_k \cap M_1 \geq \ell$. By definition of an $(\ell, m)-$cycle, this is equivalent to saying $\{M_1, \ldots, M_k\}$ is an $(\ell, m)-$cycle, where $m = \#(\bigcap_j M_j)$.

By (1), $\{M_1, \ldots, M_k\}$ is an $(\ell, m)-$cycle in G for some m if and only if ℓ is maximal such that $\{M_1, \ldots, M_k\}$ is a cycle in $\mathrm{Clq}_{\geq \ell}(G)$. Therefore, we need only show that $\mathrm{Clq}_{\geq \ell-1}(G)[M_1, \ldots, M_k]$ is complete if and only if $\ell - m \leq 1$.

To this end, notice that $\mathrm{Clq}_{\geq \ell-1}(G)[M_1, \ldots, M_k]$ is complete if and only if $M_1, \ldots, M_k$ is a subset of $V(H)$ for some maximal clique H of $\mathrm{Clq}_{\geq \ell-1}(G)$.

By the Clique-link correspondence, $V(H)$ corresponds to $\mathrm{MaxClq}(G/F)$ for some clique F in G of size $\ell - 1$. Therefore, we have that $M_1, \ldots, M_k$ is a subset of $V(H)$ for some maximal clique H in $\mathrm{Clq}_{\geq \ell-1}(G)$ if and only if F is contained in maximal cliques $M_1, \ldots, M_k$ for some clique F in G of size $\ell - 1$. The latter is true if and only if there is a clique F of size $\ell - 1$ in G such that $F \subseteq \left(\bigcap_j M_j\right)$ which is equivalent to saying $\#\left(\bigcap_j M_j\right) \geq \ell - 1$. By definition of m, we now have that $\mathrm{Clq}_{\geq \ell-1}(G)[M_1, \ldots, M_k]$ is complete if and only if $m \geq \ell - 1$, i.e. $\ell - m \leq 1$.

□

With this reinterpretation of the $\ell-$chordal condition in terms of clique graphs, we now see how this combinatorial condition effects the linear strands, and in particular, the graded Betti numbers. We are still limited in that we can relate clique graphs to links of faces in $\Delta(G^C)$, but cannot relate their homologies, and in such cannot relate clique graphs to graded Betti numbers. Therefore, we return to higher nerve complexes to help make that connection.

Remark 4.2.5. *Recall from the Eagon-Reiner formula that for $m \geq 2$,*

$$\beta_{m,n-r+m}(I^\vee) = \sum_{\#F=r-m} \dim_k H_{m-1}(\operatorname{Link} F).$$

By Theorem 3.4.8, we now have the following equivalence:

$$\beta_{m,n-r+m}(I^\vee) = \sum_{\#F=r-m} \dim_k H_{m-1}(N_{r-m+1}(\Delta(G^C))|_{H_F})$$

where $H_F = \{i : F \subseteq A_i\}$ and $\{A_1, \ldots, A_r\}$ is the set of facets of $\Delta(G^C)$.

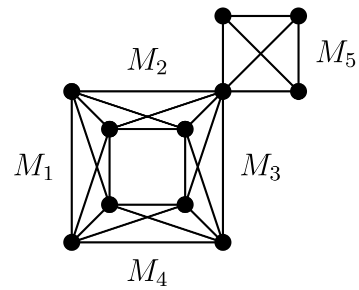
From Theorem 3.4.5, we then have:

$$\beta_{m,n-r+m}(I^\vee) = \sum_{\#F=r-m} \dim_k H_{m-1}(\Delta(\operatorname{Clq}_{\geq r-m+1}(G^C))[\operatorname{MaxClq}(G^C/F)])$$

However, it is not always the case that $\beta_{m,n-r+m}(I^\vee) = \dim_k H_{m-1}(\Delta(\operatorname{Clq}_{\geq r-m+1}(G^C))$. To see this, consider the following example.

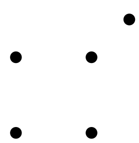
Example 4.2.6. Consider again G_2 from Example 4.2.1, which we rename here to be simply G. So G^C is the graph given below. Also, recall G^C is not 2-chordal.

G^C:

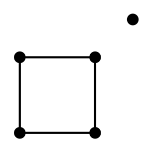


Recall that $\beta_{2,11}(I^\vee) = 1$ and $\beta_{2,j}(I^\vee) = 0$ for all $j \neq 11$. We also observed that G^C had the following clique graphs:

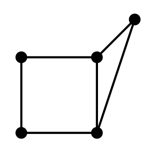
$\mathrm{Clq}_{\geq 3}(G^C)$:



$\mathrm{Clq}_{\geq 2}(G^C)$:



$\mathrm{Clq}_{\geq 1}(G^C)$:



We know $\beta_{2,10}(I^\vee) = 0$. However $\dim_k H_1(\mathrm{Clq}_{\geq 2}(G^C)) = 1$.

We look at the reinterpretation of $\beta_{2,10}(I^\vee)$ given in Remark 4.2.5 to see what causes the difference, i.e. consider

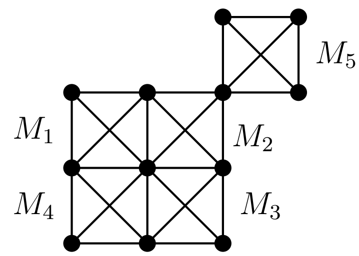
$$\sum_{\#F=1} \dim_k H_1(\Delta(\mathrm{Clq}_{\geq 2}(G^C))[\mathrm{MaxClq}(G^C/F)])$$

Notice there is no vertex in G^C that is contained in more than three maximal cliques. Equivalently, there is no maximal clique in $\mathrm{Clq}_{\geq 1}(G^C)$ of size greater than three. Therefore, there can be no cycle in $\Delta(\mathrm{Clq}_{\geq 2}(G^C))[\mathrm{MaxClq}(G^C/F)])$ for any F of size one. Therefore, $\dim_k H_1(\Delta(\mathrm{Clq}_{\geq 2}(G^C))[\mathrm{MaxClq}(G^C/F)]) = 0$ for all such F.

The issue with this graph that keeps $\beta_{m,n-r+m}(I^\vee)$ from being equal to $\dim H_{m-1}(\Delta(\mathrm{Clq}_{\geq r-m+1}(G^C))$ is that a cycle was formed in $\mathrm{Clq}_{\geq 2}(G^C)$ that did not arise from a restriction of a single maximal clique in $\mathrm{Clq}_{\geq 1}(G^C)$. If G^C were 2-chordal, this issue would be resolved. To see this, consider the following example.

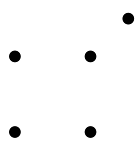
Example 4.2.7. Consider again G_1 from Example 4.2.1, which we again rename here to be G. So G^C is the graph given below. Also, recall G^C in this case is ℓ-chordal for all ℓ.

G^C:

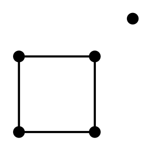


Recall that $\beta_{2,11}(I^\vee) = 1$ and $\beta_{2,j}(I^\vee) = 0$ for all $j \neq 11$. We also observed that G^C had the following clique graphs:

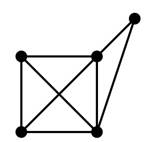
$\mathrm{Clq}_{\geq 3}(G^C)$:



$\mathrm{Clq}_{\geq 2}(G^C)$:



$\mathrm{Clq}_{\geq 1}(G^C)$:



Notice that $\dim H_1(\mathrm{Clq}_{\geq 2}(G^C)) = 1$ and $\dim H_1(\mathrm{Clq}_{\geq i}(G^C)) = 0$ for all $i \neq 2$. Therefore, $\beta_{2,n-j}(I^\vee) = \dim H_1(\mathrm{Clq}_{\geq j+1}(G^C))$ for all j.

We see in the next theorem that the condition of a graph being $\ell-$chordal is indeed enough to simplifiy the calculation of the graded Betti numbers.

Theorem 4.2.8. *Let $I = I(G)$ an edge ideal of dimension d in the polynomial ring $R = k[x_1, \ldots, x_n]$. If G^C is a $(r-m+1)-$chordal graph, then $\beta_{m,n-r+m}(I^\vee) = \dim H_{m-1}(\Delta(\mathrm{Clq}_{\geq r-m+1}(G^C)))$ for $m \geq 2$.*

Proof. First note that by Remark 4.2.5, we have

$$\beta_{m,n-r+m}(I^\vee) = \sum_{\#F=r-m} \dim H_{m-1}(\Delta(\mathrm{Clq}_{\geq r-m+1}(G^C))[\mathrm{MaxClq}(G^C/F)])$$

Now note that if C is a cycle in $\mathrm{Clq}_{\geq r-m+1}(G^C)[\mathrm{MaxClq}(G^C/F)]$, then C is also a cycle in $\mathrm{Clq}_{\geq r-m+1}(G^C)$. Therefore,

$$\begin{aligned}\dim H_{m-1}(\Delta(\mathrm{Clq}_{\geq r-m+1}(G^C)) &\leq \sum_{\#F=r-m} \dim H_{m-1}(\Delta(\mathrm{Clq}_{\geq r-m+1}(G^C))[\mathrm{MaxClq}(G^C/F)]) \\ &= \beta_{m,n-r+m}(I^\vee).\end{aligned}$$

Also if $\{M_1, \ldots, M_k\}$ is a cycle in $\mathrm{Clq}_{\geq r-m+1}(G^C)$, then $\{M_1, \ldots, M_k\}$ is complete in $\mathrm{Clq}_{\geq r-m}(G^C)$. Therefore, $\{M_1, \ldots, M_k\}$ is contained in a maximal clique, H, in $\mathrm{Clq}_{\geq r-m}(G^C)$. By the Clique-Link correspondence H corresponds to a clique F of size $r-m$ in G^C, where $\#V(H) = \#(\mathrm{MaxClq}(G^C/F))$. This shows that $\{M_1, \ldots, M_k\}$ is a cycle in $\mathrm{Clq}_{\geq r-m+1}(G^C)[\mathrm{MaxClq}(G^C/F)]$. Therefore,

$$\begin{aligned}\beta_{m,n-r+m}(I^\vee) &= \sum_{\#F=r-m} \dim H_{m-1}(\Delta(\mathrm{Clq}_{\geq r-m+1}(G^C))[\mathrm{MaxClq}(G^C/F)]) \\ &\leq \dim H_{m-1}(\Delta(\mathrm{Clq}_{\geq r-m+1}(G^C))).\end{aligned}$$

□

We further speculate that in the $\mathrm{pdim}(I^\vee) \leq 2$ case, if G^C is an $r-$chordal graph, then

$$\beta_{1,n-r+1}(I^\vee) = \#(\pi_0(\mathrm{Clq}_{\geq r}(G^C))) - \#(\pi_0(\mathrm{Clq}_{\geq r-1}(G^C))) + \dim H_1(\Delta(\mathrm{Clq}_{\geq r-1}(G^C))).$$

We now consider the $(n-r)-$linear strand of $I^\vee$ when $\mathrm{pdim}(I^\vee) \leq 2$.

$$\mathbb{L}_\bullet^{\langle n-r\rangle} : 0 \longrightarrow R^{\beta_{2,n-r+2}(I^\vee)} \xrightarrow{\phi_{n-r}} R^{\beta_{1,n-r+1}(I^\vee)} \to 0$$

We also expect, though we do not show here, that in the case where G^C is r-chordal, ϕ_{n-r} is injective. These expectations lead to the following question.

Question 4.2.9. *Let $I(G)$ be an edge ideal of a graph G such that* $\mathrm{pdim}(I(G)^\vee) \leq 2$. *Does* *the condition of G^C being $\ell-$chordal for all ℓ force $\Lambda(R/I)$ to be the following:*

$$\Lambda(R/I) = \begin{bmatrix} 0 & C_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & C_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & C_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & C_{d-2} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & J \end{bmatrix}$$

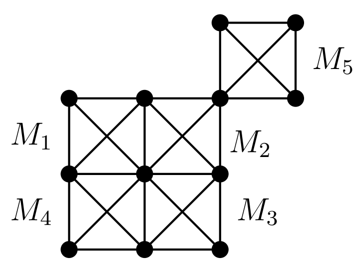
where $J = \#\pi_0(\mathrm{Clq}_{\geq d-1}(G^C))$ and for all $0 \leq i \leq d-2$,

$C_i = \beta_{1,n-i}(I^\vee) - \beta_{2,n-i+1}(I^\vee) = \#(\pi_0(\mathrm{Clq}_{\geq i+1}(G^C))) - \#(\pi_0(\mathrm{Clq}_{\geq i}(G^C)))$?

We view this speculation through an example to give evidence for its validity.

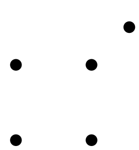
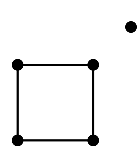
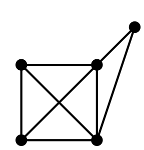
Example 4.2.10. We return once more to G_1 from Example 4.2.1, which we again rename to be G, and I to be $I(G)$. Therefore, we recall the graph G^C and the Lyubeznik table, $\Lambda(R/I)$, both given below. Also, recall G^C in this case is ℓ-chordal for all ℓ.

G^C:



$$\Lambda(R/I) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

For convenience, we list again the clique graphs of G^C.

$\mathrm{Clq}_{\geq 3}(G^C)$:  $\mathrm{Clq}_{\geq 2}(G^C)$:  $\mathrm{Clq}_{\geq 1}(G^C)$: 

Notice that $\lambda_{4,4}(R/I) = 5 = \#\pi_0(\mathrm{Clq}_{\geq d-1}(G^C))$,

$\lambda_{2,3}(R/I) = 5 - 2 = \#(\pi_0(\mathrm{Clq}_{\geq 3}(G^C))) - \#(\pi_0(\mathrm{Clq}_{\geq 2}(G^C)))$, and

$\lambda_{1,2}(R/I) = 2 - 1 = \#(\pi_0(\mathrm{Clq}_{\geq 2}(G^C))) - \#(\pi_0(\mathrm{Clq}_{\geq 1}(G^C)))$.

In an effort to validate this question, we begin checking the last column of the Lyubeznik table, which we explore in the next section.

4.2.2 Last Column of Lyubeznik Table

In this section we shift our focus to the $\langle n-d\rangle$-linear strand, i.e. the last column in the Lyubeznik table, with the purpose of gleaning information from the Hochster-Huneke graph. The hope is that the approaches taken in this section can be generalized to other columns of the Lyubeznik table using the higher Hochster-Huneke graphs.
An alternate description of the highest Lyubeznik number, i.e. $\lambda_{d,d}(R/I)$ follows fairly easily from the relationship between the Hochster-Huneke graph and clique graphs. We see this in the following lemma.

Lemma 4.2.11. *If G is a graph and $I = I(G)$ is an unmixed edge ideal in the polynomial ring $R = k[x_1, \ldots, x_n]$, then the highest Lyubeznik number is given by:*

$$\lambda_{d,d}(R/I) = \#(\pi_0(\mathrm{Clq}_{\geq d-1}(G^C))).$$

Proof. Recall that the highest Lyubeznik number is equal to the number of connected components in the Hochster-Huneke graph, i.e. $\lambda_{d,d}(R/I) = \#(\pi_0(\Gamma_1(I)))$. By Theorem 3.4.12, $\#(\pi_0(\Gamma_1(I))) = \#(\pi_0(\mathrm{Clq}_{\geq d-1}(G^C)))$. Therefore
$\lambda_{d,d}(R/I) = \#(\pi_0(\mathrm{Clq}_{\geq d-1}(G^C)))$. □

Before computing the rest of the last column of the Lyubeznik table, we state a couple more necessary lemmas.

Lemma 4.2.12. *If G is a graph and $I = I(G)$ is an edge ideal of dimension d such that* $\operatorname{pdim}(I^\vee) \leq 2$. *We have the equivalence:*

$$\beta_{1,n-d+1}(I^\vee) = \#(V(\mathrm{Clq}_{\geq d-1}(G^C))) - \sum_j (-1)^{j+1} c_j,$$

where c_j represents the number of cliques of size j in $\Delta(\mathrm{Clq}_{\geq d-1}(G^C))$.

Proof. By the Eagon-Reiner formula, we have that

$$\beta_{1,n-d+1}(I^\vee) = \sum_{\#F=d-1} \dim_k \tilde{H}_0(\mathrm{Link}_{\Delta(G^C)} F)$$

and by definition of reduced homology, we see this is equal to

$$\sum_{\#F=d-1} (\#(\pi_0(\mathrm{Link}_{\Delta(G^C)} F)) - 1).$$

By Theorem 3.2.8, we see that there is a bijection between $\mathcal{F}_{d-1}$ and $\mathcal{H}_{d-1}$ where

$$\mathcal{F}_{d-1} = \{F \in \Delta(G^C) : \#F = d-1 \text{ and } \mathrm{Link}(F) \text{ is disconnected}\},$$

$$\mathcal{H}_{d-1} = \{H \in \mathrm{MaxClq}(\mathrm{Clq}_{\geq d}) : \mathrm{Clq}_{\geq d}|_H \text{ is disconnected}\}.$$

Let F be in $\mathcal{F}_{d-1}$. By Corollary 3.2.10, we have that $\#(\pi_0(\mathrm{Link}(F))) = \#(\pi_0(\mathrm{Clq}_{\geq d}|_H))$, where H is the maximal clique in $\mathcal{H}$ that corresponds to F. Notice that for $F \notin \mathcal{F}_{d-1}$, then $\#(\pi_0(\mathrm{Link}(F))) = 1$. Therefore, for $F \notin \mathcal{F}_{d-1}$, then $\#(\pi_0(\mathrm{Link}(F))) - 1 = 0$. Similarly, for $H \notin \mathcal{H}_{d-1}$, then $\mathrm{Clq}_{\geq d}$ is connected, and so $\#(\pi_0(\mathrm{Clq}_{\geq d}|_H)) - 1 = 0$. Putting these facts together, we see that

$$\begin{aligned}\sum_{\#F=d-1}(\#(\pi_0(\mathrm{Link}_{\Delta(G^C)}F))-1) &= \sum_{F\in\mathcal{F}}(\#(\pi_0(\mathrm{Link}_{\Delta(G^C)}F))-1)\\ &= \sum_{H\in\mathcal{H}}(\#(\pi_0(\mathrm{Clq}_{\geq d-1}|_H))-1)\\ &= \sum_{\substack{H \text{ in}\\ \mathrm{MaxClq}(\mathrm{Clq}_{\geq d})}}(\#(\pi_0(\mathrm{Clq}_{\geq d}|_H))-1).\end{aligned}$$

Since $\mathrm{Clq}_{\geq d}(G^C)$ is a graph of isolated points, we have that

$$\beta_{1,n-d+1}(I^\vee) = \sum_{\substack{H \text{ in}\\ \mathrm{MaxClq}(\mathrm{Clq}_{\geq d})}}(\#V(H)-1).$$

Let s denote the size of the largest maximal clique in $\mathrm{Clq}_{\geq d-1}(G^C)$. We then have

$$\beta_{1,n-d+1}(I^\vee) = \sum_{j=1}^{s}\sum_{\substack{\#V(H)=j, H \text{ in}\\ \mathrm{MaxClq}(\mathrm{Clq}_{\geq d-1})}}(\#(\pi_0(\mathrm{Clq}_{\geq d}|_H))-1)$$

$$= \sum_{j=2}^{s}\sum_{\substack{\#V(H)=j, H \text{ in}\\ \mathrm{MaxClq}(\mathrm{Clq}_{\geq d-1})}}(j-1). \tag{4.1}$$

Now notice that the number of size ℓ cliques in a size j maximal clique is given by $\binom{j}{\ell}$. Therefore,

$$\begin{aligned}\#(V(\mathrm{Clq}_{\geq d-1}(G^C))) - \sum_j(-1)^{j+1}c_j &= \sum_{j=2}^{s}(-1)^j c_j\\ &= \sum_{j=2}^{s}\sum_{\substack{\#V(H)=j, H \text{ in}\\ \mathrm{MaxClq}(\mathrm{Clq}_{\geq d-1})}}\sum_{\ell=2}^{j}(-1)^j\binom{j}{\ell}.\end{aligned}$$

Lastly, recall the well known fact that for $j \geq 1$,

$$\sum_{\ell=0}^{j}(-1)^{\ell}\binom{j}{\ell} = 0.$$

Noting that $\binom{j}{0} = 1$ and $\binom{j}{1} = j$, we see that

$$\sum_{\ell=2}^{j}(-1)^{\ell}\binom{j}{\ell} = j-1.$$

Therefore, using this fact and Equation 4.1, we see the following equivalence which is enough to complete the proof:

$$\#(V(\mathrm{Clq}_{\geq d-1}(G^C))) - \sum_{j}(-1)^{j+1}c_j = \sum_{j=2}^{s}\sum_{\substack{\#V(H)=j, H \text{ in} \\ \mathrm{MaxClq}(\mathrm{Clq}_{\geq d-1})}}(j-1) = \beta_{1,n-d+1}(I^{\vee}).$$

□

Lemma 4.2.13. *Let I_Δ be a Stanley Reisner ideal such that $\Delta^{\vee}$ has dimension d and* $\mathrm{pdim}(I_\Delta) \leq 2$. *If*

$$\mathbb{L}_{\bullet}^{<d>}(I_\Delta) : 0 \longrightarrow R(-n+d-2)^{\beta_{2,n-d+2}(I_\Delta)} \xrightarrow{\phi_d} R(-n+d-1)^{\beta_{1,n-d+1}(I_\Delta)} \longrightarrow 0$$

is the d^{th}−linear strand of I_Δ, then the map ϕ_d is injective.

Proof. To prove the claim, consider the graded free resolution of $I^{\vee}$,

$$F_{\bullet} : 0 \to \bigoplus_{r} R(-r-2)^{\beta_{2,r+2}(I^{\vee})} \xrightarrow{\phi} \bigoplus_{r} R(-r-1)^{\beta_{1,r+1}(I^{\vee})} \to \bigoplus_{r} R(-r)^{\beta_{0,r}(I^{\vee})} \to 0.$$

Since $F_{\bullet}$ is exact, then ϕ is injective.

Now recall from the Eagon-Reiner formula that

$$\beta_{1,r+1}(I_\Delta) = \sum_{\#F=n-r-1} \dim_k \tilde{H}_0(\mathrm{Link}_{\Delta^\vee} F).$$

If $r = n-d-1$, then $\beta_{1,r+1}(I_\Delta)$ is summing over faces of size d. Since $\mathrm{Link}(F) = \varnothing$ for $F \in \Delta^\vee$ such that $\#F = d$, we see that $\dim_k \tilde{H}_0(\mathrm{Link}_{\Delta^\vee} F) = 0$.

If $r < n-d-1$, then $\beta_{1,r+1}(I_\Delta)$ is summing over faces of size $> d-1$. Since there are no faces of size $> d$ in $\Delta^\vee$, we see that $\dim_k \tilde{H}_0(\mathrm{Link}_\Delta F) = 0$.

Therefore,

$$\bigoplus_r R(-r-1)^{\beta_{1,r+1}(I_\Delta)} = \bigoplus_{r\geq n-d} R(-r-1)^{\beta_{1,r+1}(I_\Delta)}$$

Consider the restriction of the ϕ:

$$R(-n+d-2)^{\beta_{2,n-d+2}(I_\Delta)} \longrightarrow \bigoplus_{r\geq n-d} R(-r-1)^{\beta_{1,r+1}(I_\Delta)}$$

Notice if $r \geq n-d+1$, then $-r-1 \leq -n+d-2$. Therefore,

$R(-n+d-2)^{\beta_{2,n-d+2}(I_\Delta)} \longrightarrow R(-r-1)^{\beta_{1,r+1}(I_\Delta)}$ is the zero map for all $r \geq n-d+1$.

Since ϕ is injective, this shows $\phi_d : R(-n+d-2)^{\beta_{2,n-d+2}(I_\Delta)} \longrightarrow R(-n+d-1)^{\beta_{1,n-d+1}(I_\Delta)}$ is injective as well. □

We are now about to show the following theorem, which will give the last column of the Lyubeznik table.

Theorem 4.2.14. *If G is a graph and $I = I(G)$ is an edge ideal of dimension d such that $\mathrm{pdim}(I^\vee) \leq 2$ and $H_j(\Delta(\mathrm{Clq}_{\geq i}(G^C))) = 0$ for all $j \geq 3$ and for all i, then we have the following entries in the last column of $\Lambda(R/I)$:*

$\lambda_{d,d}(R/I) = \#(\pi_0(\mathrm{Clq}_{d-1}(G^C)))$,

$\lambda_{d-1,d}(R/I) = \dim H_1(\mathrm{Clq}_{\geq d-1}(G^C)) - \beta_{2,n-d+2}(I^\vee)$, *and*

$\lambda_{j,d}(R/I) = 0$ *for all $j \leq d-2$.*

Proof. First, since $\text{pdim}(I^\vee) \leq 2$, we have the $\langle n-d \rangle$-linear strand of $I^\vee$ given below:

$$\mathbb{L}_\bullet^{\langle n-d \rangle}(I^\vee)\colon 0 \xrightarrow{\gamma_d} R^{\beta_{2,n-d+2}(I^\vee)} \xrightarrow{\phi_d} R^{\beta_{1,n-d+1}(I^\vee)} \xrightarrow{\psi_d} R^{\beta_{0,n-d}(I^\vee)} \xrightarrow{\alpha_d} 0.$$

From this we apply Theorem 2.2.8 to see

$$\lambda_{d,d}(R/I) = \dim_k[H_0(\mathbb{L}_\bullet^{\langle n-d \rangle} \otimes_R K)] = \dim \text{Ker}(\alpha_d) - \dim \text{Im}(\psi_d) = \beta_{0,n-d}(I^\vee) - \dim \text{Im}(\psi_d),$$

$$\lambda_{d-1,d}(R/I) = \dim_k[H_1(\mathbb{L}_\bullet^{\langle n-d \rangle} \otimes_R K)] = \dim \text{Ker}(\psi_d) - \dim \text{Im}(\phi_d),$$

$$\lambda_{d-2,d}(R/I) = \dim_k[H_2(\mathbb{L}_\bullet^{\langle n-d \rangle} \otimes_R K)] = \dim \text{Ker}(\phi_d),$$

$$\lambda_{j,d}(R/I) = \dim_k[H_{d-j}(\mathbb{L}_\bullet^{\langle n-d \rangle} \otimes_R K)] = 0 \text{ for all } 0 \leq j \leq d-3.$$

First we recall from Lemma 4.2.11 that $\lambda_{d,d}(R/I) = \#(\pi_0(\text{Clq}_{\geq d-1}(G^C)))$.
Next we show $\lambda_{d-1,d}(R/I) = \dim H_1(\Delta(\text{Clq}_{\geq d-1}(G^C))) - \dim \text{Im}(\phi_d)$.
Since $\lambda_{d,d}(R/I) = \beta_{0,n-d}(I^\vee) - \dim \text{Im}(\psi_d)$ and $\lambda_{d,d}(R/I) = \#(\pi_0(\text{Clq}_{\geq d-1}(G^C)))$, we have that

$$\beta_{0,n-d}(I^\vee) - \dim \text{Im}(\psi_d) = \#(\pi_0(\text{Clq}_{\geq d-1}(G^C))),$$

and so

$$\dim \text{Im}(\psi) = \beta_{0,n-d}(I^\vee) - \#(\pi_0(\text{Clq}_{\geq d-1}(G^C))).$$

We know for a simplicial complex Δ that $\chi(\Delta) = \sum_j (-1)^j \operatorname{rank} H_j(\Delta)$, where
$\chi(\Delta) = \sum_j (-1)^{j+1} c_j$ and c_j represents the number of faces of dimension $j-1$ in Δ
Therefore,

$$\sum_j (-1)^j \dim H_j(\Delta(\text{Clq}_{\geq d-1}(G^C))) = \sum_j (-1)^{j+1} c_j$$

where c_j is the number of cliques of size j in $\Delta(\text{Clq}_{\geq d-1}(G^C))$.

Now since $H_j(\Delta(\mathrm{Clq}_{\geq i}(G^C))) = 0$ for all $j \geq 3$ and for all i, we see that

$$\sum_j (-1)^j \dim H_j(\Delta(\mathrm{Clq}_{\geq d-1}(G^C)) = \#(\pi_0(\Delta(\mathrm{Clq}_{\geq d-1}(G^C))) - \dim H_1(\Delta(\mathrm{Clq}_{\geq d-1}(G^C)),$$

and so

$$\#(\pi_0(\mathrm{Clq}_{\geq d-1}(G^C))) - \dim H_1(\Delta(\mathrm{Clq}_{\geq d-1}(G^C)) = \sum_j (-1)^{j+1} c_j.$$

Therefore,

$$\dim \mathrm{Im}(\psi) = \beta_{0,n-d}(I^\vee) - \dim H_1(\Delta(\mathrm{Clq}_{\geq d-1}(G^C))) - \sum_j (-1)^{j+1} c_j.$$

Recall from Proposition 4.1.1 that $\beta_{0,n-d}(I^\vee) = \#(V(\mathrm{Clq}_{\geq d-1}(G^C)))$, and so

$$\dim \mathrm{Im}(\psi_d) = \#(V(\mathrm{Clq}_{\geq d-1}(G^C))) - \dim H_1(\Delta(\mathrm{Clq}_{\geq d-1}(G^C))) - \sum_j (-1)^{j+1} c_j.$$

By Lemma 4.2.12, we have $\#(V(\mathrm{Clq}_{\geq d-1}(G^C))) - \sum_j (-1)^{j+1} c_j = \beta_{1,n-d+1}(I^\vee)$

We now have that $\dim \mathrm{Im}(\psi_d) = \beta_{1,n-d+1}(I^\vee) - \dim H_1(\Delta(\mathrm{Clq}_{\geq d-1}(G^C)))$. Also, note that $\beta_{1,n-d+1}(I^\vee) = \dim \mathrm{Im}(\psi_d) + \dim \mathrm{Ker}(\psi_d)$. Therefore, $\dim \mathrm{Ker}(\psi_d) = \dim H_1(\Delta(\mathrm{Clq}_{\geq d-1}(G^C)))$. This tells us that $\lambda_{d-1,d}(R/I) = \dim H_1(\Delta(\mathrm{Clq}_{\geq d-1}(G^C))) - \dim \mathrm{Im}(\phi_d)$.

Now recall that I is the Stanley Reisner ideal of $\Delta(G^C)$, and so $I^\vee$ is the Stanley Reisner ideal of $\Delta(G^C)^\vee$. Thus, by Lemma 4.2.13, the map ϕ_d is injective. Therefore, $\dim \mathrm{Im}(\phi_d) = \beta_{2,n-d+2}(I^\vee)$, which gives us that $\lambda_{d-1,d}(R/I) = \dim H_1(\Delta(\mathrm{Clq}_{\geq d-1}(G^C))) - \beta_{2,n-d+2}(I^\vee)$. Also, since ϕ_d is injective, we know that $\mathrm{Ker}(\phi_d) = 0$. Therefore, $\lambda_{d-2,d}(R/I) = 0$, and since $\mathrm{pdim}(I^\vee) \leq 2$, we also have $\lambda_{d-j,d}(R/I)$ for all $j \geq 3$, which is enough to complete the proof. □

From this theorem, we have a couple quick corollaries, the first showing how the statement can be strengthened in the case of $(d-1)-$chordality, and the second showing an Euler characteristic like calculation of the highest Lyubeznik number.

Corollary 4.2.15. *Let $I = I(G)$ be an edge ideal of dimension d in the polynomial ring $R = k[x_1, \ldots, x_n]$ such that $\operatorname{pdim}(I^\vee) \leq 2$. If G is a $(d-1)-$chordal graph, then $\lambda_{d,d}(R/I) = \#\pi_0(\mathrm{Clq}_{d-1}(G^C))$ and $\lambda_{j,d}(R/I) = 0$ for all $j \leq d-1$.*

Proof. By Lemma 4.2.11, we have that $\lambda_{d,d}(R/I) = \#\pi_0(\mathrm{Clq}_{\geq d-1}(G^C))$, $\lambda_{d-1,d}(R/I) = \dim H_1(\mathrm{Clq}_{\geq d-1}(G^C)) - \beta_{2,n-d+2}(I^\vee)$, and $\lambda_{j,d}(R/I) = 0$ for all $j \leq d-2$. However, by Theorem 4.2.8, we see that $\dim H_1(\mathrm{Clq}_{\geq d-1}(G^C)) = \beta_{2,n-d+2}(I^\vee)$. □

Corollary 4.2.16. *Let $I = I(G)$ be an edge ideal of dimension d in the polynomial ring $R = k[x_1, \ldots, x_n]$ such that $\operatorname{pdim}(I^\vee) \leq 2$. If G is a $(d-1)-$chordal graph, then $\lambda_{d,d}(R/I) = \beta_{0,n-d}(I^\vee) - \beta_{1,n-d+1}(I^\vee) + \beta_{2,n-d+2}(I^\vee)$.*

Proof. Consider the $\langle n-d\rangle$-linear strand of $I^\vee$:

$$\mathbb{L}_\bullet^{\langle n-d\rangle}(I^\vee)\colon 0 \xrightarrow{\gamma_d} R^{\beta_{2,n-d+2}(I^\vee)} \xrightarrow{\phi_d} R^{\beta_{1,n-d+1}(I^\vee)} \xrightarrow{\psi_d} R^{\beta_{0,n-d}(I^\vee)} \xrightarrow{\alpha_d} 0.$$

We showed in the proof of Theorem 4.2.14 that $\lambda_{d,d}(R/I) = \beta_{0,n-d}(I^\vee) - \dim \mathrm{Im}(\psi_d)$. We also showed though that $\dim \mathrm{Im}(\psi_d) = \beta_{1,n-d+1}(I^\vee) - \dim H_1(\mathrm{Clq}_{\geq d-1}(G^C))$. Lastly, from Theorem 4.2.8, we see that $\beta_{2,n-d+2}(I^\vee) = \dim H_1(\mathrm{Clq}_{\geq d-1}(G^C))$.

□

To end this section, we pose the following question that would generalize Corollary 4.2.16.

Question 4.2.17. *Let $I = I(G)$ be an edge ideal of dimension d in the polynomial ring $R = k[x_1, \ldots, x_n]$ where G is a $(\ell)-$chordal graph for all ℓ. Is it true that*

$$\lambda_{d,d}(R/I) = \sum_j (-1)^j \beta_{j,n-d+j}(I^\vee)?$$

Also, is this related to the Euler characteristic of the clique graphs, in particular $\mathrm{Clq}_{\geq d-1}(G^C)$*?*

5 Betti and Lyubeznik splittings

Finally, we utilize Betti splittings to extend all computations of Lyubeznik tables considered so far to much wider classes of graphs. Recall that a sum decomposition of an ideal, $I = J + K$, is a **Betti splitting** if for all $i, j \geq 0$, we have

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K).$$

Here, we are interested in Betti splittings of cover ideals, which we define formally below, since we calculate the Lyubeznik numbers of the edge ideal using Betti numbers of the dual of the edge ideal.
As before, throughout, we fix a graph G, $I = I(G)$ its edge ideal which is assumed to be an unmixed ideal of dimension $d \geq 1$ in a polynomial ring R of dimension n over a field k.

Definition 5.0.1. We define the **cover ideal** of G to be the Alexander dual, $I^\vee$, of I.

In particular, the goal is to find a Betti splitting of the cover ideal that arises naturally from the graph so that we can continue to utilize graph combinatorics to calcuate the Lyubeznik table. We will explore such Betti splittings in this chapter.

5.1 (G, ℓ)-splittings

To split the Lyubeznik table, we want to consider the effect on the linear strands of the cover ideal, we introduce the following definition.

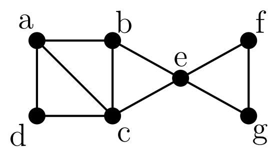
Definition 5.1.1. We say that $I(G)$ has a **Lyubeznik splittings** whenever the cover ideal has a Betti splitting.

Once again, recall that the linear strands of the cover ideal are tied to combinatorial information in the graph complement. We now describe an interesting type of splitting arising from partitions of the vertex sets that will give rise to a Lyubeznik splitting of the edge ideal.

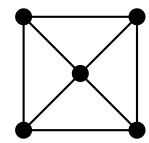
Definition 5.1.2. If there are subsets $S = \{x_1, \ldots, x_m, a_1, \ldots, a_l\}$ and $T = \{y_1, \ldots, y_n, a_1, \ldots, a_l\}$ of the vertices of a graph G with $m, n \geq 1$ such that $G = G|_S \cup G|_T$ where $G|_S \cap G|_T = K_l$, then we call $G|_S \cup G|_T$ a $(\mathbf{G}, \ell) - \textbf{splitting}$.

Example 5.1.3. Consider the simple graphs below:

G_1:



G_2:



Notice there exists a $(G_1, 1)$-splitting, namely $G_1|_S \cup G_1|_T$ where $S = \{a, b, c, d, e\}$ and $T = \{e, f, g\}$. There is also a $(G_1, 2)$-splittings namely $G_1|_S \cup G_1|_T$ where $S = \{a, c, d\}$ and $T = \{a, b, c, e, f, g\}$ and and also $G_1|_P \cup G_1|_Q$ where $P = \{a, b, c, d\}$ and $Q = \{b, c, e, f, g\}$.

However, there does not exist a (G_2, ℓ)-splittings for any ℓ.

Remark 5.1.4. *If $G|_S \cup G|_T$ is a $(G, 0)-$splitting, notice that $G|_S$ and $G|_T$ have to be disjoint. Therefore, a $(G, 0)-$splitting is simply a partition of $\pi_0(G)$, and thus only exists if G is disconnected.*

Recall that the Eagon-Reiner formula calculates Betti numbers using the homologies of links of faces in $\Delta(G^C)$. Using Remark 5.1.4 and the topological fact that the homology of a simplicial complex is isomorphic to the direct sum of the homologies of each connected component, we can begin to see a Betti splitting unfold. We will give the details of this splitting in the following theorem. However, before stating the theorem, we will use a basic example to get a better grasp of the splitting.

Example 5.1.5. Let I by the ideal (ab, bc, cd, da). Then $I^\vee$ is equal to (ac, bd).

The graded Betti numbers of $I^\vee$ are given by $\beta_{1,4}(I^\vee) = 1$ and $\beta_{0,2}(I^\vee) = 2$.

Now consider the graph G for which $I = I(G)$, given in Figure 1.

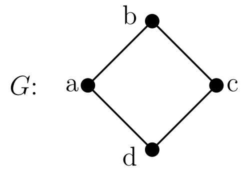


Figure 1

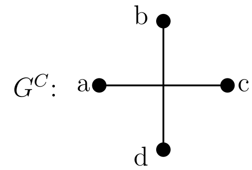


Figure 2

Notice that there is a $(G^C, 0)-$splitting of G^C, seen in Figure 2, namely $G^C|_{\{a,c\}} \cup G^C|_{\{b,d\}}$. Notice $I^\vee = (ac, bd) = (ac) + (bd)$. Consider the graded Betti numbers of ideals $(ac), (bd)$, and $(ac) \cap (bd)$, i.e. $\beta_{0,2}(ac) = 1, \beta_{1,4}(ac) = 0, \beta_{0,2}(bd) = 1, \beta_{1,4}(bd) = 0, \beta_{0,2}((ac) \cap (bd)) = 0$, and $\beta_{1,4}((ac) \cap (bd)) = 1$. Therefore, we see we indeed have a Betti splitting of $I^\vee$, using connected components in G^C.

We observe the behavior seen in this example in more generality in the Theorem 5.1.8, which we will state after giving a couple relevant lemmas.

Lemma 5.1.6. *Let $I = I(G)$ be an edge ideal. Let $G^C|_S \cup G^C|_T$ be a $(G^C, 0)-$splitting. Set $R_S = k[x_1, \ldots, x_m]$ and $R_T = k[y_1, \ldots, y_n]$. Let $K = I(G|_S)$ be an edge ideal in R_S and $J = I(G|_T)$ in R_T. The edge ideal $I = I(G)$ in $R = k[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n]$ is equal to $KR + JR + (x_1, x_2, \ldots, x_m) \cap (y_1, y_2, \ldots, y_m)$.*

Proof. First we note that ab is a generator of I for variables a, b in R if and only if $\{a, b\}$ is a edge in G which is true if and only if $\{a, b\}$ is not an edge in G^C. Since $G^C|_S$ and $G^C|_T$ are disjoint, we see that $\{a, b\}$ is not an edge in G^C if and only if $\{a, b\}$ is not an edge in $G^C|_S$ or $\{a, b\}$ is not an edge in $G^C|_T$ or $\{a, b\} = \{x_i, y_j\}$ for some $x_i \in S$ and $y_j \in T$. One of these three cases can be true if and only if $\{a, b\}$ is an edge in $G|_S$ or $\{a, b\}$ is an edge in $G^C|_T$ or $\{a, b\} \in (x_1, x_2, \ldots, x_m) \cap (y_1, y_2, \ldots, y_n)$, i.e.
$ab \in IR + JR + (x_1, x_2, \ldots, x_m) \cap (y_1, y_2, \ldots, y_n)$. □

Lemma 5.1.7. *Let $I = I(G)$ be an edge ideal. Let $G^C|_S \cup G^C|_T$ be a $(G^C, 0)-$splitting. Set $R_S = k[x_1, \ldots, x_m]$ and $R_T = k[y_1, \ldots, y_n]$. Let $K = I(G|_S)$ be an edge ideal in R_S and $J = I(G|_T)$ in R_T. We have*

$$I^\vee = KR^\vee \cap \left(\prod_{i=1}^{n} y_i\right) + JR^\vee \cap \left(\prod_{i=1}^{m} x_i\right).$$

Proof. The ideal $I^\vee$ is generated by complements of faces of $\Delta(G^C)$. Since $G^C|_S$ and $G^C|_T$ are disjoint, faces of $\Delta(G^C)$ are in $\Delta(G^C|_S)$ or $\Delta(G^C|_T)$. Let F be a face in $\Delta(G^C)$. Without loss of generality, F is a face in $\Delta(G^C_S)$. Therefore, the complement of F in $\Delta(G^C)$ consists of all vertices not in F, which is then all the vertices in $\Delta(G^C|_S)$ not in F, i.e. all the vertices in the complement of F in $\Delta(G^C|_S)$, along with all the vertices in $\Delta(G^C|_T)$. Since $KR^\vee$ is generated by complements of faces of $\Delta(G^C|_S)$ and $JT^\vee$ is generated by complements of faces of $\Delta(G^C|_T)$, we obtain the desired result. □

We are now ready to give a Lyubeznik splitting of an edge ideal. Before we state the theorem though, we first remark that Fransisco, Há, and Van Tuyl found that there is a Betti splitting of a cover ideal, i.e. a Lyubeznik splitting, in the case the G is a Cohen-Macaulay bipartite graph, [FHVT, Thm. 3.8]. Van Tuyl posed the question of finding other ways to naturally split the cover ideal, [Van13, Question 68]. Theorem 5.1.8 provides one such answer to this question. Also, in the case of the splitting Fransisco, Há, and Van Tuyl provided, the Lyubeznik table is trivial. Theorem 5.1.8 and the rest of the chapter will give splittings that split the Lyubeznik table in a non-trivial way.

Theorem 5.1.8. *Let $I = I(G)$ be an edge ideal. If $G^C = G^C|_S \cup G^C|_T$ is a $(G^C, 0)$-splitting, then I has a Lyubeznik splitting.*

Proof. Let $G^C|_S \cup G^C|_T$ be a $(G^C, 0)-$splitting. Set $R_S = k[x_1, \ldots, x_m]$ and $R_T = k[y_1, \ldots, y_n]$. Let $K = I(G|_S)$ be an edge ideal in R_S and $J = I(G|_T)$ in R_T.

Set $K^* = KR^\vee \cap (y_1 \cdots y_n)$ and $J^* = JR^\vee \cap (x_1 \cdots x_m)$. By Lemma 5.1.7, we see that $I^\vee = K^* = J^*$. We will show this is a Betti splitting.

By Eagon-Reiner,

$$\beta_{i,j}(I^\vee) = \sum_{\substack{F \in \Delta(G^C) \\ \#(F) = n+m-j}} \dim_K \tilde{H}_{i-1}(\mathrm{Link}_{\Delta(G^C)} F, K)$$

where K =Frac(R). Now if $\#(F) > 0$, then F is contained either in $G^C|_S$ or F is contained in $G^C|_T$. Therefore, for $j \neq n+m$, we have

$$\begin{aligned}
&\sum_{\substack{F \in \Delta(G^C) \\ \#(F) = n+m-j}} \dim_k \tilde{H}_{i-1}(\mathrm{Link}_{\Delta(G^C)} F, k) \\
&= \sum_{\substack{F \in \Delta(G^C|_S) \\ \#(F) = n+m-j}} \dim_k \tilde{H}_{i-1}(\mathrm{Link}_{\Delta(G^C)} F, k) + \sum_{\substack{F \in \Delta(G^C|_T) \\ \#(F) = n+m-j}} \dim_k \tilde{H}_{i-1}(\mathrm{Link}_{\Delta(G^C)} F, k) \\
&= \beta_{i,j-n}(K^\vee) + \beta_{i,j-m}(J^\vee) \\
&= \beta_{i,j}(K^*) + \beta_{i,j}(J^*).
\end{aligned}$$

If $j = n+m$, then $\#(F) = 0$, and so $\mathrm{Link}(F) = G^C$. So $\beta_{i,j} = \dim_k \tilde{H}_{i-1}(\Delta(G^C))$. We use the fact that the homology of a simplicial complex can be computed as the sum of the homology of each connected component. For all $i \neq 1$,

$$\begin{aligned}
\dim_k \tilde{H}_{i-1}(\Delta(G^C), k) = \quad & \dim_k(\tilde{H}_{i-1}(\Delta(G^C|_S), k) \oplus \tilde{H}_{i-1}(\Delta(G^C|_T), k)) \\
&= \dim_k \tilde{H}_{i-1}(\Delta(G^C|_S), k) + \dim_k \tilde{H}_{i-1}(\Delta(G^C|_T), k) \\
&= \beta_{i,j-n}(K^\vee) + \beta_{i,j-m}(J^\vee) \\
&= \beta_{i,j}(K^*) + \beta_{i,j}(J^*).
\end{aligned}$$

When $i = 1$, we have that

$$\begin{aligned}
\beta_{i,j}(I^{\vee}) &= \dim_k \tilde{H}_0(\Delta(G^C), k) \\
&= \#\pi_0(G^C) - 1 = \#\pi_0(G^C|_S) + \#\pi_0(G^C|_T) - 1 \\
&= (\#\pi_0(G^C|_S) - 1) + (\#\pi_0(G^C|_T) - 1) + 1 \\
&= \beta_{i,j-n}(K^{\vee}) + \beta_{i,j-m}(J^{\vee}) + 1 \\
&= \beta_{i,j}(K^*) + \beta_{i,j}(J^*) + 1.
\end{aligned}$$

Set $L = K^* \cap J^*$. Now notice that $L = (x_1 \cdots x_m \cdot y_1 \cdots y_n)$. So $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ is the only nonface in Δ_L. Therefore, since faces of $\Delta_{L^{\vee}}$ are the complements of nonfaces of Δ_L, we see that $\Delta_{L^{\vee}} = \varnothing$.

Notice that the only face in $\Delta_{L^{\vee}}$ is $F = \varnothing$, and therefore, $\mathrm{Link}_{\Delta_{L^{\vee}}} F = \Delta_{L^{\vee}} = \varnothing$, and $\tilde{H}_{i-1}(\varnothing) = 0$ for all $i \neq 0$ and $\tilde{H}_{0-1}(\varnothing) = 1$.

So, by Eagon-Reiner, we have that $\beta_{0,n+m}(L) = 1$ and $\beta_{i,j}(L) = 0$ for all $i \neq 0, j \neq n+m$.

Putting all this together, we see that

In the case that $j \neq n+m$, for all i and in the case that $j = n+m$, for all $i \neq 1$

$$\begin{aligned}
\beta_{i,j}(I^{\vee}) &= \beta_{i,j}(K^*) + \beta_{i,j}(J^*) \\
&= \beta_{i,j}(K^*) + \beta_{i,j}(J^*) + \beta_{i-1,j}(L)
\end{aligned}$$

In the case that $j = n+m, i = 1$

$$\begin{aligned}
\beta_{i,j}(I^{\vee}) &= \beta_{i,j}(K^*) + \beta_{i,j}(J^*) + 1 \\
&= \beta_{i,j}(K^*) + \beta_{i,j}(J^*) + \beta_{i-1,j}(L)
\end{aligned}$$

Therefore, $\beta_{i,j}(I^\vee) = \beta_{i,j}(K^*) + \beta_{i,j}(J^*) + \beta_{i-1,j}(K^* \cap J^*)$ for all i, j, which is enough to show that $I^\vee = KR^\vee \cap (y_1 \cdots y_n) + JR^\vee \cap (x_1 \cdots x_m)$ is a Betti splitting.

□

Since we showed in Chapter 4 that in the case of clique-chordal graphs, Lyubeznik numbers can be calculated using Betti numbrs, a quick corollary to Theorem 5.1.8 shows how to use a $(G^C, 0)-$ splitting to produce a splitting of the Lyubeznik table. Instead, we will revert to clique graphs to provide a more efficient proof and to work in more generality.

Before stating the next sequence of results, all of which will show a way we can split the Lyubeznik table by splitting the graph complement, we remark that Álvarez-Montaner also gave several ways you can find the Lyubeznik table by breaking it up into smaller pieces, [MS21]. However they are looking at the Lyubeznik table of cover ideals and used splittings based on the Mayer-Vietoris Theorem, [MS21, Thm. 3.3]. In contrast, we work with edge ideals here and use splittings of the graph complement. They also showed how to find the Lyubeznik numbers as a sum of smaller ideals, but these ideals had to be in disjoint variables, [MY16, Thm. 4.5]. Here we allow for ideals which have Alexander duals that are in not necessarily disjoint sets of variables, but specifically for edge ideals.

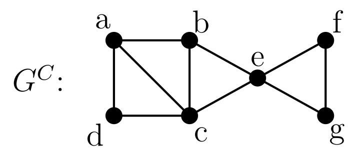
Before stating the result, we will return to Example 5.1.3 to illustrate the splitting. We first introduce the following notation.

Remark 5.1.9. *Throughout the rest of this section, let $C_{i,j}$ denote the matrix with 1 in the $i+1$ row and $j+1$ column and 0 for all other entries. For example, $C_{0,1}$ is the matrix given below:*

$$C_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that we shift the index of $C_{i,j}$ to make it match the index of the Lybeznik number $\lambda_{i,j}(R/I)$.

Example 5.1.10. Recall from Example 5.1.3 the graph G_1 and rename G_1 as G^C for the purpose of this example.



Recall that there exists a $(G^C, 1)$-splitting, namely $G^C|_S \cup G^C|_T$ where $S = \{a, b, c, d, e\}$ and $T = \{e, f, g\}$. Let $K = I(G|_S)$ and $L = I(G|_T)$. We give the Lyubeznik tables of K, L and $I = I(G)$ below.

$$\Lambda(R/K) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Lambda(R/L) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Lambda(R/I) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Notice that $\Lambda(R/I) = \Lambda(R/K) + \Lambda(R/L) + C_{1,2}$. We could have chosen any $(G^C, i)-$splitting for any i and would see a similar splitting of the Lyubeznik table.

However, we will simply show one more graph complement splitting, in particular the $(G^C, 2)$- spllitting given by $G^C|_P \cup G^C|_Q$ where $P = \{a, b, c, d\}$ and $Q = \{b, c, e, f, g\}$. Let $M = I(G|_P)$ and $N = I(G|_Q)$. We give the Lyubeznik tables for M, N and $J = I(G)$.

$$\Lambda(R/K) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Lambda(R/L) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \Lambda(R/I) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Notice that $\Lambda(R/J) = \Lambda(R/M) + \Lambda(R/N) - C_{3,3}$.

We will see both of these cases generalized in the following theorem.

Theorem 5.1.11. *Let k be a field and $I = I(G)$ an unmixed edge ideal of dimension d at least 1 in the polynomial ring $R = k[x_1, \ldots, x_n]$ where G is a clique-cochordal graph with $G^C|_S \cup G^C|_T$ a $(G^C, \ell)-$splitting for $\ell \geq 0$. Set $K = I(G|_S)$ and $L = I(G|_T)$.*

If $\ell = d - 1$, then

$$\Lambda(R/I) = \Lambda(R/KR) + \Lambda(R/LR) - C_{d,d}.$$

If $\ell < d - 1$, then

$$\Lambda(R/I) = \Lambda(R/KR) + \Lambda(R/LR) + C_{\ell,\ell+1}.$$

Proof. We know $\Lambda(R/I), \Lambda(R/KR)$, and $\Lambda(R/LR)$ from Theorem 4.1.3. We will let $J_I = \#\pi_0(\mathrm{Clq}_{d-1}(G^C))$, $J_K = \#\pi_0(\mathrm{Clq}_{d-1}(G^C|_S))$, and $J_L = \#\pi_0(\mathrm{Clq}_{d-1}(G^C|_T))$. Similarly, we set $C_{i,I} = \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C)) - \#\pi_0(\mathrm{Clq}_{\geq i}(G^C))$,

$C_{i,K} = \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C|_S)) - \#\pi_0(\mathrm{Clq}_{\geq i}(G^C|_S))$, and

$C_{i,L} = \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C|_T)) - \#\pi_0(\mathrm{Clq}_{\geq i}(G^C|_T))$.

We next work by cases based on if $\ell = d - 1$ or not.

Suppose $\ell = d - 1$. Since $G^C|_S \cup G^C|_T$ is a (G^C, ℓ)-splitting, then $G^C|_S \cap G^C|_T = K_\ell$. Therefore, there is a maximal clique in each of $G^C|_S$ and $G^C|_T$, call them A and B

respectively, containing $G^C|_S \cap G^C|_T$. So $\#(A \cap B) \geq l = d-1$, but if $\#(A \cap B) = d$, then $A = B$, which is a contradiction, and so $\#(A \cap B) = d - 1$. Therefore, there is an edge $\{A, B\}$ in $\mathrm{Clq}_{\geq d-1}(G^C)$ and so the connected component in $G^C|_S$ containing $\{A, B\}$ is connected to the connected componenet in $G^C|_T$ containing $\{A, B\}$. From this we see that $J_I = J_K + J_L - 1$.

Furthermore, since $\{A, B\}$ is also an edge in $\mathrm{Clq}_{\geq i}(G^C)$ for all $i < d-1$, we also have

$$\begin{aligned} C_i &= \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C)) - \#\pi_0(\mathrm{Clq}_{\geq i}(G^C)) \\ &= \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C|S)) + \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C|T)) - 1 \\ &\quad - (\#\pi_0(\mathrm{Clq}_{\geq i}(G^C|_S)) + \#\pi_0(\mathrm{Clq}_{\geq i}(G^C|_T)) - 1) \\ &= C_{i,K} + C_{i,L} \text{ for all } i < d-1. \end{aligned}$$

Now suppose $l < d-1$. It follows that $\mathrm{Clq}_{\geq i}(G^C) = \mathrm{Clq}_{\geq i}(G^C|_S) \sqcup \mathrm{Clq}_{\geq i}(G^C|_T)$ and so $J_I = J_K + J_L$ and $C_{i,I} = C_{i,K} + C_{i,L}$ for all $i > l$.

Also, since $\#(G^C|_S \cap G^C|_T) = l$, we use a similar argument as above to see that $\#\pi_0(\mathrm{Clq}_{\geq i}(G^C)) = \#\pi_0(\mathrm{Clq}_{\geq i}(G^C|_S)) + \#\pi_0(\mathrm{Clq}_{\geq i}(G^C|_T)) - 1$ for all $i \leq l$.

Therefore,

$$\begin{aligned} C_l &= \#(\pi_0(\mathrm{Clq}_{\geq l+1}(G^C))) - \#(\pi_0(\mathrm{Clq}_{\geq l}(G^C))) \\ &= \#\pi_0(\mathrm{Clq}_{\geq l+1}(G^C|_S)) + \#\pi_0(\mathrm{Clq}_{\geq l+1}(G^C|_T)) \\ &\quad - (\#\pi_0(\mathrm{Clq}_{\geq l}(G^C|_S)) + \#\pi_0(\mathrm{Clq}_{\geq l}(G^C|_T)) - 1) \\ &= C_{i,K} + C_{i,L} + 1, \end{aligned}$$

and

$$\begin{aligned} C_i &= \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C)) - \#\pi_0(\mathrm{Clq}_{\geq i}(G^C)) \\ &= \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C|_S)) + \#\pi_0(\mathrm{Clq}_{\geq i+1}(G^C|_T)) - 1 \\ &\qquad - (\#\pi_0(\mathrm{Clq}_{\geq i}(G^C|_S)) + \#\pi_0(\mathrm{Clq}_{\geq i}(G^C|_T)) - 1) \\ &= C_{i,K} + C_{i,L} \end{aligned}$$

for all $i < l$.

□

A consequence of Theorem 5.1.11 is that we can compute the Lyubeznik table of an edge ideal by breaking the graph complement up into two disjoint components. This occurs when considering a $(G^C, 0)-$splitting. In the following corollary, we generalize further by showing we can compute the Lyubeznik table by breaking the graph complement up into all its connected components. The only information lost in the process of adding these smaller Lyubeznik tables is the number of connected components, which we observed in Theorem 4.0.3 to be one more than $\lambda_{0,1}(R/I)$. To remedy this, we simply add $(m-1)C_{0,1}$ to the sum, where m is the number of connected components in G^C.

Corollary 5.1.12. *Let G be an unmixed, clique-cochordal graph, and $I = I(G)$ be an edge ideal. If $\{C_1, \ldots, C_m\} = \pi_0(G^C)$ and $I_i = I(G|_{\mathcal{C}_i})$ for all i, then*

$$\Lambda(R/I) = (m-1)\mathbf{C}_{0,1} + \sum_k \Lambda(R/I_k R)$$

Proof. We proceed by induction on m. The result is trivial if $m = 1$.

If $m = 2$, then $G^C|_{C_1} \cup G^C|_{C_2}$ is a $(G^C, 0)-$splitting. Therefore, by Theorem 5.1.11,

$$\Lambda(R/I) = \Lambda(R/I_1 R) + \Lambda(R/I_2 R) + C_{0,1}$$

Now if $\#(\pi_0(G^C)) = m$, assume that

$$\Lambda(R/I) = (m-1)C_{0,1} + \sum_k \Lambda(R/I_k R).$$

Consider the case where $\#(\pi_0(G^C)) = m+1$. Notice that $G^C|_{C_{m+1}} \cup G^C|_{\bigcup_{k=1}^{m} C_k}$ is a $(G^C, 0)-$splitting. Therefore, by Theorem 5.1.11,

$$\Lambda(R/IR) = \Lambda(R/I_{m+1}R) + \Lambda(R/KR) + C_{0,1}$$

where $K = I(G|_{\bigcup_{k=1}^{m} C_k})$. Notice that $\#(\pi_0(G^C|_{\bigcup_{k=1}^{m} C_k})) = m$. Therefore, by induction hypothesis,

$$\Lambda(R/K) = (m-1)C_{0,1} + \sum_{k=1}^{m} \Lambda(R/I_k R).$$

Putting these together, we see that

$$\Lambda(R/I) = (m)C_{0,1} + \sum_{k=1}^{m+1} \Lambda(R/I_k R).$$

□

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