


1951

Moment-Generating Functions and Laplace Transforms

James F. Miller

University of Arkansas, Fayetteville

Follow this and additional works at: <http://scholarworks.uark.edu/jaas>

 Part of the [Ordinary Differential Equations and Applied Dynamics Commons](#)

Recommended Citation

Miller, James F. (1951) "Moment-Generating Functions and Laplace Transforms," *Journal of the Arkansas Academy of Science*: Vol. 4 , Article 16.

Available at: <http://scholarworks.uark.edu/jaas/vol4/iss1/16>

This article is available for use under the Creative Commons license: Attribution-NoDerivatives 4.0 International (CC BY-ND 4.0). Users are able to read, download, copy, print, distribute, search, link to the full texts of these articles, or use them for any other lawful purpose, without asking prior permission from the publisher or the author.

This Article is brought to you for free and open access by ScholarWorks@UARK. It has been accepted for inclusion in Journal of the Arkansas Academy of Science by an authorized editor of ScholarWorks@UARK. For more information, please contact scholar@uark.edu, ccmiddle@uark.edu.

MOMENT-GENERATING FUNCTIONS AND LAPLACE TRANSFORMS

JAMES F. MILLER

University of Arkansas, Fayetteville, Arkansas

The purpose of this paper is to establish a relationship between *moment-generating functions* and another type of integral transform, namely the *Laplace transform*, and to show what could follow from such relationship.

If $M_x(\theta)$ denotes a *moment-generating function* of say a statistical distribution, then $M_x(\theta) = g(\theta) + h(\theta)$ where $g(\theta)$ and $h(\theta)$ are *Laplace transforms*; i. e., such a *moment-generating function* is a linear combination of two *Laplace transforms*, if the given function can be *dominated* in any sort of way.

Let a *moment-generating function* be denoted by $M_x(\theta)$ where

$$(1) \quad M_x(\theta) = \int_a^b e^{\theta x} f(x) dx; \quad f(x) \geq 0 \text{ for } a \leq x \leq b \text{ and } f(x) \equiv 0 \quad \begin{cases} x < a \\ x > b \end{cases}.$$

Now for a Riemannian integral such as $M_x(\theta)$ one may write

$$M_x(\theta) = \int_a^0 e^{\theta x} f(x) dx + \int_0^b e^{\theta x} f(x) dx,$$

in which $f(x)$ is defined so that

$$\int_a^0 e^{\theta x} f(x) dx = \int_{-\infty}^0 e^{\theta x} f(x) dx \quad \text{and} \quad \int_0^b e^{\theta x} f(x) dx = \int_0^{\infty} e^{\theta x} f(x) dx.$$

Then

$$M_x(\theta) = \int_{-\infty}^0 e^{\theta x} f(x) dx + \int_0^{\infty} e^{\theta x} f(x) dx \quad \text{or}$$

$$M_x(\theta) = \int_0^{\infty} e^{-\theta x} f(-x) dx + \int_0^{\infty} e^{-(\theta)x} f(x) dx.$$

The *Laplace transform* is defined such that

$$(2) \quad L \left\{ f(t) \right\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

By comparison with $M_x(\theta)$ one sees that $\int_0^{\infty} e^{-\theta x} f(-x) dx$ and $\int_0^{\infty} e^{-(\theta)x} f(x) dx$ are two Laplace transforms if we restrict $f(x)$ and $f(-x)$ so that they are of exponential order. We may designate them as $g(\theta)$ and $h'(-\theta)$ or $g(\theta)$ and $h(\theta)$ respectively. Now it follows that a *moment-generating function* is the sum of two *Laplace transforms*.²

¹The notation used in this paper is the same as that in references 6 and 7.

²This theorem has been generalized to the case of more than one variable.

An interesting application of this theorem is that of handling certain *moment-generating functions* by looking them up in a table of *Laplace transforms*. As an example let us consider a *moment-generating function* where $f(x) = e^{-x}$, $0 \leq x \leq \infty$ and $f(x) \equiv 0$ for $-\infty \leq x \leq 0$.

$$\text{Then } M_x(\theta) = \int_0^{\infty} e^{\theta x} 1^{-x} dx = \int_0^{\infty} e^{(\theta x - x)} dx = \int_0^{\infty} e^{x(\theta - 1)} dx = \left[\frac{e^{x(\theta - 1)}}{\theta - 1} \right]_0^{\infty}.$$

This function will exist provided that $\theta < 1$; it follows that

$$\left[\frac{e^{x(\theta - 1)}}{\theta - 1} \right]_0^{\infty} = \left[\frac{e^{-x(1 - \theta)}}{\theta - 1} \right]_0^{\infty} = -\frac{1}{\theta - 1} = \frac{1}{1 - \theta}.$$

Now by the use of *Laplace transforms*

$$M_x(\theta) = \int_0^{\infty} e^{-\theta x} [0] dx + \int_0^{\infty} e^{-(-\theta)x} e^{-x} dx.$$

The first transform is zero, and from a table of *Laplace transforms* we find that the transform of e^{ax} is $\frac{1}{s - a}$.

Here $s = -\theta$ and $a = -1$ and $\frac{1}{s - a} = \frac{1}{-\theta - (-1)} = \frac{1}{1 - \theta}$

The fact that a *moment-generating function* is the sum of two *Laplace transforms* is sufficient information for one to conclude that any *moment-generating function* represents one and only one function, and conversely.

The demonstration is as follows:

$$\text{Consider } M_x(\theta) = \int_a^b e^{\theta x} f(x) dx, \quad a \leq x \leq b \text{ and } f(x) \equiv 0 \text{ for } \begin{cases} x < a; \\ x > b; \end{cases}$$

then $M_x(\theta) = g(\theta) + h(\theta)$, where $g(\theta)$ and $h(\theta)$ are *Laplace transforms*. If both sides of the equality are operated on by L^{-1} , the inverse operator (a linear operator), we obtain

$$L^{-1} \{ M_x(\theta) \} = L^{-1} \{ g(\theta) \} + L^{-1} \{ h(\theta) \} = f(x),$$

but the *Laplace transform* links only pairs of functions, if we do not stress a *very strict* sense. Hence $M_x(\theta)$ represents one and only one function, $f(x)$, and conversely. This conclusion is explained in Churchill's book, *Modern Operational Mathematics in Engineering*, on page 11.

The author wishes to thank Dr. C. L. Perry of the University of Arkansas mathematics staff for helpful comments on this paper.

LITERATURE CITED

1. Kullback, Solomon, 1934. Characteristic functions and distribution. *Annals of Math. Statistics* 5: 264-307.
2. Curtiss, J. H., 1942. A note on the theory of Moment-Generating Functions. *Annals of Math. Statistics* 1942: 430-434.
3. Zygmund, A., 1947. A remark on characteristic functions. *Annals of Math. Statistics* 18: 272-285.

4. Kozakiewicz, W., 1947. On the convergence of sequences of Moment Generating Functions. *Annals of Math. Statistics* 18: 61-70.
5. Franklin, Philip. *Fourier Methods*, 1st edition, chap. 5.
6. Churchill. *Modern Operational Mathematics in Engineering*, 1st edition, chap. 1.
7. Hoel. *The Mathematical Theory of Statistics*, 1st edition.
8. Wilks, S. S. *Mathematical Statistics*, pp. 36.

UNIVERSITY OF ARKANSAS
LIBRARY