

5-2017

# 3-Manifold Perspective on Surface Homeomorphisms for Surfaces with Very Negative Euler Characteristic

Michael Harris

*University of Arkansas, Fayetteville*

Follow this and additional works at: <http://scholarworks.uark.edu/etd>

 Part of the [Other Applied Mathematics Commons](#)

---

## Recommended Citation

Harris, Michael, "3-Manifold Perspective on Surface Homeomorphisms for Surfaces with Very Negative Euler Characteristic" (2017). *Theses and Dissertations*. 1909.

<http://scholarworks.uark.edu/etd/1909>

This Dissertation is brought to you for free and open access by ScholarWorks@UARK. It has been accepted for inclusion in Theses and Dissertations by an authorized administrator of ScholarWorks@UARK. For more information, please contact [scholar@uark.edu](mailto:scholar@uark.edu), [ccmiddle@uark.edu](mailto:ccmiddle@uark.edu).

3-Manifold Perspective on Surface Homeomorphisms for Surfaces with Very Negative Euler  
Characteristic

A dissertation submitted in partial fulfillment  
of the requirements for the degree of  
Doctor of Philosophy in Mathematics

by

Michael Harris  
Missouri Southern State University  
Bachelor of Science in Mathematics, 2010  
University of Arkansas  
Master of Science in Mathematics, 2012

May 2017  
University of Arkansas

This dissertation is approved for recommendation to the Graduate Council.

---

Dr. Yo'av Rieck  
Dissertation Director

---

Dr. Jeremy Van-Horn Morris  
Committee Member

---

Dr. Matt Clay  
Committee Member

## **Abstract**

The goal of this paper is to show for a compact triangulated 3-manifold  $M$  with boundary which fibers over the circle that whenever  $F$  is a fiber with sufficiently negative Euler characteristic the monodromy maps an essential simple closed curve or an essential simple arc in  $F$  to be disjoint from its image (possibly after isotopy). This is shown by applying the theorem of Ichihara, Kobayashi, and Rieck in [10] to the double of  $M$  to get a pair of pants. We then find an equivariant pair of pants and use it to find an essential simple closed curve or an essential simple arc which satisfies our theorem. As a corollary, if we add the hypothesis that  $M$  is a hyperbolic manifold, we get that the translation distance of the monodromy in the arc and curve complex of  $F$  is at most 1 for all but finitely many monodromy maps.

## **Acknowledgements**

I would like to thank the Missouri Southern State University Mathematics Department for offering a friendly environment and helping me take my first steps into the world of mathematics. In particular, I would like to thank Dr. Richard Laird who was always very helpful and helped create a positive environment around the department. I would also like to thank Dr. Charles Curtis who first encouraged me to apply to different graduate programs in mathematics; without this encouragement I am uncertain if I would have ever gone to graduate school.

I would also like to express my appreciation to the Department of Mathematics at the University of Arkansas. I am especially appreciative of the people who work in the mathematics office and the hard work they do to keep everything running well and keeping me informed about important deadlines. I would like to thank both Dr. Matthew Day and Dr. Dan Rutherford who taught differential topology and knot theory classes respectively and helped spark my initial interest in studying topology. I would like to thank Dr. Matt Clay who played an important role in my initial understanding of the curve complex of a surface and also looked at some of the very early work on my dissertation. I would like to thank Dr. Jeremy Van-Horn Morris who taught a class on low dimensional topology which helped expand my knowledge of 3-manifolds, my topic of study. Lastly, I would like to thank my academic advisor, Dr. Yo'av Rieck, who has spent a considerable amount of time looking at various versions of my dissertation and helping me throughout the past few years. He has done an excellent job at pointing out flaws in my logic and has been a great person to work with.

Finally, I would like to thank my family, without whom I would not exist and would be rather lonely.

## **Table of Contents**

<b>1 Introduction</b>	<b>1</b>
<b>2 Preliminaries</b>	<b>5</b>
<b>3 Curve Complex and Arc and Curve Complex</b>	<b>28</b>
<b>4 Proof</b>	<b>34</b>
<b>References</b>	<b>57</b>

## 1 Introduction

The goal of this paper is to prove a result about fibered 3-manifolds. Our main result is the following (for standard definitions see section 2):

*Theorem 1.1.* Let  $F$  be a connected surface, let  $\mu : F \rightarrow F$  be a homeomorphism, and let  $M := (F \times [0, 1]) / \sim$  where  $(x, 0) \sim (\mu(x), 1)$  for  $x \in F$ . Let  $t(M)$  be the minimal number of tetrahedra required to triangulate  $M$ . If  $\chi(F) \leq -76t(M) + 1$  then there is an essential simple closed curve or essential simple arc which is disjoint from its image under  $\mu$  (possibly after isotopy).

The case when  $F$  is closed is due to Ichihara, Kobayashi, and Rieck in [10]. The current paper exclusively deals with the case when  $F$  has boundary.

In this section we assume the reader is familiar with the curve complex of a surface  $F$  denoted  $\mathcal{C}(F)$  and the arc and curve complex of a surface  $F$  denoted  $\mathcal{AC}(F)$ . A brief discussion on these topics is included in section 3. Notice that the homeomorphism  $\mu : F \rightarrow F$  in Theorem 1.1 induces an isometry  $\mu_* : \mathcal{AC}(F) \rightarrow \mathcal{AC}(F)$  on  $\mathcal{AC}(F)$ . We define the translation distance of  $\mu$  in  $\mathcal{AC}(F)$  to be  $\min\{d(v, \mu_*(v)) \mid v \in \mathcal{AC}(F) \text{ is a vertex}\}$  where  $d$  is the distance between vertices in the arc and curve complex. Note that  $\mu$  sends essential simple closed curves to essential simple closed curves and essential simple arcs to essential simple arcs. Since we found in Theorem 1.1 an essential simple closed curve or essential simple arc which is disjoint from its image under  $\mu$  (possibly after isotopy) we see that the translation distance for  $\mu$  in  $\mathcal{AC}(F)$  is at most 1.

We now describe two types of 3-manifolds. A 3-manifold is simple if it contains no essential surfaces with nonnegative Euler characteristic and, as in [11, p. 42], a compact orientable irreducible 3-manifold is a Haken manifold if it contains a 2-sided incompressible surface. A result due to Haken, as presented in [15], states that for a simple Haken 3-manifold there is a finite number of essential surfaces with a given Euler characteristic. This result also holds when the 3-manifold is simple and not Haken since then the manifold will have no

essential surfaces, however, this case is less interesting to us.

As stated in [8, p. 16], an irreducible 3-manifold is called atoroidal if every incompressible torus is boundary parallel. Thurston's hyperbolization theorem, as presented in [12], states that a compact atoroidal Haken 3-manifold  $M$  whose boundary consists of a possibly empty collection of tori has interior which admits a complete hyperbolic metric of finite volume. A hyperbolic manifold is a compact 3-manifold whose interior admits a complete finite volume Riemannian metric locally isometric to hyperbolic 3-space. So this theorem says that such an  $M$  is a hyperbolic manifold. For some background on hyperbolic manifolds see for example [2].

Going back to the setting in Theorem 1.1, notice that if  $F$  is orientable and  $\mu: F \rightarrow F$  is orientation preserving then  $M$  is orientable. We show in Corollary 2.20 that  $M$  must be irreducible and moreover we show in Lemma 2.15 that  $F$  associated with  $F \times \{0\}$  is essential and hence incompressible in  $M$ . We conclude in this case that  $M$  is a Haken manifold. So under these restraints  $M$  satisfies all of the hypotheses in Thurston's hyperbolization theorem except for being atoroidal. Without adding another hypothesis we cannot guarantee that  $M$  is atoroidal.

Let  $M$  be a 3-manifold. A surface  $F$  in  $M$  is called a fiber if  $M \setminus F$  is diffeomorphic to  $F \times I$ . We define for an isotopy class of a fiber  $F$  the monodromy of  $F$  in  $M$  to be the isotopy class of diffeomorphisms  $\mu: F \rightarrow F$  so that  $M := (F \times I) / \sim$  where  $(x, 0) \sim (\mu(x), 1)$  for  $x \in F$ . If  $M$  is a 3-manifold which fibers over  $S^1$  then at least one monodromy exists. In general a 3-manifold  $M$  may have more than one monodromy corresponding to distinct fibers. We consider Thurston's hyperbolization theorem for fibered manifolds presented in [16] which states: let  $M$  be a 3-manifold that fibers over  $S^1$  whose monodromy is a pseudo Anosov diffeomorphism  $\mu$ . Then the interior of  $M$  admits a complete hyperbolic metric of finite volume. That is,  $M$  is a hyperbolic manifold. For a discussion on what it means for a diffeomorphism to be pseudo Anosov see for example [4] and [6].

Let  $M$  be a compact 3-manifold which fibers over  $S^1$ . A priori it is unknown how many

isotopy classes of essential surfaces are fibers for  $M$ . Since  $M$  is a specific 3-manifold it has a fixed number of tetrahedra,  $t(M)$ , for a minimal triangulation. Applying Theorem 1.1 to  $M$  we get that only fibers,  $F$ , with sufficiently high Euler characteristic (that is  $\chi(F) > -76t(M) + 1$ ) will have monodromy with translation distance in  $\mathcal{AC}(F)$  greater than 1. Now assume that the monodromy of  $F$  in  $M$  is pseudo Anosov, Thurston's hyperbolization theorem for fibered manifolds tells us that  $M$  is hyperbolic. Additionally, since the monodromy of  $F$  is pseudo Anosov  $M$  is simple. Finally, if we require that  $M$  is orientable we get from our argument earlier that  $M$  is a Haken manifold. Applying the earlier theorem of Haken's we get that at most finitely many isotopy classes of fibers have sufficiently high Euler characteristic. From our discussion we get the following corollary:

*Corollary 1.2.* Let  $M$  be a compact orientable hyperbolic 3-manifold. Then there are at most finitely many isotopy classes of fibers whose monodromy has translation distance in  $\mathcal{AC}(F)$  greater than 1 .

The closed case for Corollary 1.2 is due to Ichihara, Kobayashi, and Rieck in [10]. Now, from just Corollary 1.2, we don't even know if a compact hyperbolic 3-manifold has a single monodromy; in general a compact hyperbolic 3-manifold need not fiber over  $S^1$ . Now, under certain conditions, we can use Corollary 1.2 to get a much stronger conclusion. Let  $H_2(M, \partial M; \mathbb{R})$  be the second homology of  $M$  relative to  $\partial M$ . Whenever  $M$  is a hyperbolic 3-manifold which fibers over  $S^1$  and  $\dim(H_2(M, \partial M; \mathbb{R})) \geq 2$ , then there are infinitely many isotopy classes of fibers  $F$  in  $M$  so that  $M$  fibers over  $S^1$  with fiber  $F$ . This result is derived from the work of Thurston in [19]. It turns out that hyperbolic 3-manifolds which fiber over  $S^1$  with  $\dim(H_2(M, \partial M; \mathbb{R})) \geq 2$  naturally arise as the covering space of a hyperbolic manifold. Indeed, in [1] Agol proves the virtual fibering conjecture which states that for a closed hyperbolic 3-manifold  $M$  there exists a finite-sheeted cover  $\widetilde{M} \rightarrow M$  such that  $\widetilde{M}$  fibers over  $S^1$  and satisfies  $\dim(H_2(\widetilde{M}, \partial \widetilde{M}; \mathbb{R})) \geq 2$ . Moreover, in [20] Wise proves the virtual fibering conjecture when  $M$  is a complete finite volume hyperbolic 3-manifold that is not closed; also see [21] and



reference within. So there are many compact hyperbolic 3-manifolds  $M$  which fiber over  $S^1$  satisfying  $\dim(H_2(M, \partial M; \mathbb{R})) \geq 2$ . In particular, our next two corollaries are not vacuous. From our discussion along with Corollary 1.2 we conclude the following:

*Corollary 1.3.* Let  $M$  be a compact orientable hyperbolic 3-manifold which fibers over  $S^1$  satisfying  $\dim(H_2(M, \partial M; \mathbb{R})) \geq 2$ . Then  $M$  has infinitely many isotopy classes of fibers of which only finitely many have monodromy with translation distance in  $\mathcal{AC}(F)$  greater than 1.

As in Corollary 1.2 the closed case for Corollary 1.3 also follows from [10], this is due to the fact that the arc and curve complex and the curve complex are equivalent for closed surfaces. We finish this section off by relating our result for the arc and curve complex to a similar result for the curve complex. Applying Lemma 3.4 to Corollary 1.3 we conclude the following:

*Corollary 1.4.* Let  $M$  be a compact orientable hyperbolic 3-manifold which fibers over  $S^1$  satisfying  $\dim(H_2(M, \partial M; \mathbb{R})) \geq 2$ . Then then  $M$  has infinitely many isotopy classes of fibers of which only finitely many have monodromy with translation distance in  $\mathcal{C}(F)$  greater than 2.

Again the closed case follows from [10]. In fact, they get the slightly better result that only finitely many isotopy classes of fibers will have monodromy with translation distance in the curve complex greater than 1.

## 2 Preliminaries

In this paper we use methods from the study of 3-dimensional manifolds. We define briefly some of our basic terms and mention some of the notation we will use. We always assume transversality unless it is clear from context otherwise.

*Definition 2.1.* An  $N$ -manifold  $M^N$  is a Hausdorff second countable topological space such that every point in  $M^N$  has a neighborhood homeomorphic to an open set of  $\mathbb{R}_{\geq 0}^N$ , where  $N$  is the dimension of  $M^N$ . The boundary of  $M^N$ , denoted  $\partial M^N$ , is the set of all points which do not have a neighborhood homeomorphic to an open set of  $\mathbb{R}^N$ . Let  $M_1$  and  $M_2$  be manifolds (not necessarily of the same dimension) and let  $f : M_1 \rightarrow M_2$ , we say that  $f$  is proper if the preimage of the boundary of  $M_2$  is equal the boundary of  $M_1$ .

We include some of our basic notation. We use  $I$  to represent the closed interval  $[0, 1]$ . We use  $S^1$  to represent the circle. For a surface  $F$  we denote the Euler characteristic of  $F$  to be  $\chi(F)$ . For a topological space  $X$  we denote  $int(X)$  to be the interior of  $X$  and  $cl(X)$  to be the closure of  $X$ . We use  $\cong$  to mean homeomorphic to. For a function  $f$  we use  $f|_D$  to mean  $f$  restricted to  $D$  and  $im(f)$  to mean the image of  $f$ .

Some of the major subjects studied in this paper include that of fiber bundles, essential surfaces, irreducible manifolds, normal surfaces and the double of a manifold. Over the next several pages we will be discussing these terms and related concepts in detail. We first introduce fiber bundles.

*Definition 2.2.* Let  $M$  and  $N$  be manifolds. A map  $f : M \rightarrow N$  is an embedding if it is a homeomorphism onto its image  $f(M)$  and  $f(M)$  is a submanifold of  $N$ . (See [18, p. 4])

Recall for a smooth manifold  $X$  and with a point  $p \in X$  that the tangent space of  $X$  at  $p$  is denoted  $T_p X$ . Also recall that the tangent bundle of  $X$  is denoted by  $TX$  where  $TX := \{(p, v) \mid p \in X, v \in T_p X\}$ .

*Definition 2.3.* Let  $F$  be a properly embedded surface in a 3-manifold  $M$ . A coorientation of  $F$

is represented by a continuous function  $f : F \rightarrow TM$  so that for all  $p \in F$ ,  $f(p) \in T_pM \setminus T_pF$ . We say that two coorientations  $g_1$  and  $g_2$  are equivalent if there exists a homotopy  $H : F \times [0, 1] \rightarrow TM$  so that  $H(s, 0) = g_1(s)$  and  $H(s, 1) = g_2(s)$  for all  $s \in F$  and, moreover, for every fixed  $t_0 \in [0, 1]$  we require that  $H(s, t_0)$  is a coorientation. Notice that for each point  $p \in F$  the space  $T_pM \setminus T_pF$  is disconnected. So  $F$  can have at most two coorientations. We say that  $F$  is 2-sided if it has two coorientations.

*Definition 2.4.* Let  $F$ ,  $E$ , and  $B$  be topological spaces, and  $p : E \rightarrow B$  a continuous surjection. We say that  $\xi = (F, E, B, p)$  is a fiber bundle (or simply a bundle) if the preimage of every  $b \in B$  is homeomorphic to  $F$ , and moreover, these preimages satisfy the local triviality condition: every  $b \in B$  has an open neighborhood  $U \subset B$ , so that there is a homeomorphism  $F \times U \rightarrow p^{-1}(U)$  that maps  $F \times \{x\}$  homeomorphically onto  $p^{-1}(x)$  for every  $x \in U$ . We call  $E$  the total space,  $B$  the base,  $F$  the fiber, and  $p$  the bundle projection. We say that  $\xi$  is an  $F$  bundle over  $B$ . A bundle  $(F, E, B, p)$  is called a trivial bundle if  $E \cong F \times B$  and  $p$  is the projection onto the second factor.

*Definition 2.5.* A covering space of a space  $X$  is a space  $\tilde{X}$  together with a map  $p : \tilde{X} \rightarrow X$  satisfying the following condition: there exists an open cover  $\{U_\alpha\}$  of  $X$  such that for each  $\alpha$ ,  $p^{-1}(U_\alpha)$  is a disjoint union of open sets in  $\tilde{X}$ , each of which is mapped by  $p$  homeomorphically onto  $U_\alpha$ . We do not require  $p^{-1}$  to be nonempty, so  $p$  need not be surjective. (See [7, p. 56])

*Definition 2.6.* Let  $M$  be an  $n$ -manifold. Let  $S$  be a submanifold of  $M$  of dimension  $m$ . A regular neighborhood of  $S$  is a submanifold  $NS$  of  $M$  of dimension  $n$  that is the total space of a bundle over  $S$  with fiber the  $n - m$  ball  $B^{n-m}$ . (See [18, p. 60])

In this paper we are primarily interested in fiber bundles with total space a 3-manifold  $M$ , base  $S^1$ , fiber a surface  $F$ , and bundle projection  $p$ . It follows from the bundle structure that  $p^{-1}(x) \cong F \times \{x\}$  is 2-sided in  $M$  for every  $x \in S^1$ . In this setting since  $F \times \{x\}$  is homeomorphic to  $F$  for all  $x \in S^1$  we often refer to  $\{F \times \{x\} \subset M \mid x \in S^1\}$  as the fibers of  $M$  without explicitly mentioning the bundle projection. We often associate  $F$  with  $F \times \{0\}$  using the association of  $S^1$  with  $[0, 1] / \sim$ , where  $0 \sim 1$ , to make sense of  $\{0\}$  in  $S^1$ .

We now introduce terminology related to essential surfaces along with related results used in this paper.

*Definition 2.7.* A properly embedded surface  $F$  in a 3-manifold  $M$  is called boundary parallel if it is isotopic, fixing  $\partial F$ , to a subsurface of  $\partial M$ . (See [8, p. 16])

*Definition 2.8.* A 2-sided connected surface  $F$ , which is not a sphere, is called incompressible if for each disk  $D \subset M$  with  $D \cap F = \partial D$  there is a disk  $D' \subset F$  with  $\partial D' = \partial D$ . If  $F$  is a sphere then it is said to be incompressible if it does not bound a 3-ball. (See [8, p. 13])

Remark. In some references, such as [9, p. 58], the disk may have additional requirements to be incompressible. We will not worry about these details.

*Definition 2.9.* A properly embedded surface  $F$  in a 3-manifold  $M$  is boundary incompressible if for each disk  $D \subset M$  such that  $\text{int}(D) \cap (F \cup \partial M) = \emptyset$ ,  $\partial D \cap F$  is an arc  $\alpha$ ,  $\partial D \setminus \alpha \subset \partial M$  is an open arc  $\beta$ , and there exists a disk  $D' \subset F$  with  $\alpha \subset \partial D'$  and  $\partial D' - \alpha \subset \partial F$ . (See [8, p. 18])

In addition to the two previous definitions we also say a surface is compressible if it is not incompressible and a surface is boundary compressible if it is not boundary incompressible. Furthermore, in the definition of an incompressible surface, the disk  $D$  is a compressing disk for  $F$  and in the definition of a boundary incompressible surface, the disk  $D$  is called a boundary compressing disk for  $F$ . We now define what it means to be an essential surface.

*Definition 2.10.* A surface  $S$  in a 3-manifold  $M$  is said to be essential if it is incompressible, boundary incompressible, and not boundary parallel.

Related to the notion of an essential surface we have the notion of an essential simple closed curve and an essential arc. Before we define these, we introduce the notation  $\setminus\setminus$  which means cut open along. In particular, for a manifold  $X$  with a properly embedded codimension-1 submanifold  $Y$ , we define  $X \setminus\setminus Y$  to be the closure of a  $X \setminus NY$ , where  $NY$  is a regular neighborhood of  $Y$ .

*Definition 2.11.* A simple closed curve  $\alpha$  in a connected surface  $F$  is said to be essential if either  $F \setminus \alpha$  is connected or  $F \setminus\setminus \alpha$  does not have a disk or an annulus component.

*Definition 2.12.* A simple arc  $\eta$  is said to be an essential arc in a connected surface  $F$  if  $\eta$  is properly embedded and either  $F \setminus \eta$  is connected or  $F \setminus \eta$  does not contain a disk component.

We now introduce the concept of an essential subsurface contained in a surface along with a related lemma.

*Definition 2.13.* Let  $H$  be an embedded subsurface of a surface  $S$ . We say that  $H$  is an essential subsurface of  $S$  if no component of  $\partial H$  bounds a disk in  $S$ . We do not worry about components of  $\partial H$  being boundary parallel in  $S$ .

*Lemma 2.14.* Let  $H$  be a connected essential subsurface of a connected surface  $S$ , where  $S$  is not a sphere, then  $\chi(S) \leq \chi(H)$ .

Proof of Lemma 2.14. Suppose for a contradiction that  $\chi(H) < \chi(S)$ . If it is not already the case we isotope  $H$  to ensure that  $\partial H$  is disjoint from  $\partial S$  while still ensuring that  $H \subset S$ . The previous sentence along with the fact that the boundary components of a surface are all simple closed curves allows us to have that  $\chi(S) = \chi(H) + \chi(S \setminus H)$  or  $\chi(S) - \chi(H) = \chi(S \setminus H)$ . Since  $\chi(S) - \chi(H) > 0$  we see that  $\chi(S \setminus H) > 0$ . Since the Euler characteristic is additive over disjoint unions  $cl(S \setminus H)$  has a component with positive Euler characteristic and boundary. Hence a component of  $cl(S \setminus H)$  is a disk. Since we are considering  $H \subset S$  we get that  $H$  has a boundary component which bounds this disk. This is a contradiction since we assumed that  $H$  was essential. We conclude that the result of the lemma.  $\square$

The following lemma is important for the main result of this paper.

*Lemma 2.15.* Let  $F$  be a connected surface. Let  $\mu : F \rightarrow F$  be a homeomorphism. Identify  $F$  with  $F \times \{0\}$ . Then  $F$  is an essential surface of  $M := (F \times [0, 1]) / \sim$  where  $(x, 0) \sim (\mu(x), 1)$  for  $x \in F$ .

Proof of Lemma 2.15. In the case that  $F$  is a sphere it is clear from the structure of  $M$  that  $F$  does not bound a ball and hence is incompressible, In the case that  $F$  is not a sphere, suppose for a contradiction that  $F$  is compressible. Then there exists a disk  $D \subset M$  such that  $D \cap F = \partial D$  and there does not exist a disk  $D' \subset F$  with  $\partial D' = \partial D$ . Since  $int(D) \cap F = \emptyset$  we conclude that  $F$  is

also compressible in  $F \times I$  with compressing disk  $D$ . We assume without loss of generality that  $\partial D \subset F = F \times \{0\}$ . Notice that  $F \times I$  deformation retracts onto  $F$  and let  $\iota : F \hookrightarrow F \times I$  be the inclusion. It follows from [7, Prop. 1.17, p. 36] that the induced map  $\iota_* : \pi_1(F, p) \rightarrow \pi_1(F \times I, p)$  is an isomorphism, where  $p \in \partial D$ . The inclusion of  $\partial D$  into  $F$  is a nullhomotopic embedding. Since  $\partial D$  represents the trivial element in  $\pi_1(F \times I, p)$  our isomorphism shows that  $\partial D$  also represents the trivial element in  $\pi_1(F, p)$ . Since  $\partial D$  is an embedded curve in the surface  $F$  and represents the trivial element in  $\pi_1(F, p)$  it must bound an embedded disk in  $F$ . We assumed that such a disk does not exist and hence have a contradiction. We conclude that  $F$  is incompressible in  $M$ .

If  $\partial F = \emptyset$  then  $F$  is trivially boundary incompressible. Otherwise if  $\partial F \neq \emptyset$  we suppose for a contradiction that  $F$  is boundary compressible. Then there exists a disk  $D$  in  $M$  such that  $\text{int}(D) \cap (F \cup \partial M) = \emptyset$ ,  $\partial D \cap F$  is an arc  $\alpha$ ,  $\partial D \setminus \alpha \subset \partial M$  is an open arc  $\beta$ , and there does not exist a disk  $D' \subset F$  with  $\alpha \subset \partial D'$  and  $\partial D' \setminus \alpha \subset \partial F$ . Notice that  $\beta$  is disjoint from  $F$ . Since we are considering when  $F$  has boundary and these boundary components are simple closed curves we get that  $\partial M$  consist of tori and Klein bottles. Let  $C$  be the component of  $\partial M$  containing  $\beta$ . Notice that  $C^* := C \setminus F$  consists of one or more disjoint open annuli. Since  $\beta$  is disjoint from  $F$  we get that  $\beta$  is contained in one of the annuli in  $C^*$ ; we call this component  $C'$ . Since  $\alpha \subset F$ ,  $\alpha \cup \beta = \partial D$ , and  $\beta \subset C'$  we get that either  $\beta$  connects distinct ends of  $C'$  or  $\beta$  connects an end of  $C'$  to the same end. Suppose for a contradiction that  $\beta$  connects distinct ends of  $C'$ , we consider the following maps:

- $\iota : \partial D = \alpha \cup \beta \cong S^1 \hookrightarrow M$ , the inclusion
- $q : M \rightarrow S^1$ , projection onto the base of the fiber bundle

Since  $\beta$  is disjoint from  $F$  and connects distinct ends of  $C'$  we get that  $q \circ \iota$  not nullhomotopic. We get the following induced homomorphisms:

- $\iota_* : \pi_1(\partial D, p_1) \rightarrow \pi_1(M, p_1)$ , the 0-map

- $q_* : \pi_1(M, p_1) \rightarrow \pi_1(S^1, p_2)$

where  $p_1 \in \alpha \subset \partial D$  and  $p_2 = q(p_1)$ . Since  $q \circ \iota$  is not nullhomotopic we get that  $(q \circ \iota)_* = q_* \circ \iota_*$  is not the 0-map. Now, since  $\iota_*$  is the 0-map  $q_* \circ \iota_*$  is also the 0-map, a clear contradiction. Thus,  $\beta$  connects an end of  $C'$  to the same end of  $C'$ . So  $\beta$  and an arc component of  $\partial F$  cobound a disk in  $cl(C')$ . So we may isotope  $\beta$ , in  $C$ , into  $\partial F$ . This isotopy will also isotope  $\partial D$  into  $F$ . Doing this isotopy of  $D$  we get  $\alpha \subset \partial D$  and  $\partial D \setminus \alpha \subset \partial F \subset F$ . Since  $F$  is incompressible there exists a disk  $D^* \subset F$  such that  $\partial D^* = \partial D$ . In particular,  $\partial D^* \setminus \alpha \subset \partial F$  and since we have assumed that such a disk  $D^*$  did not exist we have a contradiction. We conclude that  $F$  is boundary incompressible.

Since  $F$  is boundary incompressible it follows that  $F$  is not boundary parallel. We conclude that  $F$  is essential in  $M$ .  $\square$

This lemma actually shows that every fiber of 3-manifold which fibers over  $S^1$  is essential. We now introduce the definition of an irreducible manifold along with a useful definition and some results which relate to fibered 3-manifolds.

*Definition 2.16.* A 3-manifold  $M$  is irreducible if every 2-sphere  $S^2 \subset M$  bounds a 3-ball  $B^3 \subset M$ . (See [8, p. 6])

*Definition 2.17.* For a disk  $D$  and a finite collection of pairwise disjoint simple closed curves  $\{\gamma_i\}$  which are disjoint from  $\partial D$  we say that  $\gamma \in \{\gamma_i\}$  is an innermost simple closed curve of  $D$  if there exists a disk  $D' \subset D$  bounded by  $\gamma$  which is disjoint from any other element in  $\{\gamma_i\}$ . Similarly, for a 2-sphere  $S$  and a finite collection of pairwise disjoint simple closed curves  $\{\gamma_i\} \subset S$  we say that  $\gamma \in \{\gamma_i\}$  is an innermost simple closed curve of  $S$  if there exists a disk  $D' \subset S$  bounded by  $\gamma$  which is disjoint from any other element in  $\{\gamma_i\}$ .

*Lemma 2.18.* If  $F$  is a connected surface with boundary and  $\chi(F) \leq 1$ , then  $F \times I$  is irreducible.

*Proof of Lemma 2.18.* We induct on  $-\chi(F)$ . Base case: If  $\chi(F) = 1$  then since  $\partial F \neq \emptyset$  we get that  $F$  is a disk. Thus  $F \times I$  is a 3-ball and as shown by Schultens in [18, p. 66-67], which utilizes the

Schonflies Theorem which states that every sphere in  $\mathbb{R}^3$  bounds a 3-ball, we get that  $F \times I$  is irreducible.

Induction hypothesis:  $F \times I$  is irreducible for  $-\chi(F) = k$ .

Induction step: Suppose  $-\chi(F) = k + 1$ . Recall that  $\partial F \neq \emptyset$ .

We consider two cases. First consider when  $\partial F$  consists of two or more components. Let  $\eta$  be a properly embedded arc in  $F$  with  $\partial\eta$  in two distinct components of  $\partial F$ . Notice that  $F \setminus \eta$  is connected. See for example Figure 1.

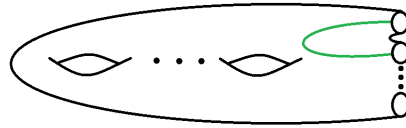


Figure 1: Non-separating essential arc for a surface with more than one boundary component.

Now consider when  $\partial F$  consists of exactly one component. Since we have already considered the case when  $\chi(F) = 1$  we may assume that  $\chi(F) \leq 0$ . It follows that  $F$  has genus. That is,  $F$  consists of a finite connect sum of tori or projective planes along with a single boundary component. It follows that there exists an essential arc  $\eta$  such that  $F \setminus \eta$  is connected. See for example Figure 2 and Figure 3.

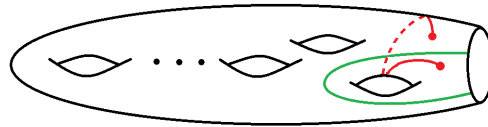


Figure 2: Non-separating essential arc with a curve connecting points on both sides of the arc for a surface with one boundary component.

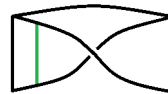


Figure 3: Non-separating essential arc in a mobius strip.



In both cases, for our choice of  $\eta$ , we get that  $F \setminus \eta$  is connected. Since  $\chi(\eta) = 1$  we get that  $\chi(F \setminus \eta) = \chi(F) + 1$ . Thus,

$$-\chi(F \setminus \eta) = -\chi(F) - 1 = (k + 1) - 1 = k$$

and so  $(F \setminus \eta) \times I$  is irreducible by our induction hypothesis. Let  $B$  be a regular neighborhood of  $\eta \times I$ . Notice that  $cl((F \times I) \setminus B)$  is homeomorphic to  $(F \setminus \eta) \times I$  and hence is irreducible.

To show that  $F \times I$  is irreducible, it is sufficient to show that every sphere in  $F \times I$  can be isotoped to be disjoint from  $\eta \times I$ . This is sufficient because then the sphere can be isotoped into  $cl((F \times I) \setminus B)$  which is irreducible and hence the sphere must bound a ball both before and after the isotopy. Suppose for a contradiction that there exist one or more spheres which cannot be isotoped to be disjoint from  $\eta \times I$ . Of all the spheres which cannot be isotoped to be disjoint from  $\eta \times I$ , let  $S$  be the sphere which, after isotopy, intersects  $\eta \times I$  in the fewest components. Let  $\mathcal{A}$  be the components of  $(\eta \times I) \cap S$ . Let  $\alpha \in \mathcal{A}$  be an innermost simple closed curve in  $\eta \times I$ . Since  $\eta \times I$  is a disk it is contractible and hence there exists a disk  $D \subset \eta \times I$  with  $\partial D = \alpha$ . Notice that there are open disks  $D_1$  and  $D_2$  such that  $S = D_1 \cup \alpha \cup D_2$ . Define  $S_1 := D_1 \cup D$  and  $S_2 := D_2 \cup D$ . So  $S_1$  and  $S_2$  are spheres which may be isotoped, in the region around  $D$ , to be disjoint from  $\eta \times I$  without introducing any new intersections. Thus  $S_1 \cap (\eta \times I)$  and  $S_2 \cap (\eta \times I)$  have fewer components than  $S \cap (\eta \times I)$  and hence we may isotope  $S_1$  and  $S_2$  to be disjoint from  $\eta \times I$  due to our assumption that  $S$  is the sphere with the fewest components of intersection with  $\eta \times I$  after isotopy. It follows that  $S_1$  bounds a 3-ball  $B_1$  and  $S_2$  bounds a 3-ball  $B_2$ . Since isotopy will not change the fact that  $S_1$  and  $S_2$  both bound 3-balls we may assume they are in their original position where  $S_1 \cap S_2 = D$ . It should be noted that  $B_1$  (respectively  $B_2$ ) may a priori be on either side of  $S_1$  (respectively  $S_2$ ), however, a 3-ball only has one boundary component and hence can only be on one side of  $S_1$  (respectively  $S_2$ ). We conclude that  $S$  bounds the 3-ball  $B_1 \cup B_2$ . Since  $S$  bounds a 3-ball we may isotope  $S$  to be disjoint from  $\eta \times I$ . From our assumption about  $S$  this is a contradiction. We get that every

sphere in  $F \times I$  may be isotoped to be disjoint from  $\eta \times I$ . We conclude that  $F \times I$  is irreducible which completes our induction step and the proof.  $\square$

*Corollary 2.19.* If  $F$  is a connected surface with boundary and  $\chi(F) \leq 1$ , then  $F \times (0, 1)$  is irreducible.

Proof of Corollary 2.19. Every sphere in  $F \times (0, 1)$  is contained in  $F \times I$  and hence by Lemma 2.18 must bound a 3-ball (disjoint from the boundary).  $\square$

*Corollary 2.20.* Let  $F$  be a connected surface with boundary and  $\chi(F) \leq 1$ . Let  $\mu : F \rightarrow F$  be a homeomorphism. Define  $M := (F \times [0, 1]) / \sim$  where  $(x, 0) \sim (\mu(x), 1)$  for  $x \in F$ . Then  $M$  is irreducible.

Proof of Corollary 2.20. Suppose for a contradiction that  $M$  is not irreducible. Then there exists a sphere  $S \subset M$  which does not bound a ball. Identify  $F$  with  $F \times \{0\}$ . We get, from Lemma 2.15, that  $F$  is essential in  $M$ . If after isotopy  $S \cap F = \emptyset$  then  $S \subset M \setminus F = F \times (0, 1)$ . We know, from Corollary 2.19 that  $F \times (0, 1)$  is irreducible and hence  $S$  must bound a ball. This is a contradiction. Thus, we may assume that  $S \cap F \neq \emptyset$  and that  $S$  cannot be isotoped to be disjoint from  $F$ . Suppose that out of all the spheres in  $M$  which cannot be isotoped to be disjoint from  $F$ , and hence don't bound a 3-ball, that  $S$  is the sphere which, after isotopy, intersects  $F$  in the fewest number of simple closed curves. Let  $\mathcal{A}$  be this collection of simple closed curves in  $S$ . Let  $\alpha \in \mathcal{A}$  be an innermost simple closed curve in  $S$ . Let  $D$  be a disk in  $S$  such that  $\text{int}(D)$  is disjoint from all simple closed curves in  $\mathcal{A}$  and  $\partial D = \alpha$ . At least one such disk  $D$  exists since  $\alpha$  is innermost. Notice that  $D$  is a compressing disk for  $F$ . Since  $F$  is essential it is also incompressible and hence there exists a disk  $D' \subset F$  such that  $\partial D' = \alpha$ . We first suppose that  $S \cap D' = \emptyset$ . In this case we see that  $D \cup D'$  is a sphere in a manifold homeomorphic to the irreducible manifold  $F \times (0, 1)$ . This follows due to the fact that  $D' \subset F$  and  $\text{int}(D)$  is disjoint from  $F$ . So  $D \cup D'$  bounds a 3-ball and hence we may isotope  $D$  into  $D'$  and then off of  $F$  in such a way as to reduce the number of intersections of  $S \cap F$ . This is a contradiction since we have assumed the number of intersections is minimal. Thus,  $S \cap D'$

consists of a finite collection of simple closed curves  $\mathcal{B}$  in  $D'$ . Let  $\beta \in \mathcal{B}$  be an innermost simple closed curve in  $D'$ . Let  $D''$  be the disk in  $D'$  such that  $\partial D'' = \beta$  and  $\text{int}(D'')$  is disjoint from the simple closed curves in  $\mathcal{B}$ . Notice that  $S \setminus \beta$  consists of two open disks, say  $D_1$  and  $D_2$ . Notice that  $S = D_1 \cup \beta \cup D_2$ . Define  $S_1 := D_1 \cup D''$  and  $S_2 := D_2 \cup D''$ . Notice that  $S_1$  and  $S_2$  are spheres and  $S_1 \cap S_2 = D''$ . Notice that  $S_1$  and  $S_2$  can be isotoped to be disjoint from each other and off of  $F$  near  $D'$ . This reduces the number of times they intersect  $F$  to be fewer than the number of times  $S$  intersects  $F$ . From our assumption about  $S$  we conclude that both  $S_1$  and  $S_2$  can be isotoped to be disjoint from  $F$  and hence are isotopic to spheres in the irreducible submanifold  $M \times (0, 1) \subset M$ . Thus,  $S_1$  must bound a 3-ball  $B_1$  and  $S_2$  must bound a 3-ball  $B_2$  before isotopy. It should be noted that  $B_1$  (respectively  $B_2$ ) may a priori be on either side of  $S_1$  (respectively  $S_2$ ), however, a 3-ball only has one boundary component and hence can only be on one side of  $S_1$  (respectively  $S_2$ ). We conclude that  $S$  bounds the 3-ball  $B_1 \cup B_2$ . This is a contradiction since we assumed that  $S$  did not bound a 3-ball. We conclude that every sphere in  $M$  can be isotoped to be disjoint from  $F$  and by Corollary 2.19 must bound a ball. We conclude that  $M$  is irreducible.  $\square$

Combining Lemma 2.15 with Corollary 2.20 we get that every 3-manifold  $M$  which is the total space of a fiber bundle with base  $S^1$  and fiber a surface  $F$  with boundary is irreducible with essential fibers. Having established some basic results about irreducible 3-manifolds we move on to normal surface theory.

We now give a brief introduction to normal surface theory. We include some key definitions and finish with a couple of major results which are critical to this paper. To learn more, one may consult for example [18, Ch. 5] which includes a more in-depth discussion. To start with we need the terminology to describe a triangulation of a manifold.

*Definition 2.21.* The standard  $k$ -simplex in  $\mathbb{R}^{k+1}$  is given by

$$\Delta_k := \{(x_1, x_2, \dots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1, x_2, \dots, x_{k+1} \geq 0, x_1 + x_2 + \dots + x_{k+1} = 1\}.$$

A  $k$ -simplex for a 3-manifold  $M$  is a continuous map  $\psi : \Delta_k \rightarrow M$  such that  $\psi|_{\text{int}(\Delta_k)}$  is a homeomorphism onto

its image.

For the next definition to make sense we note that two spaces are isometric if there exists a map between them which preserves distances. A map with this property is called an isometry.

*Definition 2.22.* For  $0 \leq i \leq k$  we define a face,  $\Delta'_i$ , of the standard  $k$ -simplex,  $\Delta_k$ , to be a subset of  $\Delta_k$  isometric to  $\Delta_i$  which fixes  $k - i$  of the coordinates  $x_1, x_2, \dots, x_{k+1}$  in  $\Delta_k$  to equal 0. A face of a  $k$ -simplex  $\psi : \Delta_k \rightarrow M$  is a map  $\psi|_{\Delta'_i} : \Delta'_i \rightarrow M$  for any face  $\Delta'_i$  of the standard  $k$ -simplex  $\Delta_k$ .

*Definition 2.23.* A simplicial complex for a 3-manifold  $M$  is a collection of simplices for  $M$ , which we denote  $K$ , such that for all simplices  $f \in K$ , all faces of  $f$  are in  $K$  and for all pairs of simplices  $f_1, f_2 \in K$  the interior of their images are either equal or disjoint. We define the underlying space of the simplicial complex  $K$  to be the union of the images the simplices of  $K$ . (See [18, p. 15])

*Definition 2.24.* Let  $K$  and  $L$  be simplicial complexes whose underlying spaces are given by  $UK$  and  $UL$  respectively. We say that a continuous map  $h : UK \rightarrow UL$  is a simplicial map if for every simplex  $f$  in  $K$ , there is a simplex  $g$  in  $L$  such that  $h \circ f = g$ . (See [18, p. 18])

*Definition 2.25.* Let  $K$  and  $L$  be simplicial complexes whose underlying spaces are given by  $UK$  and  $UL$  respectively. We say that  $h : UK \rightarrow UL$  is a simplicial isomorphism if it is simplicial and a homeomorphism. We say that the two simplicial complexes  $K$  and  $L$  are isomorphic if there is a simplicial isomorphism  $h : UK \rightarrow UL$ . (See [18, p. 19])

*Definition 2.26.* A map  $f$  defined from a convex set  $K$  into a vector space is said to be affine if  $f(\sum \alpha_j x_j) = \sum \alpha_j f(x_j)$  when  $x_j \in K$ ,  $\alpha_j \geq 0$ , and  $\sum \alpha_j = 1$ . (See [5, p. 151])

*Definition 2.27.* A triangulation of a 3-manifold  $M$  is a pair  $(M, K)$ , where  $K$  is a simplicial complex for  $M$ , so that the underlying space of  $K$  is equal to  $M$ ; for every compact subset  $C$  of  $M$ , the set  $\{f \in K \mid C \cap im(f) \neq \emptyset\}$  is finite; and for  $f, g \in K$ , restricted to the interior of their domain, the map  $g^{-1} \circ f$  is affine on its domain. We often denote the pair  $(M, K)$  as just  $\mathcal{T}$ . (See [18, p. 15])

A key component of normal surface theory is the ability to triangulate any compact 3-manifold. This fact follows from the work of Edwin E. Moise, in [14], who showed that a 3-manifold can be triangulated and R.H. Bing, in [3], who extended the result to show that all compact 3-manifolds are triangulable. We now include the basic definitions related to normal surfaces.

*Definition 2.28.* A properly embedded arc in a 2-dimensional face of the standard 3-simplex is a normal arc if its endpoints lie on distinct edges of the face. A simple closed curve  $c$  in the boundary of the standard 3-simplex is a normal curve if every component of intersection of  $c$  with a 2-dimensional face of the standard 3-simplex is a normal arc. (See [18, p. 149])

*Definition 2.29.* The length of a normal curve in the boundary of the standard 3-simplex  $\Delta_3$  is the number of times it intersects a 1-cell of  $\Delta_3$ . (See [18, p. 149])

*Definition 2.30.* A normal triangle in the standard 3-simplex is a properly embedded disk whose boundary is a normal curve of length 3. A normal quadrilateral in the standard 3-simplex is a properly embedded disk whose boundary is a normal curve of length 4. A normal disk is a normal triangle or quadrilateral. (See [18, p. 150-151])

*Definition 2.31.* Let  $\mathcal{T}$  be a triangulation of a 3-manifold  $M$ . A surface  $F$  is said to be normal with respect to  $\mathcal{T}$  if for every 3-simplex  $\psi : \Delta_3 \rightarrow M$  in  $\mathcal{T}$  we get that  $\psi^{-1}(F \cap \psi(\Delta_3))$  consists of a collection normal disks or is empty.

If  $M$  is irreducible and  $F$  is an essential, hence incompressible, surface in  $M$  we can isotope  $F$  to be normal with respect to any triangulation of  $M$ . For a proof of this when  $F$  is closed see for example [18, p. 152-153]. We will furnish a proof for the similar case when  $F$  can have boundary by adapting the proof in [18, p. 152-153]. We add a couple more definitions and a lemma before we begin.

*Definition 2.32.* For a disk  $D$  and a finite collection of properly embedded pairwise disjoint arcs  $\{\alpha_i\}$  we say  $\alpha \in \{\alpha_i\}$  is an outermost arc of  $D$  if one (or both) of the disks  $\alpha$  cobounds with  $\partial D$  is disjoint from any other element in  $\{\alpha_i\}$ .

*Definition 2.33.* Let  $\mathcal{T}$  be a triangulation of the 3-manifold  $M$ . Let  $F$  be a properly embedded surface in  $M$ . The weight of  $F$  with respect to  $\mathcal{T}$ , denoted  $w(F)$ , is the number of components in the intersection of  $F$  with the 1-skeleton of  $\mathcal{T}$ . Similarly, the measure of  $F$  with respect to  $\mathcal{T}$ , denoted  $m(F)$ , is the number of components in the intersection of  $F$  with the union of the interior of the 2-faces of  $\mathcal{T}$  in  $M$ .

*Lemma 2.34.* A closed normal curve on the boundary of a 3-simplex either has length 3 or 4 or it meets some edge more than once.

Proof of Lemma 2.34. For a proof see [18, Lemma 5.2.4, p. 149-150]  $\square$

*Lemma 2.35.* Let  $M$  be a compact irreducible 3-manifold with incompressible boundary and triangulation  $\mathcal{T}$ . If  $F$  is an incompressible and boundary incompressible surface in  $M$ , that is not a disk, then we can isotope  $F$  to be normal with respect to  $\mathcal{T}$ .

Proof of Lemma 2.35. Let  $(w(F), m(F))$  be the weight and measure of  $F$  respectively. Let  $(w_1, m_1)$  and  $(w_2, m_2)$  be different pairs of weight and measure. We say that  $(w_1, m_1)$  is greater than  $(w_2, m_2)$  if  $w_1 > w_2$  or if  $w_1 = w_2$  and  $m_1 > m_2$ . We say that  $(w_1, m_1)$  is equal  $(w_2, m_2)$  if  $w_1 = w_2$  and  $m_1 = m_2$ . Otherwise,  $(w_1, m_1)$  is less than  $(w_2, m_2)$ . We isotope  $F$  to be transverse to the triangulation  $\mathcal{T}$  and so that  $(w(F), m(F))$  is minimal. In particular,  $w(F)$  is as small as possible. Let  $\psi : \Delta_3 \rightarrow M$  be a 3-simplex of  $\mathcal{T}$ . Define  $T := \psi(\Delta_3)$ .

Suppose for a contradiction that a 2-cell  $\Delta$  of  $T$  intersects  $F$  in a nonempty collection of simple closed curves. Let  $\alpha$  be an innermost simple closed curve from this collection. Notice that  $\alpha$  is contained in  $\text{int}(\Delta)$  and hence is contained entirely in  $\partial M$  or entirely in  $\text{int}(M)$ . Let  $D$  be the disk in  $\Delta$  that  $\alpha$  bounds. Since  $F$  is incompressible there exists a disk  $D'$  in  $F$  such that  $\partial D' = \alpha = \partial D$ . Notice that  $D \cup D'$  is a sphere. Since  $M$  is irreducible  $D \cup D'$  bounds a 3-ball  $B$ . There are two cases to consider. Case 1 occurs if  $\alpha \subset \partial M$ . In this case  $\partial F = \alpha$  and  $F$  is a disk which we have assumed is not the case and hence we get a contradiction. Case 2 occurs if  $\alpha \subset \text{int}(M)$ . Then  $B$  gives an isotopy of  $D'$  into  $D$  which we can then isotope off of  $\Delta$  and this reduces  $(w(F), m(F))$  by at least  $(0, 1)$  which is a contradiction since we assumed  $(w(F), m(F))$

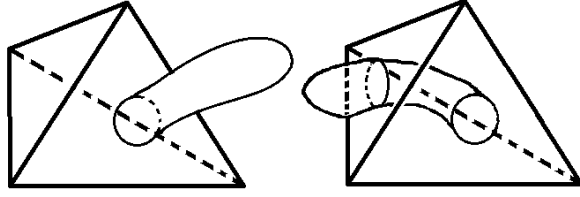


Figure 4: Examples of possible disks in  $F$  which can be isotoped off of  $\Delta$ .

was minimal. See for example Figure 4. Thus,  $\Delta \cap F$  consists of no simple closed curves. Since  $\Delta$  was an arbitrary 2-cell for  $T$  and  $T$  was arbitrary itself so we may assume that the image of every 2-simplex in  $\mathcal{T}$  does not intersect  $F$  in a simple closed curve.

Suppose for a contradiction that a 2-cell  $\Delta$  of  $T$  intersects  $F$  in a nonempty collection of arcs of which at least one has endpoints in the same 1-cell  $l$  of  $\Delta$ . Let  $\eta$  be an outermost arc in  $\Delta$  with both endpoints in  $l$ . Let  $D$  be the disk in  $\Delta$  cobounded by  $\eta$  and a component  $l'$  of  $l$ . Notice that  $\text{int}(D)$  is disjoint from  $F$ . Keeping in mind that  $\partial D = \eta \cup l'$  and  $D \setminus l' \subset \text{int}(\Delta)$  we have 3 cases. Case 1 occurs when  $D \subset \text{int}(M)$ . In this case the disk  $D$  gives us an isotopy of  $\eta$  into  $l'$  which we can then isotope off of  $l$  to reduce  $(w(F), m(F))$  by at least  $(2, 1)$  which is a contradiction since we assumed  $(w(F), m(F))$  was minimal. Case 2 occurs when  $D \subset \partial M$ . In this case  $D$  gives us an isotopy of  $\eta$  into  $l'$  which we can then isotope off of  $l$  to reduce  $(w(F), m(F))$  by at least  $(2, 1)$  which is a contradiction since we assumed  $(w(F), m(F))$  was minimal. Case 3 occurs when  $D \cap \partial M = l'$ . In this case  $D$  is a boundary compressing disk. Since  $F$  is boundary incompressible there exists a disk  $D' \subset F$  such that  $D' \cap D = \eta$  and  $\partial D' \setminus \eta \subset \partial F$ . Notice that  $D'' = D \cup D'$  is a compressing disk for  $\partial M$ . Since we have assumed that  $\partial M$  is incompressible there exists a disk  $D''' \subset \partial M$  such that  $D'' \cup D'''$  is a sphere in  $M$ . Since  $M$  is irreducible  $D'' \cup D'''$  bounds a 3-ball. This 3-ball gives us an isotopy of  $D'$  into  $D$  which we can then isotope off of  $\Delta$  and this reduces  $(w(F), m(F))$  by at least  $(2, 1)$  which is a contradiction since we assumed  $(w(F), m(F))$  was minimal. Thus,  $\Delta \cap F$  consists of no arcs

whose boundary points share the same 1-cell. Since  $\Delta$  was an arbitrary 2-cell for  $T$  and  $T$  was arbitrary itself we may assume that the image of every 2-face in  $\mathcal{T}$  does not intersect  $F$  in arcs whose boundary points share the same 1-cell in the 1 skeleton of  $\mathcal{T}$ .

So far we have shown that the intersection of  $F$  with the image of any 2-simplex in  $\mathcal{T}$  does not consist of simple closed curves nor does it consist of arcs whose boundary are in the image of the same 1-simplex. Thus, the intersection of  $F$  with the image of a 2-simplex is either empty or is the image of a collection of normal arcs. In particular, since  $F$  is properly embedded and transverse to  $\mathcal{T}$ , the intersection of  $F$  with the boundary of the image of any 3-simplex of  $\mathcal{T}$  is empty or the image of a collection of normal curves. Consider again an arbitrary 3-simplex  $\psi : \Delta_3 \rightarrow M$  in  $\mathcal{T}$  with image given by  $T := \psi(\Delta_3)$  so that  $T \cap F \neq \emptyset$ . Let  $\alpha^* \subset \Delta_3$  be a normal curve and define  $\alpha := \psi(\alpha^*) \subset \partial T$ . Since  $\alpha$  is an embedded curve and  $T$  is contractible it must bound a disk  $D$ . Suppose that  $F$  intersects  $\text{int}(D)$ . Let  $\beta$  be an innermost simple closed curve of  $D \cap F$  in  $D$ . So  $\beta$  bounds a disk  $D'$  in  $D$  and since  $F$  is incompressible there exists another disk  $D'' \subset F$  such that  $\partial D' = \partial D''$ . In particular,  $D' \cup D''$  is a sphere in  $M$ . Since  $M$  is irreducible  $D' \cup D''$  bounds a 3-ball  $B$ . Due to the existence of  $B$  we may isotope  $D''$  into  $D'$  and then off of  $D$  without increasing  $(w(F), m(F))$  and reducing the number of components in  $D \cap F$ . Continuing in this way we may isotope  $F$  to be disjoint from  $D$ . Notice that  $D$  is a compressing disk for  $F$ . Since  $F$  is incompressible there exists a disk  $D^* \subset F$  such that  $\partial D = \beta = \partial D^*$ . Since  $D \cup D^*$  is a sphere in an irreducible manifold we know that  $D \cup D^*$  bounds a 3-ball. Thus, we may isotope  $D^*$  into  $D$ . If  $\text{int}(D^*) \cap \partial T \neq \emptyset$  then this isotopy reduces  $(w(F), m(F))$  which is a contradiction since we assumed  $(w(F), m(F))$  is minimal. Thus, we may assume that  $\text{int}(D^*) \subset \text{int}(T)$ . Since  $T$  was arbitrary we may assume that the image of every normal curve in a simplex in  $\mathcal{T}$  must bound a disk in  $F$ . We would like to show that these disks are the images of a normal disk.

Let  $\psi : \Delta_3 \rightarrow M$  be an arbitrary 3-simplex in  $\mathcal{T}$  with  $T := \psi(\Delta_3)$ . Assume that  $\alpha^*$  is a normal curve in  $\Delta_3$  and define  $\alpha := \psi(\alpha^*)$ . By Lemma 2.34,  $\alpha^*$  bounds a normal disk or intersects a



1-face of  $\Delta_3$  more than once. If  $\alpha^*$  bounds a normal disk we are done, otherwise we know that  $\alpha \subset T$  intersects a 1-cell of  $T$  more than once. Let  $D \subset T$  be the disk in  $F$  with  $\partial D = \alpha$ . Since  $\alpha$  is an embedded curve in the sphere  $\partial T$  the Jordan Curve Theorem tells us that there exist disks  $D_1$  and  $D_2$  in  $\partial T$  such that  $\partial D_1 = \alpha = \partial D_2$  and  $D_1 \cup D_2 = \partial T$ . Since  $\alpha$  intersects a 1-cell, say  $l$ , of  $T$  more than once there is an arc  $\eta$  which is a component of either  $D_1 \cap l$  with  $\partial\eta \subset \partial D_1$  or  $D_2 \cap l$  with  $\partial\eta \subset \partial D_2$ . We may assume without loss of generality the former case. Since  $\text{int}(D) \subset \text{int}(T)$  and  $\partial\eta \subset \partial D$  there exists an arc  $\eta' \subset D \subset F$  such that  $\partial\eta' = \partial\eta$  and  $\text{int}(\eta') \subset \text{int}(D)$ . Since  $\eta \cup \eta'$  is a simple closed curve in the contractible space  $T$  we get that  $\eta \cup \eta'$  bounds a disk  $D^*$  such that  $D^* \setminus \eta \subset \text{int}(T)$ . Notice that  $F \cap \text{int}(D^*)$  is empty or consists of simple closed curves. If  $F \cap D^* \neq \emptyset$  let  $\zeta$  be an innermost curve in this collection. Then  $\zeta$  bounds a disk  $D_\zeta \subset D^*$ . Notice that  $D_\zeta$  is a compressing disk for  $F$ . Since  $F$  is incompressible there exists a disk  $D'_\zeta$  in  $F$  such that  $\partial D_\zeta = \partial D'_\zeta$ . Since  $M$  is irreducible  $D_\zeta \cup D'_\zeta$  is a ball bounding sphere. Thus we may isotope  $D'_\zeta$  into  $D_\zeta$  and then off of  $D^*$ . We remark that since this isotopy may reduce  $(w(F), m(F))$ , which we assumed was minimal, it must be the case that  $D_{\zeta'}$  was already contained in  $\text{int}(M)$  before isotopy. We continue this process until  $F \cap \text{int}(D^*) = \emptyset$ . Since  $\eta \subset l \subset \partial T$  and  $D^* \setminus \eta \subset \text{int}(T)$  we can break up the remainder of the problem into 2 cases. Case 1 is when  $\eta \subset \text{int}(M)$ . In this case we use  $D^*$  to provide an isotopy of  $\eta'$  into  $\eta$ . We do this isotopy followed by an isotopy which moves  $\eta'$  off of  $l$ . This will reduce  $w(F)$  by 2 and hence reduce  $(w(F), m(F))$  which is a contradiction since we assumed that  $(w(F), m(F))$  was minimal. Case 2 is when  $\eta \subset \partial M$ . In this case  $D^*$  is a boundary compressing disk. Since  $F$  is boundary incompressible there exists a disk  $D^{**}$  in  $F$  such that  $D^* \cap D^{**} = \eta'$  and  $\partial D^{**} \setminus \eta' \subset \partial F \subset \partial M$ . Notice that  $D^* \cup D^{**}$  is a compressing disk for  $\partial M$ . Since  $\partial M$  is incompressible there exists a disk  $D^{***}$  in  $\partial M$  such that  $\partial D^{***} = \partial(D^* \cup D^{**})$ . Since  $M$  is irreducible the sphere  $D^* \cup D^{**} \cup D^{***}$  bounds a 3-ball. So we may isotope  $D^{**}$  into  $D^*$ . Following this isotopy we isotope  $D^{***}$  off of  $l$ . This will reduce  $w(F)$  by at least 2 and hence reduce  $(w(F), m(F))$  which is a contradiction since we assumed that  $(w(F), m(F))$  was

minimal. We conclude for both cases that  $\alpha$  must intersect a 1-cell of  $T$  at most once and following from Lemma 2.34 we conclude that  $\alpha^*$  bounds a normal disk in  $\Delta_3$ . Since  $\psi$ ,  $\alpha^*$ ,  $T$ , and  $\alpha$  were arbitrary choices we conclude that every normal curve bounds a normal disk. In particular we get after our particular isotopy of  $F$  that  $F$  is normal with respect to the triangulation  $\mathcal{T}$ .  $\square$

The lemma which we have just proved is useful, however, it has the downside of requiring that the boundary of the 3-manifold is incompressible. Thankfully, for the particular class of manifolds that we are interested in this will not be a problem. The following lemma will clarify what is meant by this.

*Lemma 2.36.* Let  $F$  be a connected surface with boundary with  $\chi(F) \leq 0$ . Let  $\mu : F \rightarrow F$  be a homeomorphism. Define  $M := (F \times [0, 1]) / \sim$  where  $(x, 0) \sim (\mu(x), 1)$  for  $x \in F$ . Then  $\partial M$  is incompressible.

*Proof of Lemma 2.36.* Notice that  $\partial M$  consists of a collection of tori and Klein bottles. Suppose for a contradiction that  $\partial M$  is compressible, then there exists a component of  $\partial M$ , say  $S$ , so that  $S$  is compressible. Let  $D \subset M$  be a compressing disk for  $S$  so that  $D \cap S = \partial D$ , but there does not exist a  $D' \subset S$  such that  $\partial D = \partial D'$ . Consider the following maps:

- $\iota : \partial D \cong S^1 \hookrightarrow M$ , the inclusion
- $q : M \rightarrow S^1$ , projection onto the base of the fiber bundle

We get the following induced homomorphisms:

- $\iota_* : \pi_1(\partial D, p_1) \rightarrow \pi_1(M, p_1)$ , the 0-map since  $\partial D$  bounds a disk  $D \subset M$
- $q_* : \pi_1(M, p_1) \rightarrow \pi_1(S^1, p_2)$

where  $p_1 \in \partial D \subset M$  and  $p_2 = q(p_1)$ . We consider two cases. Case 1 is when

$(q \circ \iota)_* : \pi_1(\partial D, p_1) \rightarrow \pi_1(S^1, p_2)$  is a nontrivial homomorphism. Thus  $(q \circ \iota)_*$  is not the 0-map.

Now, since  $\iota_*$  is the 0-map  $(q \circ \iota)_* = q_* \circ \iota_*$  is also the 0-map, a clear contradiction. Case 2 is

when  $(q \circ \iota)_* : \pi_1(\partial D, p_1) \rightarrow \pi_1(S^1, p_2)$  is the trivial homomorphism and hence  $q \circ \iota : \partial D \rightarrow S^1$  is nullhomotopic. In this case we may homotope  $\partial D$  to be the boundary of a fiber  $F \times \{x\}$  where  $x \in [0, 1)$ . In particular, we may homotope  $\partial D$ , in  $S$ , to be in  $\partial F$  where  $F$  is identified with  $F \times \{0\}$ . We get from Lemma 2.15 that  $F$  is essential and hence incompressible, so there exists a disk  $D^* \subset F$  so that  $\partial D = \partial D^*$ . Since  $F$  is connected,  $D^* \subset F$ , and  $\partial D \subset \partial F$  we see that  $F = D^*$ . So  $\chi(F) = 1$ , however, we have assumed that  $\chi(F) \leq 0$  so we have a contradiction. Since in both case 1 and case 2 we get a contradiction, we see that  $S$  is not compressible. Thus,  $\partial M$  is incompressible.  $\square$

We have established some properties of a 3-manifold  $M$  which fibers over  $S^1$  with fiber a surface  $F$ . We have also established some of the properties of  $F$  when we identify it with  $F \times \{0\}$  with  $S^1$  thought of as  $[0, 1] / \sim$  where  $0 \sim 1$ . We would now like to understand the properties of the double of  $M$  and the double of  $F$  thought of as a submanifold of  $M$ .

*Definition 2.37.* Let  $M'$  be a copy of  $M$  and  $\phi : M \rightarrow M'$  the homeomorphism which takes a point  $x \in M$  to the corresponding point of  $M'$ . We define the double of  $M$ ,  $DM$ , as  $DM := (M \sqcup M') / \sim$ , where  $\sim$  is the equivalence relation generated by  $x \sim \phi(x)$  for all  $x \in \partial M$ . Additionally, for  $F$  a properly embedded subsurface of  $M$ , the double of  $F$ ,  $DF$ , is defined as  $DF := (F \sqcup F') / \sim$  where  $F' := \phi(F)$  and  $\sim$  is the equivalence relation generated by  $x \sim \phi(x)$  for all  $x \in F \cap \partial M$ .

We first remark that by our definition of the double, both  $DF$  and  $DM$  are closed manifolds. While not necessarily critical to our proof it is useful to have the following lemma.

*Lemma 2.38.* If  $M$  is an irreducible 3-manifold and  $\partial M$  is incompressible then  $DM$  is irreducible.

*Proof of Lemma 2.38.* Let  $S$  be a sphere in  $DM$ . Isotope  $S$  so that  $S$  intersects  $\partial M$  minimally and transversally. From our definition of  $DM$  we can see that  $M' = cl(DM \setminus M)$ . If  $S \cap \partial M$  is empty then  $S \subset int(M)$  or  $S \subset int(M')$ . Since  $M$  and  $M'$  are irreducible  $S$  must bound a ball. Now if  $S \cap \partial M$  is not empty then  $S$  intersects  $\partial M$  in a finite number of simple closed curves.

Claim. Together  $S \cap M$  and  $S \cap M'$  contain at least 2 disk components.

Proof of Claim. Firstly, since  $S$  is a sphere, none of the components of  $S \cap M$  or  $S \cap M'$  has genus. Secondly, since  $S \cap \partial M$  is not empty no component of  $S \cap M$  or  $S \cap M'$  is a sphere. Now  $\chi(S) = 2$  and so there exists at least two components of positive Euler characteristic 1 and these components must be disks.

Now take one of the disks from the claim and call it  $D$ . If  $D$  is isotopic to a disk in  $\partial M$  then we can isotope  $D$  so that it does not intersect  $\partial M$ , but this contradicts the minimality of  $S$  intersecting with  $\partial M$  and so we may assume that  $D$  is not isotopic to a disk in  $\partial M$ . Now if  $D$  is not isotopic to a disk in  $\partial M$  then either there is a disk  $D'$  in  $\partial M$  with  $\partial D = \partial D'$  and hence  $S' := D \cup D'$  is a sphere or  $D$  is a compressing disk which does not share its boundary with an embedded disk  $D' \subset \partial M$ . Now in the case that we get the sphere  $S'$  we know that since  $D$  is not isotopic to a disk in  $\partial M$  that  $S'$  does not bound a ball. Notice that  $S' \subset M$  or  $S' \subset M'$  and so by isotopy we can get  $S' \subset \text{int}(M)$  or  $S' \subset \text{int}(M')$ . We just said that  $S'$  does not bound a ball, however,  $S'$  is a sphere in  $M$  or in  $M'$  and hence by irreducibility  $S'$  bounds a ball which leads to a contradiction. Thus we only need to concern ourselves with the case in which  $D$  is a compressing disk which does not share its boundary with an embedded disk  $D' \subset \partial M$ . Since  $\partial D \subset \partial M$  and  $D$  is a compressing disk which does not share its boundary with an embedded disk  $D' \subset \partial M$  we see that  $\partial M$  is compressible which is a contradiction since we have assumed  $\partial M$  is irreducible. This argument tells us that any sphere which intersects  $\partial M$  minimally must not intersect  $\partial M$  at all and hence by irreducibility of  $M$  and  $M'$  bound a ball. From this we see that  $DM$  is irreducible.  $\square$

We now establish that  $DF$  is essential in  $DM$  whenever  $F$  is essential in  $M$ .

*Lemma 2.39.* If  $F$  is an essential surface in a 3-manifold  $M$ , then  $DF$  is essential in  $DM$ .

Proof of Lemma 2.39. By the definition of the double of a manifold given in this paper it is trivially true that  $DF$  is both boundary incompressible and not boundary parallel since  $DM$  has empty boundary. So we only need to show that  $DF$  is incompressible in  $DM$ . We assume

transversality for all of our intersections. Let  $D$  be an arbitrary disk in  $DM$  such that  $\partial D \cap DF \subset DF$ . Now  $D \cap \partial M$  is empty or consists of a finite collection of simple closed curves and properly embedded arc components. If  $D \cap \partial M$  is empty then rename  $D$  as  $D^{**}$  and go to section 3 of the proof, otherwise  $D \cap \partial M$  is nonempty and  $|D \cap \partial M| = n + k$  is nonzero with  $n$  and  $k$  natural numbers where  $n$  is the number of simple closed curve components of  $D \cap \partial M$  and  $k$  is the number of arc components of  $D \cap \partial M$ .

Section 1. If  $n = 0$  we rename  $D$  as  $D^*$  and go to section 2 of the proof, otherwise if  $n \neq 0$  then there exists an innermost simple closed curve component of  $D \cap \partial M$  in  $D$  which we will call  $\gamma_{in}$ . Now since  $\gamma_{in}$  is the boundary of a disk in  $D$ , whose intersection with  $\partial M$  is  $\gamma_{in}$ , and  $\partial M$  is incompressible it follows that  $\gamma_{in}$  bounds a disk, say  $D_{\partial M}$ , in  $\partial M$  as well. Notice now that  $int(D_{\partial M}) \cap D$  is empty or consists of simple closed curve components. Figure 5 gives a partial illustration of the following argument.

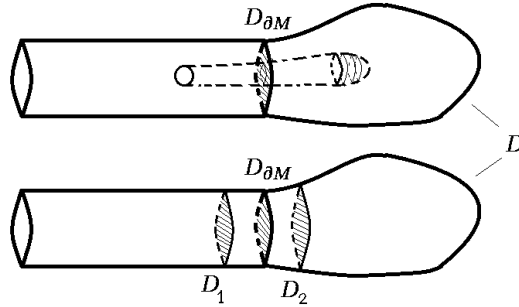


Figure 5: We give an illustration when  $D$  intersects  $int(D_{\partial M})$  above and another which includes  $D_1$  and  $D_2$  below.

If  $int(D_{\partial M}) \cap D$  is empty then we find a regular neighborhood  $ND_{\partial M}$  of  $D_{\partial M}$  such that  $ND_{\partial M} \cap D = A$  and  $\partial ND_{\partial M} = D_1 \cup D_2 \cup A$  where  $A$  is an annulus in  $D$  and  $D_1$  and  $D_2$  are disks with  $\partial D_1 \cup \partial D_2 = \partial A$  where without loss of generality  $D_1 \subset int(M)$  while  $D_2 \subset int(M')$ . Define  $D_{temp} := (D \setminus A) \cup D_1 \cup D_2$ . Notice that  $D_{temp}$  is the disjoint union of a disk and a sphere. Let  $D'$  be the disk component of  $D_{temp}$ . Notice that  $\partial D = \partial D'$ , but  $|D' \cap \partial M| = n - 1 + k$ .

If  $int(D_{\partial M}) \cap D$  is not empty then we find an innermost simple closed curve of

$\text{int}(D_{\partial M}) \cap D$  in  $D_{\partial M}$ . Let  $D'_{\partial M}$  be the disk in  $\partial M$  bounded by this simple closed curve.

Following the argument of the previous paragraph with  $D_{\partial M}$  replaced with  $D'_{\partial M}$  we again end up with a disk  $D'$  such that  $\partial D = \partial D'$ , but  $|D' \cap \partial M| \leq n - 1 + k$ .

We repeat the process above to find a disk  $D''$  such that  $\partial D' = \partial D''$  and  $|D'' \cap \partial M| \leq n - 2 + k$ . We repeat this process until we find a disk  $D^{(n)}$  such that  $\partial D^{(n-1)} = \partial D^{(n)}$  and  $|D^{(n)} \cap \partial M| = k$ . We rename  $D^{(n)}$  as  $D^*$ . Notice that  $D^* \cap \partial M$  only contains arc components and  $\partial D^* = \partial D$ .

Section 2. If  $k = 0$  we rename  $D^*$  as  $D^{**}$  and go to section 3 of the proof, otherwise if  $k \neq 0$  then there is an outermost arc component of  $D^* \cap \partial M$  in  $D^*$  which we call  $\alpha_{out}$ . Since  $\alpha_{out}$  is outermost there is at least one arc, say  $\alpha'_{out}$ , in  $\partial D^*$  such that  $\partial \alpha_{out} = \partial \alpha'_{out}$ ,  $\alpha'_{out} \subset F$  or  $\alpha'_{out} \subset F'$ , and  $\alpha'_{out}$  does not intersect any components of  $D^* \cap \partial M$  except for  $\alpha_{out}$ . We assume without loss of generality that  $\alpha'_{out} \subset F$ . Notice that  $\alpha_{out} \cup \alpha'_{out}$  form the boundary of a disk in  $D^*$ , we call this disk  $D^*_{out}$ . Since  $\alpha'_{out} \subset F$  we have  $D^*_{out} \subset M$ . Notice that  $D^*_{out}$  is a boundary compressing disk for  $F$  in  $M$  with  $D^*_{out} \cap F = \alpha'_{out}$ . Thus, since  $F$  is boundary incompressible in  $M$ , there exists a disk  $D^*_F \subset F$  with  $\partial D^*_F = \alpha'_{out} \cup \alpha_{\partial M}$  where  $\alpha_{\partial M}$  is an arc in  $\partial M$  such that  $\partial \alpha'_{out} = \partial \alpha_{\partial M}$ . Since  $D^*_F$  exists we isotope  $\partial D^*$ , in  $DF$ , so that  $\alpha'_{out} \subset \text{int}(F')$  and  $D^* \cap \partial M$  still has  $k$  components, but now exactly one of these components is a simple closed curve.

Following the result of section 1 of the proof we can find a disk  $D^{*(1)}$  such that

$|D^{*(1)} \cap \partial M| = k - 1$ ,  $D^{*(1)} \cap \partial M$  consists only of arc components and  $\partial D^{*(1)}$  is isotopic  $\partial D^*$ .

Repeating the process above we may after an appropriate isotopy of  $D^{*(1)}$  find a disk  $D^{*(2)}$  such that  $|D^{*(2)} \cap \partial M| = k - 2$ ,  $D^{*(2)} \cap \partial M$  consists only of arc components and  $\partial D^{*(2)}$  is isotopic to  $\partial D^{*(1)}$ . We repeat this process until we get a disk  $D^{*(k)}$  such that  $D^{*(k)} \cap \partial M$  is empty and  $\partial D^{*(k)}$  is isotopic to  $\partial D^{*(k-1)}$  in  $DF$ . We rename  $D^{*(k)}$  as  $D^{**}$ .

Section 3. Notice that  $\partial D^{**}$  is isotopic to  $\partial D$ , in  $DF$ , and  $D^{**} \cap \partial M$  is empty and so  $D^{**} \subset M$  or  $D^{**} \subset M'$ . Without loss of generality we assume  $D^{**} \subset M$ . Notice that  $D^{**}$  is isotopic to  $\partial D$ , in  $DF$ ;  $D^{**} \cap DF \subset F$ ; and  $D^{**} \subset M$ . Thus, since  $F$  is incompressible as a subsurface of  $M$ , we get

that  $D^{**}$  bounds a disk in  $F$  and hence  $D$  bounds a disk in  $DF$ . Since  $D$  was arbitrary we get that  $DF$  is incompressible and hence essential.  $\square$

We include two more lemmas related to the double of a manifold.

*Lemma 2.40.* If  $F$  is a 2-sided surface in a 3-manifold  $M$ , then  $DF$  is 2-sided in  $DM$ .

*Proof of Lemma 2.40.* We may assume  $F$  and  $M$  are smooth. Additionally, in this case,  $\phi : M \rightarrow M'$  from the definition of the double is a diffeomorphism. By smoothing  $DM$  we may assume that  $\phi$  extends to a diffeomorphism  $\Phi : DM \rightarrow DM$  such that  $\Phi|_M = \phi$  and  $\Phi|_{M'} = \phi^{-1}$ . Let  $\Phi_* : T_p DM \rightarrow T_{\Phi(p)} DM$  be the pushforward. Since  $F$  is 2-sided there is a trivial normal bundle  $NF := \{(p, v) | p \in F, v \in T_p M, v \notin T_p F\}$ .

To show that  $DF$  is 2-sided we take a coorientation on  $F$  and extend it a coorientation on  $DF$ . Give  $F$  a coorientation  $f : F \rightarrow TM$ . For an arbitrary point  $p \in F$  we have  $f(p) = (p, v_p)$ . For this coorientation we require that whenever  $p \in \partial F \subset \partial M$  that  $v_p \in T_p \partial M \setminus T_p F$ . We extend  $f$  to  $f_{ext} : DF \rightarrow T(DM)$  as follows:

$$f_{ext}(p) := \begin{cases} (p, v_p) & p \in F \\ (p, \Phi_*(v_{\Phi(p)})) & p \in int(F') \end{cases}$$

Notice that  $f_{ext}$  agrees with the coorientation of  $f$  on  $\partial F$ . Furthermore, since  $f_{ext}$  is continuous on  $F$  it must also be continuous on  $F'$ . Thus,  $f_{ext}$  is a coorientation for  $DF$ . Now since  $f_{ext}$  agrees with  $f$  on  $F$  and in particular on  $\partial F$  and  $F$ , itself, has two distinct coorientations it must be the case that there is another distinct coorientation for  $DF$ . We conclude that  $DF$  is 2-sided.  $\square$

For the next lemma, assume that  $M$  is a triangulated 3-manifold with triangulation  $\mathcal{T}$  and that  $F \subset M$  is normal with respect to  $\mathcal{T}$ . Give  $M'$  a triangulation  $\mathcal{T}'$  such that  $\Phi$  sends the image of every simplex of  $\mathcal{T}$  to the image of a simplex of  $\mathcal{T}'$ . Notice that  $F'$  will be normal with respect to  $\mathcal{T}'$  since  $F$  is normal with respect to  $\mathcal{T}$ . Let  $D\mathcal{T}$  denote the triangulation of  $DM$  using the tetrahedra from  $\mathcal{T}$  and  $\mathcal{T}'$ . It is easy to verify that  $D\mathcal{T}$  is in fact a triangulation of

$DM$ .

*Lemma 2.41.* Given  $F$  normal with respect to  $\mathcal{T}$  then  $DF$  is normal with respect to  $D\mathcal{T}$ .

*Proof of Lemma 2.41.* Since  $F$  is normal with respect to  $\mathcal{T}$  and  $F'$  is normal with respect to  $\mathcal{T}'$  we get for every 3-simplex  $\psi : \Delta_3 \rightarrow DM$  in  $D\mathcal{T}$  we get that  $\psi^{-1}(DF \cap \psi(\Delta_3))$  consists of a collection normal disk or is empty. This holds because every simplex in  $D\mathcal{T}$  is a simple in  $\mathcal{T}$  or in  $\mathcal{T}'$ .  $\square$

We conclude this section with two basic definitions which are used throughout the remainder of the paper.

*Definition 2.42.* Let  $h : X \rightarrow Y$ . Let  $Q$  be a subset of  $X$ . We say that  $Q$  is fixed under  $h$  if for all  $q \in Q$  we have that  $h(q) = q$ . We say that  $Q$  is invariant under  $h$  if  $h(Q) = Q$ . We say that  $Q$  is equivariant under  $h$  if  $h \circ h(Q) = Q$  and either  $h(Q) = \emptyset$  or  $h(Q) = Q$ .

*Definition 2.43.* A continuous map  $h$  is nullhomotopic if it is homotopic to the identity map.



### 3 Curve Complex and Arc and Curve Complex

In this section we give a brief introduction to the curve complex and the arc and curve complex. For more details one may consult [18] or [17]. We begin with the definition of the curve complex of a surface.

*Definition 3.1.* The curve complex of a surface  $F$ , which we denote  $\mathcal{C}(F)$ , is a simplicial complex associated with  $F$  which relates isotopy classes of essential simple closed curves in  $F$  with a 0-simplex (or vertex) in  $\mathcal{C}(F)$  and relates any collection of  $k + 1$  isotopy classes of essential simple closed curves with pairwise disjoint representatives in  $F$  to a  $k$ -simplex whose  $k + 1$  vertices in  $\mathcal{C}(F)$  each correspond to one of the pairwise disjoint isotopy classes of essential simple closed curves. We define for the surface  $F$  the complexity

$$c(F) := 3g(F) + b(F) - 4$$

where  $g(F)$  is the genus of  $F$  and  $b(F)$  is the number of components of  $\partial F$ .

We are primarily interested in a surface,  $F$ , satisfying  $c(F) > 0$ . For an example of part of a simplicial complex for a genus 3 surface see Figure 6. It is shown in [18, p. 249]) that the curve complex of a surface is connected whenever the Euler characteristic of the surface is less than or equal to  $-2$ . Thus, the types of surfaces we are considering in our paper will all have a connected curve complex. An important concept of the curve complex is distance.

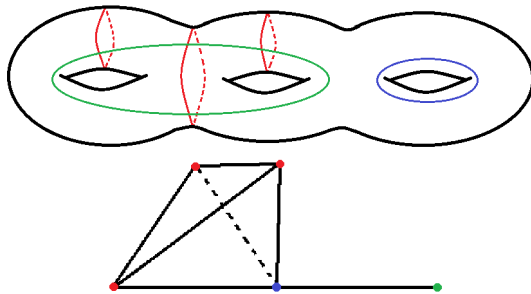


Figure 6: Part of simplicial complex for a genus 3 surface.

*Definition 3.2.* Let  $F$  be a surface. The distance between two vertices  $v$  and  $w$  in  $\mathcal{C}(F)$  is the minimal number of edges in the 1-skeleton of  $\mathcal{C}(F)$  required to connect  $v$  with  $w$ .

It has been shown in [13], for an oriented surface, that for any vertex  $v$  in the curve complex there exist another vertex which can have arbitrarily large distance from  $v$ . Due to this, we say that the curve complex has infinite diameter.

In addition to the curve complex we are also interested in the arc and curve complex.

*Definition 3.3.* The arc and curve complex of a surface  $F$ , which we denote  $\mathcal{AC}(F)$ , is a simplicial complex associated with  $F$  which relates each isotopy class of essential simple closed curves and each isotopy class of essential simple arcs in  $F$  with a 0-simplex (or vertex) in  $\mathcal{AC}(F)$ . Furthermore a  $k$ -simplex in  $\mathcal{AC}(F)$  is associated to a collection of  $k + 1$  isotopy classes of essential simple closed curves and essential simple arcs with pairwise disjoint representatives in  $F$ .

It is worth pointing out that for a surface  $F$  without boundary  $\mathcal{C}(F)$  and  $\mathcal{AC}(F)$  are equivalent. Since we are primarily interested in surfaces with boundary this fact is not so useful to us, however, we can relate the two complexes in another way. First we define the translation distance. Let  $\mu : F \rightarrow F$  be a homeomorphism and  $\mu_* : \mathcal{AC}(F) \rightarrow \mathcal{AC}(F)$  be the isometry on  $\mathcal{AC}(F)$  induced by  $\mu$ . We define the translation distance of  $\mu$  in  $\mathcal{AC}(F)$  to be  $\min\{d(v, \mu_*(v)) \mid v \in \mathcal{AC}(F) \text{ is a vertex}\}$  where  $d$  is the distance between vertices in the arc and curve complex. Translation distance also makes sense for the curve complex.

*Lemma 3.4.* Let  $F$  be a connected surface with  $\chi(F) \leq -16$  and let  $\mu : F \rightarrow F$  be a homeomorphism. If the translation distance of  $\mu$  in  $\mathcal{AC}(F)$  is at most 1 then the translation distance of  $\mu$  in  $\mathcal{C}(F)$  is at most 2.

*Proof of Lemma 3.4.* If the translation distance of  $\mu$  in  $\mathcal{AC}(F)$  is at most 1 then there is an essential simple closed curve or essential simple arc  $\alpha \subset F$  so that  $\alpha$  is disjoint from  $\mu(\alpha)$  (possibly after isotopy). It may be the case that  $\alpha$  and  $\mu(\alpha)$  are in the same isotopy class. We have 2 cases to consider.

Case 1 is when  $\alpha$  is an essential simple closed curve. Since  $\alpha$  and  $\mu(\alpha)$  are essential simple closed curve they are represented by vertices in  $\mathcal{C}(F)$  (possibly the same one). Thus, as  $\alpha$  and  $\mu(\alpha)$  are disjoint (possibly after isotopy) the translation distance of  $\mu$  in  $\mathcal{C}(F)$  is at most 1.

Case 2 is when  $\alpha$  is an essential simple arc. In this case  $\alpha$  and  $\mu(\alpha)$  are both essential simple arcs disjoint from each other (possibly after isotopy). Essential simple arcs are not represented in  $\mathcal{C}(F)$  and so we need a different strategy than in the previous case. Our goal is to show that we can use  $\alpha$  to find an essential simple closed curve  $\beta$  with the property that  $\beta$  and  $\mu(\beta)$  intersect in at most 4 points (possibly after isotopy). First we let  $c$  be the disjoint union of the boundary components (or component) of  $F$  which intersect  $\partial\alpha$ . So  $c$  contains at most 2 components. Now let  $N(c \cup \alpha)$  be a regular neighborhood of  $c \cup \alpha$  in  $F$ . At least one component of  $\partial N(c \cup \alpha)$  is not a component of  $\partial F$ ; we let  $\beta$  be one such component (it does not matter which).

We have 6 subcases to consider. Let  $c_1, c_2, c_3,$  and  $c_4$  be an arbitrary collection of components from  $\partial F$ . Note that some subcases may not apply to  $F$  if it has fewer than 4 boundary components. The following subcases are sufficient since we do not care about the arrangement of the components of  $\partial F$ . We first consider subcases in which  $\partial\alpha$  is contained in a single component of  $\partial F$ . Subcase 1 is when  $\partial\alpha \subset c_1$  and  $\partial\mu(\alpha) \subset c_1$  and as we traverse  $c_1$  we alternate intersecting  $\partial\alpha$  and  $\partial\mu(\alpha)$ , see Figure 7. Subcase 2 is when  $\partial\alpha \subset c_1$  and  $\partial\mu(\alpha) \subset c_1$  and as we traverse  $c_1$  we intersect  $\partial\alpha$  twice and then intersect  $\partial\mu(\alpha)$  twice after picking an appropriate starting point, see Figure 8. Subcase 3 is when  $\partial\alpha \subset c_1$  and  $\partial\mu(\alpha) \subset c_2$ , see Figure 9. We now consider subcases in which  $\partial\alpha$  is contained in distinct components of  $\partial F$ . Subcase 4 is when one component of  $\partial\alpha$  is contained in  $c_1$  and the second is contained in  $c_2$  while one component of  $\partial\mu(\alpha)$  is contained in  $c_1$  and the second is contained in  $c_2$ , see Figure 10. Subcase 5 is when one component of  $\partial\alpha$  is contained in  $c_1$  and the second is contained in  $c_2$  while one component of  $\partial\mu(\alpha)$  is contained in  $c_2$  and the second is contained in  $c_3$ , see Figure 11. Subcase 6 is when one component of  $\partial\alpha$  is contained in  $c_1$  and the second is

contained in  $c_2$  while one component of  $\partial\mu(\alpha)$  is contained in  $c_3$  and the second is contained in  $c_4$ , see Figure 12.

The figures associated to subcase 1 through subcase 6 show, up to isotopy, all the possible ways in which we can obtain  $\beta$  and  $\mu(\beta)$  given  $\alpha$ . The figures take advantage of the fact that  $\mu$  is a homeomorphism and hence must map the regular neighborhood  $N(c \cup \alpha)$  to a regular neighborhood of  $\mu(c \cup \alpha)$ , limiting our choice of  $\mu(\beta)$  which must be a boundary component of this regular neighborhood. The figures only show the minimum required number of boundary components from  $F$ . The possible intersections between  $\beta$  and  $\mu(\beta)$  can, after isotopy, always be contained in an annulus in  $F$  which is cobounded by a component of  $\partial F$  and a boundary parallel simple closed curve and hence we only show the interesting region near  $\partial F$  in our figures. From Figure 7 to Figure 12 we can see that depending on our  $\alpha$  it is only possible for  $\beta$  and  $\mu(\beta)$  to intersect at most 4 times after an intersection minimizing isotopy. We isotope  $\beta$  and  $\mu(\beta)$  to intersect minimally.

Let  $K = N(\beta \cup \mu(\beta))$  be a regular neighborhood of  $\beta \cup \mu(\beta)$ . We get

$$0 \geq \chi(K) = \chi(\beta \cup \mu(\beta)) \geq -4.$$

Now  $\partial K$  consists of simple closed curves which have Euler characteristic 0 and so

$$-16 \geq \chi(F) = \chi(K) + \chi(F \setminus K).$$

Therefore,

$$\chi(F \setminus K) \leq -12.$$

Now, since  $F$  is connected and  $\beta \cap \mu(\beta)$  consists of at most 4 points we see that  $F \setminus K$  consists of at most 6 components. Since the Euler characteristic is additive over disjoint unions there is a

component  $K'$  of  $F \setminus K$  so that

$$\chi(K') \leq \frac{-12}{6} \leq -2.$$

Since  $K' \leq -2$  there is an essential simple closed curve  $\beta' \subset K'$ , moreover, since  $\beta, \mu(\beta) \subset K$  it must be the case that  $\beta$  and  $\mu(\beta)$  are disjoint from  $\beta'$ . We conclude that the translation distance of  $\mu$  in  $\mathcal{C}(F)$  is at most 2.  $\square$

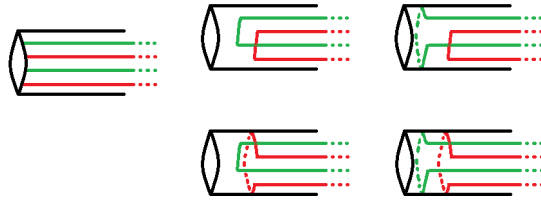


Figure 7: Subcase 1 with  $\alpha, \mu(\alpha)$  on the left and possible  $\beta, \mu(\beta)$  on right.

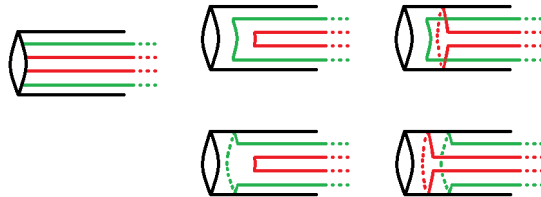


Figure 8: Subcase 2 with  $\alpha, \mu(\alpha)$  on the left and possible  $\beta, \mu(\beta)$  on right.

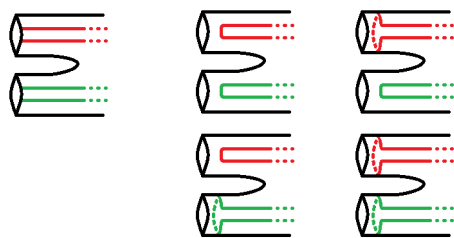


Figure 9: Subcase 3 with  $\alpha, \mu(\alpha)$  on the left and possible  $\beta, \mu(\beta)$  on right.

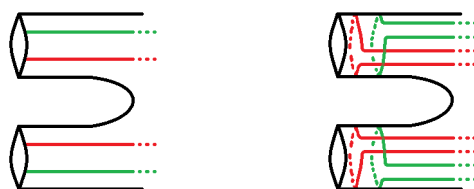


Figure 10: Subcase 4 with  $\alpha, \mu(\alpha)$  on the left and possible  $\beta, \mu(\beta)$  on right.



Figure 11: Subcase 5 with  $\alpha, \mu(\alpha)$  on the left and possible  $\beta, \mu(\beta)$  on right.

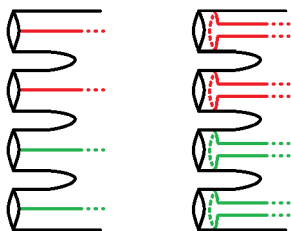


Figure 12: Subcase 6 with  $\alpha, \mu(\alpha)$  on the left and possible  $\beta, \mu(\beta)$  on right.

## 4 Proof

### Introduction

Our goal is to generalize the work of Ichihara, Kobayashi, and Rieck in [10]. The statement of their main result given in terms of the Euler characteristic is:

*Theorem 4.1.* Let  $M$  be a connected tame 3-manifold that admits a triangulation using  $t(M)$  tetrahedra and  $F \subset M$  a 2-sided connected essential closed surface. If  $\chi(F) \leq -76t(M) + 2$ , then  $F$  is strongly cylindrical.

We would like to have a similar result for an essential surface  $F$  with boundary. We cannot achieve a result as general as in [10]. Instead we focus on the case when  $M$  is the total space of a fibered manifold with base space the circle. We prove Theorem 1.1, as stated below, when  $F$  is a surface with boundary; the closed case is due to Ichihara, Kobayashi, and Rieck in [10].

*Theorem 1.1.* Let  $F$  be a connected surface, let  $\mu : F \rightarrow F$  be a homeomorphism, and let  $M := (F \times [0, 1]) / \sim$  where  $(x, 0) \sim (\mu(x), 1)$  for  $x \in F$ . Let  $t(M)$  be the minimal number of tetrahedra required to triangulate  $M$ . If  $\chi(F) \leq -76t(M) + 1$  then there is an essential simple closed curve or essential simple arc which is disjoint from its image under  $\mu$  (possibly after isotopy).

To help achieve this, we will take the double of the manifold  $M$  described in the statement of Theorem 1.1 and apply Proposition 11 from the proof of [10]. The statement of Proposition 11 is as follows: there exists a pair of pants  $X \subset \text{int}(Y)$  or  $X \subset \text{int}(B)$  so that  $\partial X$  is essential in  $F$ .

### Set up for Proof of Main Theorem

Let  $F$  and  $M$  be as in Theorem 1.1. Throughout this section we associate  $F$  with  $F \times \{0\}$ . It follows from our construction that  $F$  is 2-sided in  $M$  and we get from Lemma 2.15 that  $F$  is essential in  $M$ . Notice that  $M$  cut open along  $F$  is homeomorphic to  $F \times I$ ; this fact is important later on in the proof.

Let  $\mathcal{T}$  be a minimal triangulation of  $M$  such that the number of tetrahedra in  $\mathcal{T}$  is equal to  $t(M)$ , where  $t(M)$  represents the minimal number of tetrahedra required to triangulate  $M$ .

Using Lemma 2.36 we get that  $\partial M$  is incompressible. Applying Lemma 2.35 we isotope  $F$  to be normal with respect to  $\mathcal{T}$ . In the following discussion of tetrahedra in a triangulation we will be a bit informal in describing whether we are in the domain of a 3-simplex or in the image of a 3-simplex. Notice that the intersection of  $F$  with the tetrahedra of  $\mathcal{T}$  gives a decomposition of  $F$  into normal disks (normal triangles and normal quadrilaterals). Let  $N$  be a regular neighborhood of the 1-skeleton of  $\mathcal{T}$ . Consider the components of  $N \cap F$ . These components consist of disks disjoint from  $\partial F$  which we call vertex disks and disks having a single arc in  $\partial F$  which we call half-vertex disks. For every normal disk  $D_{normal}$  we take  $cl(D_{normal} \setminus N)$  and call this a truncated normal disk. In this way normal triangles become truncated normal triangles (which are hexagons) and normal quadrilaterals become truncated normal quadrilaterals (which are octagons). Notice that  $F$  has a decomposition which consists of truncated normal disks, vertex disks, and half-vertex disks. We shall call the components of this decomposition the faces of  $F$ .

As in [10] we use the idea of parallel families. In this case we have 4 distinct types of parallel families. First consider the edges of the 1-skeleton of  $\mathcal{T}$ . The intersection of an edge of the 1-skeleton of  $\mathcal{T}$  with the faces of  $F$  will either intersect a collection of vertex disks or half-vertex disks. When this edge is contained in  $\partial M$  then this collection is empty or consists of half-vertex disks and hence we call this a parallel family of half-vertex disks. Otherwise, for an edge whose interior is disjoint from  $\partial M$  the edge intersects a collection of vertex disks or is empty and this collection is called a parallel family of vertex disks. From normal surface theory there are four types of normal triangles and one type of normal quadrilateral in a given tetrahedron. A type of normal triangle is determined by which vertex is separated from the other three vertices by the normal triangle. We only have one type of normal quadrilateral since if we had more they would intersect each other. It should be noted that we can have many normal disks of the same type in a given tetrahedron. We have in total five types of normal disks in each tetrahedron and since our truncated normal disks are contained in our



normal disks we define the type of our truncated normal disk to correspond to the type of the normal disk it is contained in. We call a collection of truncated normal triangles of the same type a parallel family of truncated normal triangles and we call a collection of truncated normal quadrilaterals of the same type a parallel family of truncated normal quadrilaterals. Thus we have four families of truncated normal triangles and one family of truncated normal quadrilaterals in each tetrahedron. It should be noted that a parallel family may be empty.

We define a *RB**Y* coloring of a normal surface in a triangulated 3-manifold to be a coloring of the faces of the surface such that in each parallel family the outermost faces are colored red and the remaining faces are colored alternately yellow and blue. We color  $F$  in this way. Thus at most two faces are red in each parallel family. Let  $R$  be the union of red faces,  $B$  the union of blue faces, and  $Y$  the union of yellow faces of  $F$ . Let  $G$  be the union of the boundary of the faces of  $F$ . By construction  $G$  is a trivalent graph and so  $R$ ,  $B$ , and  $Y$  are subsurfaces of  $F$ .

Let  $M'$  be a copy of  $M$  and  $\phi : M \rightarrow M'$  the homeomorphism which takes a point  $x \in M$  to the corresponding point of  $M'$ . We define the double of  $M$ ,  $DM$ , as  $DM := (M \sqcup M') / \sim$ , where  $\sim$  is the equivalence relation generated by  $x \sim \phi(x)$  for all  $x \in \partial M$ . Additionally, for  $F$  in  $M$ , the double of  $F$ ,  $DF$ , is defined as  $DF := (F \sqcup F') / \sim$  where  $F' := \phi(F)$  and  $\sim$  is the equivalence relation generated by  $x \sim \phi(x)$  for all  $x \in F \cap \partial M$ . There is a natural map, which we will call the symmetric map,  $\Phi : DM \rightarrow DM$  such that  $\Phi|_M = \phi$  and  $\Phi|_{M'} = \phi^{-1}$ . Notice that both  $DM$  and  $DF$  are invariant under  $\Phi$  and that  $\partial M$  and  $\partial F = F \cap \partial M$  are fixed under  $\Phi$ . Since  $M$  has a decomposition into tetrahedra and  $F$  has a decomposition into faces we give  $M'$  a triangulation  $\mathcal{T}'$  such that  $\Phi$  sends a tetrahedron of  $\mathcal{T}$  to a tetrahedron of  $\mathcal{T}'$  and  $\Phi$  sends a face  $f \subset F$  to a face  $\Phi(f) \subset F'$ . We color the face  $\Phi(f)$  the same color as the face  $f$ . For the purpose of convenience we will use the notation  $D\mathcal{T}$  to denote the triangulation of  $DM$  using the tetrahedra from  $\mathcal{T}$  and  $\mathcal{T}'$ . It is easy to see that  $D\mathcal{T}$  is in fact a triangulation of  $DM$  since in the double of  $M$  we force the faces, edges, and vertices of the tetrahedra of  $\mathcal{T}$  which intersect  $\partial M$  to match up exactly with a corresponding face, edge, or vertex of a tetrahedra of  $\mathcal{T}'$ .

From the above argument we have a *RB**Y* coloring of the surface  $DF$ , with respect to the triangulation  $D\mathcal{T}$ , which is invariant under  $\Phi$ . For simplicity we reuse the notation  $R$ ,  $B$ , and  $Y$  to represent the union of red, blue, and yellow faces of  $DF$  respectively. It is easy to see that  $G \cup \Phi(G)$  is the union of the boundary of the faces of  $DF$ , however,  $G \cup \Phi(G)$  is not trivalent. We replace every half-vertex disk  $D_{half} \subset F$  and its corresponding half-vertex disk  $\Phi(D_{half}) \subset F'$  with the disk  $D_{half} \cup \Phi(D_{half})$ . Notice that our new disks are vertex disks. Moreover these new vertex disks are invariant under  $\Phi$  and have the same color as the two half-vertex disks they contain. Each of these new vertex disks lie in a parallel family of vertex disks where a collection of these vertex disks lie in the same parallel family if its associated half-vertex disks in  $F$  lie in the same parallel family. We consider these new vertex disks to be faces of  $DF$ . We now have a new decomposition of  $DF$  into faces which are truncated normal triangles, truncated normal quadrilaterals, and vertex disks. Our faces no longer include half-vertex disks. Let  $G'$  be the union of the boundary components of the faces of  $DF$  under our new decomposition of faces; then  $G'$  is a trivalent graph and  $R$ ,  $B$ , and  $Y$  are subsurfaces of  $DF$ . We still have a *RB**Y* coloring of  $DF$  with the new decomposition of faces.

Since the Euler characteristic is additive over disjoint unions and the intersection of the two copies of  $F$  in the double is along simple closed curves which have Euler characteristic zero we get that  $\chi(DF) = 2\chi(F)$ . Using  $t'(DM)$  to represent the number of tetrahedra in  $D\mathcal{T}$ , with  $D\mathcal{T}$  as described above, we see that  $t'(DM) = 2t(M)$ . Using the hypothesis of Theorem 1.1, that  $\chi(F) \leq -76t(M) + 1$ , along with the facts above we get the inequality  $\chi(DF) \leq -76t'(DM) + 2$ .

*Lemma 4.2.* There exists a *RB**Y* coloring of  $DF$  invariant under  $\Phi$  for which there is an essential pair of pants  $X \subset DF$  contained in  $int(B)$  (or in  $int(Y)$ ).

*Proof of Lemma 4.2.* We remark that a compact 3-manifold is always tame. Since  $\chi(DF) \leq -76t'(DM) + 2$ , as seen above;  $DM$  is a connected compact 3-manifold with a triangulation of  $t'(DM)$  tetrahedra; and  $DF \subset DM$  is a 2-sided connected essential closed

surface, which follows from Lemma 2.40 and Lemma 2.39; it is clear that  $DF \subset DM$  satisfies the hypothesis of the main theorem in [10]. We make four observations. The first observation is that  $DF$  is already normal with respect to the triangulation  $D\mathcal{T}$  and hence we do not need to isotope it, this follows from Lemma 2.41. The second observation is that any  $RBV$  coloring of  $DF$  is sufficient for the result of [10]. The third observation is that in the proof of the main theorem in [10] there is a possible need to swap the coloring of every blue and yellow vertex disk. That is, we may need to recolor every blue vertex disk to be yellow and recolor every yellow vertex disk to be blue. Since our  $RBV$  coloring of  $DF$  is invariant under  $\Phi$  we will still have an invariant  $RBV$  coloring of  $DF$  if we swap every blue and yellow vertex disk. So we do not have an issue here and can use our  $RBV$  coloring of  $DF$  or one with the blue and yellow vertex disks swapped. The fourth observation is that we do not require a minimal triangulation of  $DM$  and so  $D\mathcal{T}$  is good enough. Now, rather than take the result from the main theorem in [10] we take the result from [10, Prop. 11] to get that there is an essential pair of pants  $X \subset DF$  in  $int(B)$  (or in  $int(Y)$ ).  $\square$

We now want to consider some special pairs of pants that we can have in  $DM$ . In particular, we give a definition of the types of pairs of pants that we are interested in and prove some useful properties that they have.

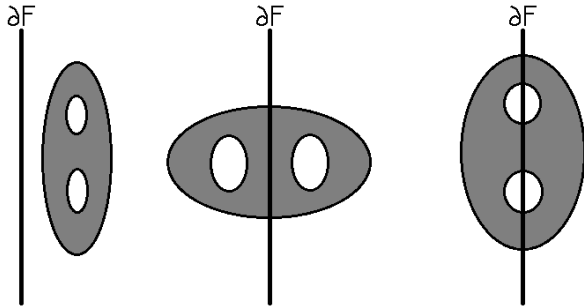


Figure 13: Example of pairs of pants of type  $T_0$ ,  $T_1$ , and  $T_3$  in that order.

*Definition 4.3.* We define three types of equivariant under  $\Phi$  pairs of pants in  $DF$ . An equivariant pair of pants  $X$  is of type  $T_0$  if  $X \cap \Phi(X)$  is empty. An equivariant pair of pants  $X$  is

of type  $T_1$  if there is exactly one component  $c$  of  $\partial X$  such that  $c = \Phi(c)$ . Finally, an equivariant pair of pants  $X$  is of type  $T_3$  if for each component  $c$  of  $\partial X$  we get  $c = \Phi(c)$ . See Figure 13.

Remark. In the case when  $X$  is a pair of pants of type  $T_1$  we have that the two components  $c_1$  and  $c_2$  of  $\partial X$  which do not map to themselves under  $\Phi$  have the property  $\Phi(c_1) = c_2$  and hence  $\Phi(c_2) = c_1$ .

Notice that if  $X$  is equivariant under  $\Phi$  then either  $\Phi(X) = X$  or  $\Phi(X) \cap X$  is empty.

*Lemma 4.4.* Let  $X$  be a pair of pants equivariant under  $\Phi$ , then  $X$  is of type  $T_0$ ,  $T_1$ , or  $T_3$ .

Proof of Lemma 4.4. Let  $c_1$ ,  $c_2$ , and  $c_3$  be the boundary components of  $X$ . We break this problem into cases based on the number of invariant boundary components of  $X$  under  $\Phi$ .

Case 1: No component of  $\partial X$  is invariant under  $\Phi$ . Suppose for a contradiction that  $\Phi(X) = X$ . Since  $\Phi|_X$  is a homeomorphism it maps boundary components to boundary components and the only way to do this without an invariant boundary component is that without loss of generality  $\Phi(c_1) = c_2$ ,  $\Phi(c_2) = c_3$ , and  $\Phi(c_3) = c_1$  which contradicts the fact that  $\Phi \circ \Phi$  is the identity. Thus  $X$  and  $\Phi(X)$  are disjoint and  $X$  is a pair of pants of type  $T_0$ .

Case 2: Exactly one component of  $\partial X$  is invariant under  $\Phi$ . Then  $X$  is by definition a pair of pants of type  $T_1$ .

Case 3: Exactly two components of  $\partial X$  are invariant under  $\Phi$ . We show that no such  $X$  exists. Suppose for a contradiction that without loss of generality  $\Phi(c_1) = c_1$  and  $\Phi(c_2) = c_2$ . Since  $X$  is equivariant under  $\Phi$  and  $\Phi|_X$  is a homeomorphism, and hence maps boundary components to boundary components, then  $\Phi(c_3) = \Phi(c_3)$ . So every component of  $\partial X$  is invariant under  $\Phi$  which is a contradiction. Thus there are no equivariant pairs of pants with exactly two invariant boundary components.

Case 4: Every boundary component of  $X$  is invariant under  $\Phi$ . By definition  $X$  is a pair of pants of type  $T_3$ . This completes the proof of the lemma.  $\square$

### **Description of Opp map**

We construct a special map, similar to the one in [10],  $Opp: V \rightarrow DF$  where  $V$  is a

connected subsurface of  $DF$  contained in  $int(B)$  or in  $int(Y)$ . We show that  $Opp$  is injective. Moreover, we show that for  $V \subset int(B)$  then  $Opp(V) \subset F \setminus B$  or for  $V \subset int(Y)$  then  $Opp(V) \subset F \setminus Y$ . We construct the map below.

This section is very technical and hence we clarify our use of terminology. A vertex disk cannot, in general, be contained in the preimage of a single 3-simplex. So when we refer to a vertex disk we are referring to a subsurface in  $DM$ . Thus a parallel family of normal disk is contained in  $DM$ . In contrast a normal disk is, by definition, contained in the preimage of a single 3-simplex. So we consider a normal disk as an object in the standard 3-simplex. We are primarily interested in the faces of  $F$ , however, and want to think about them all as subsurfaces of  $DM$ . Our decomposition of  $DF$  into faces consists of vertex disks, truncated normal triangles, and truncated normal quadrilaterals. We already consider a vertex disk as a subsurface of  $DM$ . Similarly, in this section we will consider our truncated normal disks as subsurfaces of  $DM$ . Thus, a parallel family of faces is contained in  $DM$ . A parallel family of truncated normal disks has the additional property of being properly embedded in the image of a 3-simplex of  $D\mathcal{T}$ .

Let  $\{\psi_j\}_{j=1}^{j=2t(M)}$  be the 3-simplices of  $D\mathcal{T}$ . Then  $\psi_j : \hat{T} \rightarrow M$  where  $\hat{T}$  is the standard 3-simplex in  $\mathbb{R}^4$ . If we refer to a tetrahedron of  $D\mathcal{T}$  we are referring to one of the pairs  $(\hat{T}, \psi_j)$  where  $1 \leq j \leq 2t(M)$ .

We define a partial vertex disk to be the intersection of a vertex disk with the image of  $\hat{T}$  under a 3-simplex of  $D\mathcal{T}$  and a parallel family of partial vertex disks to be the intersection of a parallel family of vertex disks with the image of  $\hat{T}$  under a 3-simplex of  $D\mathcal{T}$ . In this way we can decompose every vertex disk into partial vertex disks. Let  $[F]$  be a nonempty parallel family of truncated normal triangles, truncated normal quadrilaterals, or partial vertex disks with ordered faces  $F_1, \dots, F_n$  and  $n \geq 3$ . We define  $[\hat{F}] \subset \hat{T}$  to be  $\psi^{-1}([F])$  where  $\psi$  is a simplex in  $D\mathcal{T}$  such that  $[F] \subset \psi(\hat{T})$ . The components of  $[\hat{F}]$  are given by  $\hat{F}_1 = \psi^{-1}(F_1), \dots, \hat{F}_n = \psi^{-1}(F_n)$  which we call the faces of  $[\hat{F}]$ . Let  $\hat{T}^{(1)}$  be the 1-skeleton of  $\hat{T}$  and let  $\hat{T}^{(2)}$  be the 2-skeleton of  $\hat{T}$ . For

$1 \leq i \leq n$  we give the face  $\widehat{F}_i \subset [\widehat{F}]$  a cell decomposition as follows:

- 0-cells: components of  $\left(\partial\widehat{F}_i \setminus \left(\text{int}(\partial\widehat{F}_i \cap \widehat{T}^{(2)}) \cup \text{int}(\widehat{T})\right)\right) \cup \left(\partial\widehat{F}_i \cap \partial\widehat{T}^{(1)}\right)$
- 1-cells: components of  $\partial\widehat{F}_i \setminus \{0\text{-cells}\}$
- 2-cell:  $\text{int}(\widehat{F}_i)$

Recall from earlier that  $N$  is a regular neighborhood of the 1-skeleton of  $\mathcal{T}$ . By this we mean that  $N$  is a regular neighborhood of the underlying space of the 2-simplices of  $\mathcal{T}$ . We define  $DN := N \cup \Phi(N)$ . Notice that  $DN$  is a regular neighborhood of the underlying space of the 2-simplices of  $D\mathcal{T}$ .

When  $[F]$  is a parallel family of truncated normal triangles (or quadrilaterals), with at least 3 faces, assume  $(\widehat{T}, \psi)$  is the tetrahedron of  $D\mathcal{T}$  so that  $[F] \subset T$  where  $T := \psi(\widehat{T})$  and define  $\widehat{C}$  to be the closure of the component of  $(\widehat{T} \setminus \psi^{-1}(DN \cap T)) \setminus \{\text{outermost faces of } [\widehat{F}]\}$  which intersects  $[\widehat{F}]$  nontrivially.

When  $[F]$  is a parallel family of partial vertex disks, with at least 3 faces, assume  $(\widehat{T}, \psi)$  is the tetrahedron of  $D\mathcal{T}$  so that  $[F] \subset T$  where  $T := \psi(\widehat{T})$  and define  $\widehat{C}$  to be the closure of the component of  $\psi^{-1}(DN \cap T) \setminus \{\text{outermost faces of } [\widehat{F}]\}$  which intersects  $[\widehat{F}]$  nontrivially.

In both cases  $\widehat{C}$  is homeomorphic to  $D \times I := C$  where  $D$  is a disk simplicially isomorphic to the faces of  $[\widehat{F}]$  and  $I := [0, 1]$ . Let  $p : D \times I \rightarrow D$  be the natural projection. Let  $D^{(1)}$  be the 1-skeleton of  $D$  and let  $D^{(2)}$  be the 2-skeleton of  $D$ . Let  $h : C \rightarrow \widehat{C}$  be a homeomorphism such that for  $0 = p_1 < \dots < p_i < p_{i+1} < \dots < p_n = 1$  and  $\widehat{F}_1, \dots, \widehat{F}_i, \widehat{F}_{i+1}, \dots, \widehat{F}_n$ , the faces of  $[\widehat{F}]$ , we have  $h|_{D \times \{p_i\}} : D \times \{p_i\} \rightarrow \widehat{F}_i$  and:

- For each  $q \in D^{(1)} \setminus D^{(0)}$  we have that  $h(\{q\} \times I)$  is contained in the interior of a 2-face of  $\widehat{T}$  or in the interior of  $\widehat{T}$ .
- We require the following to hold for all 1-cells  $l$  of  $D$  and for all  $s \in I$ . Let  $\hat{l}$  be the image under  $h \circ (p|_{D \times \{s\}})^{-1}$  of a 1-cell  $l$  of  $D$  and  $\hat{v}$  a boundary point of  $\hat{l}$ . Define  $v := p \circ h^{-1}(\hat{v})$ .

Notice that  $\nu$  is a boundary component of  $l$ . We require for  $t$  a point of distance  $d$  from  $\nu$  in  $l$  that  $h \circ (p|_{D \times \{s\}})^{-1}(t)$  is of distance  $\frac{d * \text{length}(\hat{l})}{\text{length}(l)}$  from  $\hat{\nu}$  in  $\hat{l}$ . We may assume that  $C$  lies in  $\mathbb{R}^3$ . Thus, since  $\hat{T}$  is the standard 3-simplex in  $\mathbb{R}^4$  the notion of length above makes sense.

- Finally for all  $q$  in  $D^{(1)}$  define  $l_q := q \times [0, 1]$  and  $\hat{l}_q := h(l_q)$ . We require that for  $t$  a point in  $l_q$  of distance  $d$  from  $q \times \{0\}$  in  $l_q$  that  $h(t)$  is of distance  $\frac{d * \text{length}(\hat{l}_q)}{\text{length}(l_q)}$  from  $h(q \times \{0\})$  in  $\hat{l}_q$ .

Let  $U$  be a, not necessarily connected, subsurface in a blue (or yellow) face  $F_i = \psi(\hat{F}_i)$  of the parallel family  $[F]$  above and  $U \subset \text{int}(B)$  (or  $U \subset \text{int}(Y)$ ) where  $[F]$  is a parallel family of truncated normal triangles, truncated normal quadrilaterals, or partial vertex disks. We define for all  $u \in U$  the points

$$\text{opp}_{\pm}(u) := \psi \circ h \circ (p|_{D \times \{p_{i \pm 1}\}})^{-1} \circ p \circ h^{-1} \circ \psi^{-1}(u)$$

and define the map  $\text{Opp}|_U : U \rightarrow DF$  either to be  $\text{Opp}|_U : u \mapsto \text{opp}_+(u)$  for all  $u \in U$  or to be  $\text{Opp}|_U : u \mapsto \text{opp}_-(u)$  for all  $u \in U$  so that  $\text{Opp}|_U : U \rightarrow DF$  agrees with the coorientation of  $DF$  in  $DM$ . Notice that  $\text{Opp}|_U$  sends  $U \subset F_i$  either to  $F_{i-1}$  or to  $F_{i+1}$ . It is easy to see that  $\text{Opp}|_U$  is injective and hence a homeomorphism.

We would like to define an  $\text{Opp}$  map for a connected subsurface  $V \subset \text{int}(B)$  (or  $V \subset \text{int}(Y)$ ) such that  $\text{Opp} : V \rightarrow DF$  is a homeomorphism onto its image and behaves similar to  $\text{Opp}|_U$  when  $\text{Opp}$  is restricted to a face of  $DF$ . We now think of the faces of  $DF$  to be truncated normal triangles, truncated normal quadrilaterals and partial vertex disks. Let  $\tilde{F}_1, \dots, \tilde{F}_w$  be the faces of  $DF$  whose interiors have nonempty intersection with  $V$  and define for  $1 \leq i \leq w$  that  $F_i^* := \tilde{F}_i \cap V$ . Similar to  $\text{Opp}|_U$  we have for each  $F_i^*$  a map  $\text{Opp}|_{F_i^*}$  which sends  $F_i^* \subset \tilde{F}_i$  into the face directly above or below  $\tilde{F}_i$  in the parallel family containing  $\tilde{F}_i$  consistent with the

coorientation of  $DF$  in  $DM$ . We define  $Opp: V \rightarrow DF$  so that

$$Opp(x) := \begin{cases} Opp|_{F_1^*}(x) & x \in F_1^* \\ Opp|_{F_2^*}(x) & x \in F_2^* \\ \vdots & \vdots \\ Opp|_{F_w^*}(x) & x \in F_w^* \end{cases}$$

for all  $x$  in  $V$ .

By our construction of  $Opp: V \rightarrow DF$  it is clear that for  $V \subset \text{int}(B)$  (or  $V \subset \text{int}(Y)$ ) that  $Opp(V) \subset DF \setminus B$  (or  $Opp(V) \subset DF \setminus Y$ ). We finish this section by proving that  $Opp$  is injective.

Remark. Our description of  $Opp$  is defined for an arbitrary connected subsurface of  $DF$  which is contained in  $\text{int}(B)$  or in  $\text{int}(Y)$  and hence we may have multiple  $Opp$  maps so long as we specify an appropriate domain. Additionally, for a given domain  $Opp$  is well-defined.

*Lemma 4.5.* For a connected subsurface  $S \subset DF$  which is contained in  $\text{int}(B)$  or in  $\text{int}(Y)$ ,  $Opp$  is injective and hence a homeomorphism onto its image.

Proof of Lemma 4.5. Suppose without loss of generality that  $S \subset \text{int}(B)$ . By our comment above,  $S$  is disjoint from  $Opp(S) \subset DF \setminus B$ . Suppose for a contradiction that  $Opp$  is not injective. Then there exists distinct points  $s_1, s_2 \in S$  so that  $Opp(s_1) = Opp(s_2)$ . Since  $Opp(s_1) = Opp(s_2)$  it must be the case that  $s_1, s_2$ , and  $Opp(s_1)$  are in the same parallel family; let  $F_1, \dots, F_n$  be the faces of this parallel family. It is not hard to see that if  $Opp(s_1) \in F_i$  then without loss of generality  $s_1 \in F_{i-1}$  and  $s_2 \in F_{i+1}$ . Let  $Ns_1$  be a regular neighborhood of  $s_1$  in  $S \cap F_{i-1}$  and  $Ns_2$  be a regular neighborhood of  $s_2$  in  $S \cap F_{i+1}$ . Define  $Q_1 := Opp(Ns_1)$  and  $Q_2 := Opp(Ns_2)$ . Recall that  $Opp$  is injective on a single face of  $DF$  and hence  $Q_1$  and  $Q_2$  are homeomorphic to  $Ns_1$  and  $Ns_2$  respectively.

There is a natural fiber structure between adjacent parallel faces in a given parallel family; in particular we define a fiber  $I_p$  to be the fiber which has initial point  $p \in S$  and end point  $Opp(p)$ . We define an induced coorientation of  $Opp(S)$ , by  $S$ , under  $Opp$  to be the



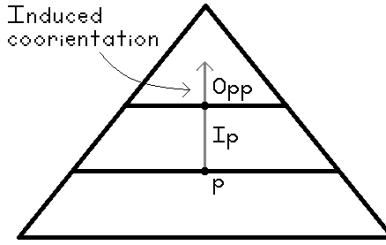


Figure 14: Illustration of induced coorientation of  $Opp(S)$ , by  $S$ , under  $Opp$  relative to  $I_p$ .

coorientation so that for every point  $p \in S$  the induced coorientation points in the direction opposite of  $I_p$ . See Figure 14. Since  $S$  is a connected subsurface and  $Opp$  is continuous, we have that the induced coorientation on  $Opp(S)$  either agrees with the coorientation of  $DF$  for all points in  $Opp(S)$  or disagrees with the coorientation of  $DF$  for all points in  $Opp(S)$ ; we assume without loss of generality that the induced coorientation on  $Opp(S)$  agrees with the coorientation on  $DF$ .

Using the notion of an induced coorientation that we just defined there is an induced coorientation of  $Q_1$ , by  $N_{S_1}$ , under  $Opp$  and an induced coorientation of  $Q_2$ , by  $N_{S_2}$ , under  $Opp$ . Notice that the induced coorientations point in opposite directions, particularly at  $Opp(s_1)$ , and hence one of our induced coorientations doesn't agree with the coorientation of  $DF$ , a contradiction. We conclude that  $Opp$  is injective and hence a homeomorphism onto its image.  $\square$

### Main Lemma

Before we prove the Main Lemma we prove a useful lemma and corollary. We use  $|\cdot|$  to represent the number of components of.

*Lemma 4.6.* A simple closed curve in  $DF$  which is invariant under  $\Phi$  intersects  $\partial F$  exactly twice.

*Proof of Lemma 4.6.* Let  $\alpha$  be a simple closed curve such that  $\Phi(\alpha) = \alpha$ . Since  $DF \setminus \partial F$  consists of the two components  $int(F)$  and  $int(F')$ , it is not connected and hence  $|\alpha \cap \partial F| = 2k$ ,  $k \geq 1$ .

Now  $\alpha \setminus (\alpha \cap \partial F)$  consists of  $2k$  components,  $k$  components in  $F$  and the other  $k$  components in  $F'$ . Since  $\Phi$  fixes  $\partial F$  it must be the case that for  $\alpha_1$  the closure of a component of  $\alpha \setminus (\alpha \cap \partial F)$  in  $F'$ , there is a corresponding  $\alpha_2$  which is the closure of a component of  $\alpha \setminus (\alpha \cap \partial F)$  in  $F$  such that  $\Phi(\alpha_1) = \alpha_2$ . Thus,  $\partial\alpha_1 = \partial\alpha_2$  and  $\alpha_1 \cup \alpha_2$  is a simple closed curve component of  $\alpha$ . Since  $\alpha$  is connected  $\alpha = \alpha_1 \cup \alpha_2$  and hence  $k = 1$ .  $\square$

*Corollary 4.7.* A disk in  $DF$  which is invariant under  $\Phi$  intersects  $\partial F$  in exactly one arc.

*Proof of Lemma 4.7.* The definition of invariance guarantees that any invariant disk must intersect  $\partial F$  at least once. Suppose for a contradiction that there exists a disk  $D \subset DF$  such that  $\Phi(D) = D$  and  $|D \cap \partial F| = k$  where  $k \geq 2$ . Notice that  $D \cap \partial F$  consists only of arc components. Indeed if  $D \cap \partial F$  contained simple closed curves then  $D$  would contain a component of  $\partial D$  and no longer be a disk. Thus,  $|D \setminus \partial F| = k + 1$  and

$$|\partial D \setminus \partial F| \geq k + 1 \geq 3$$

which is a contradiction by Lemma 4.6 since  $\partial D \subset DF$  is an invariant simple closed curve.  $\square$

In the next lemma, or Main Lemma, we make use of some terminology. We say a subsurface of  $DF$  is essentially blue (or yellow) if the subsurface is blue (or yellow) except possibly for a portion which can be contained in the interior of a disk, where the disk is in the interior of the subsurface. We also may say that a subsurface of  $DF$  is entirely blue (or yellow) if it is contained in  $\text{int}(B)$  (or in  $\text{int}(Y)$ ).

*Lemma 4.8.* There exists a *RB**Y* coloring of  $DF$  invariant under  $\Phi$  for which there is an entirely blue (or entirely yellow) essential pair of pants  $X \subset DF$  of type  $T_0$ ,  $T_1$ , or  $T_3$ .

*Proof of Lemma 4.8.* The *RB**Y* coloring of  $DF$  already exists. It remains to show that there exists  $X$  satisfying the lemma. We assume without loss of generality that  $X$ , from Lemma 4.2, is a pair of pants in  $\text{int}(B)$ . When we talk about intersections we assume transversality, as usual, and in particular transversality of  $X$  and  $\Phi(X)$ . If  $X$  does not intersect  $\partial F$  it is a pair of pants of type  $T_0$

with essential boundary components and  $X \subset \text{int}(B)$  and we are done. We assume from now on that  $X$  intersects  $\partial F$ . We construct a surface, denoted  $X^{s,d}$ , with the following properties:

- $X^{s,d}$  is a connected subsurface of  $DF$ ,
- $\Phi(X^{s,d}) = X^{s,d}$  (i.e.  $X^{s,d}$  is invariant under  $\Phi$ ),
- $\partial X^{s,d}$  is essential in  $DF$ ,
- $X \subset X^{s,d}$ ,
- $X^{s,d}$  is essentially blue, and
- $\chi(X^{s,d}) \leq -1$ .

We first define  $X^s := X \cup \Phi(X)$ . Notice that  $X^s$  is invariant under  $\Phi$ , is a subsurface of  $DF$  (this follows from transversality) containing  $X$ , and is entirely blue. Let  $D$  be the disjoint union of the disks in  $DF$  which are bounded by boundary components of  $X^s$ . We define  $X^{s,d} := X^s \cup D$ . Thus  $\partial X^{s,d}$  is essential in  $DF$ . The only part of  $X^{s,d}$  which can be yellow is contained in  $D$  and since  $X^{s,d}$  is connected we can find a single disk in  $X^{s,d}$  which contains all of  $D$ . So  $X^{s,d}$  is essentially blue. Since  $X^s$  is invariant under  $\Phi$  we get for any boundary component  $c$  of  $X^s$  which bounds a disk  $D_c$  that there is a corresponding boundary component  $\Phi(c)$  which bounds a disk  $\Phi(D_c)$ . It may be the case that  $D_c = \Phi(D_c)$ . We conclude that  $X^{s,d}$  is invariant under  $\Phi$ . Notice that  $X$  is a connected essential subsurface of  $X^{s,d}$  and hence Lemma 2.14 tells us that  $\chi(X^{s,d}) \leq \chi(X) \leq -1$ . Thus,  $X^{s,d}$  is a subsurface of  $DF$  satisfying the six properties above. Claim.  $X^{s,d}$  contains an entirely blue essential pair of pants of type  $T_1$  or  $T_3$ .

Proof of claim. Since  $X$  is entirely blue and connected we can define  $\text{Opp}: X \rightarrow DF$  and get that  $\text{Opp}(X) \subset DF \setminus B = \text{int}(R \cup Y)$ .

Subclaim. For any component  $\alpha$  of  $\partial X$  we get that  $\alpha$  is essential in  $DF$  if and only if  $\text{Opp}(\alpha)$  is essential in  $DF$ .

Proof of subclaim. We show one direction using proof by contrapositive; the other direction is similar. If  $Opp(\alpha)$  is inessential in  $DF$ , then it bounds a disk  $D'' \subset DF$ . From our construction of the  $Opp$  map,  $\alpha$  and  $Opp(\alpha)$  cobound an annulus  $A$  in  $DM$ . Thus  $\alpha$  is the boundary of the disk  $D'' \cup A \subset DM$ . Now  $int(D'' \cup A) \cap F = D''$  so we use an isotopy to push  $D''$  off of  $F$  so that  $int(D'' \cup A) \cap F = \emptyset$ . So  $D'' \cup A$  is a compressing disk for  $DF$  whose boundary is  $\alpha$ . Since  $DF$  is essential and hence incompressible in  $DM$  we get that  $\alpha$  bounds a disk in  $DF$  and hence is inessential in  $DF$ . This completes the proof of the subclaim.

We conclude that since  $X$  is essential in  $DF$  that  $Opp(X)$  is essential in  $DF$ . We now show that  $Opp(X) \cap X^{s,d}$  is empty. Suppose for a contradiction that  $Opp(X) \cap X^{s,d}$  is nonempty. Since  $Opp(X) \subset int(R \cup Y)$  it is disjoint from  $X^s \subset int(B)$ . Thus  $Opp(X)$  is contained in a disk component of  $D$ . Since  $Opp(X)$  is essential in this disk Lemma 2.14 tells us that  $\chi(Opp(X)) \geq 1$  which is a contradiction. Thus  $Opp(X)$  is disjoint from  $X^{s,d}$ . We conclude that  $X^{s,d}$  is not all of  $DF$  and hence  $X^{s,d}$  has boundary. Let  $\rho$  be a boundary component of  $X^{s,d}$ . Notice that  $\partial X^{s,d}$  is contained in  $\partial X^s$  and hence  $\rho$  is entirely blue. Since  $\partial X^{s,d}$  is invariant under  $\Phi$  we get that  $\rho$  is equivariant under  $\Phi$  and hence is disjoint from  $\partial F$  or intersects  $\partial F$  exactly twice.

In the case that  $\rho$  does not intersect  $\partial F$  then due to connectedness of  $X^s$  and the invariance under  $\Phi$  of  $X^s$  there is a simple path  $l$  in  $X^s$  starting at the point  $p$  in  $\rho$  and ending in  $\rho' := \Phi(\rho)$  such that  $l$  only intersects  $\rho$  at  $p$ . Suppose without loss of generality that  $\rho \subset F$  and let  $l'$  be the component of  $l \cap F$  containing  $p$ . Notice that  $l' \cup \Phi(l')$  is a simple path with one boundary component in  $\rho$ , the second boundary component in  $\rho'$ , and is invariant under  $\Phi$ . We define  $q := \rho \cup l' \cup \Phi(l') \cup \rho'$ . See Figure 15. Notice that  $q$  is invariant under  $\Phi$  and is contained in  $X^s$  (hence is entirely blue). Let  $N(q \cap F)$  be the regular neighborhood of  $q \cap F$ , in  $F \cap X^s$ . Notice that there is a regular neighborhood of  $q$ , in  $X^s$ , which is invariant under  $\Phi$  given by  $Q := N(q \cap F) \cup \Phi(N(q \cap F))$ . We see that  $Q$  is an entirely blue pair of pants of type  $T_1$ .

By our construction of  $Q$  we know that since  $\rho$  and  $\rho'$  are boundary components of  $X^s$  they are also boundary components of  $Q$ . Moreover,  $\rho$  and  $\rho'$  are boundary components of  $X^{s,d}$

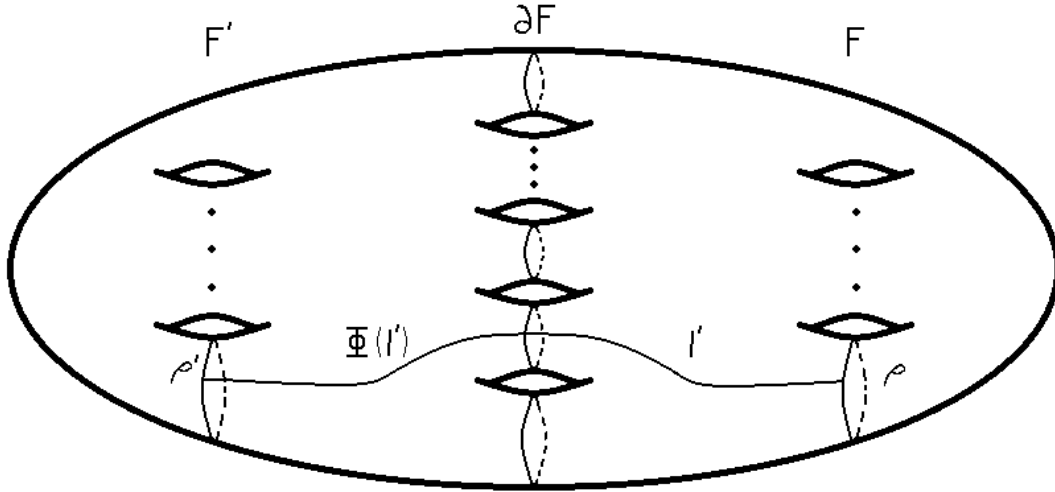


Figure 15: Example of possible decomposition for  $q$ .

and hence are essential in  $DF$ . Let  $\rho''$  be the third boundary component of  $Q$ . Since  $Q$  is a regular neighborhood of  $q$  and  $\rho''$  was not already a boundary component of  $X^{s,d}$  we get from our construction that  $\rho'' \subset \text{int}(X^s)$ .

Claim.  $\rho''$  is essential in  $DF$ .

Proof of Claim. Suppose for a contradiction that  $\rho''$  is inessential and hence bounds a disk  $D'$ . This disk either contains both  $\rho$  and  $\rho'$  or neither. If  $\rho \sqcup \rho' \subset D'$  then  $\rho$  and  $\rho'$  both bound a disk and hence are inessential which is a contradiction. Thus  $D'$  is disjoint from  $\rho$  and  $\rho'$ . Now  $\rho''$  is an inessential simple closed curve in  $X^s \subset X^{s,d}$ . We show that  $D' \subset X^{s,d}$ . Suppose for a contradiction that  $D' \not\subset X^{s,d}$ . Since  $\partial D' \subset X^{s,d}$ , but  $D' \not\subset X^{s,d}$  we get that  $\text{int}(D') \cap X^{s,d}$  is a submanifold of  $D'$  with boundary consisting of simple closed curves from  $\partial X^{s,d}$ . Each boundary component of  $X^{s,d}$  in  $D'$  bounds a disk in  $D'$ ; this contradicts the construction of  $X^{s,d}$ . We conclude that  $D' \subset X^{s,d}$ .

Since  $\rho$  and  $\rho'$  are boundary components of  $X^{s,d}$ ,  $\rho \sqcup \rho' = \partial(Q \cup D')$ ,  $Q \cup D' \subset X^{s,d}$ , and  $X^{s,d}$  is connected we get that  $X^{s,d} = Q \cup D'$ . Thus, since  $Q \cup D'$  is an annulus,  $\chi(X^{s,d}) = \chi(Q \cup D') = 0$  which is a contradiction since  $\chi(X^{s,d}) \leq -1$  and so our claim holds.

Thus  $\rho''$  is essential as desired, moreover, we get that  $Q$  is an essential pair of pants of type

$T_1$  which is entirely blue.

We now consider the case when  $\rho$  intersects  $\partial F$ . Let  $p$  be one of the two points in  $\rho \cap \partial F$ . There exists a path  $\rho_2$  contained in  $\partial F \cap X^{s,d}$  which has endpoints in  $\partial X^{s,d}$  one of which is  $p$ . The other endpoint of  $\rho_2$  we call  $p'$ . We now consider 2 separate cases:

Case 1:  $p'$  is in  $\rho$ .

Let  $\rho_l \subset F'$  and  $\rho_r \subset F$  be the oriented paths starting at  $p$  and ending in  $p'$  such that  $\rho = \rho_l \rho_r^{-1}$ . Additionally, orient  $\rho_2$  from  $p$  to  $p'$ .

Claim. The simple closed curves  $\rho_l \rho_2^{-1}$  and  $\rho_r \rho_2^{-1}$  are essential in  $DF$ .

Proof of Claim. Assume for a contradiction that  $\rho_l \rho_2^{-1}$  is inessential in  $DF$ . Since  $\Phi(\rho_l \rho_2^{-1}) = \rho_r \rho_2^{-1}$  it follows that  $\rho_r \rho_2^{-1}$  is also inessential in  $DF$ . Now with  $\sim$  standing for homotopy we get

$$(\rho_l \rho_2^{-1})(\rho_r \rho_2^{-1})^{-1} \sim (\rho_l \rho_2^{-1})(\rho_2 \rho_r^{-1}) \sim \rho_l (\rho_2^{-1} \rho_2) \rho_r^{-1} \sim \rho_l \rho_r^{-1} \sim \rho$$

Now we consider these curves in the fundamental group of  $DF$  with base point  $p$  and we get for  $e$  the identity that

$$e = e * e^{-1} = [\rho_l \rho_2^{-1}] * [\rho_r \rho_2^{-1}]^{-1} = [\rho]$$

and hence  $\rho$  is inessential giving us our contradiction. Showing  $\rho_r \rho_2^{-1}$  is essential is basically the same proof and so our claim holds.

Recall that  $X^{s,d}$  is only essentially blue. We know that  $X^{s,d} \setminus \text{int}(D) = X^s$  is entirely blue. Let  $D'$  be the disjoint union of the components of  $D$  which have nontrivial intersection with  $\rho_2$ . Since each of these components intersect  $\partial F$  and  $D$  is invariant under  $\Phi$  it is clear that  $D'$  is invariant under  $\Phi$  and hence every component of  $D'$  intersects  $\partial F$  in exactly one arc. Let  $N((D' \cup \rho_2) \cap F)$  be a regular neighborhood, in  $F \cap X^{s,d}$ , of  $(D' \cup \rho_2) \cap F$ , then  $N(D' \cup \rho_2) := N((D' \cup \rho_2) \cap F) \cup \Phi(N((D' \cup \rho_2) \cap F))$  is a regular neighborhood of  $D' \cup \rho_2$ , in  $X^{s,d}$ , which is invariant under  $\Phi$ . We define  $\rho'_2 := \text{cl}((\partial N(D' \cup \rho_2) \cap F) \setminus \rho)$ . Notice that

$\rho'_2 \subset \partial N(D' \cup \rho_2)$  is entirely blue since the construction of  $N(D' \cup \rho_2)$  was designed so that  $\partial N(D' \cup \rho_2)$  was entirely blue. We also remark that  $\rho'_2 \subset \text{int}(F)$  and  $\rho'_2$  is homotopic relative the boundary of  $X^{s,d}$  to  $\rho_2$ . Define  $q := \rho \cup \rho'_2 \cup \Phi(\rho'_2)$ . Notice that  $q$  is entirely blue and invariant under  $\Phi$ . Let  $q_{p'}$  be the component of  $q \setminus (\partial \rho'_2 \cup \partial \Phi(\rho'_2))$  containing  $p'$ . Define  $q' := q \setminus q_{p'}$ . Similar to  $q$  we get that  $q'$  is entirely blue (in fact  $q' \subset X^s$ ) and invariant under  $\Phi$ . Let  $N(q' \cap F)$  be a regular neighborhood, in  $F \cap X^s$ , of  $q' \cap F$ ; then  $Q := N(q' \cap F) \cup \Phi(N(q' \cap F))$  is a regular neighborhood, in  $X^s$ , of  $q'$ , is invariant under  $\Phi$ , and is entirely blue. Since  $Q$  is a pair of pants with boundary components homotopic to  $\rho$ ,  $\rho_l \rho_2^{-1}$ , and  $\rho_r \rho_2^{-1}$  we conclude that  $Q$  is an entirely blue essential pair of pants of type  $T_1$ .

Case 2:  $p'$  is not in  $\rho$  and hence is in a different boundary component of  $X^{s,d}$ ; we call this component  $\rho'$  and remark that since it is a boundary component of  $X^{s,d}$  it is essential.

The following argument is fairly similar to case 1 and hence we omit some details. Let  $D'$  be the disjoint union of the components of  $D$  which have nontrivial intersection with  $\rho_2$ . Like case 1,  $D'$  is invariant under  $\Phi$  and hence every component of  $D'$  intersects  $\partial F$  in exactly one arc. We are able to find a regular neighborhood  $N(D' \cup \rho_2)$ , in  $F \cap X^{s,d}$ , of  $D' \cup \rho_2$  which is invariant under  $\Phi$ . We define  $\rho'_2 := \text{cl}((\partial N(D' \cup \rho_2) \cap F) \setminus (\rho \cup \rho'))$ . By our construction  $\rho'_2$  is entirely blue,  $\rho'_2 \subset \text{int}(F)$ , and  $\rho'_2$  is homotopic relative the boundary of  $X^{s,d}$  to  $\rho_2$ . We define  $q := \rho \cup \rho' \cup \rho'_2 \cup \Phi(\rho'_2)$ . Notice that  $q$  is entirely blue and invariant under  $\Phi$ . Let  $q_{p'}$  be the component in  $q \setminus (\partial \rho'_2 \cup \partial \Phi(\rho'_2))$  containing  $p'$ . Define  $q' := q \setminus q_{p'}$ . Similar to  $q$  we get that  $q'$  is entirely blue (in fact  $q' \subset X^s$ ) and invariant under  $\Phi$ . Let  $N(q' \cap F)$  be a regular neighborhood, in  $F \cap X^s$ , of  $q' \cap F$ , then  $Q := N(q' \cap F) \cup \Phi(N(q' \cap F))$  is a regular neighborhood, in  $X^s$ , of  $q'$ , is invariant under  $\Phi$ , and is entirely blue. Now  $Q$  is an entirely blue pair of pants of type  $T_3$ .

Claim.  $Q$  is essential in  $DF$ .

Proof of claim. By construction the essential simple closed curve  $\rho$  is a component of  $\partial Q$ . Let  $q_1$  and  $q_2$  be the remaining boundary components of  $Q$ . By our construction and since  $\rho'_2$  is homotopic to  $\rho_2$  it follows that one of the boundary components, say  $q_1$ , of  $Q$  is homotopic to

$\rho_2 \rho' \rho_2^{-1}$  which is the conjugate of an essential curve and hence essential. It remains to show that  $q_2$  is essential. Suppose for a contradiction that  $q_2$  is inessential, then  $q_2$  bounds a disk  $D''$  in  $DF$ . Now  $\rho \sqcup q_1 \subset D''$  or  $(\rho \sqcup q_1) \cap D'' = \emptyset$ . If  $\rho \sqcup q_1 \subset D''$  then both  $\rho$  and  $q_1$  bound a disk in  $D''$  and hence are inessential which is a contradiction. Thus  $D''$  is disjoint from  $\rho \sqcup q_1$ .

Now  $\partial Q \subset X^s \subset X^{s,d}$  and hence  $q_2 \subset X^s \subset X^{s,d}$ . While not all of  $q'$  is disjoint from  $\partial X^s$  we know that since  $Q$  is a regular neighborhood of  $q'$ , in  $X^s$ , that  $Q \setminus q' \subset \text{int}(X^s)$ . Thus, since  $q_2$  is disjoint from  $q'$  we know that  $q_2 \subset \text{int}(X^s)$ . We show that  $D'' \subset X^{s,d}$ . Suppose for a contradiction that  $D'' \not\subset X^{s,d}$ . Since  $\partial D'' = q_2 \subset \text{int}(X^s) \subset \text{int}(X^{s,d})$  there exist at least one component of  $\partial X^{s,d} \subset \text{int}(D'')$  and each component bounds a disk and hence is inessential which is a contradiction. Thus,  $Q \cup D''$  is an essential annulus contained in  $X^{s,d}$  with  $\partial(Q \cup D'') = \rho \sqcup q_1$  where  $\rho \subset \partial X^{s,d}$  and  $q_1$  is homotopic to  $\rho' \subset \partial X^{s,d}$ . Since  $X^{s,d}$  is connected we get that  $Q \cup D''$  is homotopic, in  $X^{s,d}$  to  $X^{s,d}$  and hence  $\chi(X^{s,d}) = \chi(Q \cup D'') = 0$  which is a contradiction. Thus,  $q_2$  is essential and hence  $Q$  is essential which proves the claim.

We conclude that  $Q$  is an entirely blue essential pair of pants of type  $T_3$ .  $\square$

### Half of Double

Let  $X$  be the pair of pants found in Lemma 4.8, we assume without loss of generality that  $X \subset \text{int}(B)$ . We consider  $Opp: X \rightarrow DF$ . Let  $\alpha_1, \alpha_2$ , and  $\alpha_3$  be the boundary components of  $X$ . Using the bundle structure from the parallelism of faces we have for each  $\alpha_i \subset \partial X$  that there is an annulus  $A_i \subset DM$  with boundary components  $\alpha_i$  and  $Opp(\alpha_i)$ .

Notation: We write  $\setminus\setminus$  to mean cut open along. We let  $I$  stand for the interval  $[0, 1]$  and let  $S^1$  stand for the circle.

We want to determine what happens to  $X$  when we restrict our problem back to the case with just  $F$  in  $M$  by cutting  $DM$  open along  $\partial M$ . In other words we want to know what  $X \cap F \subset F$  is. Before we look at  $DM \setminus \setminus \partial M$ , however, we want some results about  $X$  and  $Opp(X)$  in  $DM \setminus \setminus DF$ . Using the bundle structure of  $DM$ , which is derived from  $M$ , we get that  $DM \setminus \setminus DF$  is homeomorphic to  $DF \times I$  and we use the two expressions interchangeably.



Consider  $DF \times I$ . Since  $F$  is 2-sided it follows that  $DF$  is 2-sided and hence  $\partial(DF \times I)$  has two copies of  $DF$ ,  $DF \times \{0\}$  and  $DF \times \{1\}$ . We define  $DF_0 := DF \times \{0\}$  and  $DF_1 := DF \times \{1\}$ . Additionally, there are two copies of  $X$ ; one in  $DF_0$  and the other in  $DF_1$ . Suppose without loss of generality that  $DF_0$  is characterized by having a coorientation in  $DM \setminus DF$  which points in while  $DF_1$  is characterized by having a coorientation in  $DM \setminus DF$  which points out. Choose  $X$  to be the copy of  $X \subset DF_0$  so that the  $Opp$  map makes sense. Then,  $Opp(X) \subset DF_0$  or  $Opp(X) \subset DF_1$ . Moreover,  $A_1$ ,  $A_2$ , and  $A_3$  all make sense in  $DF \times I$  and the symmetric map  $\Phi : DM \rightarrow DM$  which we defined earlier will also make sense in  $DF \times I$ .

*Lemma 4.9.* For our choice of  $X \subset DF_0$  we get that  $Opp(X) \subset DF_1$ .

*Proof of Lemma 4.9.* Suppose for a contradiction that  $Opp(X) \subset DF_0$ . For  $i = 1, 2, 3$  let  $f_i : S^1 \rightarrow DF \times I$  such that  $im f_i = \alpha_i$  and  $g_i : S^1 \rightarrow DF \times I$  such that  $im(g_i) = Opp(\alpha_i)$ . Let  $H_i : S^1 \times I \rightarrow DF \times I$  such that  $H_i(s, 0) = f_i(s)$  and  $H_i(s, 1) = g_i(s)$ . We know  $H_i$  exists because of the annulus  $A_i$ . Now let  $P : DF \times I \rightarrow DF \times \{0\}$  be the natural projection, then  $P \circ H_i : S^1 \times I \rightarrow DF \times \{0\}$  is a homotopy of  $f_i$  and  $g_i$  in  $DF_0$ . The existence of a homotopy in the surface  $DF_0$  guarantees that there is an isotopy in  $DF_0$  of  $f_i$  and  $g_i$ . Thus, for  $i = 1, 2, 3$  we get that  $\alpha_i$  is isotopic to  $Opp(\alpha_i)$  in  $DF_0$ .

Since  $X$  is in  $int(B)$  we get that  $Opp(X)$  is disjoint from  $B$  and hence  $X$  and  $Opp(X)$  are disjoint subsurfaces of  $DF_0$ . We get for  $i = 1, 2, 3$  that since  $\alpha_i$  is isotopic to  $Opp(\alpha_i)$  then there exists an annulus,  $Q_i$ , in  $DF_0$  between  $\alpha_i$  and  $Opp(\alpha_i)$  which has interior disjoint from  $X$  and  $Opp(X)$ . Thus we have that  $Q := X \cup Opp(x) \cup Q_1 \cup Q_2 \cup Q_3$  is a closed genus 2 subsurface of  $DF_0$  and hence is equal to  $DF_0$ . From our hypothesis  $\chi(F) \leq -76t(M) + 1$ . So,

$$\begin{aligned} -2 &= \chi(Q) = \chi(DF_0) = \chi(DF) = 2\chi(F) \\ &\leq -152t(M) + 2 \\ &\leq -150 \end{aligned}$$

where the last inequality holds since  $t(M) \geq 1$ . This is a clear contradiction and hence  $Opp(X) \subset DF_1$  as desired.  $\square$

We now go back to studying  $M$ . Recall that  $DM \setminus \partial M$  is homeomorphic to  $M \sqcup M'$ . Using the bundle structure of  $M$  we conclude that  $M \setminus F$  is homeomorphic to  $F \times I$  and use the two expressions interchangeably. Thus, since  $DM \setminus DF$  is homeomorphic to  $DF \times I$  we talk about  $F \times I \subset DF \times I$  instead of  $M \setminus F \subset DM \setminus DF$ . In this setting we consider separately when  $X$  is a pair of pants of type  $T_0, T_1$ , or  $T_3$  in  $DF_0 \subset DF \times I$ .

If  $X$  is a pair of pants of type  $T_0$  in  $DF_0$  then  $\Phi(X)$  is also a pair of pants of type  $T_0$  in  $DF_0$ . Either  $X$  or  $\Phi(X)$  is contained in  $F_0 := F \times \{0\}$ . If  $X \subset F_0$  then we leave it alone, otherwise if  $X \subset DF \setminus F_0$  we redefine  $X$  to be  $\Phi(X)$ . Thus  $X$  is an entirely blue essential pair of pants of type  $T_0$  contained in  $F_0$ . Recall for  $i = 1, 2, 3$  that  $A_i$  is the annulus with  $\alpha_i$  and  $Opp(\alpha_i)$  as boundary components; the  $\alpha_i$  are boundary components of  $X$ . Recall that we are associating  $F \times I$  with  $M \setminus F$ . Since  $X \subset F_0 \subset F \times I$  and  $Opp$  works with tetrahedra which are contained entirely in  $M$  or  $M'$ , it turns out that  $A_i \subset F \times I$  for all  $i$ . By Lemma 4.9 we conclude for  $i = 1, 2, 3$  that since  $\alpha_i \subset F_0$  that  $Opp(\alpha_i) \subset F_1 := F \times \{1\}$ .

*Lemma 4.10.* When  $X \subset F_0$  is an essential pair of pants of type  $T_0$ , as above, then at least one of  $A_1, A_2$ , or  $A_3$  is not boundary parallel to  $\partial(F \times I)$ .

*Proof of Lemma 4.10.* Suppose for a contradiction that  $A_1, A_2$ , and  $A_3$  are all boundary parallel in  $F \times I$ . We know for  $i = 1, 2, 3$  that  $A_i$  has the boundary components  $\alpha_i \subset F_0$  and  $Opp(\alpha_i) \subset F_1$ . Since  $A_i$  is boundary parallel in  $F \times I$  there exists for each  $i$  an annulus  $B_i$  in  $\partial(F \times I)$  with boundary components  $\alpha_i$  and  $Opp(\alpha_i)$ .

We need to show that for each  $i$  that  $int(B_i)$  is disjoint from  $X$  and  $Opp(X)$ . Suppose for now that the annulus  $B_1$  intersects  $int(X)$ ,  $int(Opp(X))$ , or both. Then it must be the case that  $int(X)$ ,  $int(Opp(X))$ , or both are contained in the annulus  $B_1$ . This is because to be boundary parallel  $A_1$  must, with fixed boundary components, be isotopic to a subsurface of  $\partial(F \times I)$  which we are calling  $B_1$  which contains everything in  $\partial(F \times I)$  which is on the same side of  $\partial A_1$

as  $B_1$ . Thus,  $B_1$  contains one or two essential pairs of pants which is a contradiction by Lemma 2.14. Since  $B_1$  was an arbitrary choice we conclude that  $\text{int}(B_i)$  is disjoint from  $X$  and  $\text{Opp}(X)$  for  $i = 2, 3$  as well. We conclude that  $Q := B_1 \cup B_2 \cup B_3 \cup X \cup \text{Opp}(X)$  is a closed genus 2 subsurface of  $\partial(F \times I)$ .

Notice that  $F$  has boundary and hence  $\partial(F \times I)$  is a connected closed surface with  $\chi(\partial(F \times I)) = 2\chi(F)$ . Since  $Q$  is a closed genus 2 subsurface of  $\partial(F \times I)$  we get that  $Q = \partial(F \times I)$ . Thus,

$$\begin{aligned}
-2 &= \chi(Q) \\
&= \chi(\partial(F \times I)) \\
&= 2\chi(F) \\
&\leq 2(-76t(M) + 1), \text{ by the hypothesis of Theorem 1.1} \\
&\leq -150, \text{ since } t \geq 1
\end{aligned}$$

This gives a clear contradiction and we conclude that at least one of  $A_1$ ,  $A_2$ , or  $A_3$  must not be boundary parallel in  $F \times I$ .  $\square$

Using the result of Lemma 4.10 we assume without loss of generality that  $A_1$  is not boundary parallel in  $F \times I$ . Let  $\alpha := \alpha_1 = A_1 \cap F_0$  and so  $\text{Opp}(\alpha) = \text{Opp}(\alpha_1) = A_1 \cap F_1$ .

*Lemma 4.11.*  $\text{Opp}(\alpha)$  is isotopic to  $\alpha \times \{1\}$  in  $F_1$ .

*Proof of Lemma 4.11.* Let  $f : S^1 \rightarrow F \times I$  such that  $\text{im}(f) = \text{Opp}(\alpha)$  and let  $g : S^1 \rightarrow F \times I$  such that  $\text{im}(g) = \alpha$ . Now let  $H : S^1 \times I \rightarrow F \times I$  such that  $H(s, 0) = f(s)$  and  $H(s, 1) = g(s)$  for all  $s \in S^1$ . We know  $H$  exists because of the annulus  $A_1$ . Now let  $P : F \times I \rightarrow F \times \{1\}$  be the natural projection, then  $P \circ H : S^1 \times I \rightarrow F \times \{1\}$  is a homotopy of  $f$  and  $P \circ g$  in  $F_1$ . The restrictions of  $P$  to  $F_0$  and  $F_1$  are homeomorphisms and since  $\text{im}(f)$  and  $\text{im}(g)$  are embeddings we get that  $\text{im}(P \circ f) = \text{im}(f) = \text{Opp}(\alpha)$  and  $\text{im}(P \circ g) = \alpha \times \{1\}$  are embedded. Thus, since  $f$  is homotopic to  $P \circ g$  and both  $\text{im}(f)$  and  $\text{im}(P \circ g)$  are embedded, we get that there exists an isotopy of  $f$

and  $P \circ g$  and hence  $Opp(\alpha)$  is isotopic to  $\alpha \times I$  in  $F_1$ .  $\square$

The next two paragraphs are used to fix some notation for when  $X$  is a pair of pants of type  $T_1$  or  $T_3$ . We use similar notation in order to prove Lemma 4.12 and Lemma 4.13 simultaneously for both types of pants.

If  $X$  is a  $T_1$  pair of pants in  $DF_0$  then in  $F \times I$  we may assume without loss of generality that  $A_1 \cap (F \times I)$  is a disk  $D_1$ ,  $A_2 \cap (F \times I) = A_2$ , and  $A_3 \cap (F \times I)$  is empty. Notice that  $\eta := D_1 \cap \alpha_1 = F_0 \cap \alpha_1$  and  $Opp(\eta) = D_1 \cap Opp(\alpha_1) = F_1 \cap Opp(\alpha_1)$  are arcs in  $F_0$  and  $F_1$  respectively.

Lastly if  $X$  is a  $T_3$  pair of pants in  $DF_0$  then for  $i = 1, 2, 3$  we get that  $D_i := A_i \cap (F \times I) \subset F \times I$  is a disk. Notice that  $a_i := D_i \cap \alpha_i = F_0 \cap \alpha_i$  and  $a'_i := D_i \cap Opp(\alpha_i) = F_1 \cap Opp(\alpha_i)$  are arcs in  $F_0$  and  $F_1$  respectively. We define  $\eta := a_1 = D_1 \cap \alpha_1 = F_0 \cap \alpha_1$  and hence  $Opp(\eta) = a'_1 = D_1 \cap Opp(\alpha_1) = F_1 \cap Opp(\alpha_1)$ .

*Lemma 4.12.*  $\eta$  and  $Opp(\eta)$  are essential arcs in  $F_0$  and  $F_1$  respectively.

*Proof of Lemma 4.12.* Suppose for a contradiction that  $\eta$  is inessential in  $F_0$ . Then there is an arc  $a$  in  $\partial F_0$  such that  $\partial\eta = \partial a$  and  $\eta \cup a$  bounds a disk  $D_\eta \subset F_0$ . We go back to the double  $DF_0$  and notice that this implies that  $\Phi(\eta)$  is inessential in  $\Phi(F_0)$  and we see that  $\Phi(\eta) \cup a$  bounds the disk  $\Phi(D_\eta) \subset \Phi(F_0)$ . We conclude, since  $a$  is a single arc in  $\partial F_0$ , that  $\alpha_1 = \eta \cup \Phi(\eta)$  bounds the disk  $D_\eta \cup \Phi(D_\eta) \subset DF_0$  and hence is inessential which is a contradiction. We can prove, similarly, that  $Opp(\eta)$  must be essential.  $\square$

*Lemma 4.13.*  $Opp(\eta)$  is isotopic to  $\eta \times \{1\}$  in  $F_1$ .

*Proof of Lemma 4.13.* Let  $f : [0, 1] \rightarrow F \times I$  such that  $im(f) = Opp(\eta)$  and let  $g : [0, 1] \rightarrow F \times I$  such that  $im(g) = \eta$ . Now let  $H : [0, 1] \times I \rightarrow F \times I$  such that  $H(s, 0) = f(s)$  and  $H(s, 1) = g(s)$  for all  $s$  in  $[0, 1]$  and such that for all  $t$  in  $I$  we have that  $H(0, t)$  and  $H(1, t)$  are in  $\partial F \times I$ . We know  $H$  exists because of the disk  $D_1$ . Now let  $P : F \times I \rightarrow F \times \{1\}$  be the natural projection, then  $P \circ H : [0, 1] \times I \rightarrow F \times \{1\}$  is a homotopy relative the boundary of  $F_1$  of  $f$  and  $P \circ g$  in  $F_1$ . Now  $P$  restricted to  $F_0$  or  $F_1$  is a homeomorphism and since  $im(f)$  and  $im(g)$  are embeddings we get

that  $im(P \circ f) = im(f) = Opp(\eta)$  and  $im(P \circ g) = \eta \times \{1\}$  are embedded. Thus, since  $f$  is homotopic to  $P \circ g$  and  $im(f)$  and  $im(P \circ g)$  are embedded we get that there exists an isotopy, relative the boundary of  $F_1$ , of  $f$  and  $P \circ g$  and hence  $Opp(\eta)$  is isotopic to  $\eta \times \{1\}$  in  $F_1$   $\square$

### Conclusion

If  $X$  is a pair of pants of type  $T_0$  then Lemma 4.10 implies that  $\alpha$  and  $Opp(\alpha)$  are essential simple closed curves. Recall  $\mu : F \rightarrow F$  from the statement of Theorem 1.1. In  $F \times I$ , which is homeomorphic to  $M \setminus F$ , we get  $\mu(\alpha) = \alpha \times \{1\}$  which, by Lemma 4.11, is isotopic to  $Opp(\alpha)$ . Since  $\alpha \subset int(B)$  (or is in  $int(Y)$ ) we know  $Opp(\alpha)$  is disjoint from  $B$  (or from  $Y$ ) and so  $\mu(\alpha)$  is (possibly after isotopy) disjoint from  $\alpha$ .

If  $X$  is a pair of pants of type  $T_1$  or of type  $T_3$  then Lemma 4.12 implies that  $\eta$  and  $Opp(\eta)$  are essential arcs in  $F_0$  and  $F_1$  respectively. Recall  $\mu : F \rightarrow F$  from the statement of Theorem 1.1. In  $F \times I$ , which is homeomorphic to  $M \setminus F$ , we get  $\mu(\eta) = \eta \times \{1\}$  which by Lemma 4.13 is isotopic relative to the boundary of  $F_0$  to  $Opp(\eta)$ . Since  $\eta \subset int(B)$  (or is in  $int(Y)$ ) we know  $Opp(\eta)$  is disjoint from  $B$  (or from  $Y$ ) and so  $\mu(\eta)$  is (possibly after isotopy) disjoint from  $\eta$ .

From the two preceeding paragraphs we conclude the result of Theorem 1.1.

## References

- [1] Ian Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.
- [2] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.
- [3] R. H. Bing. An alternative proof that 3-manifolds can be triangulated. *Ann. of Math. (2)*, 69:37–65, 1959.
- [4] Andrew J. Casson and Steven A. Bleiler. *Automorphisms of surfaces after Nielsen and Thurston*, volume 9 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1988.
- [5] John B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.
- [6] Albert Fathi, François Laudenbach, and Valentin Poénaru. *Thurston's work on surfaces*, volume 48 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2012. Translated from the 1979 French original by Djun M. Kim and Dan Margalit.
- [7] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [8] Allen Hatcher. Notes on basic 3-manifold topology. <https://www.math.cornell.edu/~hatcher/3M/3Mdownloads.html>, 2007.
- [9] John Hempel. *3-manifolds*. AMS Chelsea Publishing, Providence, RI, 2004. Reprint of the 1976 original.
- [10] Kazuhiro Ichihara, Tsuyoshi Kobayashi, and Yo'av Rieck. Strong cylindricality and the monodromy of bundles. *Proc. Amer. Math. Soc.*, 143(7):3169–3176, 2015.
- [11] William Jaco. *Lectures on three-manifold topology*, volume 43 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, R.I., 1980.
- [12] Michael Kapovich. *Hyperbolic manifolds and discrete groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 2001 edition.
- [13] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.
- [14] Edwin E. Moise. Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung. *Ann. of Math. (2)*, 56:96–114, 1952.
- [15] Ulrich Oertel. On the existence of infinitely many essential surfaces of bounded genus. *Pacific J. Math.*, 202(2):449–458, 2002.

- [16] Jean-Pierre Otal. *The hyperbolization theorem for fibered 3-manifolds*, volume 7 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. Translated from the 1996 French original by Leslie D. Kay.
- [17] Saul Schleimer. Notes on the complex of curves.  
<http://homepages.warwick.ac.uk/~masgar/Maths/notes.pdf>.
- [18] Jennifer Schultens. *Introduction to 3-manifolds*, volume 151 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2014.
- [19] William P. Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.*, 59(339):i–vi and 99–130, 1986.
- [20] Daniel Wise. The structure of groups with a quasiconvex hierarchy.  
<http://www.math.mcgill.ca/wise/papers.html>.
- [21] Daniel T. Wise. *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, volume 117 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2012.