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Asymptotic Properties and Separation Rates for Navier-Stokes Flows

Patrick Michael Phelps
University of Arkansas-Fayetteville

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Patrick Michael Phelps
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Zachary Bradshaw, Ph.D.
Dissertation Director

Ariel Barton, Ph.D.
Committee Member

Andrew Seth Raich, Ph.D.
Committee Member

ABSTRACT

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In this dissertation, we investigate asymptotic properties of local energy solutions to the Navier-Stokes equations and develop an application which controls the separation of non-unique solutions in this class. Specifically, we quantify the rate at which two, possibly unique solutions evolving from the same data may separate pointwise away from a singularity. This is motivated by recent results on non-uniqueness for forced and unforced Navier-Stokes and analytical and numerical evidence suggesting non-uniqueness in the Leray class. Our investigation begins with discretely self-similar solutions known to exist globally in time and to be regular outside a space-time paraboloid. We prove decay rates for these solutions with locally sub-critical data away from the origin and show improved decay for the 'non-linear part' of the flow. We also lower the Hölder regularity required to obtain our maximal decay rate. To achieve improved decay, we use Picard iterates to approximate solutions. We demonstrate a scale of decay rates for Picard approximations which determine upper bounds for how non-unique, discretely self-similar solutions may separate. In subsequent sections, we replace the self-similar condition with local sub-critical regularity and are able to obtain all but the maximal separation rate for Lorentz solutions, a subclass of local energy solutions.

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"You can know anything. It's all there. You just have to find it."

- Neil Gaiman, Sandman.

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* Papers Submitted and Under Preparation

• Bradshaw, Z. and Phelps, P. "Spatial decay of discretely self-similar solutions to the Navier-Stokes equations", *Accepted to Pure and Applied Analysis*, arXiv:2202.08352. – Chapters 4 and 5

• Bradshaw, Z. and Phelps, P. "Estimation of non-uniqueness and short-time asymptotic expansions for Navier-Stokes flows", *Accepted to Annales de l'Institut Henri Poincaré Analyse Non Linéaire*, arXiv:2206.03534. – Chapters 4 and 5

1 Introduction

1.1 Background

The incompressible Navier-Stokes equations are a system of partial differential equations which model the velocity, u, and pressure, p, of a viscous incompressible fluid with viscosity ν inside a domain Ω, under a body force f. In particular, $u : \Omega \times (0,T) \to \Omega$ and $p : \Omega \times (0,T) \to \mathbb{R}$ are required to satisfy

$$\begin{aligned} \partial_t u - \nu\Delta u + u \cdot \nabla u + \nabla p = f; \\ \operatorname{div} u = 0; \qquad u(\cdot, 0) = u_0. \end{aligned} \tag{1.1}$$

Intuitively, the equation says that the acceleration ($\partial_t u$) of the fluid is influenced by diffusion ($-\Delta u$), inertia $(u \cdot \nabla u)_j := \sum_i u_i \partial_i u_j$, pressure gradient ($\nabla p$), and the body force. For our investigations, we take $\nu = 1$, $f = 0$, and $\Omega = \mathbb{R}^3$. This system has applications in many fields including but not limited to aerodynamics, meteorology, oceanography, and cosmology. Open problems including well-posedness and global regularity for large data still remain.

Well-posedness consists of the existence, uniqueness, and stability of solutions. Existence and regularity of solutions constitute one of the Millennium Problems offered by the Clay Institute of Mathematics, a set of longstanding open problems; more specifically, the problem of whether singularities can form in finite time from smooth initial data remains largely open.

By definition, a solution to this system is unique in a class if no other solution in the same class with the same initial data u_0 exists. It is not known whether solutions to (1.1) with rough data are unique; in fact, non-uniqueness has been affirmed in some settings for the non-forced [18] and forced Navier-Stokes equations [2]. Within the Leray class [54], where solutions satisfy a global energy inequality, the numerical work of Guillod and Šverák

[35] simulates non-uniqueness. They simulate a pair of solutions from the same initial data and zero forcing, where one is axi-symmetric and the other breaks this symmetry. This program first proposes a scenario of non-uniqueness in a class of solutions with large data in the Lorentz space $L^{3,\infty}$ which can be truncated to give non-unique Leray-Hopf solutions. This space is critical, as it is on the borderline of well-posedness theory. To investigate this symmetry breaking, we work in a class of scaling invariant solutions.

1.2 Self-similar flows

In the analysis of partial differential equations it is useful to consider the scaling properties an equations to investigate its solutions. This has been used, for example, to show the blow-up of the semi-linear heat equation. The Navier-Stokes enjoy a scaling invariance: For (u, p), a solution to (1.1), we may define another solution (u^λ, p^λ) by

$$u^\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \qquad p^\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t),$$

with data $u_0^\lambda(x) = \lambda u_0(\lambda x)$. A space whose norm is invariant under this scaling, i.e. X such that $\|u\|_X = \lambda^p \|u_\lambda\|_X$, for $p = 0$, is called critical. Some examples include $L^s(0,T; L^q(\Omega))$ for $\frac{3}{q} + \frac{2}{s} = 1$, e.g. $L^\infty L^3$. A *sub*critical space has this property for $p < 0$ and a *super*critical space for $p > 0$.

These types of spaces are important, as the critical spaces lie on the boundary of known well-posedness, i.e. there are small data results, whereas subcritical spaces are generally well behaved, e.g. large data results, and supercritical are poorly behaved.

Definition 1.1. A solution is called self-similar if $(u, p) = (u^\lambda, p^\lambda)$ for all $\lambda > 0$ and discretely self-similar (DSS) if this is true for some $\lambda > 0$. The data u_0 is self-similar or discretely self-similar respectively if the above property holds with the time variable omitted.

We consider scaling invariant solutions to (1.1), as their symmetry makes them good candidates to exhibit non-unique solutions [35]. Motivated by this numerical work, we quantify how non-uniqueness would evolve from locally sub-critical data: first, for a class of local energy solutions with scaling invariance, and second, in the class of $L^{3,\infty}$ weak solutions with no scaling assumption.

Self-similar solutions with large C^α data were first constructed by Jia and Šverák in [40]. Tsai [67] then constructed DSS solutions with C^α datafor λ close to 1. In [10], Bradshaw and Tsai construct solutions with data in $L^{3,\infty}$. This space contains functions with multiple isolated singularities. The literature on the existence of DSS or self-similar solutions to (1.1) in $\mathbb{R}^4_+$ in a variety of function spaces is rich, including [1, 4, 10, 11, 12, 13, 21, 22, 25, 33, 40, 42, 47, 53, 67].

1.3 Asymptotics of self-similar flows

Our investigation begins with the asymptotic properties of DSS local energy solutions to the Navier-Stokes equations considered on $\mathbb{R}^3 \times (0,\infty)$. Brandolese pioneered this subject for small, smooth data in [16]. There, an asymptotic formula is given for the time-independent profile of a self-similar solution in which the dominant terms only involve the data. The remaining terms have faster decay, the worst of which is $\mathcal{O}(|x|^{-4})$. This implies spatial asymptotics for the self-similar solution for all $t > 0$.

In [67], in the rougher class of λ-DSS local energy solutions with data in $C^\alpha_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ for $0 < \alpha < 1$ and $\lambda \sim 1$ Tsai the following asymptotics.

$$|u|(x,t) \lesssim \frac{1}{|x| + \sqrt{t}}. \tag{1.2}$$

We extend this to a scale of decay rates for the wider class of DSS solutions with data in $L^q_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$, $q > 3$. Additionally, for $\lambda \sim 1$, u is globally regular, by [67]; however, we work with solutions with any λ scaling, which are only known to be regular in the region $|x| \geq R_0\sqrt{t}$, where R_0 is the radius of far-field regularity from [43, Theorem 1.8]. We work in the class of local energy solutions which can be thought of a a localized version of Leray solutions.

Theorem 1.1 (Algebraic decay for rough data)**.** *Let $q \in (3,\infty]$ and $u_0 \in L^q_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ be divergence free and DSS. Assume u is a DSS local energy solution with initial data u_0. For any $l \in \mathbb{N}_0$ and $|x| \geq R_0\sqrt{t}$,*

$$|\nabla^l u|(x,t) \lesssim_{u_0,q,\lambda} \frac{1}{\sqrt{t}^{|l|+\frac{3}{q}} \left(|x| + \sqrt{t}\right)^{1-\frac{3}{q}}}, \tag{1.3}$$

where the dependence on u_0 is via the quantities $\|u_0\|_{L^2_{\mathrm{uloc}}}$ and $\|u_0\|_{L^q_{\mathrm{loc}}(\mathbb{R}^3\setminus\{0\})}$.

We then pursue improved decay for the 'non-linear' part of these flows. Let u be the solution to (1.1), and $P_0 = e^{t\Delta}u_0$ be the solution to the homogeneous heat equation with the data u_0. The difference $u - P_0$ has the following improved decay rates compared to u shown by Tsai [67] for $\lambda \sim 1$, and by Lai, Miao, and Zheng [55, 56] for self-similar solutions, respectively,

$$|u - P_0|(x,t) \lesssim \begin{cases} \frac{\sqrt{t}}{(|x|+\sqrt{t})^2} & u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}) \text{ \& self-similar} \quad \text{[55, Theorem 1.1]} \\ \frac{t}{(|x|+\sqrt{t})^2} & u_0 \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}) \text{ \& } \lambda \sim 1 \quad \text{[67, Theorem 1.1]} \\ \frac{t\log\left(2+\frac{|x|}{\sqrt{t}}\right)}{(|x|+\sqrt{t})^3} & u_0 \in C^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}) \text{ \& self-similar} \quad \text{[56, Corollary 1.1]} \\ \frac{t}{(|x|+\sqrt{t})^3} & u_0 \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}) \text{ \& self-similar} \quad \text{[56, Corollary 1.1].} \end{cases}$$

One goal of this research program is to generalize and improve the decay rates in [55], [56] and [67]. This is done by establishing pointwise bounds for local energy solutions with large, rough initial data in the DSS class for any scaling factor $\lambda > 1$, leading to the following theorem.

Theorem 1.2 (Improved algebraic decay for rough data)**.** *Let $q \in (3,\infty]$ and $u_0 \in L^q_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ be divergence free and DSS. Assume u is a DSS local energy solution* *with initial data u_0. For $|x| \geq R_0\sqrt{t}$, the difference $u - P_0$ satisfies*

$$|u - P_0|(x,t) \lesssim_{u_0,q,\lambda} \frac{1}{\sqrt{t}^{\frac{6}{q}-1}(|x| + \sqrt{t})^{2-\frac{6}{q}}}, \tag{1.4}$$

where the dependence on u_0 is via the quantities $\|u_0\|_{L^2_{\mathrm{uloc}}}$ and $\|u_0\|_{L^q_{\mathrm{loc}}(\mathbb{R}^3\setminus\{0\})}$.

Next, we reduce the exponent of Hölder regularity required to obtain the decay in [56] and establish a finer bound on the 'non-linear' part of the flow for Hölder regular data compared to [67].

Theorem 1.3. *Let $0 < \alpha \leq 1$ and assume $u_0 \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}) \cap DSS$ is divergence free . Assume u is a DSS local energy solution with initial data u_0. Then*

$$|u - P_0|(x,t) \lesssim_{u_0,\alpha,\lambda} \begin{cases} \frac{\sqrt{t}^{1+\alpha}}{(|x|+\sqrt{t})^{2+\alpha}} & \alpha < 1 \\ \frac{t}{(|x|+\sqrt{t})^3}\log(2 + \frac{|x|}{\sqrt{t}}) & \alpha = 1 \end{cases}, \tag{1.5}$$

for $|x| \geq R_0\sqrt{t}$, where the dependence on u_0 is via the quantities $\|u_0\|_{L^2_{\mathrm{uloc}}}$ and $\|u_0\|_{C^\alpha(A_0)}$.

To avoid the logarithm in Theorem 1.3, we assume slightly more Hölder regularity and obtain the following theorem.

Theorem 1.4. *Let $0 < \alpha \leq 1$ and assume $u_0 \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}) \cap DSS$ is divergence free. Assume u is a DSS local energy solution with initial data u_0. Then for $|x| \geq R_0\sqrt{t}$,*

$$|u - P_0|(x,t) \lesssim_{u_0,\alpha,\lambda} \frac{t}{(|x| + \sqrt{t})^3}, \tag{1.6}$$

where the dependence on u_0 is via the quantities $\|u_0\|_{L^2_{\mathrm{uloc}}}$ and $\|u_0\|_{C^{1,\alpha}(A_0)}$.

We summarize the above results as the following scale of decay rates: for $|x| \geq R_0\sqrt{t}$,

$$|u - P_0|(x,t) \lesssim \begin{cases} \frac{\sqrt{t}^{1-\frac{6}{q}}}{(|x|+\sqrt{t})^{2-\frac{6}{q}}} & u_0 \in L^q_{\text{loc}}(\mathbb{R}^3 \setminus \{0\}),\ q > 3 \\ \frac{\sqrt{t}^{1+\alpha}}{(|x|+\sqrt{t})^{2+\alpha}} & u_0 \in C^\alpha_{\text{loc}}(\mathbb{R}^3 \setminus \{0\}),\ \alpha \in (0,1) \\ \frac{t\log\left(2+\frac{|x|}{\sqrt{t}}\right)}{(|x|+\sqrt{t})^3} & u_0 \in C^1_{\text{loc}}(\mathbb{R}^3 \setminus \{0\}), \\ \frac{t}{(|x|+\sqrt{t})^3} & u_0 \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\}),\ \alpha \in (0,1) \end{cases}.$$

Compared to previous literature, we work in a more general case of DSS solutions with no assumption of global regularity. These solutions are only known to be regular in the sub-paraboloid region $|x| \geq R_0\sqrt{t}$. Our L^q result is identical to [55] when $q = \infty$, but for general DSS solutions and extends down in q, almost to the critical class L^3.

To understand the limitation on q note that in [10] it is shown that $u_0 \in L^{3,\infty} \cap DSS$ if and only if $u_0 \in L^3_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$. Our decay estimates create a scale of spaces that approach, but do not reach, the critical initial data space $L^{3,\infty}$ on one end. On the other end, when $q = \infty$, L^∞_{loc} is weaker than the Hölder-type spaces considered in [40, 67]. We expect the $q = 3$ case to be excluded because, as shown in an example in [10], there is no algebraic decay rate for $e^{t\Delta}u_0$ when $u_0 \in L^{3,\infty} \cap DSS$. We do, however, pursue *approximate* decay rates for $q = 3$ and in $\dot{B}^{-1+3/p}_{p,\infty}$ in an upcoming paper with Z. Bradshaw.

The L^q_{loc} result requires fine integral estimates on a bilinear operator that defines $u - P_0$. We define mild solutions in Definition 2.10 in Section 2. For now, note that a mild solution u satisfies

$$u(t) = e^{t\Delta}u_0 - B(u,u)(t), \tag{1.7}$$

where

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\partial_j(u_jv)(s)\,ds. \tag{1.8}$$

Because local energy solutions are mild we use integral estimates to prove that $B(u,u) = -(u - P_0)$ enjoys a squared decay compared to u.

For data with Hölder regularity, we create a new scale of decay rates, depending on α. To achieve this, note that $u - P_0$ satisfies the following expansion

$$u - P_0 = -B\big(u - P_0, u - P_0\big) - B\big(u - P_0, P_0\big) + B\big(P_0, u - P_0\big) - B(P_0, P_0).$$

The first three terms have cubic decay by the result for $u_0 \in L^\infty_{\text{loc}}$. The term $B(P_0, P_0)$ fully determines our estimates for $u - P_0$. This can be written explicitly as

$$B(P_0, P_0) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (e^{t\Delta}u_0 \otimes e^{t\Delta}u_0)\, ds.$$

Because u_0 is in C^α_{loc}, $e^{t\Delta}u_0$ decays like $|x|^{-1}$ and $\nabla e^{t\Delta}u_0$ only decays like $|x|^{-1-\alpha}$. Therefore, we find that $\nabla \cdot (e^{t\Delta}u_0 \otimes e^{t\Delta}u_0)$ decays like $|x|^{-2-\alpha}$. Attempting to integrate the cubic kernel of the Oseen tensor introduces a logarithm. To avoid this, we define the fractional Laplacian $\Lambda = (-\Delta)^{\frac{1}{2}}$ to take the finer estimate $|\Lambda^\gamma e^{t\Delta}u_0| \lesssim |x|^{-1-\alpha}$ for $\alpha < \gamma < 1$. After navigating a commutator, this implies

$$|\Lambda^\gamma(e^{t\Delta}u_0 \otimes e^{t\Delta}u_0)| \lesssim |x|^{-2-\alpha}, \quad t \in [1, \lambda^2].$$

Re-writing $B(P_0, P_0)$ as

$$B(P_0, P_0) = \int_0^t \Lambda^{-\gamma}\nabla\mathbb{P}e^{(t-s)\Delta}\Lambda^\gamma(e^{t\Delta}u_0 \otimes e^{t\Delta}u_0)\, ds,$$

depletes the singularity of the Oseen kernel and maintains decay of $|x|^{-2-\alpha}$ on the product part. By an integral estimate (4.11) from [67], this has the advertised decay.

With any regularity higher than C^1, i.e. $C^{1,\alpha}$, $0 < \alpha < 1$, we avoid the logarithm in (4.11), improving the results in [67] and in [56]. We do this by taking advantage of a technique in [56] that uses mean-value theorem to introduce another gradient in the near field of the $B(P_0, P_0)$ integral. The same fractional Laplacian methods help us finish the proof without a $C^{1,1}$ assumption.

1.3.1 Picard improvements

Inspired by estimates for the 'non-linear' difference, $u - P_0$, in the literature and by work done by Albritton and Barker in [1], we investigate the decay for the difference between u and the k^{th} Picard iterate P_k which solves the iterated heat equation

$$\partial_t P_k - \Delta P_k = -\mathbb{P}\nabla \cdot (P_{k-1} \otimes P_{k-1}),$$

with the same initial data u_0. Due to Duhamel's principle and the solvability of P_0, each iterate is exactly solvable. Classically, Picard iterates converge to solutions of (1.1) whenever the system can be considered a perturbation of the heat equation, i.e. when the non-linear term $u \cdot \nabla u$ can be made small in an appropriate sense. This is not the case in the local energy class, but we do demonstrate that Picard iterates capture some asymptotic properties of DSS solutions near $t = 0$ and away from $x = 0$.

Theorem 1.5 (Improved decay using Picard iterates)**.** *Let $q \in (3, \infty]$ and $u_0 \in L^q_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ be divergence free and DSS. Assume u is a DSS local energy solution with initial data u_0. Define for $k \in \mathbb{N}_0$,*

$$a_k = (k+2)\left(1 - \frac{3}{q}\right) = a_{k-1} + 1 - \frac{3}{q}; \quad k_q = \left\lceil \frac{4q}{q-3} - 2 \right\rceil.$$

The following hold for $|x| \geq R_0\sqrt{t}$

1. *For $k < k_q$,*

$$|u - P_k|(x,t) \lesssim_{k,\lambda,R_0,u_0} \frac{\sqrt{t}^{a_k}}{\sqrt{t}(|x| + \sqrt{t})^{a_k}}, \tag{1.9}$$

where the dependence on u_0 is via the quantities $\|u_0\|_{L^2_{\mathrm{uloc}}}$ and $\|u_0\|_{L^q_{\mathrm{loc}}(\mathbb{R}^3\setminus\{0\})}$.

2. *For $k \geq k_q$,*

$$|u - P_k|(x,t) \lesssim_{k,\lambda,R_0,u_0} \frac{\sqrt{t}^3}{(|x| + \sqrt{t})^4}. \tag{1.10}$$

The essence of proof is to use the following bi-integral formula:

$$u = \underbrace{P_0 - B(P_0, P_0)}_{P_1}) + B(P_0, u - P_0) + B(u - P_0, P_0) - B(u - P_0, u - P_0). \tag{1.11}$$

For $u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}) \cap DSS$, we show P_0 is $\mathcal{O}(|x|^{-1})$ and $u - P_0$ is $\mathcal{O}(|x|^{-2})$. Then for $t \in [1, \lambda^2]$, as $|x| \to \infty$, we should have

$$B(P_0, u - P_0) + B(u - P_0, P_0) = \mathcal{O}(|x|^{-3}) \text{ and } B(u - P_0, u - P_0) = \mathcal{O}(|x|^{-4}).$$

This demonstrates an improvement, as $u - P_1$ heuristically decays faster than $u - P_0$. This improvement continues for higher iterates. This argument is an example of the "improvement property" of Picard iterates evident in the literature [30, 16, 1].

Furthermore, in Theorem 1.5, item 2 can be viewed as generalizing the small data result of Brandolese [16, Theorem 2] in the sense that the solutions satisfy the asymptotic formula $u(x,t) = F(u_0) + \mathcal{O}(|x|^{-4})$, for $t \in [1, \lambda^2]$, where $F(u_0)$ can be explicitly written as a short time asymptotic expansion depending only on u_0.

Short-time asymptotic expansions have been examined in [16, 17] for both large and small non-self-similar flows and for forced Navier-Stokes in [3]. In [49], Kukavica and Ries give an expansion for smooth solutions. A novelty in our result is, unlike all previous papers, we do not work with data strong enough to generate smooth solutions or assume solutions to be smooth and the terms in our expansion, save one, depend only on u_0, not u.

1.3.2 Estimating non-uniqueness

We can use the uniqueness of Picard iterates to quantify how non-uniqueness of DSS local energy solutions might unfold.

Theorem 1.6 (Separation rate)**.** *Let $q \in (3, \infty]$ and $u_0 \in L^q_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ be divergence free and DSS. Assume u is a DSS local energy solution with initial data u_0. Assume v is another DSS local energy solution with data u_0. Then for $|x| \geq R_0\sqrt{t}$,*

$$|u - v|(x,t) \lesssim_{q,\lambda,R_0,u_0} \frac{\sqrt{t}^3}{(|x| + \sqrt{t})^4}. \tag{1.12}$$

Proof of Theorem 1.6. If two solutions u and v satisfy the hypothesis of Theorem 1.5, then there exists a k_q such that, for $|x| \geq R_0\sqrt{t}$,

$$|u - v|(x,t) = |u - P_{k_q}(u_0)|(x,t) + |v - P_{k_q}(u_0)|(x,t) \lesssim_{q,\lambda,R_0,u_0} \frac{\sqrt{t}^3}{(|x| + \sqrt{t})^4}.$$

□

This implies the following bound:

$$|u - v|(x,t) \lesssim_x t^{3/2}.$$

We call the above an "estimation of non-uniqueness" and the right-hand side as the "separation rate".

While the work of Jia and Šverák [38] and the numerical evidence of Guillod and Šverák [35] suggest non-uniqueness within the class of self-similar solutions, not much has been done to quantify how this uniqueness might evolve over time. This separation rate is an interesting perspective on how non-uniqueness for Navier-Stokes might unfold; it implies solutions locally stay very close at short times, compared to rates t^γ, $\gamma < 1$ that are implied by current literature.

We would like to know what other classes this can be extended to, when the separation rate can be recovered, and if we may relax the DSS assumption. To this end, we investigate $L^{3,\infty}$ weak solutions away from a singularity.

1.4 Locally sub-critical flows

For the subsequent results, we impose no scaling assumptions and work with $L^{3,\infty}$ weak solutions, a subclass of local energy solutions, with data in the space $L^{3,\infty} \cap L^p(B)$ for some ball B and $p > 3$.

Theorem 1.7 (Local asymptotic expansion)**.** *Assume $u_0 \in L^{3,\infty}$ and is divergence free. Fix $x_0 \in \mathbb{R}^3$ and $p \in (3, \infty]$. Assume further $u_0|_B \in L^p(B)$ where $B = B_2(x_0)$. Then there exists $\gamma = \gamma(p) \in (0,1)$ and $T = T(p, \|u_0\|_{L^{3,\infty}}, \|u_0\|_{L^p(B)}) > 0$ such that for any $\sigma \in (0, 3/2)$, $t \in (0,T)$ and $k = 0, 1, \ldots, k_0$,*

$$\|u - P_k\|_{L^\infty(B_{1/4}(x_0))}(t) \lesssim_{p,u_0,\sigma,k} t^{a_k},$$

where $a_0 = \min\{\gamma/2, 1/2 - 3/(2p)\}$, $a_{k+1} = \min\{\sigma, k(1/2 - 3/(2p)) + a_0\}$ and k_0 is the smallest natural number such that

$$k_0\left(\frac{1}{2} - \frac{3}{2p}\right) + a_0 \geq \sigma.$$

In particular, $a_{k_0} = \sigma$ and $a_k > a_{k-1}$ for $k = 1, \ldots, k_0$. For (x,t) in $B_{\frac{1}{4}}(x_0) \times (0,T)$ and letting $a_{-1} = -3/(2p)$, it follows that

$$u(x,t) = P_0 + \sum_{k=0}^{k_0-1} \mathcal{O}(t^{a_k}) + \mathcal{O}(t^\sigma) = \sum_{k=-1}^{k_0} \mathcal{O}(t^{a_k}),$$

where the $\mathcal{O}(t^{a_k})$ terms are exactly solvable for $-1 \leq k < k_0$.

The rate $t^{3/2}$ achieved for DSS solutions is not completely recovered due to limitations of the exponents in the Gagliardo-Nirenberg-Sobolev inequality. For this reason, we expect this is optimal for $L^{3,\infty}$ weak solutions.

The short-time, space-local asymptotic expansion is of independent interest. Each term except for the highest order is *exactly* solvable because the expansion arises from Picard iterates. This expansion demonstrates the ability of the Picard iterates to locally capture the short-time asymptotic properties of solutions.

The following estimation of non-uniqueness is a corollary to the preceding theorem.

Corollary 1.8 (Estimation of non-uniqueness)**.** *Assume $u_0 \in L^{3,\infty}$ and is divergence free. Fix $x_0 \in \mathbb{R}^3$. Assume $u_0|_B \in L^p(B)$ where $B = B_2(x_0)$ and $p \in (3, \infty]$. Let u and v be $L^{3,\infty}$ weak solutions with data u_0. Then there exists $T = T(p, u_0) > 0$ such that for every $\sigma \in (0, 3/2)$ and $t \in (0, T)$,*

$$\|u - v\|_{L^\infty(B_{1/4}(x_0))}(t) \lesssim_{p,\sigma,u_0} t^\sigma,$$

where the dependence on u_0 is via the quantities $\|u_0\|_{L^p(B)}$ and $\|u_0\|_{L^{3,\infty}}$.

If $u_0 \in L^2 \cap L^{3,\infty}$, then any $L^{3,\infty}$ weak solution is also a Leray weak solution [7]; therefore, this result estimates non-uniqueness in a subset of the Leray class.

Proof. Note for each u_0 the Picard iterates $P_k(u_0)$ are unique. Therefore, by Theorem 1.7 and the triangle inequality,

$$\|u - v\|_{L^\infty(B_{1/4}(x_0))} \lesssim \|u - P_k\|_{L^\infty(B_{1/4}(x_0))} + \|P_k - v\|_{L^\infty(B_{1/4}(x_0))} \lesssim_{p,u_0,\sigma,k} t^\sigma,$$

for any chosen $0 < \sigma < \frac{3}{2}$. $\square$

2 Solution classes

In this section, we define the classes of the solutions referenced above and outline some of their existence, uniqueness and regularity properties.

2.1 Function Spaces

To begin we define important function spaces that appear in related literature and the results to follow.

The L^p and L^p_{loc} spaces are defined in the classical way. We also define uniform versions.

Definition 2.1 (Uniformly local Lebesgue space)**.** The space L^p_{uloc} is defined by finiteness of the norm

$$\|f\|_{L^p_{\text{uloc}}} := \sup_{x_0 \in \mathbb{R}^3} \|f\|_{L^p(B_1(x_0))}.$$

We denote by E^p the closure of C_c^∞ in L^p_{uloc}. This class is characterized by the condition

$$\lim_{R \to \infty} \|f\|_{L^p_{\text{uloc}}(\mathbb{R}^3 \setminus B_R)} = 0.$$

We also work with data in the critical Lorentz space $L^{3,\infty}$ and prove estimates in $L^{3/2,1}$.

Definition 2.2 (Lorentz Spaces)**.** Let m denote Lebesgue measure. The Lorentz spaces on $(\mathbb{R}^3, m)$ are defined by finiteness of the quasinorm

$$\|f\|_{L^{p,q}} = \left(p \int_0^\infty \sigma^q m\{x : \sigma < |f(x)|\}^{\frac{q}{p}} \frac{d\sigma}{\sigma} \right)^{\frac{1}{q}}.$$

The endpoint Lorentz spaces $L^{p,\infty}$ are defined by

$$\|f\|_{L^{p,\infty}} := \sup_{\sigma > 0} \sigma^p m\{x : \sigma < |f(x)|\}.$$

We define Hölder spaces as usual.

Definition 2.3 (Hölder spaces)**.** For $0 < \alpha \le 1$, we define the quasinorm

$$[f]_{C^{0,\alpha}(\Omega)} = \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

We make the abbreviation $C^\alpha = C^{0,\alpha}$ when there is no confusion. We say $f \in C^\alpha_{\mathrm{loc}}(\Omega)$ if $f \in C^\alpha(\Omega')$ for all compact subsets Ω' of Ω. Finally, we set

$$\|f\|_{C^{k,\alpha}(\Omega)} = \max_{|\beta| \le k} \sup_{x \in \Omega} |D^\beta f|(x) + \max_{|\beta| = k} [D^\beta f]_{C^{0,\alpha}(\Omega)}.$$

We define $C^{k,\alpha}_{\mathrm{loc}}$ in analogy with C^α_{loc}.

We also define a parabolic Hölder space based on the scaling of (1.1). Note that the subscript t implies a norm in the time variable, and x a norm in the space variable. For normed spaces X, Y, we write $Y([0,T]; X(\mathbb{R}^3))$ to denote a norm of Y in time, and X in space.

Definition 2.4. Let $\gamma > 0$. The space $C^\gamma_{par}(\mathbb{R}^3 \times (0,T))$ consists of $f \in C(\mathbb{R}^3 \times (0,T))$ with finite semi-norm

$$[f]_{C^\gamma_{\mathrm{par}}(\mathbb{R}^3 \times [0,T])} := [f]_{C_t^{\gamma/2}([0,T]; L^\infty(\mathbb{R}^3))} + [f]_{L^\infty([0,T]; C^\gamma_x(\mathbb{R}^3))}.$$

This space arises in the local smoothing result of [40]. We also define the Kato classes that appear in our estimates.

Definition 2.5 (Kato Spaces)**.** Let $\mathcal{K}_p$ be the Kato class defined by the finiteness of the norm

$$\|u\|_{\mathcal{K}_p} := \operatorname*{ess\,sup}_{t>0} t^{\frac{1}{2}(1-\frac{3}{p})} \|u(t)\|_{L^p}.$$

The Besov spaces relevant to DSS solutions can be defined using these Kato classes. Let $e^{t\Delta} f$ denote the heat evolution of f.

Definition 2.6 (Besov Spaces)**.** Assuming $3 < p \leq \infty$, $f \in \dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ if and only if

$$\|e^{t\Delta} f\|_{\mathcal{K}_p} < \infty,$$

the above norm being equivalent to the norm classically defined using Littlewood-Paley.

2.2 Weak solutions

We outline the weak solution theory of (1.1). Define $C^\infty_{c,\sigma}(\mathbb{R}^3 \times (0,T))$ to be the space of divergence-free test functions, i.e.

$$C^\infty_{c,\sigma}(\mathbb{R}^3 \times (0,T)) = \{\zeta \in C^\infty_c(\mathbb{R}^3 \times (0,T); \mathbb{R}^3) : \operatorname{div} \zeta = 0\}.$$

Note that the subscript σ denotes divergence free vector fields.

The weak form of (1.1) is given by

$$\begin{aligned} \iint -u \cdot (\partial_t \zeta + \Delta \zeta) + u_i u_j \partial_j \zeta_i - f \cdot \zeta \, dx \, dt = 0, \qquad & \forall \zeta \in C^\infty_{c,\sigma}(\mathbb{R}^3 \times (0,T)) \\ \int u(\cdot, t) \cdot \nabla \phi \, dx = 0, \qquad & \forall \phi \in C^\infty_c(\mathbb{R}^3 \times (0,T)). \end{aligned} \tag{2.1}$$

The following definition enumerates four types of weak solutions.

Definition 2.7 ([65])**.** Denote the homogeneous Sobolev space by $V = \dot{H}^1_{0,\sigma}(\mathbb{R}^3)$. Let $0 < T < \infty$. Assume $u_0 \in L^2_\sigma(\mathbb{R}^3)$ and $f \in L^2(0,T;V')$.

1. A vector field $u(x,t)$ is a *very weak solution* of (1.1) in $\mathbb{R}^3 \times (0,T)$ if $u \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \times (0,T))$ and satisfies the weak form (2.1).

2. A very weak solution u is a *weak solution* if $u \in L^\infty(0,T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;V)$, $u \in C_{\mathrm{wk}}([0,T); L^2_\sigma(\mathbb{R}^3))$ and $u(t) \to u_0$ weakly in L^2 as $t \to 0_+$.

3. A weak solution u is a *Leray-Hopf weak solution* if it satisfies the energy inequality

$$\int |u|(\cdot,t)^2\,dx + 2\int_0^t\int |\nabla u|^2\,dx\,dt \le \int |u_0|^2\,dx + \int_0^t\int 2u\cdot f\,dx\,dt, \qquad \forall t, \tag{2.2}$$

and $\lim_{t\to 0_+}\|u(t)-u_0\|_{L^2}=0$.

4. A Leray-Hopf weak solution is a *suitable weak solution* if there is some $p\in L^{3/2}_{\text{loc}}(\mathbb{R}^3\times(0,T))$ so that (u,p) satisfies (1.1) in the distributional sense (i.e. with an extra term $-p\nabla\cdot\zeta$ in the integrand for $\zeta\in C_c^\infty(\mathbb{R}^3\times(0,T))$) and the local energy inequality: for all $\phi\ge 0\in C_c^\infty(\mathbb{R}^3\times(0,T))$, and for all $t\in(0,T)$

$$\begin{aligned}&\int |u(\cdot,t)|^2\phi\,dx + 2\int_0^t\int |\nabla u|^2\phi\,dx\,dt\\ &\quad\le \int |u_0|^2\phi\,dx + \int_0^t\int |u|^2(\partial_t\phi+\Delta\phi)+(|u|^2+2p)u\cdot\nabla\phi + 2u\cdot f\phi\,dx\,dt.\end{aligned} \tag{2.3}$$

A weak solution is global, i.e. a solution in $\mathbb{R}^3\times(0,\infty)$, if it is a solution in $\mathbb{R}^3\times(0,T)$ for any $0<T<\infty$.

The notation $u\in C_{\text{wk}}(I,L^p)$ denotes weak continuity in the dual sense, i.e.

$$\int u(x,t)\cdot w(x)\,dx \to \int u(x,t_0)\cdot w(x)\,dx,\ \text{ as } t\to t_0,\ \text{ for } t,t_0\in I,$$

for all $w\in L^{p'}$ and $1/p+1/p'=1$. A solution u is in the energy class if

$$u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;\dot H^1(\Omega)).$$

We denote spaces such as this with the shorthand $L^\infty_tL^2_x\cap L^2_t\dot H^1_x$. By Sobolev embedding and Hölder's inequality, this implies

$$u\in L^s_tL^q_x,\ \text{ for } \frac{3}{q}+\frac{2}{s}=\frac{3}{2},\ \ 2\le q\le 6.$$

Note also that in the first three definitions of weak solutions the pressure is not mentioned. Once u is found p can be recovered as follows: for $u \in L^\infty(0,T;L^q(\mathbb{R}^3))$, $q > 2$, we can define $p \in L^\infty(0,T;L^{q/2}(\mathbb{R}^3))$ via Riesz transforms. Note that p satisfies

$$-\Delta p = \partial_j u_i \partial_i u_j.$$

This is equivalent to

$$|\xi|^2 \hat{p}(\xi) = -\xi_i \xi_j (u_i u_j)^\wedge(\xi).$$

Applying the inverse Fourier transform gives

$$p(x) = \frac{1}{(2\pi)^{3/2}} \left(\frac{-\xi_i \xi_j}{|\xi|^2} (u_i u_j)^\wedge(\xi) \right)^\vee (x) = \frac{1}{(2\pi)^{3/2}} R_i R_j (u_i u_j)(x),$$

where the Riesz transforms are given by the Fourier multiplier $(R_k f)^\wedge(\xi) = \frac{i\xi_k}{|\xi|} \hat{f}(\xi)$.

The foundational treatment for the problem of existence of global weak solutions for finite energy data is given by Leray [54]. Solutions with the properties of those constructed by Leray are referred to as Leray weak solutions.

As stated, recent work suggests non-uniqueness in this class. This has been affirmed in weaker classes [18] and within the Leray class for the forced Navier-Stokes equations [2]. For zero forcing, [40, 38, 35] support non-uniqueness, but there is not a clear picture of non-uniqueness. We will demonstrate a bound on how non-uniqueness evolves for short times.

2.2.1 Local energy solutions

We impose some assumptions that locally control the energy of solutions. These local energy solutions, also called local Leray solutions, were introduced by Lemarié-Rieusset [52] and played an important role in the proof of local smoothing in [40].

Definition 2.8 (Local energy solutions)**.** A vector field $u \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \times [0,T))$, $0 < T \le \infty$, is a local energy solution to (1.1) with divergence free initial data $u_0 \in L^2_{\mathrm{uloc}}(\mathbb{R}^3)$, if:

1. for some $p \in L^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \times [0,T))$, the pair (u,p) is a distributional solution to (1.1),

2. for any $R > 0$, u satisfies

$$\operatorname*{ess\,sup}_{0\le t<R^2\wedge T} \sup_{x_0\in\mathbb{R}^3} \int_{B_R(x_0)} \frac{1}{2}|u(x,t)|^2\,dx + \sup_{x_0\in\mathbb{R}^3} \int_0^{R^2\wedge T} \int_{B_R(x_0)} |\nabla u(x,t)|^2\,dx\,dt < \infty,$$

3. for any $R > 0$, $x_0 \in \mathbb{R}^3$, and $0 < T' < T$, there exists a function of time $c_{x_0,R} \in L^{\frac{3}{2}}_{T'}$ so that for every $0 < t < T'$ and $x \in B_{2R}(x_0)$,

$$\begin{aligned} p(x,t) = {} & c_{x_0,R}(t) - \Delta^{-1}\operatorname{div}\operatorname{div}[(u\otimes u)\chi_{4R}(x-x_0)] \\ & - \int (K(x-y) - K(x_0-y))(u\otimes u)(y,t)(1-\chi_{4R}(y-x_0))\,dy, \end{aligned} \tag{2.4}$$

in $L^{\frac{3}{2}}(B_{2R}(x_0)\times(0,T'))$, where $K(x)$ is the kernel of $\Delta^{-1}\operatorname{div}\operatorname{div}$, $K_{ij}(x) = \partial_i\partial_j \frac{-1}{4\pi|x|}$, and $\chi_{4R}(x)$ is the characteristic function for B_{4R},

4. for all compact subsets $K \Subset \mathbb{R}^3$, $u(t) \to u_0$ in $L^2(K)$ as $t \to 0^+$,

5. u is suitable in the sense of Caffarelli-Kohn-Nirenberg, i.e., for all cylinders $Q := B_r(x_0) \times (t_0 - r^2, t_0) \subset \mathbb{R}^3 \times (0,T)$ and all non-negative $\phi \in C_c^\infty(Q)$, we have the *local energy inequality*

$$2\iint |\nabla u|^2\phi\,dx\,dt \le \iint |u|^2(\partial_t\phi + \Delta\phi)\,dx\,dt + \iint (|u|^2 + 2p)(u\cdot\nabla\phi)\,dx\,dt, \tag{2.5}$$

6. the function

$$t \mapsto \int u(x,t)\cdot w(x)\,dx, \tag{2.6}$$

is continuous in $t \in [0,T)$, for any compactly supported $w \in L^2(\mathbb{R}^3)$.

Local energy solutions were shown to satisfy the following *a priori* bound in [40]: Let $u_0 \in E^2$, $\operatorname{div} u_0 = 0$ and assume u is a local energy solution with data u_0. For all $r > 0$ we have

$$\operatorname*{ess\,sup}_{0 \le t \le \sigma r^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_r(x_0)} \frac{|u|^2}{2}\, dx\, dt + \sup_{x_0 \in \mathbb{R}^3} \int_0^{\sigma r^2} \int_{B_r(x_0)} |\nabla u|^2\, dx\, dt < C A_0(r), \tag{2.7}$$

where

$$A_0(r) = r N_r^0 = \sup_{x_0 \in \mathbb{R}^3} \int_{B_r(x_0)} |u_0|^2\, dx,$$

and

$$\sigma = \sigma(r) = c_0 \, \min\left\{ (N_r^0)^{-2}, 1 \right\}, \tag{2.8}$$

for a small universal constant $c_0 > 0$.

The class of of DSS functions in $L^{3,\infty}$ can be identified with the critical weak Lebesgue space L^3_{w}. This space embeds continuously in L^2_{uloc}. This is a natural class to investigate self-similar and DSS solutions as the existence of DSS local energy solutions with large data in L^3_{w} was proven in [10]. We define solutions with data in $L^{3,\infty}$ introduced by Barker, Seregin and Šverák in [7].

Definition 2.9 ($L^{3,\infty}$ weak solution)**.**) Let $T > 0$ be finite. Assume $u_0 \in L^{3,\infty}$ is divergence free. We say that u and an associated pressure p comprise a $L^{3,\infty}$ weak solution if

1. (u,p) satisfies (1.1) distributionally,
2. u satisfies the local energy inequality (2.5),
3. for every $w \in L^2$, the function

$$t \to \int u(x,t) \cdot w(x)\, dx,$$

is continuous on $[0,T]$,

4. $\tilde{u} := u - e^{t\Delta}u_0$ satisfies, for all $t \in (0,T)$,

$$\sup_{0<s<t} \|\tilde{u}\|_{L^2}^2(s) + \left(\int_0^t \|\nabla \tilde{u}\|_{L^2}^2(s)\, ds \right)^{\frac{1}{2}} < \infty, \tag{2.9}$$

and

$$\|\tilde{u}\|_2^2 + 2\int_0^t \int |\nabla \tilde{u}|^2 \, dx\, ds \leq 2\int_0^t \int (e^{t\Delta}u_0 \otimes \tilde{u} + e^{t\Delta}u_0 \otimes e^{t\Delta}u_0)(\nabla \tilde{u})\, dx\, ds \tag{2.10}$$

The fact that $L^{3,\infty}$ weak solutions are a sub-class of local energy solutions is proven in Appendix A.

2.3 Mild solutions

To define mild solutions, we begin with some background from [65]. For a moment, consider the nonstationary stokes system on $\mathbb{R}^3 \times (0,T)$ given by

$$\begin{aligned} \partial_t u - \Delta u + \nabla p = f, \\ \operatorname{div} u = 0, \qquad u|_{t=0} = u_0. \end{aligned} \tag{2.11}$$

This system has a fundamental solution known as the Oseen tensor [60]

$$S_{ij}(t,x) = \Gamma(t,x)\delta_{ij} + \frac{1}{4\pi}\frac{\partial^2}{\partial_{x_i}\partial_{x_j}} \int \frac{\Gamma(t,y)}{x-y}\, dy,$$

where Γ is the fundamental solution of the heat equation. Next, denote the Helmholtz projection by $\mathbb{P} : L^q \to L^q_\sigma$, for $1 < q < \infty$. This is a bounded operator which sends L^q functions to their divergence free counterparts. The Stokes operator A generates a contraction semigroup $\{e^{tA} : t \geq 0\}$, so the solution to (2.11) is formally given by

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathbb{P}f(s)\,ds. \tag{2.12}$$

Because we work on $\mathbb{R}^3$ with no boundary we adopt the notation $A = -\Delta$. The Navier-Stokes equations with zero forcing can be written as

$$\partial_t u + \Delta u + \mathbb{P}(u \cdot \nabla u) = 0, \;\; u(0) = u_0. \tag{2.13}$$

This is formally equivalent to

$$u(t) = e^{t\Delta}u_0 - B(u,u)(t), \tag{2.14}$$

where

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\partial_j(u_j v)(s)\,ds. \tag{2.15}$$

We now define mild solutions.

Definition 2.10 ([65]). Let $0 < T \leq \infty$. Let X be a Banach space of vector valued functions defined in $\mathbb{R}^3 \times (0,T)$ such that $P_0(t) = e^{t\Delta}u_0 \in X$ for certain u_0 and the bilinear map $B(u,v)$ can be interpreted as a bounded bilinear map $B : X \times X \to X$. A *mild solution* of (2.14) in X with initial data u_0 is some $u \in X$ such that $u = P_0 - B(u,u)$ in X.

Mild solutions are known to exist on $\mathbb{R}^3 \times (0,T)$ for L^q data with $q \geq 3$ with T dependent on the initial data. These solutions have properties of regularity and uniqueness outlined in [65]. We have a short-time unique existence result: for any $u_0 \in L^q_\sigma(\mathbb{R}^3)$, $q > 3$, there is a unique mild solution u of (2.14) satisfying

$$u \in BC([0,T); L^q_\sigma), \qquad T \geq C\|u_0\|_q^{-\frac{2q}{q-3}}. \tag{2.16}$$

For L^3 data, global existence holds because $L^\infty L^3$ and $L^s_t L^q_x$ with $3/q + 2/s = 1$ are critical norms. Let $3 < q_2 < \infty$. For any $u_0 \in L^3_\sigma(\mathbb{R}^3)$, there is a $T > 0$ and a mild solution u of (2.14) in the class

$$t^{1/s(q)} u \in BC([0,T); L^q_\sigma), \qquad \frac{1}{s(q)} = \frac{1}{2} - \frac{3}{2q}, \qquad \forall q \in [3, q_2]. \tag{2.17}$$

We also have as $t \to 0$, $t^{1/s(q)} u(t) \to 0$ in L^q for $q > 3$ and to u_0 for $q = 3$. The solution is unique in the class (2.17). There also exists a universal constant $\varepsilon > 0$ such that if $\|u_0\|_{L^3} < \varepsilon$, the solution u is global, i.e. we can take $T = \infty$.

The last result on mild solutions we cite pertains to properties inherited by weak solutions that are also mild. For $u_0 \in L^2_\sigma \cap L^3_\sigma$, the mild solution above is a Leray-Hopf weak solution satisfying the energy identity on $[0,T]$ [65].

All of the self-similar and DSS solutions constructed in [40, 67, 53, 10, 25, 1] are mild due to sufficient conditions in [52, 15]. In fact, the local pressure expansion (2.4) in the definition of local energy solutions is equivalent to being mild. Therefore, local energy *and* $L^{3,\infty}$ weak solutions are mild.

Lastly, we present the following lemma used to bound the heat evolution of L^r data for $r > 1$.

Lemma 2.1 ([65]). *Let $1 < r \leq q < \infty$ and $\sigma = \sigma(q,r) = \frac{3}{2}(\frac{1}{r} - \frac{1}{q}) \geq 0$. We have*

$$\begin{aligned} \|e^{-t\Delta}\mathbb{P}u_0\|_{L^q} &\leq Ct^{-\sigma}\|a\|_{L^r} \\ \|\nabla e^{-t\Delta}\mathbb{P}u_0\|_{L^q} &\leq Ct^{-\sigma-1/2}\|a\|_{L^r} \\ \sup_j \|e^{-t\Delta}\mathbb{P}\partial_{x_j}u_0\|_{L^q} &\leq Ct^{-\sigma-1/2}\|a\|_{L^r}, \end{aligned} \tag{2.18}$$

for all $t > 0$.

This implies a standard mild solution estimate from [26, 42] and [65, Ch. 5],

$$\|B(f,g)\|_{L^p}(t) \lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}} \|f \otimes g\|_{L^q}(s)\, ds. \tag{2.19}$$

3 Literature survey

We survey some literature essential to the results that follow.

3.1 A new proof of the Caffarelli-Kohn-Nirenberg theorem

We begin by introducing the following regularity criteria of Lin [57]. This lemma states that a suitable solution to (1.1) is regular given certain 0-dimensional weighted integrals are small over a parabolic cylinder.

Lemma 3.1 (ε-Regularity [57]). *There exists a universal small constant $\varepsilon_* > 0$ such that if the pair (u, p) is a suitable weak solution of (NS) in $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0)$, $B_r(x_0) \subset \mathbb{R}^3$, and*

$$\varepsilon^3 = \frac{1}{r^2} \int_{Q_r} (|u|^3 + |p|^{\frac{3}{2}})\, dx\, dt < \varepsilon_*, \tag{3.1}$$

then $u \in L^\infty(Q_{\frac{r}{2}})$. Moreover,

$$\|\nabla^k u\|_{L^\infty(Q_{\frac{r}{2}})} \leq C_k \varepsilon r^{-k-1}, \tag{3.2}$$

for universal constants C_k where $k \in \mathbb{N}_0$.

For DSS solutions with data in sub-critical spaces, we can control these 0 dimensional integrals via interpolation. Then ε-regularity gives a L^∞ bound in an annulus which may be extended to the sub-paraboloid region by scaling. If a flow is self-similar, globally regular, or $\lambda \sim 1$ as in [67], then this region is all of $\mathbb{R}^4_+$.

3.2 On stability of weak Navier-Stokes solutions with large $L^{3,\infty}$ initial data

In [7], Barker, Seregin and Šverák develop the theory of $L^{3,\infty}$ weak solutions. This extends ideas in [63] and has since been extended to non-endpoint critical Besov spaces of negative smoothness in [1], which is discussed below.

These solutions are shown to exist globally in time and are known to be mild. It is also shown in [7] that $L^{3,\infty}$ weak solutions satisfy the following dimensionless estimate

$$\sup_{0<s<t} \|u - P_0\|_{L^2}^2(s) + \int_0^t \|\nabla(u - P_0)\|_{L^2}^2(s)\, ds \lesssim_{u_0} t^{\frac{1}{2}}, \tag{3.3}$$

where the dependence on u_0 is in terms of $\|u_0\|_{L^{3,\infty}}$. This is established in [1] for higher Picard iterates. We use this result to deplete a time singularity at $t = 0$, as this energy vanishes at $t = 0$. We derive other dimensionless estimates in Theorem 4.12 which serve as new *a priori* estimates for these solutions which we use to control the growth of solutions in the far-field.

3.3 Global Weak Besov Solutions of the Navier-Stokes Equations and Applications

Weak Besov solutions are a weaker class than local energy solutions which generalize $L^{3,\infty}$ weak solutions. In [1], it is shown that for $u_0 \in \dot{B}^{-1+3/p}_{p,\infty}$, $p > 3$, the weak Besov solution u to (1.1) and the Picard iterate $P_k = P_k(u_0)$ satisfy

$$\sup_{0<s<t} \|u - P_k\|_{L^2}^2(s) + 2\int_0^t \|\nabla(u - P_k)\|_{L^2}^2(s)\, ds \lesssim_{u_0,k} t^{\frac{1}{2}}, \tag{3.4}$$

for $k \geq \lceil \frac{p}{2} \rceil - 2$. This is a consequence of [1, Lemma 2.2].

These estimates are used as the basis of our Picard improvement techniques. It is also shown that Picard iterates belong to the Kato classes, i.e. if $u_0 \in \dot{B}^{-1+3/p}_{p,\infty}$ for some $p > 3$, then for $k \geq 0$, $P_k(u_0) \in \mathcal{K}_p$. Because $L^{3,\infty} \hookrightarrow \dot{B}^{-1+3/p}_{p,\infty}$ for all $p > 3$, when $u_0 \in L^{3,\infty}$ we have $P_k(u_0) \in \mathcal{K}_p$ for all $p > 3$. It follows that (3.4) holds for local energy solutions for all $k \geq 0$.

We also have the following bound by [1, (2.32)]

$$
\begin{aligned}
\|P_k\|_{\mathcal{X}_T} &:= \|P_k\|_{\mathcal{K}_\infty} + \sup_{x \in \mathbb{R}^3} \sup_{R \in (0,\sqrt{T})} R^{-3/2} \|P_k\|_{L^2(B_R(x)\times(0,R^2))} \\
&\leq C(k, p, \|u_0\|_{\dot{B}^{-1+3/p}_{p,\infty}}).
\end{aligned}
\tag{3.5}
$$

Thus, $u_0 \in \dot{B}^{-1+3/p}_{p,\infty}$ implies $P_k \in (L^2_{x,t})_{\mathrm{loc}}$.

Note that both of (3.3) and (3.4) imply the following separation rate.

For two solutions, u, v in the local energy class evolving from the same data, we have the following global bound

$$
\|u - v\|_{L^2} \lesssim \|u - P_0\|_{L^2} + \|P_0 - v\|_{L^2} \lesssim t^{\frac{1}{4}},
$$

which allows rapid separation at small times.

3.4 Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self similar solutions

Our foundation for the proof of the main results is local smoothing [40]. Since $L^{3,\infty}$ weak solutions are local energy solutions and $L^{3,\infty} \subset L^2_{\mathrm{uloc}}$, we may use the following theorem.

Theorem 3.2 (Local smoothing [40, Theorem 3.1])**.** *Let $u_0 \in E^2$ be divergence free. Suppose $u_0|_{B_2(0)} \in L^p(B_2(0))$ for $p > 3$. Decompose $u_0 = U_0 + U_0'$ with $\operatorname{div} U_0 = 0$, $U_0|_{B_{4/3}} = u_0$, $\operatorname{supp} U_0 \Subset B_2(0)$ and $\|U_0\|_{L^p(\mathbb{R}^3)} < C(p, \|u_0\|_{L^p(B_2(0))})$. Let U be the locally-in-time defined mild solution to (1.1) with initial data U_0. Then, there exists a positive $T = T(p, \|u_0\|_{L^2_{\mathrm{uloc}}}, \|u_0\|_{L^p(B_2(0))})$ such that any local energy solution u with data u_0*

satisfies

$$\|u - U\|_{C^{\gamma}_{\text{par}}(\overline{B}_{\frac{1}{2}}\times[0,T])} \le C(p, \|u_0\|_{L^2_{\text{uloc}}}, \|u_0\|_{L^p(B_2(0))}), \tag{3.6}$$

for some $\gamma = \gamma(p) \in (0,1)$.

This gives us another preliminary perspective on separation rates. For any two local energy solutions u, v satisfying the above,

$$\|u - v\|_{L^\infty(B_{\frac{1}{2}})} \lesssim \|u - U\|_{L^\infty(B_{\frac{1}{2}})} + \|U - v\|_{L^\infty(B_{\frac{1}{2}})} \lesssim t^{\frac{\gamma}{2}}.$$

This still allows a very large gap for separation at small times, and our goal is to increase the exponent on t as large as possible by using Picard iteration to create finer estimates.

3.5 An ϵ-regularity criterion and estimates of the regular set for Navier-Stokes flows in terms of initial data

The following theorem defines the radius of far-field regularity R_0. This appears throughout our DSS results as a restriction on the domain where asymptotics hold.

Theorem 3.3 (Far-field regularity [43, Theorem 1.8]). *Fix* $\lambda > 1$. *Let* u *be a* λ*-DSS local energy solution of the Navier-Stokes equations in* $\mathbb{R}^3 \times (0,T)$ *with divergence free,* λ*-DSS initial data* $u_0 \in E^2$. *Then there exists* $R_0 = R_0(u_0) > 0$ *so that* u *is smooth and bounded on*

$$\{(x,t) : |x| \ge R_0\sqrt{t}; 1 \le t \le \lambda^2\}.$$

This implies smoothness on $\{(x,t) : |x| \ge R_0\sqrt{t}\}$. *Furthermore, there exists* $\lambda_0(u_0) > 1$ *so that if* $1 \le \lambda \le \lambda_0$, *then* u *is globally smooth.*

The proof of the theorem and our usage of R_0 both arise from taking a DSS solution far enough out ($|x| \ge R_0$) so that (u,p) become sufficiently small to apply ε-regularity

(Lemma 3.1). Then we can extend our asymptotics using using DSS scaling to the sub-paraboloid region.

3.6 Estimates for solutions of a non-stationary linearized system of Navier-Stokes equations

We prove an integral estimate (Lemma 4.1) which is an central tool in our work which helps us demonstrate an improvement from the operator B. To use this estimate in our L^q_{loc} results, we need the following bounds on the derivatives of the Oseen kernel.

Lemma 3.4 (Bound for derivatives of the Oseen tensor,[64])**.** *The operator $e^{t\Delta}\mathbb{P}\nabla\cdot$ (where $\mathbb{P}$ is the Helmholtz projection) in $\mathbb{R}^3$ with kernel $\nabla S(x,t)$ has the following bound:*

$$|D_x^l \partial_t^m S_{j,k}(x,t)| \le C(t^{\frac{1}{2}} + |x|)^{-3-l-2m}, \ \forall l, m \in \mathbb{Z}^+.$$

3.7 Forward discretely self-similar solutions of the Navier-Stokes equations

In [67], Tsai extends the work in [40] and proves the existence of large, forward DSS solutions. He also quantifies decay rates for the solution and its non-linear part for data in C^α_{loc} and $C^{1,\alpha}_{\text{loc}}$.

Theorem 3.5 ([67], Theorem 1.1)**.** *For any $0 < \alpha < 1$ and $C_* > 0$ there is a $\lambda_* = \lambda_*(\alpha, C_*) \in (1,2)$ such that the following hold. Suppose $u_0 \in C^\alpha_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$, $\|u_0\|_{C^\alpha(\overline{B_2}\setminus B_1)} \le C_*$, $\operatorname{div} u_0 = 0$, and u_0 is DSS with factor $\lambda \in (1, \lambda_*]$. There is a local Leray solution u of (1.1) with initial data u_0 that is DSS with factor λ, and for $v(\cdot,t) := u(\cdot,t) - e^{t\Delta}u_0$*

$$|u(x,t)| \le \frac{C}{|x| + \sqrt{t}}, \qquad |v(x,t)| \le \frac{C\sqrt{t}}{|x|^2 + t}, \tag{3.7}$$

in $\mathbb{R}^4_+$ with $C = C(\alpha, C_*)$. It is also a mild solution in the class (3.7). If, furthermore, $\|u_0$

$_{C^{1,\alpha}(\overline{B_2}\setminus B_1)} \leq C_*$, *then*

$$|v(x,t)| \leq \frac{Ct^{\frac{3}{2}}}{(|x|+\sqrt{t})^3}\log\left(1+\frac{|x|}{\sqrt{t}}\right), \qquad |D_x v(x,t)| \leq \frac{Ct^{\frac{3}{2}}}{(|x|+\sqrt{t})^3}, \tag{3.8}$$

in $\mathbb{R}^4_+$ *with* $C = C(\alpha, C_*)$.

We improve these results by obtaining (3.7) for $L^\infty_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ data and creating a scale of results depending on $q > 3$ for $L^q_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ data. We also improve on (3.8) by developing a scale of results depending on $0 < \alpha < 1$ for $C^\alpha_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ data and drop the logarithm for $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ data without D_x.

3.8 Optimal local smoothing and analyticity rate estimates for the generalized Navier-Stokes equations

In [24], pointwise bounds for the Oseen tensor involving fractional powers of the Laplacian are established and we recall the details modified slightly to match our notation.

First, define the fractional Laplacian $\Lambda = (-\Delta)^{\frac{1}{2}}$ by the singular integral operator

$$\Lambda^\alpha f(x) = c_\alpha \int \frac{f(x) - f(y)}{|x-y|^{3+\alpha}}\, dy,$$

where

$$c_\alpha = \frac{4^\alpha \Gamma(\frac{3+\alpha}{2})}{\pi^{\frac{3}{2}}\Gamma(-\frac{\alpha}{2})}.$$

This is equivalent to the definition via Fourier multipliers, i.e.

$$(\Lambda^\alpha f)^\wedge(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

Proposition 3.6 ([24], Proposition 3.1). *Let $S_{j,k}$ denote the Oseen kernel. For any integer*

$m \geq 0$ *and* $-1 < \alpha \leq 1$,

$$\left|(|x|+1)^{3+m+\alpha} D^m \Lambda^\alpha S_{j,k}\right|(x,1) \lesssim_{m,\alpha} 1, \qquad \forall x \in \mathbb{R}^d. \tag{3.9}$$

We use this estimate in the following form

$$\left|\nabla \Lambda^{-\beta} S_{j,k}\right|(x,t) \lesssim_\beta (|x|+\sqrt{t})^{-4+\beta}, \tag{3.10}$$

where $0 < \beta < 1$. This allows us to apply a fractional power of the Laplacian to the kernel and the 'other half' to P_0. This is essential in avoiding a cubic power on the Oseen kernel in the proof of decay for C^α_{loc} and $C^{1,\alpha}_{\mathrm{loc}}$ data.

3.9 Forward Self-Similar Solutions of the Fractional Navier-Stokes Equations

We also look to improve the results in the papers of Lai, Miao, and Zheng [55, 56] which we presently state, omitting the details of the fractional Navier-Stokes considered therein.

Theorem 3.7 ([55, Theorem 1.1])**.** *Let* $u_0 = \frac{\sigma(x)}{|x|}$ *with* $\sigma(x) = \sigma(x/|x|) \in L^\infty(S^2)$ *which satisfies* $\operatorname{div} u_0 = 0$ *in* $\mathbb{R}^3 \setminus \{0\}$. *Then* (1.1) *admits at least one forward self-similar solution* $u \in BC_w([0,\infty), L^{3,\infty}(\mathbb{R}^3))$ *such that*

1. *for each* $p \in [2,6]$, $\|u(t) - e^{t\Delta}u_0\|_{L^p(\mathbb{R}^3)} \leq Ct^{\frac{1}{2}(1+\frac{3}{p})-1}$,

2. $u(x,t)$ *is smooth in* $\mathbb{R}^4_+$,

3. *we have the following pointwise estimates*

$$|u(x,t)| \lesssim \frac{C}{|x|+\sqrt{t}}, \qquad \text{and} \qquad |u(x,t) - e^{t\Delta}u_0| \leq \frac{Ct}{|x|^3 + t^{\frac{3}{2}}} \log\left(1 + \frac{|x|}{\sqrt{t}}\right), \tag{3.11}$$

for all $(x,t) \in \mathbb{R}^4_+$.

3.10 Global regularity of weak solutions to the generalized Leray equations and its applications

Theorem 3.7 is expanded in [56] with the following theorem.

Theorem 3.8 ([56, Corollary 1.1]). *Let* $u_0 = \frac{\sigma(x)}{|x|}$ *with* $\sigma(x) = \sigma(x/|x|) \in C^{0,1}(S^2)$ *which satisfies* $\operatorname{div} u_0 = 0$ *in* $\mathbb{R}^3$ *in the distribution sense. Then* (1.1) *admits at least one forward self-similar solution* $u \in BC_w([0,\infty), L^{3,\infty}(\mathbb{R}^3))$ *such that*

$$|u(x,t)| \lesssim \frac{C}{(|x| + \sqrt{t})} \qquad \text{and} \qquad |u(x,t) - e^{t\Delta}u_0| \le \frac{C\sqrt{t}}{(|x| + \sqrt{t})^2} \tag{3.12}$$

for all $(x,t) \in \mathbb{R}^4_+$. *Moreover if* $\sigma(x) = \sigma(x/|x|) \in C^{1,1}(S^2)$, *we have*

$$|u(x,t) - e^{t\Delta}u_0| \le \frac{Ct}{(|x| + \sqrt{t})^3} \tag{3.13}$$

for all $(x,t) \in \mathbb{R}^4_+$.

The proofs of these results rely on the fact that a self-similar solution u is completely determined by its profile $U(x) = u(x,1)$. They investigate the decay properties of the profile $V(x) := U(x) - e^{\Delta}u_0(x)$.

We make specific use of their proof technique in the proof of [56, Proposition 4.4] in which they decompose the integral form of V into a near-, mid-, and far-field. The mid- and far-field have cubic decay, and the near-field is treated separately. Here, we use mean value theorem to introduce another gradient into the integral of $B(P_0, P_0)$ to make full use of the $C^{1,1}$ assumption and obtain cubic decay.

We utilize this technique along with fractional powers of the Laplacian to obtain this cubic decay with a $C^{1,\alpha}$ assumption, as any space stronger than $C^{1,0}$ is enough to avoid the logarithm obtained by integrating the cubic power of the Oseen kernel.

4 Preliminaries

In this section, we work through all estimates, Lemmas, and Corollaries needed to prove our main results. This section constitutes joint work with Dr. Zachary Bradshaw.

4.1 Integral estimates

The following estimate is inspired by [67]. We use this to improve the exponent of the decay at each iterative step up to a limit of $a = 4$.

Lemma 4.1. *For $a \in [0,5)$ and $b \in [0,2)$, where $a+b<5$*

$$\int_0^t \int_{\mathbb{R}^3} \frac{1}{(|x-y|+\sqrt{t-s})^4} \frac{1}{(|y|+\sqrt{s})^a} \frac{1}{\sqrt{s}^b}\, dy\, ds \le \frac{C\sqrt{t}^{\min\{a,4\}-1}}{(|x|+\sqrt{t})^{\min\{a,4\}}}. \tag{4.1}$$

Proof. Fix $t=1$ and let $R = |x|+2$. Using partial fraction decomposition, we show the above integral is bounded. Provided $a+b<5$ and $b<2$,

$$\begin{aligned}
&\sup_{R\le 8} \int_0^1 \int \frac{1}{(|x-y|+\sqrt{1-s})^4} \frac{1}{(|y|+\sqrt{s})^a} \frac{1}{\sqrt{s}^b}\, dy\, ds \\
&\quad\lesssim \int_0^1 \int \left(\frac{1}{(|y|+\sqrt{1-s})^4} + \frac{1}{(|y|+\sqrt{s})^a} \right) \frac{1}{\sqrt{s}^b}\, dy\, ds \\
&\quad\lesssim \int_0^1 \frac{1}{\sqrt{1-s}} \frac{1}{\sqrt{s}^b} + \frac{1}{\sqrt{s}^{a-3}} \frac{1}{\sqrt{s}^b}\, ds \\
&\quad\lesssim \int_0^1 \frac{1}{\sqrt{1-s}} + \frac{1}{\sqrt{s}^b}\, ds + 1 \\
&\quad\lesssim 1 \lesssim \frac{1}{R^{\min\{a,4\}}}.
\end{aligned} \tag{4.2}$$

Hence, we only need to verify the bound for $R>8$. First, we consider the far-field, $|y|>2R$, where, for $b<2$,

$$\begin{aligned}
\int_0^1 \int_{|y|>2R} \frac{1}{(|x-y|+\sqrt{1-s})^4} \frac{1}{(|y|+\sqrt{s})^a} \frac{1}{\sqrt{s}^b}\, dy\, ds &\lesssim \int_0^1 \frac{1}{\sqrt{s}^b}\, ds \int_{|y|>2R} \frac{1}{|y|^{4+a}}\, dy \\
&\lesssim \frac{1}{R^{1+a}}.
\end{aligned} \tag{4.3}$$

Next, we consider the region where $|y| < R/2$. Because $R > 8$ we find

$$\begin{aligned}
&\int_0^1 \int_{|y|<\frac{R}{2}} \frac{1}{(|x-y|+\sqrt{1-s})^4} \frac{1}{(|y|+\sqrt{s})^a} \frac{1}{\sqrt{s}^b}\, dy\, ds \\
&\qquad \lesssim \int_0^1 \int_{|y|<\frac{R}{2}} \frac{1}{R^4} \frac{1}{(|y|+\sqrt{s})^a} \frac{1}{\sqrt{s}^b}\, dy\, ds \\
&\qquad \lesssim \frac{1}{R^4} \int_0^1 \frac{1}{\sqrt{s}^b} \int_{|y|<\frac{R}{2}} \frac{1}{(|y|+\sqrt{s})^a}\, dy\, ds,
\end{aligned} \tag{4.4}$$

for $a \neq 3$. Passing to spherical coordinates yields

$$\begin{aligned}
\frac{1}{R^4} \int_0^1 \frac{1}{\sqrt{s}^b} \int_{|y|<\frac{R}{2}} \frac{1}{(|y|+\sqrt{s})^a}\, dy\, ds &\lesssim \frac{1}{R^4} \int_0^1 \frac{1}{\sqrt{s}^b} \int_0^{\frac{R}{2}} (r+\sqrt{s})^{-a} r^2\, dr\, ds \\
&\lesssim \frac{1}{R^4} \int_0^1 \frac{1}{\sqrt{s}^b} \left(\left(\frac{R}{2} + \sqrt{s} \right)^{3-a} - \sqrt{s}^{3-a} \right) ds \\
&\lesssim \left(\frac{1}{R^{1+a}} + \frac{1}{R^4} \right) \int_0^1 \frac{1}{\sqrt{s}^{a+b-3}}\, ds \\
&\lesssim \frac{1}{R^{1+a}} + \frac{1}{R^4},
\end{aligned} \tag{4.5}$$

for $a + b < 5$. If $a = 3$, then, similarly, passing to spherical coordinates yields

$$\int_0^1 \int_{|y|<\frac{R}{2}} \frac{1}{(|x-y|+\sqrt{1-s})^4} \frac{1}{(|y|+\sqrt{s})^3} \frac{1}{\sqrt{s}^b}\, dy\, ds \lesssim \frac{\ln(R+1)}{R^4} + \frac{1}{R^4} \lesssim \frac{1}{R^a} + \frac{1}{R^4}, \tag{4.6}$$

for $b < 2$ since $a = 3 < 4$.

We treat the final region, $R/2 < |y| < 2R$, with the substitution $z = x - y$ where $|z| \leq |x| + |y| \leq 3R$. We find

$$\begin{aligned}
&\int_0^1 \int_{R/2<|y|<2R} \frac{1}{(|x-y|+\sqrt{1-s})^4} \frac{1}{(|y|+\sqrt{s})^a} \frac{1}{\sqrt{s}^b}\, dy\, ds \\
&\qquad \lesssim \frac{1}{R^a} \int_0^1 \frac{1}{\sqrt{s}^b} \int_{|z|<3R} \frac{1}{(|z|+\sqrt{1-s})^4}\, dz\, ds \\
&\qquad \lesssim \frac{1}{R^a} \int_0^1 \frac{1}{\sqrt{s}^b} \left(-\frac{1}{3R+\sqrt{1-s}} + \frac{1}{\sqrt{1-s}} \right) ds \lesssim \frac{1}{R^a} + \frac{1}{R^{a+1}},
\end{aligned} \tag{4.7}$$

where we passed to spherical coordinates to evaluate the spatial integral.

Then for all x,

$$\int_0^1 \int \frac{1}{(|x-y|+\sqrt{1-s})^4} \frac{1}{(|y|+\sqrt{s})^a} \frac{1}{\sqrt{s}^b}\, dy\, ds \le \frac{C}{(|x|+2)^a} + \frac{C}{(|x|+2)^4}, \tag{4.8}$$

To conclude the proof we make the change of variables $x = \sqrt{t}\tilde{x}$, $y = \sqrt{t}\tilde{y}$ and $s = t\tilde{s}$. This substitution and the above bound lead to

$$\begin{aligned} \int_0^t \int \frac{1}{(|x-y|+\sqrt{t-s})^4} \frac{1}{(|y|+\sqrt{s})^a} \frac{1}{\sqrt{s}^b}\, dy\, ds &\le \frac{1}{\sqrt{t}} \left(\frac{C}{(|\tilde{x}|+2)^a} + \frac{C}{(|\tilde{x}|+2)^4} \right) \\ &\le \frac{C}{\sqrt{t}^{1-a}(|x|+\sqrt{t})^a} + \frac{Ct^3}{(|x|+\sqrt{t})^4}. \end{aligned} \tag{4.9}$$

□

Note this resembles a result from [67, Lemma 2.1]. Let $a, b \in (0,5)$ and $a+b>3$. Then

$$\phi(x,a,b) = \int_0^1 \int_{\mathbb{R}^3} (|x-y|+\sqrt{1-t})^{-a}(|y|+\sqrt{t})^{-b}\, dy\, dt, \tag{4.10}$$

is well defined for $x \in \mathbb{R}^3$, and

$$\phi(x,a,b) \lesssim R^{-a} + R^{-b} + R^{3-a-b}[1 + (1_{a=3} + 1_{b=3}) \log R], \tag{4.11}$$

where $R = |x|+2$. We utilize this result, as well, re-scaled to all times.

4.2 Estimates for the heat equation

In this subsection, we state and prove a variety of results on the decay of scaling invariant solutions to the heat equation. These are foundational for the proof of the decay of $u - P_0$. Let $A_k = \{x \in \mathbb{R}^3 : \lambda^k \le |x| < \lambda^{k+1}\}$ and $A_k^* = \{x : \lambda^{k-1} \le |x| < \lambda^{k+2}\}$.

Lemma 4.2. *Assume $f \in L^q_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ where $3 < q \le \infty$ and satisfies for some $\lambda > 1$,*

$$\lambda^{\sigma k} f(\lambda^k x) = f(x),$$

where $\sigma \in \left(\frac{3}{q}, \infty\right)$ ($\sigma = 1$ corresponds to being DSS). Then

$$\sup_{t\in[1,\lambda^2]} \|e^{t\Delta} f\|_{L^\infty(B_R^c)} \lesssim_\lambda \|f\|_{L^q(A_1)} R^{\frac{3}{q}-\sigma}. \tag{4.12}$$

We allow different scaling factors in order to apply this theorem to derivatives of DSS data. This conclusion is discussed without proof in [10] where it is shown for $q = 3$, there is no algebraic decay rate available.

Proof. By the assumed scaling property, we only need to show, for $t \in [1, \lambda^2]$,

$$\|e^{t\Delta} f\|_{L^\infty(A_k)} \lesssim \lambda^{k(\frac{3}{q}-\sigma)},$$

where the suppressed constants are independent of k. Fix $x \in A_k$, and decompose the heat evolution into a near-, mid-, and far-field

$$\begin{aligned} e^{t\Delta} f &= \frac{c}{t^{3/2}} \int e^{-\frac{|x-y|^2}{4t}} f(y)\,dy \\ &\lesssim \int_{|y|<\lambda^{k-1}} e^{-\frac{|x-y|^2}{4\lambda^2}} f(y)\,dy + \int_{y\in A_k^*} e^{-\frac{|x-y|^2}{4\lambda^2}} f(y)\,dy \\ &\quad + \int_{|y|\ge\lambda^{k+2}} e^{-\frac{|x-y|^2}{4\lambda^2}} f(y)\,dy \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{4.13}$$

To bound I_1, we use Hölder's inequality and $|x-y| \geq \lambda^k - \lambda^{k-1}$ to find

$$\begin{aligned}
\|I_1\|_{L^\infty(A_k)} &\leq \|e^{-\frac{|x-y|^2}{4\lambda^2}}\|_{L^\infty(|y|<\lambda^{k-1})}\|f\|_{L^1(|y|<\lambda^{k-1})} \\
&\leq e^{-\frac{(\lambda^k-\lambda^{k-1})^2}{4\lambda^2}}\|f\|_{L^1(|y|<\lambda^{k-1})} \\
&\lesssim_\lambda e^{-\lambda^{2k-4}} \sum_{k'\in\mathbb{Z}:k'\leq k-2} \|f\|_{L^1(A_{k'})}.
\end{aligned} \tag{4.14}$$

By the scaling for $f \in L^q(A_1)$,

$$\begin{aligned}
\|f\|_{L^1(A_k)} \lesssim m(A_k)^{1-\frac{1}{q}}\|f\|_{L^q(A_k)} &\lesssim m(A_k)^{1-\frac{1}{q}}\lambda^{\frac{3k}{q}}\left(\int_{A_1}|f(\lambda^k z)|^q\,dz\right)^{1/q} \\
&\lesssim m(A_k)^{1-\frac{1}{q}}\lambda^{\left(\frac{3}{q}-\sigma\right)k}\|f\|_{L^q(A_1)}.
\end{aligned} \tag{4.15}$$

This implies

$$\begin{aligned}
\|I_1\|_{L^\infty(A_k)} &\lesssim e^{-\lambda^{2k-4}} \sum_{k'\in\mathbb{Z}:k'\leq k-2} m(A_{k'})^{1-\frac{1}{q}}\lambda^{\left(\frac{3}{q}-\sigma\right)k'}\|f\|_{L^q(A_1)} \\
&\lesssim_\lambda e^{-\lambda^{2k-4}} \sum_{k'\in\mathbb{Z}:k'\leq k-2} \lambda^{3k'(1-\frac{1}{q})}\lambda^{\left(\frac{3}{q}-\sigma\right)k'}\|f\|_{L^q(A_1)} \\
&\lesssim_\lambda e^{-\lambda^{2k-4}} \frac{\lambda^{(3-\sigma)(k-2)}}{1-\lambda^{3-\sigma}}\|f\|_{L^q(A_1)},
\end{aligned} \tag{4.16}$$

by taking the limit of the geometric series. As $k \to \infty$, the Gaussian dominates any algebraic growth. Hence,

$$\|I_1\|_{L^\infty(A_k)} \lesssim_\lambda \lambda^{k(\frac{3}{q}-\sigma)}\|f\|_{L^q(A_1)}.$$

For I_2, by Young's inequality and DSS scaling,

$$\|I_2\|_{L^\infty(A_k)} \leq \|e^{-\frac{|\cdot|^2}{4\lambda^2}}\|_{L^{\frac{q}{q-1}}}\|f\chi_{A_k}\|_{L^q} \lesssim_\lambda \lambda^{k(\frac{3}{q}-\sigma)}\|f\|_{L^q(A_1)}, \tag{4.17}$$

via a similar scaling computation as in (4.15). This determines the power of R in the lemma's statement.

Finally, for I_3, we sum over the annuli $A_{k'}$, $k' \geq k+2$, and find

$$\begin{aligned}
\|I_3\|_{L^\infty(A_k)} &\leq \sum_{k' \geq k+2} e^{-\frac{\lambda^{2k'}}{4\lambda^2}} m(A_{k'})^{1-\frac{1}{q}} \|f\|_{L^q(A_{k'})} \\
&\lesssim_\lambda \sum_{k' \geq k+2} e^{-\frac{\lambda^{2k'}}{4\lambda^2}} \lambda^{3k'(1-\frac{1}{q})} \lambda^{k'(\frac{3}{q}-\sigma)} \|f\|_{L^q(A_1)} \\
&\lesssim_\lambda \|f\|_{L^q(A_1)} \sum_{k' \geq k+2} e^{-\frac{\lambda^{2k'}}{4\lambda^2}} \lambda^{(3-\sigma)k'},
\end{aligned} \tag{4.18}$$

using (4.15). Again, the Gaussian dominates any algebraic growth, and we conclude

$$\|I_3\|_{L^\infty(A_k)} \lesssim_\lambda \lambda^{k(\frac{3}{q}-\sigma)} \|f\|_{L^q(A_1)}.$$

Therefore,

$$\|e^{t\Delta} f\|_{L^\infty(A_k)} \lesssim_\lambda \lambda^{k(\frac{3}{q}-\sigma)},$$

with the suppressed constants independent of k. This implies (4.12). □

Our next lemma states fractional derivatives of Hölder space data are bounded up to, but not including, the Hölder exponent.

Lemma 4.3. *Let $0 < \alpha < 1$ and $m \in (\mathbb{N}_0)^3$ be a multi-index with $|m| \leq 1$. If $u_0 \in C^{|m|,\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ is DSS, then for all $\beta \in (0, \alpha)$,*

$$\Lambda^\beta D^m u_0 \in L^\infty_{\text{loc}}(\mathbb{R}^3 \setminus \{0\}).$$

Proof. By scaling, it suffices to show $\Lambda^\beta D^m u_0 \in L^\infty(A_1)$. To begin, we fix $x \in A_1$ and decompose the integral operator into near- and far-fields

$$\begin{aligned}
\Lambda^\beta D^m u_0(x) &\lesssim \left(\int_{|x-y| \leq \lambda^{-1}} + \int_{|x-y| > \lambda^{-1}} \right) \frac{1}{|x-y|^{3+\beta}} (D^m u_0(x) - D^m u_0(y))\, dy \\
&=: J_1(x,t) + J_2(x,t).
\end{aligned} \tag{4.19}$$

For the near-field, J_1, for $\beta < \alpha$,

$$
\begin{aligned}
|J_1| &= \int_{|x-y|\le\lambda^{-1}} \frac{1}{|x-y|^{3+\beta}}(D^m u_0(x) - D^m u_0(y))\,dy \\
&\lesssim \|u_0\|_{C^{|m|,\alpha}(A_1^*)} \int_{|x-y|\le\lambda^{-1}} \frac{1}{|x-y|^{3+\beta-\alpha}}\,dy \\
&\lesssim_{\lambda,u_0,\alpha,\beta} 1.
\end{aligned}
\tag{4.20}
$$

Next, because $u_0 \in C^{|m|}_{\mathrm{loc}}$ is DSS it decays like $|x|^{-|m|-1}$. This allows us to bound J_2 by

$$
\begin{aligned}
|J_2| &\le \int_{|x-y|>\lambda^{-1}} \frac{|D^m u_0(y)| + |D^m u_0(x)|}{|x-y|^{3+\beta}}\,dy \\
&\lesssim_{u_0,\lambda} \int_{|x-y|>\lambda^{-1}} \frac{1}{|x-y|^{3+\beta}}\frac{1}{|y|^{|m|+1}}\,dy + \lambda^{-|m|-1}\int_{|x-y|>\lambda^{-1}} \frac{1}{|x-y|^{3+\beta}}\,dy \\
&\lesssim_{u_0,\lambda} \left(\int_{|x-y|>\lambda^{-1},|y|>\lambda^{-1}} + \int_{|y|\le\lambda^{-1}}\right) \frac{1}{|x-y|^{3+\beta}}\frac{1}{|y|^{|m|+1}}\,dy + 1 \\
&\lesssim_{u_0,\lambda} \lambda^{-|m|-1}\int_{|x-y|>\lambda^{-1},|y|>\lambda^{-1}} \frac{1}{|x-y|^{3+\beta}}\,dy + \lambda^{-3-\beta}\int_{|y|\le\lambda^{-1}} \frac{1}{|y|^{|m|+1}}\,dy + 1 \\
&\lesssim_{u_0,\lambda} 1,
\end{aligned}
\tag{4.21}
$$

for $\beta > 0$ and $|m| \in \{0,1\}$. Hence, $D^m\Lambda^\beta u_0(x) \in L^\infty(A_1)$, and our proof is complete by scaling. $\square$

Now, we may prove the following corollary to Lemma 4.2. These decay rates are central to our proof of $C^{1,\alpha}$ decay.

Corollary 4.4. *Fix a multi-index $m \in (\mathbb{N}_0)^3$, $|m| \le 1$, and $\beta \in (0,2)$. If $\Lambda^\beta D^m u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$, then*

$$
\sup_{t\in[1,\lambda^2]} |\Lambda^\beta D^m e^{t\Delta} u_0(x,t)| \lesssim_{\lambda,u_0,\alpha,\beta} \frac{1}{(|x|+1)^{1+|m|+\beta}}. \tag{4.22}
$$

Proof. By Lemma 4.3, we have $\Lambda^\beta D^m u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$. We show $\Lambda^\beta D^m u_0$ has scaling factor $\sigma = 1 + |m| + \beta$, and apply Lemma 4.2 with $q = \infty$ and $f = \Lambda^\beta D^m u_0$. This scaling is clear from the calculation

$$\begin{aligned}\Lambda^\beta D^m u_0(x) &= \lambda^{3k+\beta k}\int \frac{\lambda^{(1+|m|)k}(D^m u_0)(\lambda^k x) - \lambda^{(1+|m|)k}(D^m u_0)(\lambda^k y)}{|\lambda^k x - \lambda^k y|^{3+\beta}}\,dy \\ &= \lambda^{(1+|m|+\beta)k}\int \frac{(D^m u_0)(\lambda^k x) - D^m u_0(y)}{|\lambda^k x - y|^{3+\beta}}\,dy \\ &= \lambda^{(1+|m|+\beta)k}(\Lambda^\beta D^m u_0)(\lambda^k x).\end{aligned} \tag{4.23}$$

Then we rewrite $\Lambda^\beta D^m e^{t\Delta}u_0 = e^{t\Delta}\Lambda^\beta D^m u_0$ by the boundedness of these derivatives. Applying Lemma 4.2 leads us to

$$\sup_{t\in[1,\lambda^2]}|\Lambda^\beta D^m e^{t\Delta}u_0(x,t)| \lesssim_{\lambda,u_0,\alpha,\beta} \frac{1}{(|x|+1)^{1+|m|+\beta}}.$$

□

The next lemma states that derivatives of the heat evolution enjoy improved decay when the initial data is Hölder continuous.

Lemma 4.5. *Assume $f \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^3\setminus\{0\})$ where $0<\alpha<1$ and satisfies, for some $\lambda>1$,*

$$\lambda^{\sigma k} f(\lambda^k x) = f(x),$$

where $\sigma < 3$. Then

$$\sup_{t\in[1,\lambda^2]}\|\nabla e^{t\Delta}f\|_{L^\infty(B_R^c)} \lesssim_\lambda \|f\|_{C^\alpha(A_0)}R^{-(\sigma+\alpha)}.$$

Proof. It suffices to prove, for $t\in[1,\lambda^2]$,

$$\sup_{t\in[1,\lambda^2]}\|\nabla e^{t\Delta}f\|_{L^\infty(A_k)} \lesssim \lambda^{-(\sigma+\alpha)k}\|f\|_{C^\alpha(A_0)},$$

where the suppressed constants are independent of k. Let $x\in A_k$. Note that

$$\partial_i e^{t\Delta}f(x) = \int_{\mathbb{R}^3}\frac{c}{t^{\frac52}}(x_i-y_i)e^{-\frac{|x-y|^2}{4t}}f(y)\,dy. \tag{4.24}$$

Since $(x_i - y_i)e^{-\frac{|x-y|^2}{4t}}$ is mean zero on spheres centered at x,

$$\begin{aligned}\partial_i e^{t\Delta} f(x) &\lesssim \int_{|x-y|<\lambda^{k-1}} (x_i - y_i)e^{-\frac{|x-y|^2}{4\lambda^2}} (f(y) - f(x))\, dy \\ &\quad + \int_{x-y\in A_k^*} (x_i - y_i)e^{-\frac{|x-y|^2}{4\lambda^2}} f(y)\, dy \\ &\quad + \int_{|x-y|\geq\lambda^{k+2}} (x_i - y_i)e^{-\frac{|x-y|^2}{4\lambda^2}} f(y)\, dy \\ &=: I_1(x) + I_2(x) + I_3(x).\end{aligned} \tag{4.25}$$

For I_1, because $x, y \in B_{\lambda^{k-1}}$,

$$|f(x) - f(y)| = \frac{\lambda^{-\sigma k}|f(\lambda^{-k}x) - f(\lambda^{-k}y)|}{\lambda^{k\alpha}|\lambda^{-k}x - \lambda^{-k}y|^\alpha}|x-y|^\alpha \lesssim [f]_{C^\alpha(A_0)}\frac{1}{\lambda^{k(\sigma+\alpha)}}|x-y|^\alpha.$$

Therefore,

$$|I_1|(x) \lesssim \frac{[f]_{C^\alpha(A_0)}}{\lambda^{k(\sigma+\alpha)}}\int_{|x-y|<\lambda^{k-1}} |x-y|^{1+\alpha}e^{-\frac{|x-y|^2}{4\lambda^2}}\, dy.$$

By the growth of the Gaussian, the integral above is bounded independently of k for $t \in [1, \lambda^2]$. So, we have

$$\|I_1\|_{L^\infty(A_k)}(t) \lesssim_{\alpha,\lambda} \frac{[f]_{C^\alpha(A_0)}}{\lambda^{k(\sigma+\alpha)}}.$$

This determines the power of R in the lemma's statement.

For I_2, because $x \in A_k$ and $x - y \in A_k^*$, $|y| < \lambda^{k+2} + \lambda^{k+1}$,

$$\|I_2\|_{L_x^\infty(A_k)} \leq \lambda^{k+2}e^{-\frac{\lambda^{2k-2}}{4\lambda^2}}\|f\|_{L^1\left(|y|<\lambda^{k+1}(\lambda+1)\right)}. \tag{4.26}$$

Because $|f(y)| \lesssim_\lambda \frac{\|f\|_{L^\infty(A_0)}}{|y|^\sigma}$

$$\|f\|_{L^1\left(|y|<\lambda^{k+1}(\lambda+1)\right)} \lesssim_\lambda \|f\|_{L^\infty(A_0)} \int_{|y|<\lambda^{k+1}(\lambda+1)} \frac{1}{|y|^\sigma}\, dy \lesssim_\lambda \lambda^{(3-\sigma)k} \|f\|_{L^\infty(A_0)}, \tag{4.27}$$

for $\sigma < 3$. Therefore,

$$\|I_2\|_{L_x^\infty(A_k)} \lesssim_\lambda \lambda^{(4-\sigma)k} \|f\|_{L^\infty(A_0)} e^{-\frac{\lambda^{2k-2}}{4\lambda^2}}. \tag{4.28}$$

As $k \to \infty$, the Gaussian dominates any algebraic growth. Hence,

$$\|I_2\|_{L^\infty(A_k)} \lesssim_\lambda \lambda^{-(\sigma+\alpha)k} \|f\|_{C^\alpha(A_0)}.$$

Finally, for I_3, we sum over the annuli $A_{k'}$, $k' \geq k+2$, and find

$$\begin{aligned} \|I_3\|_{L^\infty(A_k)}(t) &\leq \sum_{k' \geq k+2} \lambda^{k'} e^{-\frac{\lambda^{2k'}}{4\lambda^2}} \|f\|_{L^1(A_{k'}^*)} \\ &\leq \sum_{k' \geq k+2} e^{-\frac{\lambda^{2k'}}{4\lambda^2}} \lambda^{(4-\sigma)k'} \|f\|_{L^\infty(A_0)}, \end{aligned} \tag{4.29}$$

where we used the σ-scaling. Again, the Gaussian dominates any algebraic growth so the preceding series is summable. We conclude

$$\|I_3\|_{L^\infty(A_k)} \lesssim_\lambda \lambda^{-(\sigma+\alpha)k} \|f\|_{L^\infty(A_0)}.$$

$\square$

Lemma 4.5 suggests a similar result should hold for fractional derivatives of order between α and 1.

Lemma 4.6. *Assume* $f \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ *where* $0 < \alpha < 1$ *and satisfies, for some* $\lambda > 1$,

$$\lambda^{\sigma k} f(\lambda^k x) = f(x),$$

where $\sigma < 3$. *Fix* $\gamma \in (\alpha, 1)$. *Then*

$$\sup_{t\in[1,\lambda^2]} \|\Lambda^\gamma e^{t\Delta} f\|_{L^\infty(B_R^c)} \lesssim_\lambda \|f\|_{C^\alpha(A_0)} R^{-(\sigma+\alpha)}.$$

Proof. It suffices to prove for $t \in [1, \lambda^2]$,

$$\sup_{t\in[1,\lambda^2]} \|\Lambda^\gamma e^{t\Delta} f\|_{L^\infty(A_k)} \lesssim \lambda^{-(\sigma+\alpha)k} \|f\|_{C^\alpha(A_0)},$$

where the suppressed constants are independent of k. Let $x \in A_k$. Note that

$$\Lambda^\gamma e^{t\Delta} f(x) = \int_{\mathbb{R}^3} \frac{c}{t^{\frac{3}{2}}} \Lambda^\gamma_x e^{-\frac{|x-y|^2}{4t}} f(y)\, dy. \tag{4.30}$$

Since $\left(\Lambda^\gamma_x e^{-\frac{|x-y|^2}{4t}}\right)^\wedge (0) = 0$, $\Lambda^\gamma_x e^{-\frac{|x-y|^2}{4t}}$ is mean zero on $\mathbb{R}^3$. Hence,

$$\begin{aligned} \Lambda^\gamma e^{t\Delta} f(x) &\lesssim \int_{|x-y|<\lambda^k} \Lambda^\gamma_x e^{-\frac{|x-y|^2}{4\lambda^2}} (f(y) - f(x))\, dy \\ &\quad + \int_{|x-y|\geq\lambda^k} \Lambda^\gamma_x e^{-\frac{|x-y|^2}{4\lambda^2}} (f(y) - f(x))\, dy \\ &=: I_1(x) + I_2(x). \end{aligned} \tag{4.31}$$

For I_1, because $x, y \in B_{\lambda^{k-1}}$

$$|f(x) - f(y)| = \frac{\lambda^{-\sigma k}|f(\lambda^{-k}x) - f(\lambda^{-k}y)|}{\lambda^{k\alpha}|\lambda^{-k}x - \lambda^{-k}y|^\alpha} |x-y|^\alpha \lesssim [f]_{C^\alpha(A_0)} \frac{1}{\lambda^{k(\sigma+\alpha)}} |x-y|^\alpha.$$

Therefore,

$$|I_1|(x) \lesssim \frac{[f]_{C^\alpha(A_0)}}{\lambda^{k(\sigma+\alpha)}} \int_{|x-y|<\lambda^k} |x-y|^\alpha \Lambda_x^\gamma e^{-\frac{|x-y|^2}{4\lambda^2}}\, dy.$$

We claim that

$$\Lambda_x^\gamma e^{-\frac{|x|^2}{4\lambda^2}} \lesssim_\lambda \frac{1}{|x|^{3+\gamma}}. \tag{4.32}$$

This implies that

$$\|I_1\|_{L^\infty(A_k)} \lesssim_{\alpha,\lambda} \frac{[f]_{C^\alpha(A_0)}}{\lambda^{k(\sigma+\alpha)}} \int_{|x-y|<\lambda^k} \frac{1}{|x-y|^{3+\gamma-\alpha}}\, dy \lesssim_{\alpha,\lambda} \frac{[f]_{C^\alpha(A_0)}}{\lambda^{k(\sigma+\alpha)}},$$

which determines the power of R in the lemma's statement.

For I_2, by the scaling of f and claim (4.32),

$$\begin{aligned} I_2(x) \lesssim_\lambda & \int_{|x-y|\geq\lambda^k} \Lambda_x^\gamma e^{-\frac{|x-y|^2}{4\lambda^2}} f(x)\, dy + \int_{|x-y|\geq\lambda^k, |y|\geq\lambda^k} \Lambda_x^\gamma e^{-\frac{|x-y|^2}{4\lambda^2}} f(y)\, dy \\ & + \int_{|x-y|\geq\lambda^k, |y|<\lambda^k} \Lambda_x^\gamma e^{-\frac{|x-y|^2}{4\lambda^2}} f(y)\, dy \\ \lesssim_\lambda & \lambda^{-k\sigma} \int_{|x-y|\geq\lambda^k} \frac{1}{|x-y|^{3+\gamma}}\, dy + \lambda^{-k(3+\gamma)} \int_{|x-y|\geq\lambda^k, |y|<\lambda^k} \frac{1}{|y|^\sigma}\, dy \\ \lesssim_\lambda & \lambda^{-k(\sigma+\gamma)}, \end{aligned} \tag{4.33}$$

for $\sigma < 3$. Because $\gamma > \alpha$ we conclude that

$$\|\Lambda^\gamma e^{t\Delta} f\|_{L^\infty(A_k)} \lesssim_\lambda \lambda^{-(\sigma+\alpha)k} \|f\|_{C^\alpha(A_0)}.$$

To show the claim (4.32), for $x \in A_k$, we write

$$\begin{aligned}
\Lambda^\gamma e^{-\frac{|x|^2}{4\lambda^2}} &\lesssim \int_{\mathbb{R}^3} \frac{1}{|x-y|^{3+\gamma}} \left(e^{-\frac{|y|^2}{4\lambda^2}} - e^{-\frac{|x|^2}{4\lambda^2}} \right) dy \\
&\lesssim_\lambda \int_{|x-y|<\lambda^{k-1}} \frac{1}{|x-y|^{3+\gamma}} \left(e^{-\frac{|y|^2}{4\lambda^2}} - e^{-\frac{|x|^2}{4\lambda^2}} \right) dy \\
&+ \int_{x-y\in A_k^*} \frac{1}{|x-y|^{3+\gamma}} \left(e^{-\frac{|y|^2}{4\lambda^2}} - e^{-\frac{|x|^2}{4\lambda^2}} \right) dy \\
&+ \int_{|x-y|\geq\lambda^{k+2}} \frac{1}{|x-y|^{3+\gamma}} \left(e^{-\frac{|y|^2}{4\lambda^2}} - e^{-\frac{|x|^2}{4\lambda^2}} \right) dy \\
&=: J_1(x) + J_2(x) + J_3(x).
\end{aligned} \tag{4.34}$$

For J_1, since x and y are close, we can write

$$\begin{aligned}
J_1(x) &\lesssim \int_{|x-y|<\lambda^{k-1}} \frac{1}{|x-y|^{2+\gamma}}\, dy \|\nabla e^{-\frac{|y|^2}{4\lambda^2}}\|_{L^\infty(A_k)} \\
&\lesssim \lambda^{(1-\gamma)(k-1)}\lambda^{k+2} e^{-\frac{\lambda^{2k-2}}{4\lambda^2}} \lesssim_\lambda e^{-\lambda^{2k}}.
\end{aligned} \tag{4.35}$$

For the mid-field, J_2, we find

$$J_2(x) \lesssim \||z|^{-3-\gamma}\|_{L^1(A_k^*)} e^{-\frac{\lambda^{2k}}{4\lambda^2}} \lesssim_\lambda e^{-\lambda^{2k}}. \tag{4.36}$$

Lastly, for J_3,

$$J_3(x) \lesssim \int_{|x-y|\geq\lambda^{k+2}} \frac{1}{|x-y|^{3+\gamma}} \left(e^{-\frac{|y|^2}{4\lambda^2}} - e^{-\frac{|x|^2}{4\lambda^2}} \right) dy \lesssim_\lambda \lambda^{-(k+2)(3+\gamma)}. \tag{4.37}$$

Because $\lambda^{k-1} \leq |x| < \lambda^{k+2}$ we can rewrite these bounds in terms of decay in x to get (4.32). $\square$

Remark 1. We expect $\alpha < \gamma$ is necessary in Lemma 4.6. If this were true for $\alpha = \gamma$, this result would imply for DSS function f, that $f \in C^\alpha_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ which is equivalent to $\Lambda^\alpha f \in L^\infty_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ because f would have decay as in (4.22) with $|m| = 0$.

This should not be true for general elements of $C^\alpha_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ because C^α is equivalent to $B^\alpha_{\infty,\infty}(\mathbb{R}^3 \setminus \{0\})$, i.e. $\hat{f}$ decays like $2^{-\alpha k}$ in annuli $2^k \le \xi < 2^{k+1}$ on the Fourier side. It follows that $\Lambda^\alpha f$ is locally bounded on the Fourier side, i.e. in $(B^0_{\infty,\infty})_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$. This is *strictly* weaker than $L^\infty_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$.

4.3 A commutator estimate

The following section is dedicated to proving estimates for $\Lambda^\beta(P_0 \otimes P_0)$. The following lemmas allow us to navigate a commutator to obtain better decay for solutions with C^α data and eliminate logarithms in our decay results for solutions with $C^{1,\alpha}$ data.

First, we show the heat evolution of locally Hölder data remains in the same Hölder space, locally.

Lemma 4.7. *Fix a multi-index $m \in (\mathbb{N}_0)^3$, $|m| \le 1$. If $u_0 \in C^{|m|,\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ is DSS, then*

$$\|P_0\|_{C^{\max\{|m|,\alpha\}}(A_1)}(t) \lesssim \|u_0\|_{C^{\max\{|m|,\alpha\}}(A_1)}, \tag{4.38}$$

for $t > 0$.

Proof. For $u_0 \in C^{|m|,\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ it follows that $e^{t\Delta}u_0 \in C^{|m|,\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$. In fact, choosing a cutoff ϕ, a smooth function such that $\operatorname{supp}\phi \subset B_{\lambda^{-2}}$, $\phi|_{B_{\lambda^{-3}}} \equiv 1$ and
$\operatorname{supp}\nabla\phi \subset \{x : \lambda^{-3} \le |x| \le \lambda^{-2}\}$, we see that

$$\|e^{t\Delta}(u_0(1-\phi))\|_{C^{\max\{|m|,\alpha\}}(A_1)} \lesssim \|u_0\|_{C^{\max\{|m|,\alpha\}}(A_1)},$$

for $t > 0$, by Lemma 2.1. For the remaining part, $e^{t\Delta}(u_0\phi)$, fix $x \in A_1$. We bound $e^{t\Delta}(u_0\phi)$

in $C^1(A_1)$ for $t > 0$

$$\begin{aligned} \partial_i e^{t\Delta}(u_0\phi)(x) &\lesssim \int_{\mathbb{R}^3} \frac{(x_i - y_i)}{t^{\frac{5}{2}}} e^{-\frac{|x-y|^2}{4t}} (u_0\phi)(y)\, dy \\ &\lesssim \frac{\lambda + \lambda^{-2}}{t^{\frac{5}{2}}} e^{-\frac{(\lambda+\lambda^{-2})^2}{4t}} \int_{\mathbb{R}^3} (u_0\phi)(y)\, dy \\ &\lesssim \|u_0\|_{L^\infty_{\mathrm{loc}}} \frac{\lambda + \lambda^{-2}}{t^{\frac{5}{2}}} e^{-\frac{(\lambda+\lambda^{-2})^2}{4t}} \int_{B_{\lambda^{-2}}} \frac{1}{|y|}\, dy \\ &\lesssim \|u_0\|_{L^\infty_{\mathrm{loc}}} \frac{\lambda + \lambda^{-2}}{t^{\frac{5}{2}}} e^{-\frac{(\lambda+\lambda^{-2})^2}{4t}} \lambda^{-4} \\ &\lesssim_\lambda \|u_0\|_{L^\infty_{\mathrm{loc}}}. \end{aligned} \tag{4.39}$$

Hence,

$$\|\nabla e^{t\Delta}(u_0\phi)\|_{L^\infty(A_1)}(t) \lesssim \|u_0\|_{L^\infty_{\mathrm{loc}}},$$

for $t > 0$. Thus $e^{t\Delta}(u_0\phi) \in C^1(A_1) \subset C^\alpha(A_1)$ for $t > 0$, with

$$\|\nabla e^{t\Delta}(u_0\phi)\|_{C^{\max\{|m|,\alpha\}}(A_1)}(t) \lesssim \|u_0\|_{C^{\max\{|m|,\alpha\}}(A_1)}.$$

This implies the conclusion(4.38). $\square$

Now we may prove an estimate to navigate the commutator $\Lambda^\beta(\cdot \otimes \cdot)$.

Lemma 4.8 (Commutator decay)**.** *Fix $\alpha \in (0,1)$ and a multi-index $m \in (\mathbb{N}_0)^3$, $|m| \leq 1$. If $u_0 \in C^{|m|,\alpha}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ is DSS, then for $t \in [1, \lambda^2]$ and $\beta \in \big(0, (2-|m|)\alpha\big)$,*

$$\Lambda^\beta D^m \cdot (P_0 \otimes P_0)(x,t) \lesssim_{u_0,\lambda,\alpha,\beta} P_{0i}\Lambda^\beta D^m P_{0j}(x,t) + P_{0j}\Lambda^\beta D^m P_{0i}(x,t) + \mathcal{O}\bigg(\frac{1}{|x|^{2+|m|+\beta}}\bigg),$$

for $1 \leq i,j \leq 3$, where P_{0i} is the i^{th} component of P_0.

We use this in the proof of decay for flows with C^α_{loc} data with $|m| = 0$ and $\beta < \alpha$ and the $C^{1,\alpha}_{\mathrm{loc}}$ proof with $|m| = 1$ and $\beta = \gamma > \alpha$, but we combine the two results here due to the similarity.

Proof. For $|m| = 0$, we can rewrite each term as

$$\begin{aligned}
\Lambda^\beta(P_{0i}P_{0j})(x,t) &= \int \frac{P_{0i}(y)P_{0j}(y) - P_{0i}(x)P_{0j}(x)}{|x-y|^{3+\beta}}\,dy \\
&= \int \frac{P_{0i}(y)P_{0j}(y) - P_{0i}(x)P_{0j}(y) + P_{0i}(x)P_{0j}(y) - P_{0i}(x)P_{0j}(x)}{|x-y|^{3+\beta}}\,dy \\
&= \int P_{0i}(x)\frac{P_{0j}(y) - P_{0j}(x)}{|x-y|^{3+\beta}} + \frac{P_{0j}(y)(P_{0i}(y) - P_{0i}(x))}{|x-y|^{3+\beta}}\,dy \\
&= P_{0i}(x,t)\Lambda^\beta P_{0j}(x,t) + \int P_{0j}(x)\frac{(P_{0i}(y) - P_{0i}(x))}{|x-y|^{3+\beta}}\,dy \\
&+ \int \frac{(P_{0i}(y) - P_{0i}(x))(P_{0j}(y) - P_{0j}(x))}{|x-y|^{3+\beta}}\,dy \\
&= P_{0i}\Lambda^\beta P_{0j}(x,t) + P_{0j}\Lambda^\beta P_{0i}(x,t) \\
&\quad + \int \frac{(P_{0i}(x) - P_{0i}(y))(P_{0j}(x) - P_{0j}(y))}{|x-y|^{3+\beta}}\,dy,
\end{aligned}$$

where we have suppressed the dependence of P_0 on t in the integrals for readability.

For $|m| = 1$, because the operator $\Lambda^\beta\partial_k P_0(x)$ integrates $P_0(x) - P_0(y)$ against the kernel $\frac{x_k - y_k}{|x-y|^{5+\beta}}$ we find, similarly, that

$$\begin{aligned}
\Lambda^\beta\partial_k(P_{0i}P_{0j})(x,t) &= P_{0i}\Lambda^\beta\partial_k P_{0j}(x,t) + P_{0j}\Lambda^\beta\partial_k P_{0i}(x,t) \\
&\quad + \int \frac{(x_k - y_k)}{|x-y|^{5+\beta}}(P_{0i}(x) - P_{0i}(y))(P_{0j}(x) - P_{0j}(y))\,dy.
\end{aligned}$$

Hence, the asymptotics of the final term follow from bounding an integral of the form

$$\int \frac{1}{|x-y|^{3+|m|+\beta}}(P_{0i}(x) - P_{0i}(y))(P_{0j}(x) - P_{0j}(y))\,dy, \qquad |m| \le 1.$$

We break the above integral into three domains: $|x-y| < \frac{|x|}{2}$, $|x-y| > \frac{|x|}{2}$ and $|y| > \frac{|x|}{2}$, and $|y| < \frac{|x|}{2}$. First, we consider $|x-y| < \frac{|x|}{2}$, i.e.,

$$\int_{|x-y|<\frac{|x|}{2}} \frac{1}{|x-y|^{3+|m|+\beta}}(P_{0i}(x) - P_{0i}(y))(P_{0j}(x) - P_{0j}(y))\,dy. \tag{4.40}$$

By Lemma 4.7,

$$\|P_0\|_{C^{\max\{|m|,\alpha\}}(A_1)}(t) \lesssim \|u_0\|_{C^{\max\{|m|,\alpha\}}(A_1)}, \tag{4.41}$$

for $t > 0$. By DSS scaling, we have

$$\sup_{y\in B_{|x|/2}(x)} \frac{|P_0(x,t)-P_0(y,t)|}{|x-y|^\alpha} \lesssim_\lambda \frac{1}{|x|^{1+\alpha}}[P_0]_{C^\alpha(A_1)},$$

and for $|m|=1$,

$$\sup_{y\in B_{|x|/2}(x)} \frac{|P_0(x,t)-P_0(y,t)|}{|x-y|} \lesssim_\lambda \frac{1}{|x|^2}[P_0]_{C^1(A_1)},$$

uniformly in t. Hence,

$$\begin{aligned}
&\left|\int_{|x-y|<\frac{|x|}{2}} \frac{1}{|x-y|^{3+|m|+\beta}}(P_{0i}(x)-P_{0i}(y))(P_{0j}(x)-P_{0j}(y))\,dy\right| \\
&\lesssim \int_{|x-y|<\frac{|x|}{2}} \frac{1}{|x-y|^{3+|m|+\beta-2\max\{|m|,\alpha\}}} \frac{|P_{0i}(x)-P_{0i}(y)|}{|x-y|^{\max\{|m|,\alpha\}}} \frac{|P_{0j}(x)-P_{0j}(y)|}{|x-y|^{\max\{|m|,\alpha\}}}\,dy \\
&\lesssim_{\lambda,u_0} \frac{1}{|x|^{2+2\max\{|m|,\alpha\}}} \int_{|x-y|<\frac{|x|}{2}} \frac{1}{|x-y|^{3+|m|+\beta-2\max\{|m|,\alpha\}}}\,dy \\
&\lesssim_{\lambda,u_0} \frac{1}{|x|^{2+|m|+\beta}},
\end{aligned} \tag{4.42}$$

provided $\beta < \max\{|m|, 2\alpha-|m|\}$.

Next, we consider the region where $|x-y| > \frac{|x|}{2}$ and $|y| > \frac{|x|}{2}$. Because $|P_0(y,t)| \lesssim_{\lambda,u_0} \frac{1}{|y|}$ by Lemma 4.2 for $|y| > |x|/2$, we find

$$\begin{aligned}
&\left|\int_{|x-y|>\frac{|x|}{2};|y|>\frac{|x|}{2}} \frac{1}{|x-y|^{3+|m|+\beta}}(P_{0i}(x)-P_{0i}(y))(P_{0j}(x)-P_{0j}(y))\,dy\right| \\
&\lesssim_{\lambda,u_0} \frac{1}{|x|^2}\int_{|x-y|>\frac{|x|}{2};|y|>\frac{|x|}{2}} \frac{1}{|x-y|^{3+|m|+\beta}}\,dy \lesssim_{\lambda,u_0} \frac{1}{|x|^{2+|m|+\beta}},
\end{aligned} \tag{4.43}$$

for $\beta > 0$. The final region is $|y| < \frac{|x|}{2}$. By the same decay for P_0, we find

$$\begin{aligned}
&\left|\int_{|y|<\frac{|x|}{2}} \frac{1}{|x-y|^{3+|m|+\beta}}(P_{0i}(x)-P_{0i}(y))(P_{0j}(x)-P_{0j}(y))\,dy\right| \\
&\lesssim_{\lambda,u_0} \int_{|y|<\frac{|x|}{2}} \frac{1}{|x-y|^{3+|m|+\beta}}\frac{1}{|y|^2}\,dy \\
&\lesssim_{\lambda,u_0} \frac{1}{|x|^{3+|m|+\beta}}\int_{|y|<\frac{|x|}{2}} \frac{1}{|y|^2}\,dy \\
&\lesssim_{\lambda,u_0} \frac{1}{|x|^{2+|m|+\beta}},
\end{aligned} \tag{4.44}$$

and our proof is complete. □

This lemma applies to both $\beta < \alpha$ and $\beta \geq \alpha$. For clarity, we use γ for fractional derivatives greater than α, as is done in Lemma 4.6.

4.4 Properties of Picard iterates

In the follow subsection, we prove properties for higher Picard iterates of data in L^q_{loc} or $L^{3,\infty} \cap L^q(B)$, for some ball $B \subset \mathbb{R}^3$. We begin by showing the pointwise decay of P_0 evolving from sub-critical data is inherited by higher Picard iterates.

Lemma 4.9. *Fix $3 < q \leq \infty$. Assume, for all $(x,t) \in \mathbb{R}^3 \times (0,\infty)$,*

$$P_0(x,t) \lesssim \frac{1}{\sqrt{t}^{\frac{3}{q}}(|x|+\sqrt{t})^{1-\frac{3}{q}}}.$$

Then for all $k \in \mathbb{N}$ and all $(x,t) \in \mathbb{R}^3 \times (0,\infty)$,

$$P_k(x,t) \lesssim_{k,P_0} \frac{1}{\sqrt{t}^{\frac{3}{q}}(|x|+\sqrt{t})^{1-\frac{3}{q}}}.$$

Proof. Note that

$$P_k - P_0 = B(P_{k-1},P_{k-1}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(P_{k-1}\otimes P_{k-1})(\cdot,s)\,ds,$$

decays algebraically faster than P_{k-1} by Lemma 4.1. Namely, using Lemma 4.1,

$$
\begin{aligned}
|P_k - P_0| &\lesssim_{k,P_0} \int_0^t \int_{\mathbb{R}^3} \frac{1}{(|x-y|+\sqrt{t-s})^4} \frac{1}{\sqrt{s}^{\frac{6}{q}}(|y|+\sqrt{s})^{2-\frac{6}{q}}}\, dy\, ds \\
&\lesssim_{k,P_0} \frac{1}{\sqrt{t}^{\frac{6}{q}-1}(|x|+\sqrt{t})^{2-\frac{6}{q}}}.
\end{aligned}
\tag{4.45}
$$

This implies P_k and P_0 have the same decay. $\square$

Now, we show an estimate of the heat kernel in Lorentz space which will be used later in the proof of local sub-critical inclusion of Picard iterates.

Lemma 4.10. *Let $B = B_R(x_0)$ and $B' := B_r(x_0)$ where $0 < r < R < \infty$. Then for $0 < t < \infty$,*

$$
\left\| \| e^{-\frac{|x-y|^2}{4t}} (1-\chi_B) \|_{L_y^{\frac{3}{2},1}} \right\|_{L_x^\infty(B')} \lesssim_{R,r} e^{\frac{-(R-r)^2}{4t}}. \tag{4.46}
$$

Proof. First, assume without loss of generality that $x_0 = 0$. Then letting $x \in B'$,

$$
\| e^{-\frac{|x-y|^2}{4t}} (1-\chi_B) \|_{L_y^{\frac{3}{2},1}} = \frac{3}{2} \int_0^\infty m \left\{ y : e^{-\frac{|x-y|^2}{4t}} (1-\chi_B(y)) \geq s \right\}^{\frac{2}{3}} ds, \tag{4.47}
$$

where m is Lebesgue measure. The above set can be written as

$$
A(x,t) = \{ y : |x-y| \leq \sqrt{-4t\ln(s)},\ |y| > R \} = B(x, (-4t\ln(s))^{\frac{1}{2}}) \setminus B_R(0).
$$

This is well-defined because $t \geq 0$ and $s \leq 1$.

We, therefore, conclude

$$\begin{aligned}
\left\| \| e^{-\frac{|x-y|^2}{4t}} (1-\chi_B(y)) \|_{L^{\frac{3}{2},1}_y} \right\|_{L^\infty_x(B')} &\lesssim \left\| \int_0^\infty m(A(x,t))^{\frac{2}{3}}\, ds \right\|_{L^\infty_x(B)} \\
&\lesssim \int_0^{e^{\frac{-(R-r)^2}{4t}}} |-4t\ln(s)|\, ds \\
&\lesssim 4t \left(e^{\frac{-(R-r)^2}{4t}} \frac{(R-r)^2}{4t} + e^{\frac{-(R-r)^2}{4t}} \right) \\
&\lesssim_{R,r} e^{\frac{-(R-r)^2}{4t}}.
\end{aligned} \tag{4.48}$$

□

This leads to a local *a priori* inclusion for Picard iterates.

Lemma 4.11. *Let* $B = B_R(x_0)$ *and* $B' = B_r(x_0)$ *where* $0 < r < R < \infty$. *Let* $u_0 \in L^{3,\infty}$ *with* $u_0|_B \in L^q(B)$, *for some* $3 < q \leq \infty$. *It follows that* $P_k \in L^\infty(0,\infty; L^q(B'))$ *with*

$$\|P_{k_0}\|_{L^\infty(0,\infty;L^q(B_0))} \leq C(k_0, R, \|u_0\|_{L^{3,\infty}}).$$

Proof. Because $P_k \in \mathcal{K}_q$ when $q > 3$ for $\tau > 0$,

$$\sup_{\tau<s<\infty} \|P_k\|_{L^q}(s) \lesssim_{\tau,k} \|u_0\|_{L^{3,\infty}}.$$

So, we need only prove the estimate for a short time. Let $\{B_k\}_{k=0}^{k_0}$ be a collection of concentric balls about x_0 of shrinking radii $\alpha^{k+1}R$ for some $\alpha \in (0,1)$. Fix $k_0 \in \mathbb{N}_0$. Choose α such that $r = \alpha^{k_0+1}R$.

We can estimate $P_0 = e^{t\Delta} u_0$ as a near-field and a far-field term by decomposing

$$
\begin{aligned}
\|P_0\|_{L^q(B_0)}(t) &= \left\| \int_{\mathbb{R}^3} \frac{1}{t^{\frac{3}{2}}} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy \right\|_{L^q(B_0)} \\
&= \left\| \left(\int_{B^c} + \int_B \right) t^{-\frac{3}{2}} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy \right\|_{L^q(B_0)} \\
&\lesssim \|u_0\|_{L^{3,\infty}} t^{-\frac{3}{2}} \left\| \|e^{-\frac{|x-y|^2}{4t}} (1-\chi_B(y))\|_{L_y^{\frac{3}{2},1}} \right\|_{L_x^q(B_0)} \\
&\quad + \left\| e^{t\Delta} (\chi_B(y) u_0) \right\|_{L^q(\mathbb{R}^3)}.
\end{aligned}
\tag{4.49}
$$

We apply Lemma 4.10 to find

$$
\begin{aligned}
\left\| \|e^{-\frac{|x-y|^2}{4t}} (1-\chi_B(y))\|_{L_y^{\frac{3}{2},1}} \right\|_{L_x^q(B_0)} &\lesssim_{R,\alpha,q} \left\| \|e^{-\frac{|x-y|^2}{4t}} (1-\chi_B(y))\|_{L_y^{\frac{3}{2},1}} \right\|_{L_x^\infty(B_0)} \\
&\lesssim_{R,\alpha} e^{\frac{-(R(1-\alpha))^2}{4t}}.
\end{aligned}
\tag{4.50}
$$

For the near field term, by [65, Lemma 5.1],

$$
\|e^{t\Delta}(u_0 \chi_B)\|_{L^q(\mathbb{R}^3)} \lesssim \|u_0 \chi_B\|_{L^q(\mathbb{R}^3)} \lesssim \|u_0\|_{L^q(B)}. \tag{4.51}
$$

Because these estimates are uniform in time for $t \le \tau$ it follows that

$$
\|P_0\|_{L^\infty(0,t;L^q(B'))} \lesssim_{R,\alpha} \|u_0\|_{L^{3,\infty}} + \|u_0\|_{L^q(B)}. \tag{4.52}
$$

If $k_0 = 0$, we are done. Otherwise, for $k_0 > 0$, we continue by induction. Observe

$$
B(P_{k-1}, P_{k-1}) = B(P_{k-1}\chi_{B_{k-1}}, P_{k-1}) + B(P_{k-1}(1-\chi_{B_{k-1}}), P_{k-1}).
$$

In the near-field, by (2.19),

$$
\begin{aligned}
\|B(P_{k-1}\chi_{B_{k-1}}, P_{k-1})\|_{L^q(B_k)}(t) &\lesssim_q \|P_k\|_{\mathcal{K}_\infty} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{1}{2}}} \|P_{k-1}\|_{L^q(B_{k-1})}(s)\, ds \\
&\lesssim_{k,q} \|u_0\|_{L^{3,\infty}} \|P_{k-1}\|_{L^q(B_{k-1})}(t).
\end{aligned}
$$

For far-field, by the pointwise estimate for the kernel K of the Oseen tensor in Lemma 3.4 and $P_{k-1} \in \mathcal{K}_4$, we have

$$
\begin{aligned}
\|B(P_{k-1}(1-\chi_{B_{k-1}}), P_{k-1})\|_{L^q(B_k)}(t) &\lesssim \left\| \int_0^t \int_{B_{k-1}^c} \frac{P_{k-1} \otimes P_{k-1}(y,s)}{(|x-y| + \sqrt{t-s})^4}\, dy\, ds \right\|_{L^q(B_k)} \\
&\lesssim_{R,\alpha,k,q} \| |\cdot|^{-4} \|_{L^2(|\cdot| > R(\alpha^k - \alpha^{k+1}))} \int_0^t \|P_{k-1}\|_{L^4}^2\, ds \\
&\lesssim_{R,\alpha,k,q} \int_0^t s^{(-1+\frac{3}{4})}\, ds \lesssim_{R,\alpha,k,q} t^{\frac{3}{4}}.
\end{aligned}
$$

This shows $B(P_{k-1}, P_{k-1}) \in L^\infty(0,\infty; L^q(B_0))$ whenever P_{k-1} is in $L^\infty(0,T; L^q(B_{k-1})$. Hence,

$$
P_k = P_0 - B(P_{k-1}, P_{k-1}) \in L^\infty(0,\infty; L^q(B_k)).
$$

This extends up to k_0 so $P_{k_0} \in L^\infty(0,\infty; L^q(B'))$, with

$$
\|P_{k_0}\|_{L^\infty(0,\infty;L^q(B_0))} \leq C(k_0, R, \|u_0\|_{L^{3,\infty}}).
$$

$\square$

Remark 2. Together, (4.49) and (4.50) imply

$$
\|e^{t\Delta}(u_0(1-\chi_B))\|_{L^\infty(B')}(t) \lesssim_T \|u_0\|_{L^{3,\infty}} t^{-\frac{3}{2q}}, \tag{4.53}
$$

provided $t < T$, for any given time T. Combining this with the classical estimates for the heat evolution of $u_0 \in L^p$ (Lemma 2.1) we also have $\|e^{t\Delta} u_0 \chi_B\|_{L^\infty(B')} \lesssim t^{-\frac{3}{2q}} \|u_0\|_{L^q(B)}$.

Hence,

$$\|e^{t\Delta}u_0\|_{L^\infty(B')}(t) \lesssim_{u_0,T} t^{-\frac{3}{2q}}.$$

We use this fact in the asymptotic expansion in Theorem 1.7.

4.5 A-priori estimate for $L^{3,\infty}$ weak solutions

In this subsection, we prove Proposition 4.12 which we use to control $u - P_k$ in the far-field in the proof of Theorem 1.7.

Proposition 4.12. *Fix $q \in (3/2, 3)$, $T > 0$ and $k \in \mathbb{N}$. Assume $u_0 \in L^{3,\infty}$ and is divergence free. Let u be an $L^{3,\infty}$ weak solution with initial data u_0. It follows that, for $r = \frac{2q}{2q-3}$,*

$$\|u - P_k\|_{L^r(0,T;L^q)} \lesssim_{k,q,u_0} T^{\frac{1}{2}}.$$

Proof. First, we decompose

$$u - P_k = -B(u - P_{k-1}, u - P_{k-1}) - B(P_k, u - P_{k-1}) - B(u - P_{k-1}, P_k),$$

for $k \geq 1$. We show this estimate holds for each term. By Yamazaki [66, Theorem 2.2] each of these terms has the bound

$$\begin{aligned}\|B(u - P_k, u - P_k)\|_{L^{q,1}} &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|(u - P_k)^2\|_{L^{q,1}}(s)\, ds \\ &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|u - P_k\|_{L^{2q,2}}^2(s)\, ds,\end{aligned} \tag{4.54}$$

and

$$
\begin{aligned}
\|B(P_k, u - P_{k-1}) + B(u - P_{k-1}, P_k)\|_{L^{q,1}} &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|P_k(u - P_k)\|_{L^{q,1}}(s)\, ds \\
&\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \left(\|P_k\|_{L^{2q,\infty}}^2 + \|u - P_k\|_{L^{2q,1}}^2 \right) ds \\
&\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \left(\|P_k\|_{L^{2q}}^2 + \|u - P_k\|_{L^{2q,1}}^2 \right) ds.
\end{aligned}
\tag{4.55}
$$

We first consider P_k in L^{2q}. Because $P_k \in \mathcal{K}_{2q}$,

$$
\begin{aligned}
\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|P_k\|_{L^{2q}}^2\, ds &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{1-\frac{3}{2q}}} \|P_k\|_{\mathcal{K}_{2q}}^2\, ds \\
&\lesssim_{k,q,u_0} t^{\frac{3}{2q}-\frac{1}{2}}.
\end{aligned}
\tag{4.56}
$$

Then

$$
\|B(P_k, u - P_k)\|_{L^r(0,T;L^{q,1})} \lesssim_{k,q,u_0} \left(\int_0^T \left(t^{\frac{3}{2q}-\frac{1}{2}} \right)^r dt \right)^{\frac{1}{r}} \lesssim_{k,q,u_0} T^{\frac{3}{2q}-\frac{1}{2}+\frac{1}{r}} \lesssim_{k,q,u_0} T^{\frac{1}{2}},
$$

for $r = \frac{2q}{2q-3}$.

To bound $\|u - P_k\|_{L^{2q,\beta}}$ we use the extension of the Gagliardo-Nirenberg inequality to the Lorentz scale [23, Corollary 2.2]. For $\beta > 0$ and

$$
\frac{1}{\tilde{p}} = \frac{\theta}{\tilde{q}} + (1-\theta)\left(\frac{1}{2} - \frac{1}{3} \right),
$$

we have

$$
\|f\|_{L^{\tilde{p},\beta}} \lesssim_{\tilde{p},\tilde{q},\beta} \|f\|_{L^{\tilde{q},\infty}}^{\theta} \|\nabla f\|_{L^2}^{1-\theta},
\tag{4.57}
$$

for $1 \le \tilde{q} < \tilde{p} < \infty$ and $3/2 - 3/\tilde{p} < 1$. Fix $\tilde{p} = 2q < 6$ and $\tilde{q} = 2$. Then θ is given by

$$\frac{3}{2q} - \frac{1}{2} = \theta.$$

Because $1 < q < 3$ the other conditions above are met. Also note $L^2 \subset L^{2,\infty}$ embeds continuously. This gives

$$\begin{aligned} \|B(u-P_k, u-P_k)\|_{L^{q,1}}(t) &\lesssim_q \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|u-P_k\|_{L^2}^{2\theta} \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)}\, ds \\ &\lesssim_{k,q,u_0} t^{\frac{\theta}{2}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)}\, ds, \end{aligned} \tag{4.58}$$

where we used (3.4). For $t \in (0,T)$,

$$\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)}\, ds = \int_{\mathbb{R}} \frac{1}{|t-s|^{\frac{1}{2}}} \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)}(s) \chi_{(0,T)}(s)\, ds,$$

and the right-hand side can be viewed as $I_{\frac{1}{2}}(\|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)}\chi_{(0,T)})$ where $I_{\frac{1}{2}}$ is a Riesz potential in 1D. The Hardy-Littlewood-Sobolev inequality states

$$\left\| I_{\frac{1}{2}} \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)} \chi_{(0,T)} \right\|_{L^r(\mathbb{R})} \lesssim_r \left\| \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)} \chi_{(0,T)} \right\|_{L^{\tilde{r}}(\mathbb{R})},$$

where

$$\frac{1}{r} = \frac{1}{\tilde{r}} - \frac{1}{2}.$$

The selection

$$\tilde{r} = \frac{1}{1-\theta}; \qquad r = \frac{2}{1-2\theta},$$

is valid for the Hardy-Littlewood-Sobolev inequality provided $3/2 < q$. The choice $q = 3/2$, $\theta = 1/2$ and $r = \infty$ is not permitted in the Hardy-Littlewood-Sobolev inequality. Letting

$r = \frac{2q}{2q-3}$ and putting the above observations together leads to

$$
\begin{aligned}
\|B(u-P_k,u-P_k)\|_{L^r(0,T;L^{q,1})} &\lesssim_{k,q,u_0} T^{\frac{\theta}{2}} \big\| \|\nabla(u-P_k)\|_{L^2}^{2(1-\theta)} \chi_{(0,T)} \big\|_{L^{\tilde r}(\mathbb{R})} \\
&\lesssim_{k,q,u_0} T^{\frac{\theta}{2}} \|\nabla(u-P_k)\|_{L^2(0,T;L^2)}^{\frac{2}{\tilde r}} \\
&\lesssim_{k,q,u_0} T^{\frac{\theta}{2}} T^{\frac{1}{2\tilde r}} = T^{\frac{1}{2}},
\end{aligned}
\tag{4.59}
$$

by (3.4). The result follows. $\square$

4.6 Decay for locally sub-critical (u,p)

The following lemma achieves the desired decay in Theorem 1.1 assuming u and p are locally sub-critical.

Lemma 4.13. *Let $\varepsilon > 0$ be given and $q > 3$. Suppose u is a λ-DSS local energy solution to* (1.1) *with divergence free, DSS data in E^2. Assume u satisfies*

$$\max\{\sup_{0<s<T} \|u\|_{L^q(A_0^*)}, \sup_{0<s<T} \|p\|_{L^{q/2}(A_0^*)}\} < \varepsilon,$$

for some $T > 0$. Then

$$|\nabla^l u|(x,t) \lesssim_{\lambda,l,u_0} \frac{\varepsilon}{\sqrt{t}^{\frac{3}{q}+l}(|x|+\sqrt{t})^{1-\frac{3}{q}}} \text{ for } |x| \geq R_0\sqrt{t}.$$

Proof. Let $x_0 \in A_k$, then

$$\int_0^1 \int_{B_1(x_0)} |u|^3(y,s)\,dy\,ds \lesssim_\lambda |x_0|^{3(\frac{3}{q}-1)} \sup_{0<s\lesssim_\lambda |x_0|^{-2}} \int_{B_{\frac{1}{\lambda^k}}(\frac{x_0}{\lambda^k})} |u|^q\,dy \lesssim |x_0|^{3(1-\frac{3}{q})}\varepsilon^3,$$

and, likewise, for p,

$$\int_0^1 \int_{B_1(x_0)} |p|^{3/2}(y,s)\,dy\,ds \lesssim_\lambda |x_0|^{3(\frac{3}{q}-1)} \sup_{0<s\lesssim_\lambda |x_0|^{-2}} \int_{B_{\frac{1}{\lambda^k}}(\frac{x_0}{\lambda^k})} |p|^{q/2}\,dy \lesssim |x_0|^{3(1-\frac{3}{q})}\varepsilon^3.$$

Then by Lemma 3.1 we have

$$|\nabla^l u|(x_0,t) \lesssim_\lambda |x_0|^{\frac{3}{q}-1}\varepsilon,$$

for $t \in [0,\frac{1}{2}]$, by taking x_0 sufficiently large so that $|x_0|^{3(1-\frac{3}{q})}\varepsilon^3 < \varepsilon_*$. We may repeat this over parabolic cylinders of the form $B_1(x_0) \times [\frac{1}{2}, \frac{1}{2}+\delta^2]$ to cover all of $t \in [\frac{3}{4}, \frac{3}{4}+\lambda^2]$, a full period of t. Then

$$|\nabla^l u|(x_0,t) \lesssim_\lambda (|x_0|+1)^{\frac{3}{q}-1}\varepsilon,$$

for $|x_0| >> 1$. Now let $t > 0$ and $|x| \geq R_0\sqrt{t}$, and $\tilde{t} = \lambda^{-2k}t \in [1,\lambda^2]$ and $\tilde{x} = \lambda^{-k}x$ By DSS scaling, we conclude

$$\begin{aligned} |\nabla^l u|(x,t) = \lambda^{k(1-l)}|\nabla^l u|(\tilde{x},\tilde{t}) &\lesssim_\lambda \frac{\varepsilon\lambda^{k(1-l)}}{(|\tilde{x}|+1)^{1-\frac{3}{q}}} \\ &\lesssim_\lambda \frac{\varepsilon(\lambda^k)^{\frac{3}{q}-l}}{(|x|+\lambda^k)^{1-\frac{3}{q}}} \\ &\lesssim_\lambda \frac{\varepsilon\sqrt{t}^{\frac{3}{q}-l}}{(|x|+\sqrt{t})^{1-\frac{3}{q}}}. \end{aligned} \tag{4.60}$$

□

5 Results

We now prove the main results introduced in Section 1. This section constitutes joint work with Dr. Zachary Bradshaw.

5.1 Properties of DSS local energy solutions

5.1.1 Decay when $u_0 \in L^q_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$, $q > 3$

Proof. We will show the assumptions of Theorem 4.13 hold for DSS local energy solutions with data $u_0 \in L^q_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$, $q > 3$.

Fix $x_0 \in A_0$. By Theorem 3.2 we can find U, a local-in-time mild solution on $B_2(x_0) \times (0, T')$, with data U_0, such that $u - U \in C^\gamma_{\mathrm{par}}(B_{\frac{1}{2}}(x_0) \times (0, T'))$ and $u - U \in L^\infty(0, T'; L^q(B_{\frac{1}{2}}(x_0))$.

Because U is mild with data in $L^q(B_2(x_0))$ by sub-critical local well-posedness [26], $U \in L^\infty(0, T'; L^q(B_{\frac{1}{2}}(x_0))$. This implies that $u \in L^\infty(0, T'; L^q(B_{\frac{1}{2}}(x_0))$. We repeat this process of local smoothing to cover A_0 and find $u \in L^\infty(0, T'; L^q(A_0))$.

Next, we show the associated pressure p is in $L^\infty(0, T'; L^{q/2}(A_0))$. To achieve this, we decompose

$$p(x,t) = [(-\Delta)^{-1} \operatorname{div} \operatorname{div}]_{ij}(u_i u_j) =: p_1 + p_2 + p_3,$$

where

$$\begin{aligned} p_1(x,t) &:= [(-\Delta)^{-1} \operatorname{div} \operatorname{div}]_{ij}(u_i u_j \chi_{B_{\lambda^{-1}}(x_0)}), \\ p_2(x,t) &:= [(-\Delta)^{-1} \operatorname{div} \operatorname{div}]_{ij}(u_i u_j \chi_{A_0^*}), \text{ and} \\ p_3(x,t) &:= [(-\Delta)^{-1} \operatorname{div} \operatorname{div}]_{ij}(u_i u_j (1 - \chi_{B_{\lambda^2}(x_0)})). \end{aligned} \tag{5.1}$$

By a priori estimates for local energy solutions [15, Theorem 1.4], we have

$$\|p_1\|_{L^\infty(A_0 \times (0,T))} \lesssim \|u_0\|^2_{L^2_{\mathrm{uloc}}}.$$

Next, $[(-\Delta)^{-1}\operatorname{div}\operatorname{div}]_{ij}$ is bounded on $L^{q/2}(\mathbb{R}^3)$ because it is a Calderon Zygmund operator. Therefore $u \in L^\infty(0,T;L^q(A_0))$ implies $p_2 \in L^\infty(0,T;L^{\frac{q}{2}}(A_0))$.

For the term p_3, we write explicitly

$$p_3(x,t) = \int K_{ij}(x)(u_i u_j(1-\chi_{B_{\lambda^2}(x_0)}))(x-y)\,dy,$$

where K_{ij} decays like $|x|^{-3}$. This is bounded by the sum

$$|p_3| \le \sum_{k\ge 2} \lambda^{-3k}\left(\int_{A_k} |u|^2\,dy\right)^{1/2}.$$

To bound this using $u \in L^\infty(0,T;L^q(A_0))$, we apply DSS scaling to scale the integral into A_0 and use Hölder's inequality introduce the L^q-norm to find

$$\begin{aligned}
|p_3| &\le \sum_{k\ge 2} \lambda^{-3k}\left(\int_{A_k} |u|^2\,dy\right)^{1/2} \le \sum_{k\ge 2} \lambda^{-3k}\lambda^{3k(1-2/q)}\left(\int_{A_k} |u|^q\,dy\right)^{2/q} \\
&\le \sum_{k\ge 2} \lambda^{-6k/q}\left(\lambda^{(3-q)k}\int_{A_0} |u|^q(y,\lambda^{-k}t)\,dy\right)^{2/q} \le \sum_{k\ge 2}\lambda^{-2k}\sup_t \|u\|^2_{L^q(A_0)} \\
&\le \frac{1}{1-\lambda^{-2}}\sup_t \|u\|^2_{L^q(A_0)}.
\end{aligned} \tag{5.2}$$

This shows $p_3 \in L^\infty(A_0\times(0,T))$. Thus $p \in L^\infty(0,T;L^{q/2}(A_0))$. The decay (1.3) follows from Lemma 4.13. $\square$

Next, we prove an asymptotic improvement for the 'non-linear' part of the flow.

Proof of Theorem 1.2. Fix $t \in [1,\lambda^2]$ and $|x| \ge R_0\sqrt{t}$. For the improved decay of $u - P_0$, first note $u - P_0 = -B(u,u)$. We decompose

$$B(u,u) = \int_0^t \left(\int_{|y|\ge R_0\sqrt{t}} + \int_{|y|<R_0\sqrt{t}}\right)\nabla S(x-y,t-s)\cdot \mathbb{P}(u\otimes u)\,dy\,ds = I_1(x,t) + I_2(x,t).$$

For the sub-parabolioid region, using the local boundedness of u in L^2, the decay for the

Oseen kernel(Lemma 3.4), and Hölder's inequality we find

$$|I_1|(x,t) \lesssim \int_0^t \int_{|y|\geq R_0\sqrt{t}} \frac{1}{(|x-y|+\sqrt{t-s})^4}|u|^2(y,s)\,dy\,ds \lesssim_\lambda \frac{1}{|x|^4}\|u\|^2_{L^2(B_{\lambda R_0})}. \tag{5.3}$$

For the region above the paraboloid, we use the decay for u (1.3), the bound for derivatives of the Oseen kernel (Lemma 3.4) and the product structure of B to find

$$|I_2|(x,t) \lesssim \int_0^t \int_{\mathbb{R}^3} \frac{1}{(|x-y|+\sqrt{t-s})^4} \frac{\varepsilon\sqrt{t}^{-\frac{6}{q}}}{(|y|+\sqrt{s})^{2-\frac{6}{q}}} \lesssim_\lambda \frac{\varepsilon}{(|x|+1)^{2-\frac{6}{q}}}. \tag{5.4}$$

Together (5.3),(5.4) imply

$$|u-P_0|(x,t) \lesssim_\lambda \frac{\varepsilon}{(|x|+1)^{2-\frac{6}{q}}},$$

for $t\in[1,\lambda^2]$, $|x|\geq R_0\sqrt{t}$.

Now, let $t>0$ and $|x|\geq R_0\sqrt{t}$, and $\tilde{t}=\lambda^{-2k}t\in[1,\lambda^2]$ and $\tilde{x}=\lambda^{-k}x$. By DSS scaling, we conclude

$$\begin{aligned}
|u-P_0|(x,t) = \lambda^k|u-P_0|(\tilde{x},\tilde{t}) &\lesssim_\lambda \frac{\varepsilon\lambda^k}{(|\tilde{x}|+1)^{2-\frac{6}{q}}} \\
&\lesssim_\lambda \frac{\varepsilon(\lambda^k)^{\frac{6}{q}-1}}{(|x|+\lambda^k)^{2-\frac{6}{q}}} \\
&\lesssim_\lambda \frac{\varepsilon\sqrt{t}^{\frac{6}{q}-1}}{(|x|+\sqrt{t})^{2-\frac{6}{q}}}.
\end{aligned} \tag{5.5}$$

□

5.1.2 Picard improvement for $u_0 \in L^q_{\mathrm{loc}}(\mathbb{R}^3\setminus\{0\})$, $q>3$

Next, we develop decay rates for the difference with the Picard iterates, $u-P_k$, using techniques similar to those above, and the inherited decay of P_k (Lemma 4.9).

Proof of Theorem 1.5. Denote by a_k the following

$$a_k = (k+2)\left(1 - \frac{3}{q}\right) = a_{k-1} + 1 - \frac{3}{q}.$$

Note $a_k < 4$ precisely when $k < \frac{4q}{q-3} - 2$. This limits our improvement. Our base case is

$$|u - P_0|(x,t) \lesssim_{\lambda, R_0, u_0} \frac{\sqrt{t}^{2-\frac{6}{q}}}{\sqrt{t}(|x| + \sqrt{t})^{2-\frac{6}{q}}},$$

which follows by Theorem 1.2. For induction, assume for $|x| \geq R_0\sqrt{t}$,

$$|u - P_k|(x,t) \lesssim_{k,\lambda, R_0, u_0} \frac{\sqrt{t}^{a_k}}{\sqrt{t}(|x| + \sqrt{t})^{a_k}}. \tag{5.6}$$

Now, fix $t \in [1, \lambda^2]$. We can expand $u - P_{k+1}$ as

$$\begin{aligned} u - P_{k+1} &= P_0 - B(u,u) - P_{k+1} \\ &= -B(u,u) + B(P_k, P_k) \\ &= -B(u, u - P_k) - B(u, P_k) + B(P_k, P_k) \\ &= -B(u - P_k, u - P_k) - B(P_k, u - P_k) - B(u - P_k, P_k), \end{aligned} \tag{5.7}$$

where we used the bilinearity of B and the definition of P_{k+1}. Next, using the bound for the Oseen kernel in Lemma 3.4, we estimate each term by breaking each integral into near-field and far-field terms as follows

$$\begin{aligned} |u - P_{k+1}| \lesssim &\int_0^t \int_{B_{R_0\sqrt{t}}} \frac{1}{(|x-y| + \sqrt{t-s})^4} \big(|u - P_k|^2 + |u - P_k||P_k|\big)\, dy\, ds \\ &+ \int_0^t \int_{B^c_{R_0\sqrt{t}}} \frac{1}{(|x-y| + \sqrt{t-s})^4} \big(|u - P_k|^2 + |u - P_k||P_k|\big)\, dy\, ds \\ =:& I_{k+1}(x,t) + J_{k+1}(x,t). \end{aligned} \tag{5.8}$$

Using the decay for P_k in Lemma 4.9 and Lemma 4.1 we find

$$\begin{aligned} J_{k+1}(x,t) \lesssim &\int_0^t \int_{B^c_{R_0\sqrt{t}}} \frac{1}{(|x-y|+\sqrt{t-s})^4} \frac{\sqrt{s}^{\min\{2a_k,4\}-2}}{(|y|+\sqrt{s})^{\min\{2a_k,4\}}} \\ &+ \int_0^t \int_{B^c_{R_0\sqrt{t}}} \frac{1}{(|x-y|+\sqrt{t-s})^4} \frac{\sqrt{s}^{a_k-1-\frac{3}{q}}}{(|y|+\sqrt{s})^{a_{k+1}}}\,dy\,ds \\ \lesssim & \frac{1}{(|x|+1)^{\min\{2a_k,4\}}} + \frac{1}{(|x|+1)^{a_{k+1}}}. \end{aligned} \tag{5.9}$$

Because the second term decays more slowly we conclude

$$J_{k+1}(x,t) \lesssim_k \frac{1}{(|x|+1)^{a_{k+1}}},$$

in the sub-paraboloid region.

For I_{k+1}, by a priori estimates for local energy solutions, we have

$$\|u\|_{L^\infty(0,\lambda^2;L^2(B_{\lambda R_0}))} \lesssim_{\lambda,R_0} \|u_0\|_{L^2_{\mathrm{uloc}}}.$$

By [58], we also have

$$\|P_0\|_{L^\infty(0,\lambda^2;L^2(B_{\lambda R_0}))} \lesssim_{\lambda,R_0} \|u_0\|_{L^2_{\mathrm{uloc}}}.$$

Then by Hölder's inequality,

$$\begin{aligned} I_{k+1}(x,t) &\int_0^t \int_{B_{R_0\sqrt{t}}} \frac{1}{(|x-y|+\sqrt{t-s})^4} \big(|u-P_k|^2 + |P_k|^2\big)\,dy\,ds \\ &\lesssim \frac{1}{(|x|+1)^4} \big(\|u\|^2_{L^2(B_{\lambda R_0})} + \|P_k\|^2_{L^2(B_{\lambda R_0})}\big). \end{aligned} \tag{5.10}$$

This presents a limitation on a_k. Together, these estimates imply

$$|u-P_{k+1}| \lesssim_{k,\lambda,R_0,u_0} \frac{1}{(|x|+1)^{\min\{a_{k+1},4\}}}, \tag{5.11}$$

for $|x| \ge R_0\sqrt{t}$.

Hence, by DSS scaling,

$$|u - P_{k+1}| \lesssim_{k,\lambda,R_0,u_0} \frac{\sqrt{t}^{\min\{a_{k+1},4\}-1}}{(|x| + \sqrt{t})^{\min\{a_{k+1},4\}}}, \tag{5.12}$$

for $|x| \ge R_0\sqrt{t}$.

For $k \ge k_q := \lceil \frac{4q}{q-3} - 3 \rceil$, we find that $a_k \ge 4$ and so

$$|u - P_k| \lesssim_{k_q,\lambda,R_0,u_0} \frac{\sqrt{t}^3}{(|x| + \sqrt{t})^4}. \tag{5.13}$$

□

5.1.3 *Decay when* $u_0 \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$, $0 < \alpha < 1$

In this subsection, we pursue improved decay rates for solutions with more regular data in Hölder spaces.

Proof of Theorem 1.3. We begin by expanding

$$u - P_0 = -B(u, u) = -B(u, u - P_0) - B(u - P_0, P_0) - B(P_0, P_0). \tag{5.14}$$

Note P_0 decays algebraically slower than $u - P_0$, so the last term is expected to decay the slowest. We begin by showing $B(u, u - P_0)$ and $B(u - P_0, P_0)$ have cubic decay.

Fix $t \in [1, \lambda^2]$ and $|x| \ge 2R_0\sqrt{t}$. Using Lemma 3.4 to bound the kernel and breaking the integrals into near- and far-field, we find

$$\begin{aligned} |B(u, u - P_0)|(x,t) &\lesssim \left(\int_0^t \int_{|y| \ge R_0\sqrt{t}} + \int_0^t \int_{|y| < R_0\sqrt{t}} \right) \frac{|u|(y,s)|u - P_0|(y,s)}{(|x-y| + \sqrt{t-s})^4} \, dy \, ds \\ &=: I_1(x,t) + I_2(x,t). \end{aligned} \tag{5.15}$$

By Theorems 1.1 and 1.2, with $q = \infty$

$$I_1(x,t) \lesssim \int_0^t \int \frac{1}{(|x-y| + \sqrt{t-s})^4} \frac{\sqrt{s}}{(|y| + \sqrt{s})^3}\, dy\, ds \lesssim \frac{1}{(|x|+1)^3}. \tag{5.16}$$

We extend this from $t \in [1, \lambda^2]$ to all t by scaling. On the other hand, by the *a priori* bounds of u and P_0 in L^2_{loc} and noting $|x| \sim |x-y|$ due to our choice of x and y, we have

$$I_2(x,t) \lesssim \frac{1}{|x|^4} \|u_0\|^2_{L^2_{\text{uloc}}}.$$

Since $|x| > 1$ when $t \in [1, \lambda^2]$, by re-scaling we obtain

$$|B(u, u - P_0)| \lesssim \frac{t^{3/2}}{(|x| + \sqrt{t})^4}.$$

Since P_0 and u have the same decay properties and bound in L^2_{uloc}, the same bound follows for $B(u - P_0, P_0)$ using Lemma 4.2 alongside Theorem 1.2 in the far-field integral.

To treat the last term $B(P_0, P_0)$, we consider two cases: $\alpha < 1$ and $\alpha = 1$. First, we let $\alpha < 1$ and fix $\gamma \in (\alpha, \min(1, 2\alpha))$ to apply Lemma 4.8. We rewrite $B(P_0, P_0)$ using the fractional Laplacian $\Lambda = (-\Delta)^{\frac{1}{2}}$ and properties of Fourier multipliers as

$$\begin{aligned} B(P_0, P_0)(x,t) &= \int_0^t \int_{\mathbb{R}^3} S(x-y, t-s) \nabla \cdot (P_0 \otimes P_0)(y,s)\, dy\, ds \\ &= \int_0^t \int_{\mathbb{R}^3} \nabla \Lambda^{-\gamma} S(x-y, t-s) \Lambda^{\gamma} (P_0 \otimes P_0)(y,s)\, dy\, ds. \end{aligned} \tag{5.17}$$

We bound the fractional derivative of the Oseen Kernel using Theorem 3.6 by

$$\left|\nabla \Lambda^{-\gamma} S(x-y, t-s)\right| \lesssim \frac{1}{(|x-y| + \sqrt{t-s})^{4-\gamma}}.$$

We navigate the commutator using Lemma 4.8 applied with $|m| = 0$, which states, for

$0 < \gamma < 2\alpha$,

$$\Lambda^\gamma(P_0 \otimes P_0)(y,s) = P_{0i}\Lambda^\gamma P_{0j}(y,s) + P_{0j}\Lambda^\gamma P_{0i}(y,s) + O\left(\frac{1}{(|y|+1)^{2+\gamma}}\right).$$

By scaling and using the decay for fractional derivatives of P_0 in Lemma 4.6, and taking $\gamma > \alpha$, we find

$$\Lambda^\gamma P_0(y,s) \lesssim \frac{1}{(|y|+\sqrt{s})^{1+\alpha}}, \tag{5.18}$$

and using $|P_0|(y,s) \lesssim (|y|+\sqrt{s})^{-1}$ gives

$$\Lambda^\gamma(P_0 \otimes P_0)(y,s) \lesssim \left(\frac{1}{(|y|+\sqrt{s})^{2+\alpha}} + \frac{1}{(|y|+\sqrt{s})^{2+\gamma}}\right) \lesssim \frac{1}{(|y|+\sqrt{s})^{2+\alpha}}. \tag{5.19}$$

Therefore,

$$\begin{aligned} |B(P_0,P_0)|(x,t) &\lesssim \int_0^t \int_{\mathbb{R}^3} \frac{1}{(|x-y|+\sqrt{t-s})^{4-\gamma}} \frac{C}{(|y|+\sqrt{s})^{2+\alpha}}\,dy\,ds \\ &\lesssim \frac{t^{\frac{1+\alpha}{2}}}{(|x|+\sqrt{t})^{2+\alpha}}, \end{aligned} \tag{5.20}$$

using a re-scaled version of (4.11).

Next, for $\alpha = 1$, we apply Lemma 4.5 in place of Lemma 4.6. We cannot deplete the singularity of the Oseen kernel by moving derivatives over to P_0 as before. This yields a logarithm when applying (4.11), and we obtain

$$|B(P_0,P_0)|(x,t) \lesssim \frac{t}{(|x|+\sqrt{t})^3}\log(|x|+\sqrt{t}). \tag{5.21}$$

$\square$

5.1.4 Decay when $u_0 \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$, $0 < \alpha < 1$

Proof of Theorem 1.4. Let $t \in [1, \lambda^2]$, $|x| > 2R_0\sqrt{t}$ and decompose $u - P_0$ as in (5.14). In the proof of Theorem 1.3, we showed the first three terms have cubic decay for C^α data.

We currently establish cubic decay for $B(P_0, P_0)$. To accomplish this, we rewrite the integral for $B(P_0, P_0)$ as

$$\int_0^t \int_{\mathbb{R}^3} \nabla S(x-y, t-s)(P_0 \otimes P_0)(y, s)\, dy = \int_0^t \int_{\mathbb{R}^3} \Lambda^{-\beta} \nabla S(x-y, t-s) \Lambda^\beta (P_0 \otimes P_0)(y, s)\, dy\, ds,$$

where $\beta \in (0, 1)$ will be specified momentarily. Note

$$\int_{\mathbb{R}^3} \nabla \Lambda^{-\beta} S\, dx = 0,$$

since $(\nabla \Lambda^{-\beta} S)^\wedge(0) = 0$ for $\beta < 1$. Using this, we re-write the above integral as

$$B(P_0, P_0) = J_1(x, t) + J_2(x, t) + J_3(x, t), \tag{5.22}$$

where

$$\begin{aligned}
J_1(x, t) &= \int_0^t \int_{|x-y| \le \frac{|x|}{2}} \Lambda^{-\beta} \nabla S(x-y, t-s) \Big(\Lambda^\beta (P_0 \otimes P_0)(y, s) - \Lambda^\beta (P_0 \otimes P_0)(x, s) \Big)\, dy\, ds, \\
J_2(x, t) &= -\int_0^t \int_{|x-y| > \frac{|x|}{2}} \Lambda^{-\beta} \nabla S(x-y, t-s) \Lambda^\beta (P_0 \otimes P_0)(x, s)\, dy\, ds, \text{ and} \\
J_3(x, t) &= \int_0^t \int_{|x-y| > \frac{|x|}{2}} \Lambda^{-\beta} \nabla S(x-y, t-s) \Lambda^\beta (P_0 \otimes P_0)(y, s)\, dy\, ds.
\end{aligned}$$

We first bound J_1. Using the mean value theorem and Theorem 3.6, we have

$$\begin{aligned}
|J_1|(x, t) \lesssim & \int_0^t \int_{|x-y| \le \frac{|x|}{2}} \frac{|x-y|}{(|x-y| + \sqrt{t-s})^{4-\beta}} \|\nabla \Lambda^\beta (P_0 \otimes P_0)\|_{L^\infty(B_{\frac{|x|}{2}}(x))}(t)\, dy\, ds \\
\lesssim & |x|^\beta \sup_{0<s<t} \|\nabla \Lambda^\beta (P_0 \otimes P_0)\|_{L^\infty(B_{\frac{|x|}{2}}(x))(s)}.
\end{aligned} \tag{5.23}$$

We further assume $\beta \in (0, \alpha)$. Since $u_0 \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$, by Corollary 4.4 we have

$$\sup_{s\in[1,\lambda^2]} |\nabla \Lambda^\beta P_0(x,s)| \lesssim (|x|+1)^{-2-\beta} \qquad \text{and} \qquad \sup_{s\in[1,\lambda^2]} |\Lambda^\beta P_0(x,s)| \lesssim (|x|+1)^{-1-\beta}. \tag{5.24}$$

By Lemma 4.8, (5.24) and the decay for P_0,

$$\begin{aligned} |\nabla \Lambda^\beta (P_0 \otimes P_0)|(x,t) &\le |P_{0i} \nabla \Lambda^\beta P_{0j}|(x,t) + |P_{0j} \nabla \Lambda^\beta P_{0i}|(x,t) + \mathcal{O}\left(\frac{1}{|x|^{3+\beta}}\right) \\ &\lesssim_\lambda (|x|+1)^{-3-\beta}. \end{aligned} \tag{5.25}$$

Therefore, (5.23) is bounded by

$$|x|^\beta (|x|+1)^{-3-\beta} \lesssim (|x|+1)^{-3}.$$

Likewise, using (5.18),

$$\begin{aligned} |\Lambda^\beta (P_0 \otimes P_0)|(x,t) &\le |P_{0i} \Lambda^\beta P_{0j}|(x,t) + |P_{0j} \Lambda^\beta P_{0i}|(x,t) + \mathcal{O}\left(\frac{1}{|x|^{2+\beta}}\right) \\ &\lesssim_\lambda (|x|+1)^{-2-\beta}. \end{aligned} \tag{5.26}$$

So, for J_2, by using Theorem 3.6,

$$\begin{aligned} |J_2| &\lesssim \int_0^t \int_{|x-y|>\frac{|x|}{2}} \frac{1}{(|x-y|+\sqrt{t-s})^{4-\beta}} \Lambda^\beta (P_0 \otimes P_0)(x,s)\, dy\, ds \\ &\lesssim (|x|+1)^{-2-\beta} \int_0^t \int_{|x-y|>\frac{|x|}{2}} \frac{1}{(|x-y|+\sqrt{t-s})^{4-\beta}}\, dy\, ds \\ &\lesssim_\lambda (|x|+1)^{-2-\beta} |x|^{-1+\beta} \lesssim_\lambda (|x|+1)^{-3}. \end{aligned} \tag{5.27}$$

Lastly, for J_3, we need to handle the near- and far-field separately. We break J_3 into

$$\begin{aligned} |J_3| &\lesssim \int_0^t \int_{|x-y|>\frac{|x|}{2}} \frac{1}{(|x-y|+\sqrt{t-s})^{4-\beta}} \Lambda^\beta (P_0 \otimes P_0)(y,s)\,dy\,ds \\ &\lesssim_\lambda \left(\int_{|x-y|>\frac{|x|}{2}, |y|>|x|/2} + \int_{|y|\le |x|/2} \right) \frac{1}{|x-y|^{4-\beta}} (|y|+1)^{-2-\beta}\,dy \\ &=: J_{31} + J_{32}. \end{aligned} \tag{5.28}$$

For the near-field,

$$\begin{aligned} J_{32} &\lesssim_\lambda \frac{1}{|x|^{4-\beta}} \int_{|y|\le|x|/2} (|y|+1)^{-2-\beta}\,dy\,ds \\ &\lesssim_\lambda \frac{1}{|x|^{4-\beta}} (|x|+1)^{1-\beta} \lesssim_\lambda (|x|+1)^{-3}, \end{aligned} \tag{5.29}$$

because $\beta < 1$. Also,

$$\begin{aligned} J_{31} &\lesssim_\lambda \int_{|x-y|>\frac{|x|}{2}, |y|>|x|/2} \frac{1}{(|x-y|+\sqrt{t-s})^{4-\beta}} (|y|+1)^{-2-\beta}\,dy \\ &\lesssim (|x|+1)^{-2-\beta} \int_{|x-y|>\frac{|x|}{2}} \frac{1}{|x-y|^{4-\beta}}\,dy\,ds \\ &\lesssim (|x|+1)^{-2-\beta} \frac{1}{|x|^{1-\beta}} \lesssim (|x|+1)^{-3}. \end{aligned} \tag{5.30}$$

We, therefore, find that

$$|u - P_0|(x,t) \lesssim (|x|+1)^{-3}. \tag{5.31}$$

To extend to all $t > 0$ and $|x| > R_0\sqrt{t}$, we appeal to DSS scaling. $\square$

5.2 Local separation rates for weak $L^{3,\infty}$ solutions

Lastly, we prove Theorem 1.7. The separation rates for $L^{3,\infty}$ weak solutions are a corollary to this result.

Before we begin, we introduce O'Neil's inequality which extends Hölder's inequality for convolutions to Lorentz spaces.

Proposition 5.1 (O'Neil's inequality, [59]). *Let k and f be measurable function on a given set Ω, and let $D \subset \Omega$. Define $Af(y) = \int_D k(y-x)f(x)\,dx$. For $1 < p < q < r < \infty$ and $0 < h_1, h_2, h_3 \le \infty$, $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $\frac{1}{h_1} = \frac{1}{h_2} + \frac{1}{h_3}$, one has*

$$\|Af(y)\|_{L^{q,h_1}(\Omega)} \le c\|f\|_{L^{p,h_2}(D)}\|k\|_{L^{r,h_3}(\Omega-D)}, \tag{5.32}$$

where $\Omega - D = \{x - y : x \in \Omega, y \in D\}$.

We may now proceed with the proof.

Proof of Theorem 1.7. Without loss of generality, assume $B := B_2(x_0)$ is centered at $x_0 = 0$. Assume $u_0|_B \in L^p(B)$. Let U_0 be a localization of the data to B such that $u_0 = U_0$ in $B_{4/3}(0) \subset B$, $\operatorname{supp} U_0 \Subset B$. This is done as per the decomposition in Theorem 3.2. Let U be the locally-in-time defined mild solution to (1.1) with data U_0. Define $\{B_k\}_{k=0}^{\infty}$ to be a collection of nested balls centered at 0 with radii $\alpha^k/2$ for some $\alpha \in (0,1)$ to be specified later. Recalling $P_0 = e^{t\Delta}u_0$, we find

$$\begin{aligned}|u - P_0|(x,t) \le& |u - U|(x,t) + |U - e^{t\Delta}U_0|(x,t) + |e^{t\Delta}(U_0 - u_0)|(x,t)\\ =:& I_1(x,t) + I_2(x,t) + I_3(x,t).\end{aligned} \tag{5.33}$$

In the definition of $C^{\gamma}_{par}(\overline{B}_{\frac{1}{2}} \times [0,T])$, the exponent in the time-variable modulus of continuity is $\gamma/2$. By Theorem 3.2, there exists $T = T(p, u_0) > 0$ such that

$$I_1(x,t) \lesssim_{p,u_0} t^{\frac{\gamma}{2}},$$

for some $\gamma = \gamma(p) \in (0,1)$, $x \in B_0$ and $0 < t < T$.

For I_2, by (2.19), for any $p \in (3, \infty]$ and $0 < t < T$,

$$\begin{aligned} I_2(x,t) &\le \|B(U,U)\|_{L^\infty(\mathbb{R}^3)}(t) \\ &\lesssim t^{\frac{1}{2}-\frac{3}{2p}} \|U\|^2_{L^\infty(0,T;L^p)} \lesssim t^{\frac{1}{2}-\frac{3}{2p}} \|U_0\|^2_{L^p}, \end{aligned} \tag{5.34}$$

where we possibly re-define T to make it smaller than the timescale of existence for the strong solution to (1.1), i.e. $T \lesssim \|U_0\|_{L^p}^{-2p/(p-3)}$, and the time-scale coming from Theorem 3.2.

Noting $U_0 - u_0 = 0$ in $B_{4/3}$, the last part, I_3, is broken into integrals over a shell and a far-field region

$$I_3(x,t) \lesssim \left(\int_{\frac{4}{3} \le |y| < 2} + \int_{|y| \ge 2} \right) t^{-\frac{3}{2}} e^{-\frac{|x-y|^2}{4t}} |U_0 - u_0|(y)\, dy =: I_{31}(x,t) + I_{32}(x,t). \tag{5.35}$$

For I_{31}, because $\|U_0\|_{L^p(\mathbb{R}^3)} \lesssim \|u_0\|_{L^p(B)}$, we have for all $0 < t < T$ and $x \in B_0$ that

$$I_{31}(x,t) \lesssim t^{-\frac{3}{2}} e^{-\frac{(\frac{4}{3}-\frac{1}{2})^2}{4t}} \|U_0 - u_0\|_{L^p(\frac{4}{3} \le |y| < 2)}(t) \lesssim_{u_0,p} t^{\frac{\gamma}{2}}. \tag{5.36}$$

For I_{32}, by Lemma 4.10, $U_0(y) \equiv 0$ for $|y| \ge 2$, and taking $x \in B_0$, $0 < t < T$, we have

$$\begin{aligned} I_{32}(x,t) &\lesssim \int_{|y| \ge 2} t^{-\frac{3}{2}} e^{-\frac{|x-y|^2}{4t}} |u_0|(y)\, dy \\ &\lesssim t^{-\frac{3}{2}} \|u_0\|_{L^{3,\infty}} \left\| \left\| e^{-\frac{|x-y|^2}{4t}} (1 - \chi_B(y)) \right\|_{L_y^{\frac{3}{2},1}} \right\|_{L_x^\infty(B_0)} \\ &\lesssim_{u_0} t^{-\frac{3}{2}} e^{\frac{-(2-\frac{1}{2})^2}{4t}} \lesssim_{u_0} t^{\frac{\gamma}{2}}, \end{aligned} \tag{5.37}$$

by the growth of the Gaussian. Therefore,

$$\|u - P_0\|_{L^\infty(B_0)}(t) \lesssim_{p,u_0} t^{\min\{\frac{\gamma}{2}, \frac{1}{2} - \frac{3}{2p}\}},$$

where the dependence on u_0 is via the quantities $\|u_0\|_{L^p(B)}$ and $\|u_0\|_{L^{3,\infty}}$.

We inductively extend this estimate to higher Picard iterates. Fix σ as in the statement of the theorem. Recursively define the sequence $\{a_k\}$ by $a_{k+1} = \min\{\sigma, 1/2 - 3/(2p) + a_k\}$ and $a_0 = \min\{\gamma/2, 1/2 - 3/(2p)\}$. Assume for induction that

$$\|u - P_k\|_{L^\infty(B_k)} \lesssim_{k,\alpha,p,u_0} t^{a_k},$$

for $0 < t < T$ and the dependence on u_0 is via the same quantities as above. Note that

$$\begin{aligned} |u - P_{k+1}|(x,t) \leq& |B(u - P_k, u - P_k)| + |B(u - P_k, P_k)| + |B(P_k, u - P_k)| \\ =:& J(x,t) + K(x,t) + L(x,t). \end{aligned} \tag{5.38}$$

We split J further as

$$\begin{aligned} J(x,t) &\leq |B((u - P_k)\chi_{B_k}, u - P_k)| + |B((u - P_k)(1 - \chi_{B_k}), u - P_k)| \\ &=: J_1(x,t) + J_2(x,t). \end{aligned} \tag{5.39}$$

For the near-field, J_1, we use the inductive hypothesis to obtain, for $0 < t < T$,

$$\begin{aligned} \|J_1\|_{L^\infty(B_{k+1})}(t) &\lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|u - P_k\|^2_{L^\infty(B_k)}\, ds \\ &\lesssim_{k,\alpha,p,u_0} t^{\frac{1}{2}+2a_k} \lesssim_{k,\alpha,p,u_0} t^{\frac{1}{2}-\frac{3}{2p}+a_k}, \end{aligned} \tag{5.40}$$

by using (2.19). Considering J_2 for $0 < t < T$, we have by (3.4),

$$\begin{aligned} \|J_2\|_{L^\infty(B_{k+1})}(t) &\lesssim \int_0^t \int_{|x-y| > \frac{1}{2}\alpha^k - \frac{1}{2}\alpha^{k+1}} \frac{1}{|x-y|^4} |u - P_k|^2(y,s)\, dy\, ds \\ &\lesssim \frac{t}{(\alpha^k - \alpha^{k+1})^4} \|u - P_k\|^2_{L^2}(t) \lesssim_{\alpha,k,u_0} t^{\frac{3}{2}}, \end{aligned} \tag{5.41}$$

where we used the version of (3.3) for higher Picard iterates [1].

The terms K and L are treated identically, and we only consider K. We begin by further splitting K as

$$K(x,t) \le |B((u-P_k)\chi_{B_k}, P_k)| + |B((u-P_k)(1-\chi_{B_k}), P_k)| =: K_1(x,t) + K_2(x,t).$$

For the near-field K_1 and for $0 < t < T$, we have, using (2.19),

$$\|K_1\|_{L^\infty(B_{k+1})}(t) \lesssim \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2p}}} \|u-P_k\|_{L^\infty(B_k)}(s) \|P_k\|_{L^p(B_k)}(s)\, ds.$$

By Lemma 4.11, $\sup_{0<t<\infty} \|P_k\|_{L^p(B_k)} < \infty$. Note $1/2 + 3/(2p) < 1$ for $3 < p$. Hence,

$$\|K_1\|_{L^\infty(B_{k+1})}(t) \lesssim_{k,\alpha,p,u_0} t^{\frac{1}{2}-\frac{3}{2p}+a_k}, \tag{5.42}$$

for $0 < t < T$, by the inductive hypothesis.

For the far-field K_2, note that $(1-\chi_{B_k}(y))|x-y|^{-4}$ is bounded for $x \in B_{k+1}$. Using Proposition 4.12, taking $x \in B_{k+1}$, $0 < t < T$ and $q \in (3/2, 3)$, we have, by (3.4) and O'Neil's inequality (5.32),

$$\begin{aligned} K_2(x,t) &\lesssim \int_0^t \int_{B_k^c} \frac{1}{(|x-y|+\sqrt{t-s})^4} |u-P_k||P_k|\, dy\, ds \\ &\lesssim \left\| \frac{1-\chi_{B_k}(\cdot)}{|x-\cdot|^4} \right\|_{L^{r'}(0,T;L^{q',q''})} \|P_k\|_{L^\infty(0,T;L^{3,\infty})} \|u-P_k\|_{L^r(0,T;L^q)} \\ &\lesssim_{k,q,u_0} t^{\frac{1}{r'}+\frac{1}{2}}, \end{aligned} \tag{5.43}$$

where

$$1 = \frac{1}{q} + \frac{1}{q'} + \frac{1}{3}, \quad 1 = \frac{1}{q} + \frac{1}{q''} \quad \text{and} \quad 1 = \frac{1}{r} + \frac{1}{r'}.$$

By our choice of $r = \frac{2q}{2q-3}$ in Proposition 4.12, we have $\frac{1}{r'} = \frac{3}{2q}$. Then for any $\sigma < 3/2$, by taking $q > 3/2$ sufficiently close to $3/2$, we have

$$K_2(x,t) \lesssim_{k,\sigma,u_0,q} t^\sigma. \tag{5.44}$$

Altogether, (5.40),(5.41),(5.42) and (5.44) imply, for $0 < t < T$,

$$\|u - P_{k+1}\|_{L^\infty(B_{k+1})}(t) \lesssim_{k,\alpha,p,u_0,q} t^{a_{k+1}},$$

for $k \geq 0$ and any $\sigma < 3/2$. Choosing k_0 sufficiently large so that

$$k_0\left(\frac{1}{2} - \frac{3}{2p}\right) + a_0 \geq \sigma,$$

i.e. for $a_{k_0} = \sigma$ and α such that $1/4 = \alpha^{k_0}/2$, we arrive at

$$\|u - P_{k_0}\|_{L^\infty(B_{1/4}(x_0))}(t) \lesssim_{p,\sigma,u_0} t^{\sigma}.$$

To prove the asymptotic expansion, note

$$|P_k - P_{k-1}|(x,t) \leq |u - P_k|(x,t) + |u - P_{k-1}|(x,t) = \mathcal{O}(t^{a_{k-1}}).$$

Then we can expand u as

$$\begin{aligned} u(x,t) &= \underbrace{P_0 + \sum_{k=1}^{k_0}(P_k - P_{k-1})(x,t)}_{=P_{k_0}} + \mathcal{O}(t^{\sigma}) \\ &= \mathcal{O}(t^{-\frac{3}{2p}}) + \sum_{k=0}^{k_0-1}\mathcal{O}(t^{a_k}) + \mathcal{O}(t^{\sigma}) = \sum_{k=-1}^{k_0}\mathcal{O}(t^{a_k}), \end{aligned} \tag{5.45}$$

for short time $t \in (0,T)$. We let $a_{-1} = -3/(2p)$ using Remark 2 to justify the asymptotics for P_0. $\square$

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6 Appendix A

6.1 $L^{3,\infty}$ weak solutions are local energy solutions

Proof. We start by assuming (u,q) is a $L^{3,\infty}$ weak solution to (1.1) with data $u_0 \in L^{3,\infty}$. First, by [14, Lemma 6.3], $L^{3,\infty} \hookrightarrow L^2_{\mathrm{uloc}}$, so $u_0 \in L^2_{\mathrm{uloc}}(\mathbb{R}^3)$.

Next, (u,p) constitutes a distributional solution to (1.1), so Item 1 in the definition of local energy solutions is satisfied.

Item 2 is clearly satisfied for $u - e^{t\Delta}u_0 \in L^\infty(0,T;L^2) \cap L^2(0,T;\dot{H}^1)$, and we just need to show it is satisfied for $e^{t\Delta}u_0$.

By [7, Proposition 2.4], for $q > 3$

$$\|\nabla^k e^{t\Delta} u_0\|_{L^q}(t) \lesssim \frac{C\|u_0\|_{L^{3,\infty}}}{t^{k/2+3/2(1/3-1/q)}}.$$

Therefore, for $q < 6$

$$\begin{aligned}
&\sup_{x_0\in\mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla e^{s\Delta} u_0|^2 \, dx \, ds \\
&\lesssim \sup_{x_0\in\mathbb{R}^3} \int_0^{R^2} \||\nabla e^{s\Delta} u_0|^2\|_{L^{q/2}} \|\chi_{B_R(x_0)}\|_{L^{q/(q-2)}} \, ds \\
&\lesssim \sup_{x_0\in\mathbb{R}^3} \int_0^{R^2} \|\nabla e^{s\Delta} u_0\|^2_{L^q} \|\chi_{B_R(x_0)}\|_{L^{q/(q-2)}} \, ds \\
&\lesssim R^{3(q-2)/q} \int_0^{R^2} s^{-2+6/q} \, ds \\
&\lesssim R^{3(q-2)/q} (R^2)^{-1+6/q} \lesssim R^{1+6/q} < \infty,
\end{aligned} \tag{6.1}$$

and

$$\lim_{|x_0|\to\infty} \int_0^{R^2} \int_{B_R(x_0)} |e^{s\Delta} u_0|^2(x,s) \, dx \, ds = 0,$$

for any $R < \infty$.

Item 4 is satisfied as it is shown in [7, Proposition 2.4] that

$$\lim_{t\to 0} \|u(\cdot,t) - e^{t\Delta}u_0\|_{L^2} = 0$$

and

$$\lim_{t\to 0} \|e^{t\Delta}u_0 - u_0\|_{L^s_{\text{unif}}} = 0,$$

for $s < 3$. Therefore $u(t) \to u_0$ in $L^2(K)$ for and $K \Subset \mathbb{R}^3$.

Item 5 and Item 6 are common to both definitions.

Therefore any $L^{3,\infty}$ weak solution is also a local *Leray* solution. It remains to show Item 3, the local pressure expansion, to prove (u,p) is a local energy solution. By [7], weak $L^{3,\infty}$ weak solutions are mild, which by [15, Theorem 1.4], implies the local pressure expansion. $\square$