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The purpose of this paper is to establish a relationship between *moment-generating functions* and another type of integral transform, namely the *Laplace transform*, and to show what could follow from such relationship.

If $M_x(\theta)$ denotes a *moment-generating function* of say a statistical distribution, then $M_x(\theta) = g(\theta) + h(\theta)$ where $g(\theta)$ and $h(\theta)$ are *Laplace transforms;* i.e., such a *moment-generating function* is a linear combination of two *Laplace transforms*, if the given function can be *dominated* in any sort of way.

Let a *moment-generating function* be denoted by $M_x(\theta)$ where

$$(1) \qquad M_x(\theta) = \int_a^b e^{\theta x} f(x)dx; \quad f(x) \geq O \text{ for } a \leq x \leq b \text{ and } f(x) \equiv O \begin{cases} x < a \\ x > b \end{cases}.$$

Now for a Riemannian integral such as $M_x(\theta)$ one may write

$$M_x(\theta) = \int_a^o e^{\theta x} f(x)dx + \int_o^b e^{\theta x} f(x)dx,$$

in which f(x) is defined so that

$$\int_a^o e^{\theta x} f(x)dx = \int_{-\infty}^o e^{\theta x} f(x)dx \text{ and } \int_o^b e^{\theta x} f(x)dx = \int_o^\infty e^{\theta x} f(x)dx.$$

Then

$$M_x(\theta) = \int_{-\infty}^o e^{\theta x} f(x)dx + \int_o^\infty e^{\theta x} f(x)dx \text{ or}$$

$$M_x(\theta) = \int_o^\infty e^{-\theta x} f(-x)dx + \int_o^\infty e^{-(-\theta)x} f(x)dx.$$

The *Laplace transform* is defined such that

$$(2) \qquad L\left\{f(t)\right\} = F(s) = \int_o^\infty e^{-st} f(t)dt.$$

By comparison with $M_x(\theta)$ one sees that $\int_o^\infty e^{-\theta x} f(-x)dx$ and $\int_o^\infty e^{-(-\theta)x} f(x)dx$ are two Laplace transforms if we restrict f(x) and f(–x) so that they are of exponential order. We may designate them as $g(\theta)$ and $h'(-\theta)$ or $g(\theta)$ and $h(\theta)$ respectively Now it follows that a *moment-generating function* is the sum of two *Laplace transforms.*[2]

[1] The notation used in this paper is the same as that in references 6 and 7.
[2] This theorem has been generalized to the case of more than one variable.

An interesting application of this theorem is that of handling certain *moment-generating functions* by looking them up in a table of *Laplace transforms.* As an example let us consider a *moment-generating function* where $f(x) = e^{-x}$, $o \leq x \leq \infty$ and $f(x) \equiv o$ for $-\infty \leq x \leq o$.

Then $M_x(\theta) = \int_o^\infty e^{\theta x} 1^{-x}\,dx = \int_o^\infty e^{(\theta x - x)}\,dx = \int_o^\infty e^{x(\theta - 1)}\,dx = \left[\frac{e^{x(\theta-1)}}{\theta-1}\right]_o^\infty$.

This function will exist provided that $\theta < 1$; it follows that

$$\left[\frac{e^{x(\theta-1)}}{\theta-1}\right]_o^\infty = \left[\frac{e^{-x(1-\theta)}}{\theta-1}\right]_o^\infty = -\frac{1}{\theta-1} = \frac{1}{1-\theta}.$$

Now by the use of *Laplace transforms*

$$M_x(\theta) = \int_o^\infty e^{-\theta x}\,[o]\,dx + \int_o^\infty e^{-(-\theta)x} e^{-x} dx.$$

The first transform is zero, and from a table of *Laplace transforms* we find that the transform of e^{ax} is $\frac{1}{s-a}$.

Here $s = -\theta$ and $a = -1$ and $\frac{1}{s-a} = \frac{1}{-\theta-(-1)} = \frac{1}{1-\theta}$

The fact that a *moment-generating function* is the sum of two *Laplace transforms* is sufficient information for one to conclude that any *moment-generating function* represents one and only one function, and conversely.

The demonstration is as follows:

Consider $$M_x(\theta) = \int_a^b e^{\theta x} f(x)dx,\ a \leq x \leq b \text{ and } f(x) \equiv o \text{ for } \begin{cases} x < a; \\ x > b' \end{cases}$$

then $M_x(\theta) = g(\theta) + h(\theta)$, where $g(\theta)$ and $h(\theta)$ are *Laplace transforms.* If both sides of the equality are operated on by L^{-1}, the inverse operator (a linear operator), we obtain

$$L^{-1}\left\{M_x(\theta)\right\} = L^{-1}\left\{g(\theta)\right\} + L^{-1}\left\{h(\theta)\right\} = f(x),$$

but the *Laplace transform* links only pairs of functions, if we do not stress a *very strict* sense. Hence $M_x(\theta)$ represents one and only one function, $f(x)$, and conversely. This conclusion is explained in Churchill's book, *Modern Operational Mathematics in Engineering*, on page 11.

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